THE AMERICAN MATHEMATICAL MONTHLY

THE OFFICIAL JOURNAL OF
THE MATHEMATICAL ASSOCIATION OF AMERICA
(INCORPORATED)

DEVOTED TO THE INTERESTS OF COLLEGIATE MATHEMATICS

THE AMERICAN MATHEMATICAL MONTHLY, FOUNDED IN 1894 BY BENJAMIN F. FINKEL, WAS PUBLISHED BY HIM UNTIL 1913. FROM 1913 TO 1916 IT WAS OWNED AND PUBLISHED BY REPRESENTATIVES OF FOURTEEN UNIVERSITIES AND COLLEGES IN THE MIDDLE WEST

VOLUME 81

1974

PUBLISHED BY THE ASSOCIATION
MENASHA, WIS., AND WASHINGTON, D. C.

THE AMERICAN MATHEMATICAL MONTHLY

(FOUNDED IN 1894 BY BENJAMIN F. FINKEL)

THE OFFICIAL JOURNAL OF

THE MATHEMATICAL ASSOCIATION OF AMERICA

VOLUME 81 NUMBER 1

CODEN: AMMYAE

CONTENTS

(Continued on inside cover)

JANUARY 1974

NOTICE TO AUTHORS

Research papers per se are unsuitable; see statement of policy (Vol. 81, p. 1).

Please follow the format in current issues of the MONTHLY. Manuscripts must be legibly typewritten or reproduced from typewritten copy, double spaced with wide margins. Three copies should be submitted to the appropriate editor and one kept by the author as protection against loss. The author's full address *must* appear at the end of the manuscript.

Backlog: Main Articles 16 months, Math. Notes 30 months, Research Problems 26 months, Classroom Notes 24 months, Math. Education 18 months.

EDITORIAL CORRESPONDENCE AND MAIN ARTICLES: to ALEX ROSENBERG, Department of Mathematics, Cornell University, Ithaca, N.Y. 14850; NOTES, etc.: to the corresponding Associate Editor; ADVERTISING CORRESPONDENCE: to RAOUL HAILPERN, Mathematical Association of America, SUNY at Buffalo, Buffalo, N. Y. 14214; CHANGE OF ADDRESS and SUBSCRIPTIONS: to A. B. WILLCOX, Mathematical Association of America, 1225 Connecticut Ave., N.W., Washington, D.C. 20036.

Annual dues for members of the Association (including a subscription to the American Mathematical Monthly) are $14.00 for each of the first two years of membership and $18.00 thereafter. Student Membership is available with annual dues of $9.00. For nonmembers the subscription price is $18.00.

PUBLISHED BY THE ASSOCIATION at Washington, D. C., and Menasha, Wisconsin, during the months of January, February, March, April, May, June–July, August–September, October, November, December.

Second-class postage paid at Washington, D. C., and additional mailing offices.

PRINTED IN THE UNITED STATES OF AMERICA

EDITORIAL

With this issue the MONTHLY editorship changes hands. Hopefully, the high standards of our predecessors can and will be maintained. If the quality of the MONTHLY continues to please the readership this will be due to the high calibre of the authors' contributions and the hard and unselfish work of the Associate and Collaborating Editors and the referees.

At all times the Editor invites letters expressing opinions about the contents of the MONTHLY. Criticisms from the readership will certainly be taken into account, but with such a large readership many shades of opinion on all questions are represented amongst our readership.

In this issue a new department, **Queries**, is beginning on an experimental basis. After a period of 6 to 12 months a final decision on whether to make this department permanent will be made.

ALEX ROSENBERG, *Editor*

STATEMENT OF POLICY

Since its beginning, the MONTHLY has tried to be neither a research journal nor a journal devoted primarily to education. The founders of the MONTHLY pledged themselves to advance and promote collegiate mathematics. Following our predecessors, we again endorse this pledge.

More than half of those that receive the MONTHLY teach college mathematics; another large segment of the readership consists of people who apply mathematics in their daily work. It is our view that the readership can best be served by keeping in mind the following points in arriving at editorial decisions:

MAIN ARTICLES. These articles should be of wide interest and preferably of an expository nature. All kinds of mathematical subject matter are welcome but a special effort will be made to obtain articles dealing with various applications of mathematics. Articles that describe how mathematics solved, or can solve, a real-world important problem, or was essential in the construction of an actual physical object will be especially sought after. Of course, articles describing recent progress in pure mathematics and illustrating the great advances of our science will also be featured. An effort will be made to make all articles accessible to readers who have had a good first-year graduate mathematical education, have sufficient Sitzfleisch, and are willing to work.

The MONTHLY will definitely not publish research papers of interest only to the specialists, nor will it deal with what is best described as "minor research." However, an occasional research paper which can be followed by non-specialists and which can be expected to be of wide appeal will be printed.

MATHEMATICAL NOTES. These should in general be short papers of one to four printed pages which give new insights, new proofs of old theorems, or brief bits of mathematical folklore which have not found a home in the literature. Again, in order to merit publication the topics dealt with should be of wide current interest.

RESEARCH PROBLEMS. These should be unsolved problems whose statement uses only those concepts used in undergraduate mathematics and where there is some expectation that progress can be made without using terribly advanced methods.

CLASSROOM NOTES. For this department we hope to obtain brief papers dealing with the subject matter currently taught in *undergraduate* mathematics. Papers dealing with material usually taught in the first year of graduate study will also be accepted occasionally.

MATHEMATICAL EDUCATION. At the present time there is a growing and very healthy concern with mathematicians' role as teachers. This department will feature reports of experiments in novel methods of teaching and also discussions of all educational aspects of our profession.

PROBLEMS. From some points of view this department is the backbone of the MONTHLY. It will continue to publish meritorious elementary and advanced problems in both the classical and modern mathematical sciences.

REVIEWS. Through the telegraphic reviews the MONTHLY will continue to attempt to provide as complete a coverage as possible of the current textbook literature. At present the MONTHLY is the only journal that performs this service for the mathematical community. Extended reviews will, of course continue to appear and hopefully the number of classroom reviews will increase.

The discerning reader will note our indebtedness for the text of this Statement to our immediate predecessor. It is hoped that the future issues of the MONTHLY will continue to reflect the high standards set by him and those that came before him.

ALEX ROSENBERG, *Editor*

COEFFICIENT IDENTITIES FOR POWERS OF TAYLOR AND DIRICHLET SERIES

H. W. GOULD

1. Introduction. Let p be an arbitrary real or complex number and put

$$(1.1)\qquad f(x) = \sum_{n=0}^{\infty} A_n x^n, \; [f(x)]^p = \sum_{n=0}^{\infty} B_n x^n,$$

so that with a function f we associate a Taylor (or power) series with $A_n = f^{(n)}(0)/n!$. The study of properties of B_n has occupied the attention of many scholars, and one of the oldest and most elegant formulas is the recurrence relation

$$(1.2)\qquad \sum_{k=0}^{n} [k(p+1) - n] A_k B_{n-k} = 0, \qquad n \geqq 0.$$

If we suppose that $A_0 \neq 0$, then this yields

$$(1.3)\qquad B_n = \frac{1}{nA_0} \sum_{k=1}^{n} [k(p+1) - n] A_k B_{n-k}, \qquad n \geqq 1,$$

giving a way to calculate as many of the B's as desired. Formula (1.2), stated in tha way or in the form (1.3) or in various other forms, has been known a long time. Thus, in 1748 Euler gave (1.3) (obscurely expressed) in section 76 of his *Introductio* [**4**], [**5**]. Adams and Hippisley [**1**] gave the formula in the form (1.2) (their formula 6.361). Formula (1.3) is given as formula (0.314) in any of the several editions (Russian original, German, or English) of Ryshik and Gradstein [**16**], where, however, p is restricted to be a natural number. Thinking to remove this restriction of Ryshik and Gradstein, von Holdt [**17**] published still another derivation using properties of double sums to establish that (1.2) holds true for rational real p. His derivation avoids use of differentiation of the series in (1.1). We mention this because differentiation of formal power series affords a quick proof of (1.2) and has been used before.

The basic recurrence relation is not as widely known as it should be, and has been rediscovered repeatedly. Thus Barrucand [**2**] found (1.2) again and made applications of it. Cappellucci [**18**] found it in the form (1.3) but attributed it to Hansted [**19**]. Hindenburg [**12**], in his treatise on the multinomial theorem (p. 291) gave (1.3) in a perfectly obscure notation. Many other references could be cited.

Actually, the recurrence is implicit in still another class of formulas widespread in the literature, stemming from the early work of Hindenburg's student Rothe [**15**]. I myself have written a number of papers, e.g., [**6**]–[**11**] having to do with a formula of Rothe and its consequences for combinatorics, special functions and number theory. What we shall do here is to derive the formulas again and put them in a

logical order, and then pass on to a related matter, namely, that very similar recurrences hold for powers of Dirichlet series. These seem not at all to be known in any readily accessible literature. Yet they are just as easily derived from the same identity we shall use for finding (1.2) and the more general formula of Rothe. In passing we make some remarks about formal power series. The last section raises conjectures only.

2. Formal power series. Since the sum of an infinite series is really not used, our viewpoint can be either rigorous or formal. The reader may wish to consult Niven's elegant expository paper [**14**] for the theory of formal power series. This is one of the few readily available papers giving a derivation of the properties and showing that one may manipulate formal power series to determine identities between the coefficients of them. Most books and papers on combinatorics, special functions, etc., merely avoid the matter by waving of hands and perhaps give a remark that it was all justified by E. T. Bell in various papers and a book (1923, 1927, 1940). Actually, the first question about how to justify the techniques of formal power series seems to have occurred in a problem [**20**] posed by Bell and not solved until eight years later by Otto Dunkel. Dunkel's solution made a nice application of the formula of Leibniz for higher derivatives of a product of two functions. Here was Bell's problem: Given an identity $A(x)B(x) = C(x)$ and formal power series

$$A(x) = \sum_{n=0}^{\infty} a_n x^n,\ B(x) = \sum_{n=0}^{\infty} b_n x^n,\ C(x) = \sum_{n=0}^{\infty} c_n x^n.$$

How do we justify comparison of coefficients in the products of the series to obtain the identity

$$c_n = \sum_{k=0}^{n} a_k b_{n-k}, \tag{2.1}$$

even when the series fails to converge? The solution arises as follows. By supposing that we have formal power series we suppose that A, B, C, have derivatives of all orders at 0, and what is more $a_n = A^{(n)}(0)/n!$, $b_n = B^{(n)}(0)/n!$, $c_n = C^{(n)}(0)/n!$. Now we examine $D^n(AB)|_{x=0} = D^nC|_{x=0}$. We apply the Leibniz formula on the left, i.e.,

$$\sum_{k=0}^{n} \binom{n}{k} A^{(k)}(0)B^{(n-k)}(0) = C^{(n)}(0),$$

and simplify, getting formula (2.1). So the only way infinite series might enter in is that they afford a quicker way to manipulate the identity and write down the result without going through the work with the Leibniz formula. The objection to this method is that one must start from a known identity $AB = C$. In other words, we can start with given functions having derivatives of all orders and proceed as above, but suppose we start with the numbers a_n, b_n, c_n and imagine we have functions A, B, C. How do we justify the work then? One way to do it is to invoke a little-known

theorem of Borel [24] who proved in his dissertation that for any given sequence of real numbers $\{a_n\}$ there exists a function f for which the $\{a_n\}$ are the Taylor coefficients, i.e., such that $D_x^n f(x)|_{x=0} = n!\, a_n$. This amounts to saying that the derivatives of a real function can be assigned arbitrarily at the origin. Incidentally, the corresponding statement for an arbitrary given sequence of complex numbers is false and counterexamples have been given! One of the few analysis texts citing Borel's theorem is Boas [25]. The theorem is surely not widely known. Using Borel's theorem it is possible to set up a correspondence between sets of arbitrary real sequences and functions that must exist, whether their corresponding Taylor series converge or not, and show that the usual operations on the series (formal) correspond to the expected operations on the functions, using such operations as addition, multiplication, differentiation, and integration.

Here is an example. Let $a_n = n!$ and

$$f(x) = \sum_{n=0}^{\infty} n!\, x^n.$$

The series converges nowhere except at $x = 0$. However, Borel's theorem shows that a function exists with $n!$ being the Taylor coefficient. Then one can consider $D_x[f(x)]^2 = 2f(x)Df(x)$ and obtain

$$D_x \sum_{n=0}^{\infty} x^n \sum_{k=0}^{n} k!\,(n-k)! = 2 \sum_{n=0}^{\infty} n!\, x^n \sum_{k=0}^{\infty} k\,k!\, x^{k-1}$$

or

$$\sum_{n=1}^{\infty} n x^{n-1} \sum_{k=0}^{n} k!\,(n-k)! = 2 \sum_{n=1}^{\infty} x^{n-1} \sum_{k=0}^{n} k\,k!\,(n-k)!,$$

which yields the correct identity

$$\frac{n}{2} \sum_{k=0}^{n} 1\Big/\binom{n}{k} = \sum_{k=0}^{n} k\Big/\binom{n}{k}.$$

It is intended to develop this approach to formal power series in a later paper. Rosenthal [26] has shown how to construct such Borel functions.

We also mention Carlitz's paper [21] for its exhibit of interesting uses of the theory of formal power series and the elegant paper of Fine [22] which makes generating functions and formal power series accessible to advanced high school students.

3. Derivation of (1.2). We start with the identity

$$D_x[f(x)]^{p+1} = (p+1)[f(x)]^p D_x f(x) = D_x\{f(x)\cdot[f(x)]^p\}. \tag{3.1}$$

On the one hand we have

$$(p+1)f^pDf = (p+1)\sum_{n=0}^{\infty} B_nx^n \cdot \sum_{k=0}^{\infty} kA_kx^{k-1} = \sum_{n=1}^{\infty} x^{n-1}\sum_{k=0}^{n} k(p+1)A_kB_{n-k},$$

while on the other hand we have

$$f\cdot f^p = \sum_{n=0}^{\infty} A_nx^n \cdot \sum_{k=0}^{\infty} B_kx^k = \sum_{n=0}^{\infty} x^n \sum_{k=0}^{n} A_kB_{n-k},$$

whence

$$D(f\cdot f^p) = \sum_{n=1}^{\infty} n\,x^{n-1}\sum_{k=0}^{n} A_kB_{n-k}.$$

Equating coefficients, we have proved (1.2).

We saw that (1.3) followed upon supposing $A_0 \neq 0$. But suppose $A_0 = 0$, and $A_1 \neq 0$. If we also suppose p to be a positive integer, we can obtain the following. First of all, it becomes evident that $B_0 = B_1 = \cdots = B_{p-1} = 0$, and $B_p = A_1^p$. Therefore (1.2) becomes

$$\sum_{k=1}^{n-p} [k(p+1) - n]\,A_kB_{n-k} = 0,$$

and on putting $n - k$ for k this says

$$\sum_{k=p}^{n-1} [(n-k)(p+1) - n]A_{n-k}B_k = 0.$$

Next, replace n by $n+1$ and solve for B_n. We get

$$(3.2)\quad B_n = \frac{1}{(n-p)A_1}\sum_{k=p}^{n-1} [(n+1-k)(p+1) - (n+1)]\,A_{n+1-k}B_k, \qquad n \geqq p+1.$$

This formula also is not new.

We may extend this by next supposing that $A_0 = A_1 = \cdots = A_{r-1} = 0$, $A_r \neq 0$, whence $B_0 = B_1 = \cdots = B_{rp-1} = 0$, $B_{rp} = A_r^p$, for p a positive integer. The result this time is that we have

$$(3.3)\qquad B_n = \frac{1}{(n-rp)A_r}\sum_{k=rp}^{n-1} [(n+r-k)(p+1) - (n+r)]\,A_{n+r-k}B_k.$$

Observe that when $r = 0$ this agrees with (1.3) again. This formula also is not new.

If we now adopt a slightly more suggestive notation we may find a tie-in with other interesting relations in the mathematical literature.

4. A more general formula. We now write

$$(4.1)\qquad [f(x)]^p = \sum_{n=0}^{\infty} A_n(p)\,x^n,$$

so that $A_n(p) = B_n$, except that $A_n(1) = A_n$, in the previous notation.

From the identity $[f(x)]^a \cdot [f(x)]^c = [f(x)]^{a+c}$, we have by multiplying and using the Cauchy product,

$$\sum_{k=0}^{n} A_k(a)A_{n-k}(c) = A_n(a+c), \tag{4.2}$$

so that here is a fundamental addition theorem which the coefficients must satisfy. This of course is also an old formula.

Now consider the identity

$$aD_x[f(x)]^{a+c} = (a+c)[f(x)]^c D_x[f(x)]^a. \tag{4.3}$$

This yields

$$aD_x \sum_{n=0}^{\infty} A_n(a+c)x^n = (a+c)\sum_{n=0}^{\infty} A_n(c)x^n \cdot D_x \sum_{k=0}^{\infty} A_k(a)x^k,$$

or

$$a\sum_{n=1}^{\infty} n\,A_n(a+c)x^{n-1} = (a+c)\sum_{n=1}^{\infty} x^{n-1}\sum_{k=0}^{n} k\,A_k(a)A_{n-k}(c),$$

or consequently the identity

$$(a+c)\sum_{k=0}^{n} k\,A_k(a)A_{n-k}(c) = an\,A_n(a+c), \qquad n \geqq 0. \tag{4.4}$$

Recalling (4.2), we combine them and the result is the elegant addition formula (quoted by me in [7])

$$\sum_{k=0}^{n} (p+qk)A_k(a)A_{n-k}(c) = \frac{p(a+c)+qan}{a+c}A_n(a+c), \qquad n \geqq 0. \tag{4.5}$$

The formula in this form was found by Rothe [15] and its application to series of binomial coefficients was an inspiration of much of my own work. Two special cases of (4.5) occur most commonly.

The first case is the Rothe-type series

$$z^a = \sum_{n=0}^{\infty} \frac{a}{a+bn}\binom{a+bn}{n}x^n, \; x = \frac{z-1}{z^b}. \tag{4.6}$$

This gives the generalized Vandermonde formula

$$\begin{aligned}\sum_{k=0}^{n} \frac{a}{a+bk}\binom{a+bk}{k}\frac{c}{c+b(n-k)}\binom{c+b(n-k)}{n-k}\\ = \frac{a+c}{a+c+bn}\binom{a+c+bn}{n},\end{aligned} \tag{4.7}$$

with a corresponding form for general p and q in (4.5).

The other most common case is the Abel-type series

$$(4.8)\qquad e^{az} = \sum_{n=0}^{\infty} a\frac{(a+bn)^{n-1}}{n!}x^n, \qquad x = ze^{-bz},$$

which gives Abel's extension of the binomial theorem (occurring when $b = 0$)

$$(4.9)\quad \sum_{k=0}^{n} a\frac{(a+bk)^{k-1}}{k!}c\frac{(c+b(n-k))^{n-k-1}}{(n-k)!} = (a+c)\frac{(a+c+bn)^{n-1}}{n!}$$

with a corresponding more general form by (4.5).

Relations of type (4.7) and (4.9) occur frequently in combinatorics, graph theory, statistics, special functions, etc. See references [6]–[11] and [27]–[29]. Ettingshausen [3] finds (4.2) and (4.5) and generalizes.

To see that (4.5) is more general than (1.2), we have but to set $p = -n$, $aq = a + c$, so that $p(a+c) + qan = 0$, and then when $a + c \neq 0$, we have by (4.5), when $a \neq 0$,

$$(4.10)\qquad \sum_{k=0}^{n}\left[-n + k\left(\frac{c}{a}+1\right)\right] A_k(a)A_{n-k}(c) = 0, \qquad n \geqq 0.$$

This says then that when $a = 1$

$$(4.11)\qquad \sum_{k=0}^{n} [k(c+1) - n]A_k(1)A_{n-k}(c) = 0, \qquad n \geqq 0, \qquad c \neq -1.$$

In this sense then, (4.5) includes (1.2).

Here are special instances of (1.2) or (4.11):

$$(4.12)\qquad \sum_{k=0}^{n} [k(a+1) - n]A_k(r,b)A_{n-k}(ra,b) = 0, \qquad n \geqq 0,$$

where

$$A_k(r,b) = \frac{r}{r+kb}\binom{r+kb}{k};$$

$$(4.13)\quad \sum_{k=0}^{n} [k(a+1) - n]\binom{r}{k}\binom{ra}{n-k} = 0; \qquad \text{(case } b = 0 \text{ in above).}$$

This is related to formula (3.152) in [29].

$$(4.14)\qquad \sum_{k=0}^{n} [k(a+1) - n]\, B_k(r,b)B_{n-k}(ra,b) = 0, \qquad n \geqq 0,$$

where $B_k(r,b) = r(r+bk)^{k-1}/k!$;

$$(4.15)\qquad \sum_{k=0}^{n} [k(a+1) - n]\, g_k^r(x,h)g_{n-k}^r(ax,ah) = 0, \qquad n \geqq 0,$$

where the g's are generalized Hermite polynomials defined by

$$e^{tx+ht^r} = \sum_{n=0}^{\infty} \frac{t^n}{n!} g_n^r(x, h).$$

These were introduced by Bell and studied in [**27**] and several dozen papers have appeared treating these or related functions.

$$(4.16)\quad \sum_{k=0}^{n} [k(p+1) - n] P_k(m, x, y, 1, C) P_{n-k}(m, x, y, p, C) = 0, \qquad n \geqq 0,$$

where the P's are generalized Humbert polynomials defined in [**28**].

5. Dirichlet Series. Let us now define (with s a real number)

$$(5.1)\qquad f(s) = \sum_{n=1}^{\infty} \frac{A_n}{n^s},\ [f(s)]^p = \sum_{n=1}^{\infty} \frac{A_n(p)}{n^s},\ A_n = A_n(1).$$

The necessary theory of convergence for Dirichlet series may be found in Landau [**13**]. A Dirichlet series may converge everywhere ($A_n = 1/n!$) or nowhere ($A_n = n!$). Between these extremes, there exists for other Dirichlet series a constant m such that the series diverges for $s < m$ and converges for $s > m$. See Landau [**13**, Vol. I, p. 104] for proof. If the series for $f(s)$ in (5.1) converges, then the series with A_n replaced by $A_n \log n$ also converges. This means that the series can be differentiated term by term in its interval of convergence. More important now, for purposes of coefficient comparisons, we need the theorem in Vol. I, p. 133 of Landau. This states that if for $s > s_0$,

$$\sum_{n=1}^{\infty} \frac{A_n}{n^s} = \sum_{n=1}^{\infty} \frac{B_n}{n^s},$$

then $A_k = B_k$ identically. This uniqueness of expansion allows us to carry through the same kind of calculations we used in finding (1.2) and (4.5).

First of all, using the identity (3.1) in the form we need with x replaced by s (to conform to the notational traditions), we find

$$D_s[f(s)]^{p+1} = (p+1) \sum_{n=1}^{\infty} \frac{A_n(p)}{n^s} D_s f(s) = -(p+1) \sum_{n=1}^{\infty} \frac{A_n(p)}{n^s} \sum_{k=1}^{\infty} \frac{A_k \log k}{k^s},$$

$$= -(p+1) \sum_{n=1}^{\infty} \frac{1}{n^s} \sum_{d|n} A_d (\log d) A_{n/d}(p).$$

On the other hand we have

$$D_s\{f(s)[f(s)]^p\} = D_s \sum_{n=1}^{\infty} \frac{1}{n^s} \sum_{d|n} A_d A_{n/d}(p) = -\sum_{n=1}^{\infty} \frac{\log n}{n^s} \sum_{d|n} A_d A_{n/d}(p).$$

Equating coefficients, we have

$$\log n \sum_{d|n} A_d A_{n/d}(p) = (p+1) \sum_{d|n} A_d A_{n/d}(p) \log d,$$

which we can write in the desired form

$$(5.2) \qquad \sum_{d|n} [(p+1)\log d - \log n] A_d A_{n/d}(p) = 0, \qquad n \geqq 1,$$

which is the exact analogue of (1.2).

We shall not take the space to exhibit the formulas, but the reader can easily verify that formulas analogous to (1.3), (3.2) and (3.3) may be written down at once.

Now, in exactly the same way that the Cauchy product of power series gave us (4.2), we see that the Dirichlet product gives us at once

$$(5.3) \qquad \sum_{d|n} A_d(a) A_{n/d}(c) = A_n(a+c)$$

as an elegant addition theorem. This too is a well-known formula.

The identity (4.3) (again with x replaced by s) may be used now. We find

$$\sum_{k=1}^{\infty} \frac{A_k(c)}{k^s} \cdot D_s \sum_{n=1}^{\infty} \frac{A_n(a)}{n^s} = \frac{a}{a+c} D_s \sum_{n=1}^{\infty} \frac{A_n(a+c)}{n^s},$$

whence

$$\sum_{n=1}^{\infty} \frac{1}{n^s} \sum_{d|n} (\log d) A_d(a) A_{n/d}(c) = \frac{a}{a+c} \sum_{n=1}^{\infty} \frac{1}{n^s} A_n(a+c)(\log n),$$

and consequently, we have the identity

$$(5.4) \qquad \sum_{d|n} (\log d) A_d(a) A_{n/d}(c) = \frac{a}{a+c} A_n(a+c) \log n,$$

which parallels (4.4). Next, we complete the argument by combining (5.3) and (5.4) to obtain the most general formula

$$(5.5) \quad \sum_{d|n} (p + q\log d) A_d(a) A_{n/d}(c) = \frac{p(a+c) + qa\log n}{a+c} A_n(a+c), \qquad n \geqq 1,$$

which is the exact analogue of (4.5). The writer has not seen this formula explicitly stated in the literature, but can make no absolute claim of novelty. However, such formulas certainly seem unknown.

To give an interesting example of (5.5) parallel to examples (4.7), (4.9), etc., we consider the following number-theoretic function: Let $\tau(n,a)$ = the number of ways of expressing n as the product of a positive factors (of which any number may be unity), expressions in which the order of the factors is different being regarded as distinct. In particular, $\tau(n,2) = \tau(n)$, the number of divisors of n. The function is defined in this manner by Hardy and Wright [**30**, p. 254]. For this function Hardy and Wright prove that

$$(5.6) \qquad \zeta^a(s) = \sum_{n=1}^{\infty} \frac{\tau(n,a)}{n^s}, \qquad s > 1,$$

where ζ is the ordinary Zeta function defined by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \qquad s > 1.$$

Formula (5.5) applies then with $A_n(a) = \tau(n, a)$. Here is an example of its application. Let $a = c = 2$. We have then

$$\sum_{d|n} (p + q \log d)\tau(d)\tau(n/d) = \frac{2p + q \log n}{2} \tau(n, 4). \tag{5.7}$$

Or again, with $a = 1$, $c = 3$, $n = 12$, it is easy to verify that each side of (5.5) yields $40p + 10q \log 12$.

It is a bit hard to cite a variety of examples where we know the explicit coefficients in a power of a Dirichlet series. The reason seems to be in the nature of the Dirichlet series. In contrast to the ease with which we may write down formal Taylor series by just dashing off the higher derivatives of functions, there is no correspondingly simple way to calculate the coefficients in a Dirichlet series. For the more general type of Dirichlet series, which reduces to (5.1) when $\lambda_n = \log n$,

$$f(s) = \sum_{n=1}^{\infty} A_n \exp(-\lambda_n s), \tag{5.8}$$

Landau [**13**, Vol. 2, Satz 35, p. 788] proves that if this series is convergent for $R(s) > a$, and if $b > a$ are fixed real numbers, then for integral $m \geqq 1$ we have, if we assume $1/(\lambda_{n+1} - \lambda_n) = O(\exp\exp \lambda_n \delta)$ for $\delta > 0$,

$$\lim_{\omega \to \infty} \frac{1}{2\omega} \int_{-\omega}^{\omega} \exp(\lambda_m t i) f(b + ti)\, dt = A_m \exp(-\lambda_m b),$$

and that this limit exists uniformly for all integers $m \geqq 1$. Thus, in principle, we have a way to relate the Dirichlet coefficients to the function, but the way involves contour integration.

Speaking of the difficulties attendant to working with Dirichlet series, Helson [**23**] has written "In Fourier series people stopped long ago trying to sum series to functions, and began instead to use functions to generate series. This change in point of view set a new direction that the subject has followed ever since. The classical subject of Dirichlet series, on the contrary, has never escaped from its dependence on contour integration, and could not make that shift. My optimistic hope is that I have found a way round the difficulty so that Dirichlet series can follow power series to the center of algebraic analysis."

6. Interplay of Taylor and Dirichlet? The similarities between the formulas we have examined might lead us to think that there exists some general result which includes each. Again we come to a question raised by E. T. Bell [**31**]. Bell wanted to find out if there is some kind of abstract multiplication of series that is neither

additive nor multiplicative only. We know that for power series we have

$$\sum_{n=0}^{\infty} A_n x^n \cdot \sum_{k=0}^{\infty} B_k x^k = \sum_{n=0}^{\infty} x^n \sum_{k+j=n} A_k B_j, \tag{6.1}$$

where the sum is over all non-negative integers k,j such that $k+j=n$. For the Dirichlet series, however, we have

$$\sum_{n=1}^{\infty} \frac{A_n}{n^s} \cdot \sum_{k=1}^{\infty} \frac{B_k}{k^s} = \sum_{n=1}^{\infty} \frac{1}{n^s} \sum_{kj=n} A_k B_j, \tag{6.2}$$

where the sum is now over all positive integers k,j such that $kj=n$. Bell asked whether one could find $f(x,n)$ so that the series

$$\sum_{n=0}^{\infty} A_n f(x,n)$$

contains both the power series and the Dirichlet series type behavior. He was led to the functional equation

$$f(x,a)f(x,b) = f(x,\phi(a,b)),$$

ϕ being an associative operation, where $\phi(a,b)$ is an integer $\geqq 0$ whenever a,b are integers $\geqq 0$, and $\phi(a,b)=\phi(b,a)$. He considered the case that ϕ is a polynomial, but so far as I know no one has resolved the general questions he asked. As Bell remarked "throughout the theory of numbers there is a sharp and generally impassable break between additive and multiplicative properties." Goldbach's conjecture is one of those intractable problems combining additive and multiplicative properties. It would be interesting if any new light could be shed on Bell's problem.

We mention this for still another reason. A. H. Clifford, who is eminently well-known for his work with G. B. Preston [**34**] in the algebraic theory of semigroups, took his doctorate under Bell and Morgan Ward, both of whom were inveterate combinatorialists, number theorists, persons well versed in operations with series. Bell characterized his problem of Taylor-Dirichlet interplay by use of functional equations. Functional equations, as everyone knows, are quite important in the study of abstract algebraic systems, from the abstract equation of associativity $\phi(a,\phi(b,c)) = \phi(\phi(a,b),c)$ onward. A statement of the results in Clifford's dissertation is found in [**33**]. What is more, Bell, in his book [**32**] used what he called a *semigroup* (what we today call a right-cancellative semigroup) to study the algebra of formal power series and the umbral calculus of Blissard and Lucas. It would be interesting indeed if the results of the modern theory of algebraic semigroups could be applied to weld together the Taylor and Dirichlet multiplications into a useful tool such as may (or may not) exist, such is the state of knowledge on the subject. At any rate, we suggest it as a territory still to be explored. New examples of operations satisfying properties like (4.5) and (5.5) would surely be of interest. Mettauer [**35**], when a student at West Virginia, studied the Bell problem of functional equations for a

Taylor-Dirichlet interplay, and though he did not obtain the general solution, did relax Bell's restriction to polynomial ϕ and obtain other kinds of multiplication.

The formulas we have collected together here have been of use to the author for actual numerical calculations with series and also in the study of combinatorial identities.

References

1. E. P. Adams and R. L. Hippisley, Smithsonian Mathematical Formulae and Tables of Elliptic Functions, Smithsonian Miscellaneous Collections, Vol. 74, No. 1, Publication No. 2672, Washington D.C., 1922.

2. Pierre Barrucand, Sur une formule générale de récurrence et quelques-unes de ses applications, C. R. Acad. Sci. Paris, 264 (1967) A792–A794.

3. A. von Ettingshausen, Die combinatorische Analysis, Wien, 1826.

4. L. Euler, Introductio in Analysin Infinitorum, Lausanne, 1748.

5. ———, Einleitung in die Analysis des Unendlichen, aus dem Lateinischen übersetzt von J. A. C. Michelsen, Berlin, 2 Vols,. 1788.

6. H. W. Gould, Some generalizations of Vandermonde's convolution, this MONTHLY, 63 (1956) 84–91.

7. ———, Final analysis of Vandermonde's convolution, this MONTHLY, 64 (1957) 409–415.

8. ———, Generalization of a theorem of Jensen concerning convolutions, Duke Math. J., 27 (1960) 71–76.

9. ———, A series transformation for finding convolution identities, Duke Math. J., 28 (1961) 193–202.

10. ———, A new convoiution formula and some new orthogonal relations for inversion of series, Duke Math. J., 29 (1962) 393–404.

11. H. W. Gould and J. Kaucký, Evaluation of a class of binomial coefficient summations, J. Combinatorial Theory, 1(1966) 233–247. Errata, ibid., Ser. A, 12 (1972) 309–310.

12. C. F. Hindenburg, Der polynomische Lehrsatz, das wichtigste Theorem der ganzen Analysis, Leipzig, 1796.

13. E. Landau, Handbuch der Lehre von der Verteilung der Primzahlen, Leipzig, 1909. Chelsea, New York, Reprint, 1953, 2 Vols.

14. I. Niven, Formal power series, this MONTHLY, 76 (1969) 871–889. See Math. Reviews, 40 (1970), # 5606.

15. H. A. Rothe, Formulae de serierum reversione demonstratio universalis signis localibus combinatorio-analyticorum vicariis exhibita, Ph. D. Diss., Leipzig, 1793.

16. I. M. Ryshik and I. S. Gradstein, Summen-, Produkt- und Integral-Tafeln, VEB Deutscher Verlag der Wissenschaften, Berlin, 1963.

17. R. E. von Holdt, Rational powers of power series, this MONTHLY, 72 (1965) 740–743.

18. T. J. Cappellucci, Automatic computation of Taylor's series which are powers of Taylor's series, M. S. Thesis, Agric. and Mech. Coll. of Texas, 1963.

19. B. Hansted, Nogle bemaerkninger om bestemmelsen af koefficienterne i m^{te} potens af en potensraekke, Tidskrift for Matematik, (4) 5 (1881), 12–16.

20. Problem 2672, this MONTHLY, 25 (1918) 74; 32 (1925), 385; Solution, 33 (1926), 525–526.

21. L. Carlitz, Generating functions, Fibonacci Quarterly, 7(1969), No. 4, 359–393. See Math. Reviews, 41(1971), # 8254.

22. N. J. Fine, Generating functions, pp. 355–367 in 28th Yearbook, 1963, of the National Council of Teachers of Mathematics.

23. Henry Helson, Foundations of the theory of Dirichlet series, Acta Mathematica, 118 (1967) 61–77.

24. É. Borel, Sur quelques points de la théorie des fonctions, Ann. École Norm., Paris, (3) 12 (1895) 9–55. Note esp. p. 44.

25. R. P. Boas, A Primer of Real Functions, Carus Monograph No. 13, Math. Assoc. of America, 1960. Note esp. p. 154.

26. A. Rosenthal, On functions with infinitely many derivatives, Proc. Amer. Math. Soc., 4 (1953), 600–602.

27. H. W. Gould and A. T. Hopper, Operational formulas connected with two generalizations of Hermite polynomials, Duke Math., J., 29 (1962) 51–63.

28. H. W. Gould, Inverse series relations and other expansions involving Humbert polynomials, Duke Math. J., 32 (1965) 697–711.

29. ———, Combinatorial Identities, A Standarized Set of Tables Listing 500 Binomial Coefficient Summations, Morgantown, W. Va., Revised Edition, 1972.

30. G. H. Hardy and E. M. Wright, An Introduction to the Theory of Numbers, Oxford, 3rd Edition, 1954.

31. E. T. Bell, Possible types of multiplication of series, this MONTHLY, 37 (1930) 484–485.

32. ———, Algebraic arithmetic, Amer. Math. Soc. Colloq. Publ. VII, 1927, esp. pp. 124–134.

33. A. H. Clifford, Arithmetic and ideal theory of abstract multiplication, Bull. Amer. Math. Soc., 40 (1934) 326–330.

34. A. H. Clifford and G. B. Preston, The Algebraic Theory of Semigroups, Mathematical Surveys, No. 7, Amer. Math. Soc., Providence, R. I., Vol. I, 1961; Vol. II, 1967.

35. J. C. Mettauer, Unpublished remarks on Bell's functional equation, 1966.

DEPARTMENT OF MATHEMATICS, WEST VIRGINIA UNIVERSITY, MORGANTOWN, W.V. 26506.

COMPLEX ITERATED RADICALS

HANNS-WALTER ROHDE

1. Introduction. Let D be a simply-connected domain of the z-plane, $\{a_n(z)\}_1^\infty$ a sequence of functions which are holomorphic in D, and $\{r_n\}_1^\infty$ a sequence of complex numbers.

Our interest is in the limiting behavior, as $n \to \infty$, of an associated sequence of functions $\{f_n(z)\}_1^\infty$, $z \in D$, where

$$f_0(z) = 0,\ f_n(z) = \{a_n(z) + f_{n-1}(z)\}^{r_n} \quad (n \geqq 1), \tag{1}$$

which we call a **sequence of iterated powers.**

We will assume that

$$a_n(z) + f_{n-1}(z) \neq 0 \qquad (n \geqq 1,\ z \in D). \tag{2}$$

For some $z_0 \in D$ we prescribe one of the values of the power $\{a_n(z_0) + f_{n-1}(z_0)\}^{r_n}$, $(n \geqq 1)$, to be the value of $f_n(z)$ at the point z_0. Then $f_n(z)$ is uniquely determined as a single-valued holomorphic function in D according to the Monodromy theorem

(cf. [**1**, p. 335]). We note that (2) need not be assumed for those n for which r_n is a nonnegative integer.

If we set $r_n = 1$ we obtain $f_n(z) = \sum_{m=1}^{n} a_m(z)$; hence $f_n(z)$ is the nth partial sum of an infinite series. If $r_n = -1$ we have a limit process similar to continued fractions. Hence iterated powers are a generalization of well-known limit processes.

Our main interest, however, is in the case $r_n = \frac{1}{2}$, where we want to solve some problems raised by C. S. Ogilvy [**3**] concerning the convergence of such complex iterated radicals.

If we replace the holomorphic functions $a_n(z)$ in (1) by positive real numbers a_n and choose $r_n = \frac{1}{2}$, always taking the positive square root, the corresponding problem was handled by A. Herschfeld [**2**], who proved that for the convergence of $\{f_n\}_1^\infty$ it is necessary and sufficient that $\{a_n\}_1^\infty$ converges, in which case

$$\lim_{n\to\infty} f_n = \begin{cases} \frac{1}{2}(1+\sqrt{1+4a}) & \text{if } a \neq 0 \\ 1 & \text{if } a = 0 \text{ and at least one } a_n \neq 0 \\ 0 & \text{if all } a_n = 0, \end{cases}$$

where $a = \lim_{n\to\infty} a_n$. (Here $\sqrt{w}$ always means the nonnegative square root of the real number $w \geqq 0$.)

The general case does not seem to have been attacked. A. Herschfeld [**2**, p. 428] wrote: "It is also of interest to generalize infinite radicals to include negative or complex elements. Convergence questions appear to become very difficult in such cases." We shall see, however, that such problems may be treated in a simple way by means of a classical theorem due to Vitali (see Section 2) and the Monodromy theorem.

We begin our considerations by studying the convergence of general iterated powers, especially iterated radicals. These tools enable us to attack several of Ogilvy's problems in Section 3. This part might also provide a good exercise in the technique of choosing suitable branches of the root. Finally we show by an example that we can use our method even to study divergent sequences of iterated radicals.

I should like to thank Professor D. Drasin and the late Professor G. R. MacLane for their valuable suggestions.

2. Vitali's theorem and convergence of iterated powers. The following result is known as Vitali's theorem (cf. [**4**, p. 169]).

THEOREM. *Let $\{f_n(z)\}_1^\infty$ be holomorphic in a domain D and $\{|f_n(z)|\}_1^\infty$ be uniformly bounded on compact subsets of D. Suppose*

$$\lim_{n\to\infty} f_n(z_k) \qquad (k = 1, 2, 3, \cdots) \tag{3}$$

exists on a sequence of points $z_k \in D$ having an accumulation point in D.

Then $\{f_n(z)\}_1^\infty$ converges uniformly on compact subsets of D to a limit function

$f(z)$, holomorphic in D, which, of course, is determined by its values at $z = z_k$:

$$f(z_k) = \lim_{n\to\infty} f_n(z_k).$$

This theorem immediately gives the following result concerning our problem:

Convergence principle. *To prove the convergence of the sequence of functions $\{f_n(z)\}_1^\infty$ defined in* (1) *we only have to show that these functions are uniformly bounded on every compact subset of D and satisfy* (2) *and* (3).

To find the limit-function $f(z)$, we only have to seek the holomorphic function which has the values $\lim_{n\to\infty} f_n(z_k)$ at the points z_k.

For definiteness we now suppose that $r_n = \frac{1}{2}$, $(n \geqq 1)$, although the reader should be able to weaken this restriction in many cases.

In this special case we may replace the assumption that $\{|f_n(z)|\}_1^\infty$ is bounded by one in terms of the original sequence $\{a_n(z)\}_1^\infty$. Indeed, with $r_n = \frac{1}{2}$ in (1) we get

$$a_n(z) = f_n^2(z) - f_{n-1}(z).$$

Hence the condition

(4) $\qquad a_n(z) \to a(z)$ *uniformly on every compact subset K of D*

is necessary for the uniform convergence of $f_n(z)$ on K.

From (4) follows

$$|a_n(z)| \leqq A_K = \text{const for } z \in K,\ n = 1,2,3,\cdots$$

By induction we see

$$|f_n(z)| \leqq F_n \ (z \in K), \text{ where } F_n = \sqrt{A_K + F_{n-1}}$$

Since $\{F_n\}_1^\infty$ converges on account of A. Herschfeld's result cited above, $\{f_n(z)\}_1^\infty$ is uniformly bounded on compact subsets of D. Thus we can replace this assumption by (4).

Furthermore (2) may be rewritten as

(5) $$f_n(z) \neq 0 \qquad (z \in D,\ n = 1,2,3,\cdots).$$

Hence we have the following result:

COROLLARY 1. *Let $r_n = \frac{1}{2}$. Then $\{f_n(z)\}_1^\infty$ converges—uniformly on compact subsets —to a holomorphic limit function if the assumptions* (3), (4), (5) *are fulfilled.*

Concluding this section we wish to underline an important difference between A. Herschfeld's results in the real situation and the complex case:

The proposition

$$\lim_{n\to\infty} f_n = 0 \text{ implies } a_n = 0 \text{ for every } n,$$

which is a consequence of Herschfeld's theorem, is no longer valid for complex a_n.

Example: Let R be the plane cut along the negative axis, let D be the right half-plane and consider

$$a_1(z) = z^2, \ a_n(z) = \left(\frac{z}{n}\right)^2 - \frac{z}{n-1} \qquad (n = 2,3,4,\cdots, z \in D).$$

In (1) we always choose the branch of $\{w\}^{\frac{1}{2}}$ in R which coincides with $\sqrt{w}$ for positive w. Then an easy computation yields $f_n(z) = z/n$. Hence $\lim_{n\to\infty} f_n(z) = 0$ but $a_n(z) \neq 0$ for $z \neq n^2/(n-1)$.

3. Some problems of C. S. Ogilvy. In order to treat these problems we discuss the case $a_n(z) = z$, i.e.,

$$f_0(z) = 0, \ f_n(z) = \{z + f_{n-1}(z)\}^{\frac{1}{2}} \qquad (n = 1,2,3,\cdots). \tag{6}$$

LEMMA 1. *Given any values of the roots in* (6), *we have for all complex z and* $n = 1,2,3\cdots$

$$\text{If } |z| \leqq 2, \text{ then } |f_n(z)| < 2, \tag{7}$$

$$|z| \geqq 2 \text{ implies } |f_n(z)| < |z| \text{ and therefore } f_n(z) \neq 0, \tag{8}$$

$$0 < |z| < \tfrac{1}{4} \text{ implies } |f_n(z)| > 2|z| \text{ and therefore } f_n(z) \neq 0, \tag{9}$$

$$|f_n(z)| < 2\sqrt{|z|} \text{ if } |z| > \tfrac{4}{9}, \tag{10}$$

$$f_n(z) \neq 0 \text{ if } \operatorname{Re}\{z\} \leqq -1. \tag{11}$$

The first four inequalities follow immediately by induction, by using only the triangle inequalities and the trivial identity $|\{w\}^{\frac{1}{2}}| = \sqrt{|w|}$, valid for every complex w and every branch of the square root. As an example we prove (8) in detail: Since $|\sqrt{z}| < |z|$ for $|z| \geqq 2$, we have $|f_1(z)| < |z|$. Assuming $|f_n(z)| < |z|$ for $n \geqq 1$, we obtain

$$|f_{n+1}(z)| = \sqrt{|z + f_n(z)|} \leqq \sqrt{|z| + |f_n(z)|} < \sqrt{|z| + |z|} = \sqrt{2|z|} \leqq |z|,$$

since $|z| \geqq 2$. This proves the first assertion of (8). The second one follows from

$$|f_n(z)|^2 = |z + f_{n-1}(z)| \geqq |z| - |f_{n-1}|$$

and the first statement of (8).

To prove (11) we assume that there exist numbers n and z_0 with $f_n(z_0) = 0$ and $\operatorname{Re}\{z_0\} \leqq -1$. From (8) and the equation $f_1^2(z) = z$, it follows necessarily that $|z_0| < 2$ and $n \geqq 2$. Hence $f_{n-2}(z_0) = z_0^2 - z_0$ or

$$|f_{n-2}(z_0)|^2 = |z_0|^2 |z_0 - 1|^2 \geqq (\operatorname{Re}\{z_0\})^2(\operatorname{Re}\{z_0 - 1\})^2 \geqq 4$$

in contradiction to (7). This proves Lemma 1.

Let R be the complex plane cut along the negative real axis again. We regard $a_n(z) = z$ in the domain $D = R$:

LEMMA 2. *It is possible to define $f_n(z)$ using for each n the branch of $\{w\}^{\frac{1}{2}}$ on R which is positive for positive w. (Therefore it is clear that $f_n(z) \in R$ and consequently $f_n(z) \neq 0$.)*

Proof. We need only to show that

$$z + f_{n-1}(z) \in R$$

holds. By induction, however, we readily even see $z + f_{n-1}(z) > 0$ for real $z = x > 0$, and $\mathrm{Im}\{z + f_{n-1}(z)\} > 0$ ($\mathrm{Im}\{z + f_{n-1}(z)\} < 0$, resp.), if $\mathrm{Im}\,\{z\} > 0$ ($\mathrm{Im}\{z\} < 0$).

Using Lemma 2 and A. Herschfeld's theorem for real $z = x > 0$ our method gives the following result.

COROLLARY 2. *$\{f_n(z)\}_1^\infty$ converges to $f(z) = \frac{1}{2}(1 + \{1+4z\}^{\frac{1}{2}})$ uniformly on every compact subset of R. Here $\{1 + 4z\}^{\frac{1}{2}}$ denotes the branch which is positive for positive $z = x$.*

Now we consider the functions $f_n(z)$ in the upper half-plane $D = \{z \mid \mathrm{Im}\{z\} > 0\}$ only and continue them analytically into the half-plane $D_1 = \{z \mid \mathrm{Re}\{z\} < -1\}$ and the half-circle $D_2 = \{z \mid \mathrm{Re}\{z\} < 0,\ |z| < \frac{1}{4}\}$, respectively. This is possible because (11) and (9) ensure the validity of (5). On account of Corollary 2 hypothesis (3) even holds for all points z_k of D_1 and D_2 lying in the upper half-plane. Therefore the analytic continuations of $f_n(z)$ into D_1 and D_2 converge to the corresponding analytic continuations of $f(z)$.

Quite analogously we obtain the analytic continuation into D_1 and D_2 beginning from the lower half-plane.

Now we restrict ourselves to the case of negative real $z = x$.

COROLLARY 3. (a) $\lim_{n\to\infty} f_n(x) = \frac{1}{2}(1 + \sqrt{1+4x})$ *for* $-\frac{1}{4} < x < 0$, *no matter whether the square root in* (6) *is consistently taken as the one with the least positive amplitude or consistently taken as the other one.*

(b) $\lim_{n\to\infty} f_n(x) = \frac{1}{2}(1 + i\sqrt{-4x-1})$ *for* $-\infty < x < -1$, *if the square root in* (6) *is consistently taken as the one with the least positive amplitude, and*

$$\lim_{n\to\infty} f_n(x) = \tfrac{1}{2}(1 - i\sqrt{-4x-1}) \textit{ for } -\infty < x < -1,$$

if we always take the other one.

The convergence is always uniform on every compact subset of the intervals.

Corollary 3(a) yields a solution of one of C. S. Ogilvy's problems. To study another we define for real $x > 1$

(12) $$g_0(x) = 0,\ g_n(x) = -\sqrt{x + g_{n-1}(x)} \qquad (n = 1,2,3,\cdots).$$

This definition is reasonable, for an induction argument readily yields that $x + g_{n-1}(x) > 0$.

In order to examine the convergence of $\{g_n(x)\}_1^\infty$ we consider our functions $f_n(z)$ in a neighborhood of the positive real axis and continue them analytically along circles with centre in the origin and radius > 2; this is possible on account of (8). Moreover we have

$$f_n(z) - f_1(z) = f_1(z)h_n(z),$$

where

$$h_n(z) = \left\{1 + \frac{f_{n-1}(z)}{z}\right\}^{\frac{1}{2}} - 1 = \sum_{\nu=1}^{\infty} \binom{\frac{1}{2}}{\nu} \left(\frac{f_{n-1}(z)}{z}\right)^{\nu}.$$

Since the elementary estimate

$$\left| \sum_{\nu=1}^{\infty} \binom{\frac{1}{2}}{\nu} w^{\nu} \right| < |w| \left| \sum_{\nu=1}^{\infty} \binom{\frac{1}{2}}{\nu} (-1)^{\nu} \right| = |w| \text{ for } |w| < 1$$

gives

$$|h_n(z)| < \left|\frac{f_{n-1}(z)}{z}\right|,$$

we obtain $|f_n(z) - f_1(z)| < 2$ for $|z| > 2$ using (10).

Hence, returning to the positive real axis after one circulation, the analytic continuation of $f_n(z)$ coincides with the function that we get if we have chosen instead $\{w\}^{\frac{1}{2}} = -\sqrt{w}$ in (6). This latter function, however, is $g_n(x)$. Thus we have proved the following assertion.

COROLLARY 4. $\lim_{n\to\infty} g_n(x) = \frac{1}{2}(1 - \sqrt{1 + 4x})$ *for* $x_0 < x < \infty$ *uniformly on every compact subset of the interval where* $x_0 = 2$.

This result may be improved to $x_0 = \frac{1}{2}(1 + \sqrt{5}) = 1.618\cdots$, since our analytic continuations $f_n(z)$ may also be continued analytically into the rectangles

$$D_3 = \{z \mid z = x + iy, \tfrac{1}{2}(1 + \sqrt{5 + 2\varepsilon}) < x < 3, \ |y| < \varepsilon\}$$

$(0 < \varepsilon < 1$ arbitrary). This is a consequence of the following statement.

LEMMA 3. $\mathrm{Re}\{f_n(z)\} < 0, \mathrm{Re}\{f_n(z) + z\} > 0$ *if* $z \in D_3, |z| < 2,$ *and* $n = 1, 2, 3, \cdots$.

The proof is readily carried out by induction, where we only need the formula

$$\mathrm{Re}\{f_{n+1}(z)\} = -\tfrac{1}{2}\sqrt{\mathrm{Re}\{z + f_n(z)\} + |z + f_n(z)|}$$

and the inequality (7).

Replacing (7) by the better estimate

$$|f_n(z)| \leqq \sqrt{2 + |z|} \qquad \text{if } |z| \leqq 2,$$

we get a minor improvement of this result, but the computation becomes somewhat lengthy.

To conclude this section, we consider the analytic continuation of $f_n(z)$ in $0 < |z| < \frac{1}{4}$ along circles with centre in the origin and define $g_n(x)$ by the analytic continuation of $f_n(z)$ after one circulation. We get

$$\lim_{n\to\infty} g_n(x) = \tfrac{1}{2}(1 + \sqrt{1+4x}) \qquad \text{if } 0 < x < \tfrac{1}{4}.$$

The convergence is uniform on every compact subset of the interval.

4. An example for divergence. If we take arbitrary branches of the root in (6), we cannot expect that $\{f_n(z)\}_1^\infty$ converges. To see this we consider

$$\text{(13)} \qquad \begin{aligned} &h_0(x) = 0, \quad h_{2n-1}(x) = -\sqrt{x + h_{2n-2}(x)}, \\ &h_{2n}(x) = \sqrt{x + h_{2n-1}(x)} \qquad (x > 2,\ n = 1,2,3,\cdots), \end{aligned}$$

where, again, we use the symbol $\sqrt{w}$ to denote that branch of the square root which is positive when w is positive.

Because of (8) the sequence is well defined.

For even indices we have $0 < h_{2n}(x) < x$ and $h_{2n}(x) - h_{2n-2}(x) = H_1(h_{2n-2}(x))$, where

$$H_1(u) = \sqrt{x - \sqrt{x+u}} - u \qquad (0 < u < x).$$

Since $H_1'(u) < 0$, $H_1(u)$ has at most one zero. A simple computation leads to

$$H_1(u_0) = 0, \text{ where } u_0 = -\tfrac{1}{2} + \sqrt{x - \tfrac{3}{4}}.$$

Therefore

$$\text{(14)} \qquad H_1(u) > 0 \text{ if } u < u_0, \ H_1(u) < 0 \text{ if } u > u_0.$$

Assume $h_{2n-2}(x) > u_0$; then $H_1(h_{2n-2}(x)) < 0$ or $h_{2n}(x) < h_{2n-2}(x)$. Hence

$$h_{2n+2}(x) = \sqrt{x - \sqrt{x + h_{2n}(x)}} > \sqrt{x - \sqrt{x + h_{2n-2}}} = h_{2n}(x)$$

and therefore $H_1(h_{2n}(x)) > 0$, $h_{2n}(x) < u_0$. In the same way we get $h_{2n}(x) > u_0$ assuming $h_{2n-2}(x) < u_0$. Observing $h_0(x) = 0$ we obtain

$$\text{(15)} \qquad h_{4n}(x) < u_0 < h_{4n+2}(x) \qquad (n = 0,1,2\cdots).$$

Let $\{j_n(x)\}_1^\infty$ be any one of the sequences $\{h_{4n}(x)\}_1^\infty$, $\{h_{4n-2}(x)\}_1^\infty$. Then

$$\text{(16)} \qquad j_{n+1}(x) - j_n(x) = H_2(j_n(x)),$$

where

$$H_2(u) = H_1(H_1(u) + u) + H_1(u) \qquad (0 < u < x).$$

An easy computation shows $H_2'(u) < 0$ if $x > 8$, and $H_2(u_0) = 0$. Thus for $x > 8$

$$H_2(u) > 0 \text{ if } u < u_0, \ H_2(u) < 0 \text{ if } u > u_0.$$

Therefore it follows from (15) and (16) that $\{h_{4n}(x)\}_1^\infty$ and $\{h_{4n-2}(x)\}_1^\infty$ are monotonic and consequently convergent. According to (16) their limits must be equal to the solution of $H_2(u) = 0$ so that

$$\lim_{n\to\infty} h_{2n}(x) = u_0 = -\tfrac{1}{2} + \sqrt{x - \tfrac{3}{4}} \qquad \text{if} \quad x > 8. \tag{17}$$

On account of (8) we can apply our method in any simply-connected domain excluding the circle $|z| < 2$ and including the points $z = x > 2$, and obtain (17) even for $x > 2$.

For odd indices we get from (17) and definition (13)

$$\lim_{n\to\infty} h_{2n-1}(x) = -(\tfrac{1}{2} + \sqrt{x - \tfrac{3}{4}}) \quad (x > 2).$$

Hence, $\lim_{n\to\infty} h_{2n}(x) \neq \lim_{n\to\infty} h_{2n-1}(x)$, i.e., $\{h_n(x)\}_1^\infty$ diverges in $x > 2$.

References

1. J. D. Depree and C. C. Oehring, Elements of Complex Analysis, Addison-Wesley, Reading, Mass.; Menlo Park, Cal.; London; Don Mills, Ontario.
2. A. Herschfeld, On infinite radicals, this MONTHLY, 42 (1935) 419–429.
3. C. S. Ogilvy, To what limits do complex iterated radicals converge? this MONTHLY, 77 (1970) 388–389.
4. E. C. Titchmarsh, The Theory of Functions (2nd ed.), Oxford University Press, Oxford, 1958.

MATHEMATISCHES INSTITUT, TECHNISCHE HOCHSCHULE ,51 AACHEN, W. GERMANY.

ARE THERE $n+2$ POINTS IN E^n WITH ODD INTEGRAL DISTANCES?

R. L. GRAHAM, B. L. ROTHSCHILD AND E. G. STRAUS*

In this note we answer the question posed in the title.

THEOREM 1. *For the existence of $n + 2$ points in E^n so that the distance between any two of them is an odd integer, it is necessary and sufficient that $n + 2 \equiv 0$* (mod 16).

There are analogous results concerning integral distances relatively prime to 3 or 6 which we mention at the end of this work.

The main tool in the proof of the necessity part of Theorem 1 is a theorem of Cayley (see, e.g. [**1**], p. 122).

* Presented to the Southern California Section on March 11, 1972, by E. G. Straus, whose work was supported by NSF Grant No. GP-28696. The work of B. Rothschild was supported by NSF Grant No. GP-23482.

THEOREM 2. *Let the set of* $\binom{n+2}{2}$ *nonnegative numbers* d_{ij}; $1 \leqq i < j \leqq n+2$ *be a set of distances* $d_{ij} = d(p_i, p_j)$ *of points* $p_1, p_2, \cdots, p_{n+2}$ *in* E^n. *Then*

$$\Delta = \begin{vmatrix} 0 & d_{12}^2 & d_{13}^2 \cdots & d_{1\,n+2}^2 & 1 \\ d_{21}^2 & 0 & d_{23}^2 \cdots & d_{2\,n+2}^2 & 1 \\ \cdot & \cdot & \cdot \cdots & \cdot & \cdot \\ d_{n+2\,1}^2 & d_{n+2\,2}^2 & \cdot \cdots & 0 & 1 \\ 1 & 1 & \cdot \cdots & 1 & 0 \end{vmatrix} = 0,$$

where $d_{ij} = d_{ji}$.

Proof. We consider the points p_i as vectors in R^n and assume without loss of generality that $p_{n+2} = \mathbf{0}$, the origin. Then

$$d_{ij}^2 = |p_i - p_j|^2 = |p_i|^2 + |p_j|^2 - 2(p_i, p_j)$$

and

$$\Delta = \begin{vmatrix} (|p_i|^2 + |p_j|^2 - 2(p_i, p_j)) & & & & & \begin{matrix} 1 \\ \vdots \\ 1 \end{matrix} \\ 1 & 1 & \cdot & \cdot & \cdot \quad 1 & 0 \end{vmatrix}.$$

Subtracting $|p_i|^2$ times the last row from the ith row and $|p_j|^2$ times the last column from the jth column we get

$$\Delta = \begin{vmatrix} & & \begin{matrix} 1 \\ \vdots \end{matrix} \\ -2(p_i, p_j) & & \\ & & 1 \\ 1 \cdots & 0 & 1 \end{vmatrix} = \begin{vmatrix} -2(p_1, p_1) & \cdots & -2(p_1, p_{n+1}) & 0 & 1 \\ \vdots & & & & \\ -2(p_{n+1}, p_1) & \cdots & -2(p_{n+1}, p_{n+1}) & 0 & 1 \\ 0 & \cdots & 0 & 0 & 1 \\ 1 & \cdots & 1 & 1 & 0 \end{vmatrix}$$

$$= (-1)^n 2^{n+1} \begin{vmatrix} (p_1, p_1) & \cdots & (p_1, p_{n+1}) \\ \vdots & & \vdots \\ (p_{n+1}, p_1) & \cdots & (p_{n+1}, p_{n+1}) \end{vmatrix}$$

$$= (-1)^n 2^{n+1} \det(P \cdot P^{tr}),$$

where the $n \times (n+1)$ matrix

$$P = \begin{bmatrix} p_1 \\ \vdots \\ p_{n+1} \end{bmatrix}$$

has rank $P \leqq n$, and hence rank $(P \cdot P^{tr}) \leqq n$, so that $\det(P \cdot P^{tr}) = 0$.

REMARK. For an alternate proof, consider the linear mapping $(a_1, \cdots, a_{n+2}) \to (\Sigma a_j p_j, \Sigma a_j)$ on R^{n+2} into the $(n+1)$-dimensional space $E^n \oplus R$. It has non-zero kernel, so there is a vector $(a_1, \cdots, a_{n+2}) \neq 0$ such that $\Sigma a_j p_j = 0$ and $\Sigma a_j = 0$. Set $c = -\Sigma a_j \, |p_j|^2$. By a short direct calculation,

$$\sum_{j=1}^{n+2} a_j \, |p_i - p_j|^2 + c = 0, \qquad \Sigma a_j = 0.$$

This is a system of $n+3$ equations and it has the non-trivial solution $(a_1, \cdots, a_n, c)$, so its determinant is zero. That is Theorem 2.

The necessity of Theorem 1 now follows from a lemma.

LEMMA 1. *Let d_{ij}; $1 \leqq i < j \leqq n+2$ be a set of odd integers. Then*

$$\Delta \equiv (-1)^n (n+2) (\operatorname{mod} 16).$$

Proof. Since d_{ij} is an odd integer we get $c_{ij} = d_{ij}^2 - 1 \equiv 0 \pmod 8$. Subtracting the last column of Δ from all other columns we have

$$\Delta = \begin{vmatrix} -1 & c_{12} & \cdots & c_{1\,n+2} & 1 \\ c_{21} & -1 & \cdots & c_{2\,n+2} & 1 \\ \vdots & & & \vdots & \\ c_{n+2\,1} & & \cdots & -1 & 1 \\ 1 & & \cdots & 1 & 0 \end{vmatrix}$$

By first adding the first $n+2$ columns to the last column and then adding the first $n+2$ rows to the last row, we get

$$\Delta = \begin{vmatrix} -1 & c_{12} & \cdots & c_{1\,n+2} & a_1 \\ c_{21} & -1 & \cdots & c_{2\,n+2} & a_2 \\ \vdots & \vdots & & \vdots & \\ c_{n+2\,1} & \cdot & \cdots & -1 & a_{n+2} \\ 1 & \cdot & \cdots & 1 & n+2 \end{vmatrix} = \begin{vmatrix} -1 & c_{12} & \cdots & c_{1\,n+2} & a_1 \\ c_{21} & -1 & \cdots & c_{2\,n+2} & a_2 \\ \vdots & & & & \\ c_{n+2\,1} & & \cdots & -1 & a_{n+2} \\ a_1 & & \cdots & a_{n+2} & n+2+a \end{vmatrix},$$

where $a_i = \sum_{j \neq i} c_{ij}$ and $a = \sum_{i=1}^{n+2} a_i = 2 \sum_{i<j} c_{ij}$.

Since all the terms off the main diagonal in the last expression of Δ are divisible by 8; and each product in the expansion of Δ other than the main diagonal term contains at least two off-diagonal factors; we have

$$\Delta \equiv (-1)^{n+2}(n+2+a) = (-1)^n(n+2+a) \pmod{64}.$$

But $a = 2\sum_{i<j} c_{ij} \equiv 0 \pmod{16}$ so that

$$\Delta \equiv (-1)^n(n+2) \pmod{16}.$$

We can now complete the proof of Theorem 1 by making a suitable construction.

Let $n = 16s - 2$ and choose $p_1, \cdots, p_n$ as vertices of a regular $(n-1)$-simplex of edge length $8s - 1$ in the hyperplane $x_n = 0$ so that its centroid is at the origin. Choose the remaining two points p_{n+1}, p_{n+2} as $(0, \cdots, 0, \pm(2s - \frac{1}{2}))$ on the x_n-axis. In this set there are only three distinct distances, $d(p_i, p_j) = 8s - 1$ for $1 \leqq i < j \leqq n$; $d(p_{n+1}, p_{n+2}) = 4s - 1$ and

$$d(p_i, p_{n+k})^2 = |p_i|^2 + (2s - \tfrac{1}{2})^2; \qquad 1 \leqq i \leqq n;\ k = 1, 2. \tag{1}$$

In order to compute this last distance we need the following:

LEMMA 2. *The distance from the centroid of a unit simplex in E^k to a vertex* is $d_k = \sqrt{k/(2k+2)}$.

Proof. The unit vectors in E^{k+1} form the vertices of a regular k-simplex of edge length $\sqrt{2}$ with centroid $(1/(k+1))(1, 1, \cdots, 1)$. Thus the distance from a vertex to the centroid is

$$\begin{aligned}\sqrt{2}d_k &= \sqrt{\left(1 - \frac{1}{k+1}\right)^2 + \left(\frac{1}{k+1}\right)^2 + \cdots + \left(\frac{1}{k+1}\right)^2} \\ &= \sqrt{\frac{k^2}{(k+1)^2} + \frac{k}{(k+1)^2}} = \sqrt{\frac{k}{k+1}}.\end{aligned}$$

Thus the value of $|p_i|^2$ in (1) is

$$|p_i|^2 = (8s-1)^2 d_{16s-3}^2 = (8s-1)^2 \cdot \frac{16s-3}{2(16s-2)} = \frac{128s^2 - 40s + 3}{4},$$

and

$$\begin{aligned}d(p_i, p_{n+k})^2 &= \tfrac{1}{4}(128s^2 - 40s + 3 + 16s^2 - 8s + 1) \\ &= (6s-1)^2.\end{aligned}$$

We have thus constructed a set with $n + 2 = 16s$ points and only three distinct distances, $4s - 1$, $6s - 1$ and $8s - 1$, all of which are odd, attained respectively once, $2n$ and $\binom{n}{2}$ times.

There are many other examples of constructing $(n+2)$-tuples of points with only three distances, all odd in case $n = 16s - 2$. For example we could construct regular simplices in complementary orthogonal subspaces E^{14s-2} and E^{2s} with edge lengths $14s - 1$ and $2s + 1$ respectively. The third distance d satisfies

$$d^2 = (14s-1)^2 d_{14s-2}^2 + (2s+1)^2 d_{2s}^2 = (10s-1)^2$$

REMARK. It is impossible to have $n + 3$ points in E^n so that all distances are odd integers since by Theorem 1 this would imply both $n + 2 \equiv 0 \pmod{16}$ and $(n + 1) + 2 \equiv 0 \pmod{16}$.

The reasoning in Lemma 1 can be applied equally well in the case of integral distances relatively prime to 3.

LEMMA 3. *Let d_{ij}; $1 \leqq i < j \leqq n + 2$ be a set of integers relatively prime to* 3. *Then* $\Delta \equiv (-1)^n(n + 2) \pmod 3$.

Proof. Since $d_{ij}^2 \equiv 1 \pmod 3$ we get

$$\Delta \equiv \left| J - I \right|_{n+3} = (-1)^n(n + 2) \pmod 3.$$

THEOREM 3. *There exist $n + 2$ points in E^n whose distances are integers relatively prime to* 3 *if and only if* $n \equiv 1 \pmod 3$.

There exist $n + 2$ points in E^n whose distances are integers relatively prime to 6 *if and only if* $n \equiv -2 \pmod{48}$.

Proof. The necessity of the two congruences follows from Lemma 3 and Theorem 1.

For sufficiency in the second case we can use the same construction used in the proof of Theorem 1. Set $n = 48s - 2$ and construct the set of $n + 2$ points with distances $12s - 1$, $18s - 1$ and $24s - 1$ respectively.

For sufficiency in the first case, set $n = 3s + 1$ and construct a regular simplex in a hyperplane E^{3s} of side length $4(3s + 1)$ with centroid at the origin; then add two more points on the axis perpendicular to E^{3s} at distances $3s - 1$ from the origin. We then get three distances $4(3s + 1)$, $6s - 2$, $9s + 1$ since

$$(9s + 1)^2 = (3s - 1)^2 + \frac{3s}{2(3s + 1)} \cdot 16(3s + 1)^2.$$

These distances are attained respectively $\binom{3s+1}{2}$ times, once and $6s + 2$ times.

Our examples involve sets of points determining three distinct distances. One might ask whether there are examples involving $(n + 2)$-tuples of points with only two distinct distances. The answer appears to be in the negative for odd distances, while there are certain dimensions in which there are examples of $(n + 2)$-tuples with only two distinct distances both prime to 3.

One could generalize the above results to conditions on integral distances of the form $d_{ij}^2 \equiv 1 \pmod m$ for general moduli m. However this does not appear as attractive as the above treated problems.

Reference

1. L. M. Blumenthal, Theories and Applications of Distance Geometry, Oxford, 1953.

R. L. GRAHAM: BELL TELEPHONE LABORATORIES, MURRAY HILL, N.J. 07974.

B. L. ROTHSCHILD AND E. G. STRAUS: DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA, LOS ANGELES, CALIFORNIA 90024.

BUFFON'S NEEDLE EXPERIMENT

E. F. SCHUSTER

1. Introduction. Of central importance in probability and statistics is the notion of independence. Although a precise mathematical definition of independence can be given, we often interpret independence in keeping with our intuitive ideas. There is a natural inclination to consider events which seem unrelated as being independent of one another. In this regard we present a variation of Buffon's needle problem which we hope will be useful in the classroom.

2. Buffon's needle experiment: A needle of length L is thrown in a random fashion onto a smooth table ruled with parallel lines separated by a distance of $2L$. An observer records whether or not the needle intersects a ruled line.

Let E be the event "the needle intersects one of the ruled lines." Suppose you ask a class of students to empirically estimate $P(E)$ (it is well known that $P(E) = 1/\pi$) by performing 200 independent trials of this experiment. Their estimate of $P(E)$ is, of course, the proportion of times E occurs in the 200 trials. One ingenious student (E. P. Del Norte) covers his table with congruent squares $2L$ inches on a side (similar to a large checkerboard or a tile floor). Using a needle L inches long he performs his experiment 100 times, recording the result of each trial with respect to each set of ruled lines, and thereby coming up with his 200 trials of the experiment. Has he in fact come up with 200 independent trials of the experiment? Do you expect his estimate of $P(E)$ to be better than, as good as, or worse than that of another student randomly selected from the remaining members of the class (hereafter referred to as the typical student)? What does your intuition say? Ask your class.

3. Solution. Let A be the event "the needle intersects a ruled line parallel to the X axis" and let B be the event "the needle intersects a ruled line parallel to the Y axis." Then a single trial of Del Norte's experiment is equivalent to two independent trials of Buffon's experiment if and only if the events A and B are independent. Now A and B are independent if and only if the probability of the intersection of A and B is the product of their respective probabilities, i.e., if and only if $P(AB) = P(A)P(B)$. Since $P(A) = P(B) = 1/\pi$, one could empirically test the hypothesis of independence in the usual manner. However, we note that the computation of $P(AB)$ can be obtained as a special case of Laplace's problem as given on pages 255 and 256 of the classic by Uspensky [**2**]. Using this solution (we give a generalization in section 4) we find that $P(AB) = 1/4\pi \neq P(A)P(B) = 1/\pi^2$. Hence A and B are not independent and the 100 trials by Del Norte are not equivalent to 200 trials of Buffon's experiment.

We now proceed to the answer to our second question: Since our student E. P.

Del Norte uses the statistic $\hat{P}$ = (number of times A occurred in the 100 trials + number of times B occurred in the 100 trials)/200 as his estimate of $P(E) = 1/\pi$, do we expect his estimate to be better than or worse than that of the typical student in the class? It is clear that the expected value of $\hat{P}$ is $P(E) = 1/\pi$. Since

$$1/\pi = P(A) = P(AB) + P(AB') = 1/4\pi + P(AB'), P(AB') = 3/4\pi.$$

Similarly $P(A'B) = 3/4\pi$ and hence $P(A'B') = 1 - 7/4\pi$. Using these probabilities one finds that the variation in the estimator $\hat{P}$ equals $(5\pi-8)/800\pi^2 \simeq 0.000976$. The variance of the proportion of times the needle will intersect a line in M independent trials of Buffon's experiment is $(\pi-1)/\pi^2 M \times 0.217/M$. If we equate these two variances we find that $M \simeq 222$, i.e., 100 independent observations with respect to both grids contains approximately the same amount of information about $P(E)$ as 222 observations, with respect to one set of grid lines. Hence, we expect E. P. Del Norte's estimate to be better than that of the typical student in the class. As a word of explanation we note that Del Norte's method is a case of estimation by "antithetic variables" (see for example chapter five of Hammersley and Handscomb's monograph [**1**]). The idea is that if several random variables are negatively correlated, then their average has variance much less than that of the individual variables.

Another question now arises. Suppose one varies the angle of intersection of the X and Y grid lines. Is it possible to find an angle α such that independence is achieved, i.e., an angle of intersection for which $P(AB;\alpha) = P(A;\alpha)P(B;\alpha) = 1/\pi^2$? We answer this question in the next section.

4. The angle of independence. The following heuristic argument indicates that there is in fact an angle for which A and B are independent. If the angle of intersection α is $\pi/2$ then we have found that $P(AB; \alpha = \pi/2) = 1/4\pi$. If we start with the X and Y grids perpendicular and rotate the X grids about a fixed center that is a point of intersection of the two grids, then in the limiting case $\alpha = 0$, the X and Y grids coincide. Hence the needle intersects neither or both grids, i.e., $P(AB;\, \alpha = 0) = P(A) = 1/\pi$. Now $1/4\pi < 1/\pi^2 < 1/\pi$, so that as α decreases from $\pi/2$ to 0, one would expect $f(\alpha) = P(AB;\alpha)$ to go from $1/4\pi$ to $1/\pi$ in a continuous fashion so that by the intermediate value theorem there must exist some $\alpha = \alpha^*$ for which $f(\alpha^*) = 1/\pi^2$. This seems like a plausible argument. However, it is a fallacious one, for, as noted below, continuity breaks down at $\alpha = 0$. When one stops to think about it, this is really not surprising. If we rotate the X and Y grids about a fixed point which is not a point of intersection of the two grids, then in the limiting case the grids will be parallel but not coincident. Hence the probability in the limiting case $\alpha = 0$ will depend on the manner of rotation to the limit. So is there an angle of independence or isn't there? To answer this question, we shall now set up a probability model, compute $f(\alpha)$, and see if there exists an α^* for which $f(\alpha^*) = 1/\pi^2$.

Refer to figure 1: Let α be the smaller angle of intersection of the X and Y grid lines and let $ACDE$ be the parallelogram which contains the center of the needle.

Taking AB and AE as coordinate axes, the position of the needle is determined by the coordinates (x,y) of its center and the angle θ formed by the needle with the AB axis. We postulate that θ is uniformly distributed over $(0,\pi)$, (x,y) is uniformly distributed over the parallelogram $ACDE$, and that θ is independent of (x,y).

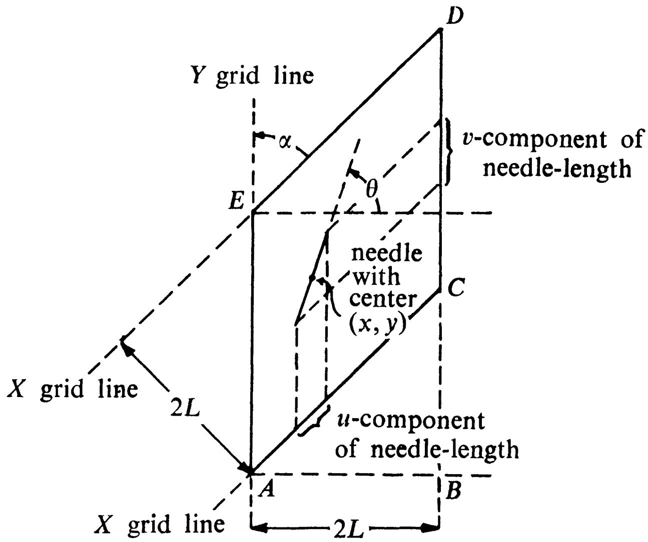


FIG. 1

The transformation

$$u = x\csc\alpha, \quad v = y - x\cot\alpha$$

then transforms from rectangular coordinates (x,y) for the center to oblique ones (u,v) measured parallel to the sides of the fundamental parallelogram $ACDE$. Since the Jacobian of the inverse of the above transformation equals $\sin\alpha$, one can easily show that u and v are independent and identically distributed as the uniform distribution over the interval $(0,2L\csc\alpha)$. We can then compute $f(\alpha) = P(AB;\alpha)$ as follows: For a given orientation $\theta = s$ of the needle, the occurrence of A is governed solely by the value of v; similarly for B and u; thus A and B are conditionally independent given θ. Now $P(A \mid \theta = s) =$ (v-component of needle length)/($2L\csc\alpha$) $= |\cos(\alpha + s)|/2$ (there are three cases to be considered,

$$\pi/2 - \alpha < \theta \leqq \pi/2, 0 \leqq \theta \leqq \pi/2 - \alpha, \text{ and } \pi/2 < \theta \leqq \pi);$$

similarly $P(B \mid \theta = s) = |\cos s|/2$. Thus $P(AB \mid \theta = s) = P(A \mid \theta = s)P(B \mid \theta = s) = |\cos(\alpha + s)\cos s|/4$. If we integrate this against ds/π on $0 < s < \pi$ we obtain

$$f(\alpha) = P(AB;\alpha) = \{(\pi/2 - \alpha)\cos\alpha + \sin\alpha\}/4\pi.$$

It is clear that $f(\alpha)$ is continuous on $(0, \pi/2)$ with $f(0^+) = 1/8$ and $f((\pi/2)^-) = 1/4\pi$. Note that $1/4\pi < 1/\pi^2 < 1/8$, so that there is an angle for which A and B are independent. Now $f'(\alpha) = \{(\alpha - \pi/2)\sin\alpha\}/4\pi < 0$, so that $f(\alpha)$ is strictly decreasing for $0 < \alpha < \pi/2$ and the angle of independence is unique. This angle α must satisfy the equation

$$f(\alpha) = 1/\pi^2.$$

Simplifying, this becomes

$$\sin(\alpha) + (\pi/2 - \alpha)\cos(\alpha) = 4/\pi$$

or

$$\cos(\beta) + \beta\sin(\beta) = 4/\pi,$$

where $\beta = \pi/2 - \alpha$. Using Newton's method with an initial guess of 0.75 we find that $\beta \simeq 0.80475$ radians ($\simeq 46.11°$) or $\alpha \simeq 0.76605$ radians ($\simeq 43.89°$).

We close this article by asking the reader to examine the question of independence when the length of the needle is a, where $a < 2L$. Does there always exist an angle $\alpha = \alpha(a)$ for which A and B are independent? Are there any needle lengths a for which we can solve explicitly for the independence angle $\alpha = \alpha(a)$? How should we rotate the X grid lines so that $f(\alpha)$ is continuous at the limit position $\alpha = 0$? What happens when the distance between X grid lines is different from the distance between Y grid lines? Can we find three sets of grid lines: X, Y, Z, separated by distances a, b, and c, respectively, and a needle length d such that the corresponding events A, B, and C are pairwise independent yet mutually dependent?

Acknowledgement. The author is indebted to the referee for his perspicacious comments and suggestions.

References

1. J. M. Hammersley and D. C. Handscomb, Monte Carlo Methods, Methuen, London, 1964.
2. J. V. Uspensky, Introduction to Mathematical Probability, McGraw-Hill, New York, 1937.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF TEXAS AT EL PASO, EL PASO, TEXAS 79968.

INNER PRODUCT SPACES

STANLEY GUDDER

1. Introduction. One frequently encounters incomplete inner product spaces in mathematics. For example, the continuous functions on $[0,1]$, the infinitely differentiable functions on R^n with compact support, and the analytic functions on the unit disk all with inner product $\langle f, g\rangle = \int f\bar{g}\,d\mu$ are important cases of such spaces.

The space of complex sequences (a_i) such that $a_i = 0$, except for finitely many i's with inner product $\langle (a_i), (b_i) \rangle = \Sigma\, a_i \bar{b}_i$ is another common example. What Hilbert space techniques can be employed in such incomplete spaces? Of course, one can always complete these spaces to form a Hilbert space but this is sometimes undesirable since the result will then frequently contain elements outside the original space. For example, a maximal orthonormal set for an inner product space V need not be maximal in its completion, so to form a basis for V we might have to go outside of V.

In this article we shall consider three types of questions. (1) What "goes wrong" in incomplete inner product spaces? In other words, for what kinds of properties is completeness indispensable? (2) What properties characterize completeness for inner product spaces? That is, can we tell if an inner product space is complete without testing every Cauchy sequence? (3) What can we do in incomplete inner product spaces? In other words, are there important results that can be obtained without completeness? We shall try to illustrate these questions using little-known results some of which are part of the folk-lore of the theory, some only available in the research literature, and some new. Two relevant papers which give results different from those considered here are [**3**] and [**4**].

2. Bases. In the sequel we shall consider real or complex inner product spaces with inner product $\langle \cdot, \cdot \rangle$ and norm $\| x \| = \langle x, x \rangle^{\frac{1}{2}}$. A set of vectors $\{x_\alpha\}$ in a normed linear space X such that any $x \in X$ can be uniquely expanded in the form $x = \Sigma\, c_\alpha\, x_\alpha$ (the sum converging in norm to x) is called a **basis** for X. It has been shown by Per Enflo that there are separable Banach spaces possessing no countable basis. It is, however, a very useful and important fact that in a (separable) Hilbert space there always exists a (countable) orthonormal basis. A natural question to ask is whether every inner product space possesses an orthonormal basis. Suprisingly enough, we shall see that the answer to this question is no.

Now there are three good properties that an orthonormal set in an inner-product space X may have: it may be maximal, total, or basic. **Maximal** is self-explanatory. **Total** means that it spans the space (i.e., the smallest closed subspace that includes it is X). **Basic** means it forms a basis for X. In a Hilbert space these three concepts coalesce.

THEOREM 2.1. *If $\{x_\alpha\}$ is an orthonormal set in a Hilbert space H the following statements are equivalent*:

(1) *$\{x_\alpha\}$ is basic,*

(2) *$\{x_\alpha\}$ is total,*

(3) *$\{x_\alpha\}$ is maximal,*

(4) *$\| x \|^2 = \Sigma\, |\langle x, x_\alpha \rangle|^2$ for every $x \in H$ (Parseval's equality).*

Since the proof is simple and since we are interested in knowing exactly where completeness is used we shall outline it. First of all, $x = \Sigma\, c_\alpha\, x_\alpha$ means that $c_\alpha = 0$,

except for countably many α's and that the resulting countable sum converges in norm to x. Now it is clear that (1) $\Rightarrow$ (2) $\Rightarrow$ (3) and we don't even need completeness for these results. If $\{y_i\}$ is any finite orthonormal set then expanding the inequality $\langle x - \Sigma \langle x, y_i \rangle y_i,\ x - \Sigma \langle x, y_i \rangle y_i \rangle \geqq 0$ gives Bessel's inequality $\Sigma |\langle x, y_i \rangle|^2 \leqq \|x\|^2$. It follows that $\langle x, x_\alpha \rangle = 0$, except for countably many x_α's, which we denote by $x_1, x_2, \cdots$, and $\Sigma |\langle x, x_i \rangle|^2 < \infty$. Now for $n > m$

$$\left\| \sum_{i=1}^{n} \langle x, x_i \rangle x_i - \sum_{i=1}^{m} \langle x, x_i \rangle x_i \right\|^2 = \sum_{i=m+1}^{n} |\langle x, \langle x_i \rangle|^2 \to 0,$$

as $m, n \to \infty$ so $\Sigma_{i=1}^{n} \langle x, x_i \rangle x_i$ is Cauchy and hence converges (this is the only place that completeness is used). Now $x - \Sigma \langle x, x_i \rangle x_i$ is orthogonal to every element x_α of a maximal orthonormal set so $x = \Sigma \langle x, x_i \rangle x_i$ (this is sometimes called the Fourier series expansion). Thus (3) $\Rightarrow$ (4) (also, we have shown (3) $\Rightarrow$ (1)). That (4) $\Rightarrow$ (1) is proved by showing that $\lim_{n \to \infty} \| x - \Sigma_{i=1}^{n} \langle x, x_i \rangle x_i \| = 0$ using (4).

We have also proved the following result.

COROLLARY 2.2 *If* $\{x_\alpha\}$ *is an orthonormal set in an inner product space then* (1), (2) *and* (4) *of Theorem* 2.1 *are equivalent.*

Theorem 2.1 also provides the existence of an orthonormal basis for any Hilbert space since a simple Zorn's lemma argument gives a maximal orthonormal set. We shall show later that (3) need not imply (1), (2) or (4) in an arbitrary inner product space.

We now consider the dimension of an inner product space.

LEMMA 2.3. *Any two maximal orthonormal sets in an inner product space have the same cardinality.*

Proof. Let A and B be maximal orthonormal sets with cardinality α and β respectively. If either α or β is finite the space is finite dimensional and the conclusion is familiar from algebra. We thus suppose that both α and β are infinite. For $e \in A$, let $B(e) = \{f \in B : \langle e, f \rangle \neq 0\}$. By Bessel's inequality, $B(e)$ is countable. Since A is maximal, $B \subseteq \bigcup_{e \in B} B(e)$ and hence $\beta \leqq \alpha \cdot \aleph_0 = \alpha$. The desired conclusion follows from symmetry.

Although Lemma 2.3 is commonly proved for Hilbert spaces it is not always recognized that it holds for arbitrary inner product spaces. We define the **dimension** of an inner product space X as the cardinality of any maximal orthonormal set in X. (Such a set always exists by Zorn's lemma.) There are Hilbert spaces (and hence inner product spaces) of any desired dimension. Indeed, let α be a cardinal number and let A be a set with cardinality α. Denote by $l_2(A)$ the set of complex-valued functions f on A which vanish except on a countable subset of A and $\Sigma |f(\alpha)|^2 < \infty$. If we define addition and multiplication be scalars pointwise, then $l_2(A)$ is an inner

product space under the inner product $\langle f,g\rangle = \Sigma f(\alpha)\bar{g}(\alpha)$. It is straightforward to check that $l_2(A)$ is complete and has dimension α. Indeed, the set $x_\beta(\alpha) = \delta_{\alpha\beta}, \alpha, \beta \in A$ is a maximal orthonormal set. If A is the set of natural numbers we usually write $l_2(A)$ simply as l_2.

It is instructive and also useful for our purposes to compare this notion of dimension with another commonly used notion. If V is any vector space, a **Hamel basis for** V is a linearly independent set B such that the linear manifold generated by B is V (i.e., any vector in V is a finite linear combination of vectors in B). By Zorn's lemma any vector space has a Hamel basis and using similar techniques to those employed in Lemma 2.3 one can show that any two Hamel bases have the same cardinality. We then define the **Hamel dimension** of V as the cardinality of any Hamel basis for V. It is clear that the Hamel dimension is the same as the dimension for any finite dimensional inner product space. For infinite dimensional spaces this need not be so. For example, let B_1 be a Hamel basis for the $\aleph_0$-dimensional Hilbert space l_2. Then card $B_1 \leqq c$ since card $l_2 = c$. Assume the continunum hypothesis, for simplicity, and suppose card $B_1 = \aleph_0$. Orthonormalize B_1 to get a countable orthonormal Hamel basis $B = \{x_i\}$. Now $x = \sum_{i=1}^{\infty} i^{-1} x_i \in l_2$ but x is not a finite linear combination of x_i's. Indeed, if $x = \sum_{i=1}^{n} c_i x_i$, then $\langle x, x_{n+1}\rangle = 0$ but we know that $\langle x, x_{n+1}\rangle = i^{-n-1}$. Thus card $B_1 = c$, so the Hamel dimension of l_2 is c.

Hamel bases are frequently useful for theoretical purposes, however, in practical situations involving infinite dimensional spaces they are almost impossible to construct. Orthonormal bases, on the other hand, are readily constructed in concrete Hilbert spaces and hence much more important in practice. As an example of a theoretical use for Hamel bases, we give an example of a linear transformation defined on all of an infinite dimensional Hilbert space H which is not bounded. Let $\{x_\alpha\}$ be a normalized Hamel basis for H and let $\{y_1, y_2, \cdots\} \subseteq \{x_\alpha\}$ be a countably infinite subset. Define $T y_n = n y_n$, $T x_\alpha = 0$, $x_\alpha \notin \{y_1, y_2, \cdots\}$ and extend by linearity. Other uses for Hamel bases will appear in this section.

We now ask whether the dimension of an inner product space is the same as the dimension of its completion. This interesting question is a problem in [**2**]. The following elegant solution is due to P. R. Halmos (private communication). If E and F are Hilbert spaces (over the same field) the **direct sum** of E and F is the Hilbert space $E \oplus F = \{(e,f): e \in E, f \in F\}$ of ordered pairs with componentwise operations and inner product

$$\langle (e_1,f_1), (e_2,f_2)\rangle = \langle e_1, e_2\rangle + \langle f_1, f_2\rangle.$$

It is easy to show that dim $E \oplus F =$ dim $E +$ dim$F$. If $T: E \to F$ is a linear transformation (not necessarily bounded), the **graph of** T is the linear manifold

$$G = \{(e, Te): e \in E\} \subseteq E \oplus F.$$

We denote the closure of G in $E \oplus F$ by $\bar{G}$.

LEMMA 2.4. *Let E and F be Hilbert spaces, and let $T: E \to F$ be a linear transformation that vanishes on some orthonormal basis B in E and has dense range in F. If G is the graph of T and $H = E \oplus F$ then $\bar{G} = H$ and dim G = dim E.*

Proof. Since $(b,0) \in G$ for each $b \in B$, it follows that $(e,0) \in \bar{G}$ for each $e \in E$. This implies that $(e, Te) - (e,0) = (0, Te) \in \bar{G}$ for each $e \in E$. Since the range of T is dense in F we conclude that $\bar{G} = H$. If $(e, Te) \perp (b,0)$ for all $b \in B$, then $e \perp b$ for all $b \in B$, whence $e = 0$, and therefore $Te = 0$. Thus $\{(b,0): b \in B\}$ in a maximal orthonormal set in G and hence dim G = dim E.

Now take $E = l_2$ and let B be an orthonormal basis for E. Let $B_1 \subseteq E$ be such that $B \cup B_1$ is a Hamel basis for E. As we have already seen, the cardinality of B_1 is c. Let F be a Hilbert space of dimension c and $T: E \to F$, so that T maps B_1 one-to-one onto an orthonormal basis for F and is 0 on B. Then T satisfies the conditions of Lemma 2.4 so dim $G = \aleph_0$ and dim $\bar{G}$ = dim E + dim $F = c$.

Thus G is an inner product space with completion H with the following properties:

(a) dim $G <$ dim H.

(b) G has no orthonormal basis B_0.

Proof. Since $\bar{G} = H$, B_0 is an orthonormal basis for H so dim G = dim H, a contradiction.

(c) A maximal orthonormal set in G is not basic.

(d) dim $G = \aleph_0$, yet G is not separable.

Proof. If G were separable, then H would be.

(e) If B_2 is a maximal orthonormal set in G the Fourier series of an element $x \in G$ need not converge to x.

(f) A maximal orthonormal set in G is not maximal for the completion of G.

The previous example relied upon the graph of a linear transformation in the direct sum of two Hilbert spaces. Since this may not be particularly easy to visualize we now give another example which does not rely on these concepts or on Lemma 2.4. Let $H_1 = l_2$ ([0,1]), $Z = \{z_1, z_2, \cdots\}$ the rationals in [0,1] and Q the irrationals in [0,1]. Extend the linearly independent vectors $y_{z_1}(n) = \delta_{1n}$, $y_{z_2}(n) = \delta_{2n}, \cdots$ to a Hamel basis $y_{z_1}, y_{z_2}, \cdots, y_\lambda$, $\lambda \in Q$, for l_2. Define vectors $f_\alpha \in H_1$, $\alpha \in [0,1]$, as follows: if $\alpha \in Z$, $f_\alpha(x) = \delta_{\alpha x}$; if $\alpha \in Q$, $f_\alpha(x) = 1$ if $x = \alpha$, $f_\alpha(x) = y_\alpha(n)$ if $x = z_n$ and $f_\alpha(x) = 0$ otherwise. Let G_1 be the linear manifold generated by the f_α's, $\alpha \in [0,1]$. We leave it as an exercise to show that $\bar{G}_1 = H_1$ and that $\{f_\alpha: \alpha \in Z\}$ is a maximal orthonormal set in G_1. Thus we have constructed an inner product space G_1 with completion H_1 such that dim $G_1 <$ dim H_1 so this example exhibits the behaviors (a)–(f) of the previous one.

One can illustrate the property that a maximal orthonormal set in an inner product space need not be basic by a simpler example than the previous ones. This example does not exhibit some of the other properties however. Let H be a separable Hilbert space with orthonormal basis $\{e_1, e_2, \cdots\}$, and let X be the linear manifold

generated by the vectors

$$\left\{ f = \sum_{i=1}^{\infty} i^{-1} e_i, e_2, e_3, \cdots \right\}.$$

Then it is easy to see that $B = \{e_2, e_3, \cdots\}$ is a maximal orthonormal set in X. But B is not basic since f is not of the form $f = \sum_{i=2}^{\infty} c_i e_i$.

3. Completeness. We now ask for criteria which ensure the completeness of an inner product space. Now two of the most important elementary properties of a Hilbert space H are the projection theorem and the Riesz theorem. The projection theorem states that if M is a closed subspace of H then $M \oplus M^{\perp} = H$ ($M^{\perp} = \{x: \langle x, y \rangle = 0$ for all $y \in M\}$). This may be weakened to the following: if M is a proper closed subspace of H then $M^{\perp} \neq \{0\}$. The Riesz theorem states that if f is a continuous linear functional on H, there exists $y \in H$ such that $f(x) = \langle x, y \rangle$ for all $x \in H$. We now show that these results need not hold in an inner product space and, in fact, characterize completeness for such spaces.

THEOREM 3.1. *For an inner product space V the following statements are equivalent*:

(1) *V is complete.*

(2) *If M is a closed subspace of V then $M \oplus M^{\perp} = V$.*

(3) *If M is a closed subspace of V then $M = M^{\perp\perp}$.*

(4) *If M is a proper closed subspace of V then $M^{\perp} \neq \{0\}$.*

(5) *If f is a continuous linear functional on V there exists $y \in V$ such that $f(x) = \langle x, y \rangle$ for all $x \in V$.*

Proof. That (1) $\Rightarrow$ (2) is the projection theorem. To show (2) $\Rightarrow$ (3) it is first clear that $M \subseteq M^{\perp\perp}$. If $x \in M^{\perp\perp}$ then by (2) $x = x_1 + x_2$ where $x_1 \in M$, $x_2 \in M^{\perp}$. Now $x_2 = x - x_1 \in M^{\perp\perp}$ so $x_2 = 0$ and hence $x = x_1 \in M$. To show (3) $\Rightarrow$ (4) let M be a proper closed subspace and suppose $M^{\perp} = \{0\}$. By (3), $M = M^{\perp\perp} = \{0\}^{\perp} = V$ which is a contradiction. For (4) $\Rightarrow$ (5) let f be a continuous linear functional on V If $f = 0$ let $y = 0$. Otherwise $M = \{x : f(x) = 0\}$ is a proper closed subspace of V, hence by (4) there exists $y_1 \in M^{\perp}$ with $\| y_1 \| = 1$. For $x \in V$, $x - f(y_1)^{-1} f(x) y_1 \in M$. Therefore, $\langle x, y_1 \rangle - f(y_1)^{-1} f(x) = 0$ and if we let $y = \overline{f(y_1)} y_1$, then $f(x) = \langle x, y \rangle$ for all $x \in V$. For (5) $\Rightarrow$ (1) let $\hat{V}$ be the completion of V and let $y \in \hat{V}$. Since $x \mapsto \langle x, y \rangle$ is a continuous linear functional on V by (5) there is a $z \in V$ such that $\langle x, z \rangle = \langle x, y \rangle$ for all $x \in V$. Since $\bar{V} = \hat{V}$, $\langle \hat{x}, z - y \rangle = 0$ for every $\hat{x} \in \hat{V}$ and hence $y = z \in V$. Therefore, $V = \hat{V}$ and V is complete.

It follows from Theorem 3.1 that if V is an incomplete inner product space, then there exists a proper closed subspace M of V which does not satisfy (2), (3) and (4) of the theorem. Such a subspace is constructed in the following example. Let $V \subseteq l_2$ be the inner product space consisting of those sequences (a_i) such that $a_i = 0$ except for finitely many i's. Let $M = \{(a_i) \in V : \Sigma\, i^{-1} a_i = 0\}$. Then M is a proper closed

subspace of V and $M^{\perp} = \{0\}$. Indeed, M is closed in V since M is the intersection of the null space of the continuous linear functional $f((b_i)) = \Sigma i^{-1} b_i = \langle (b_i), (i^{-1}) \rangle$ with V. To show $M^{\perp} = \{0\}$ let $y = (a_1, a_2, \cdots, a_N, 0, \cdots) \in M^{\perp}$. For $1 \leqq j \leqq N$ let $x = (b_1, b_2, \cdots)$ satisfy $b_j = -j$, $b_{N+1} = N+1$, $b_i = 0$ for $i \neq j$, $N+1$. Then $x \in M$ and hence $\langle y, x \rangle = 0$. It follows that $y = 0$. Since (2) $\Rightarrow$ (3) $\Rightarrow$ (4) in Theorem 3.1, M does not satisfy (2) or (3) either. Also if V is not complete it is easy to construct a continuous linear functional f on V which is not of the form $f(x) = \langle x, y \rangle$, $y \in V$. Indeed, let $f(x) = \langle x, \hat{y} \rangle$ for some $y \in \hat{V}$, $\hat{y} \notin V$.

Amameyi and Araki [1] have given a characterization of completeness which is stronger than (2) of Theorem 3.1. This characterization is interesting in that it is an algebraic property which is equivalent to the topological property of completeness. Their result states that an inner product space V is complete if and only if for any subset $S \subseteq V$ we have $S^{\perp\perp} \oplus S^{\perp\perp\perp} = V$. The proof of their result is more complicated than the proof of Theorem 3.1 and we refer the reader to their original paper. Using this result, S. Holland has characterized completeness in terms of lattice properties of the closed subspaces of V [5].

In Section 2 we showed that in an inner product space maximal orthonormal sets need not be basic (or total). We now show that this condition characterizes incomplete inner product spaces.

Theorem 3.2. *An inner product space V is complete if and only if every maximal orthonormal set in V is basic.*

Proof. Necessity has already been proved. For sufficiency assume every maximal orthonormal set in V is basic and suppose M is a proper closed subspace of V with $M^{\perp} = \{0\}$. Let B a maximal orthonormal set for M and extend B to a maximal orthonormal set $B \cup B_1$ for V. If $x_1 \in B_1$ there exists $y \in M$ such that $\langle y, x_1 \rangle \neq 0$. Since $B \cup B_1$ is basic $y = \Sigma c_i y_i + \Sigma d_i x_i$, $y_i \in B$, $x_i \in B_1$. Now $z = \Sigma d_i x_i = y - \Sigma c_i y_i \in M$ but since $x_i \perp B$, $i = 1, 2, \cdots$, we have $z \perp B$. Since B is maximal in M, $z = 0$. It follows that $\langle y, x_1 \rangle = d_1 = 0$ a contradiction. Hence $B_1 = \phi$ and B is a maximal orthonormal set for V, so B is basic in V. Therefore, $M = V$ contradicting the fact that M is proper. It follows that $M^{\perp} \neq \{0\}$ so by Theorem 3.1 (4), V is complete.

Acknowledgements. The author is indebted to two colleagues who influenced this paper considerably. In fact, many of the ideas in this paper are due to them. For Section 2 the author is indebted to P. R. Halmos who was kind enough to send him a copy of his notes "Dimension in Inner Product Spaces" (1967). D. Strawther contributed help in proving some of the results in Section 3.

References

1. I. Amemeyi and H. Araki, A remark on Piron's paper, Publ. Res. Inst. Math. Sci. Ser. A, No. 12 (1966/67) 423–427.

3. N. Bourbaki, Espaces Vectoriels Topologiques, Chap. V, § 2 No. 4, Hermann, Paris, 1955.

3. J. Byrnes, A complete set which is not a basis, this MONTHLY, 79 (1972) 510–512.
4. M. Golomb, Complete orthonormal systems in pre-Hilbert spaces, this MONTHLY, 79 (1972) 263–267.
5. S. Holland, Partial solution to Mackey's problem about modular pairs and completeness, Canad. J. Math., 21 (1969) 1518–1525.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF DENVER, DENVER, COLORADO 80210.

CONVEX REGIONS WHICH COVER ARCS OF CONSTANT LENGTH

JOHN GERRIETS AND GEORGE POOLE

1. Introduction. In this paper we shall report the details of the results announced in [4] concerning the famous and elusive "Worm Problem" of Leo Moser [3]: What is the (convex) region of smallest area which will accommodate every arc of length L? It will be shown that every arc of length L can be placed by appropriate rigid motions within the closed region whose boundary is a rhombus with major diagonal L and minor diagonal $L/\sqrt{3}$. Furthermore, this region can be truncated to obtain a region which also covers all arcs of constant length L. We shall indicate briefly some developments toward the solution of this problem which have brought it to its current status.

2. The worm problem. There is an interesting way to describe the problem posed by Moser which will set the stage for our discussion. Given a worm of length L (having no other dimension) which is placed on a table, what is the size and shape of the flat surface of minimum area that can be used as a hammer head which upon striking the worm will "smash" the worm simultaneously from "stem to stern" no matter what shape (planar) he wiggles into? For example, if the surface of the hammer head is the closed region bounded by a circle of diameter L, then by hitting the worm with the center of the hammer head at his midpoint one can be assured of obliterating all of the worm simultaneously. The area of this region is approximately $.78539L^2$.

Amram Meir showed some years ago that a semidisc of diameter L will accommodate all curves of length L. His elegant proof appears for the first time in Wetzel's paper [6]. The area of this region is approximately $.39269L^2$.

One approach to improving the result above is to consider a class of regions whose boundary is a fixed geometrical figure. For example, Schaer and Wetzel [5] showed that among all closed regions whose boundary is a square, the one with diagonal of length L is the smallest which covers all arcs of length L. However, in this particular case, the area is $.50000L^2$ and does not improve the result of Meir.

Similarly one may consider the class of all closed regions whose boundary is an equilateral triangle. But here again one side must have length at least L so that the minimum area is at least $.43301L^2$ (in fact, the side must exceed L [**1**]).

Wetzel [**6**] used this approach on the class of all sectors to obtain a result which generalizes, in some sense, the result of Meir. He showed that for the sector $S(r,2\theta)$ with radius r and angle 2θ, if $r \geqq (L/2)\csc(\theta)$, then $S(r,2\theta)$ covers all arcs of length L. When $r = (L/2)\csc(\theta)$ and the area is minimized with respect to θ, the resulting region has area less than $.34510L^2$, a significant improvement to Meir's solution. Wetzel went on to show that this sector may be truncated at the vertex to obtain a region with area less than $.34423L^2$ which also covers each arc of length L. It is not known whether a sector $S(r,2\theta)$ with $r < (L/2)\csc(\theta)$ can accommodate all arcs of length L when $\theta < \pi/6$. Concerning this remark, we now make the following observation to which we shall refer later. To cover all arcs of length L the sector $S(r,2\theta)$ must always contain a straight line segment of length L. When $r < (L/2)\csc(\theta)$ the chord of the sector is smaller than L so that 2θ must be less than $\pi/3$. This allows for r to equal or exceed L.

Gerriets [**2**] followed up Wetzel's work with a region whose area is less than $.32140L^2$ and covers any arc of length L. This region is somewhat similar to the sectors considered by Meir and Wetzel for it is the union of two regions whose boundaries are an isosceles triangle with altitude $L/4$ and base L and a semiellipse with major axis length L and minor axis length $L/2$.

We close this section by mentioning that Wetzel [**6**] has shown that any region which accommodates all arcs of length L must have area at least $.21946L^2$.

3. The main theorem. By slightly modifying the region considered by Gerriets we obtain a region with area less than $.28870L^2$ which covers any arc of length L. We must admit that even though the result is rather striking the proof is quite simple.

THEOREM 1. *The closed region whose boundary is a rhombus with major diagonal L and minor diagonal $L/\sqrt{3}$ covers any arc of length L.*

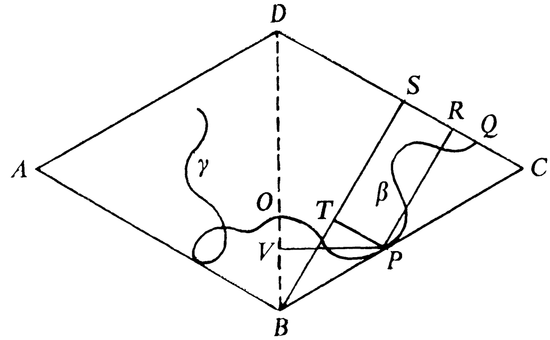


FIG. 1.

Proof. Let α be any arc of length L with midpoint O which divides α into two subarcs β and γ. $ABCD$ will denote the rhombus described in the theorem with BD having length $L/\sqrt{3}$. Suppose first that there is an orientation of α with O on BD, all points of α on or above angle ABC, and both β and γ contiguous with angle ABC. Suppose also that β meets BC at point P and meets DC at point Q with P lying between O and Q. Construct BS and PR perpendicular to DC and construct PT and PV perpendicular to BS and BD, respectively. Then the length of $\beta \geqq OP + PQ \geqq VP + PR = BT + TS = L/2$ which shows that β cannot cross DC. If Q lies between 0 and P, the argument is similar by symmetry. Similarly, γ is covered by the rhombus in this case.

Secondly, suppose in all possible orientations of α with O on BD as described above that only one arc, say β, is contiguous with angle ABC. Therefore, in any of these positions, as argued above, the rhombus covers β and by symmetry of the rhombus and the assumption that γ cannot touch angle ABC, γ cannot pass through angle ADC. For suppose γ even so much as touches angle ADC, then there is an orientation where it also touches angle ABC (since the figure is symmetric about line AC), contrary to the conditions of this case. Thus the rhombus covers all arcs of length L.

THEOREM 2. *The region described in Theorem* 1 *may be truncated to obtain a region with area less than* $.28610L^2$ *which covers all arcs of length* L.

Proof. Let $ABCD$ denote the rhombus described in Theorem 1 where EH and FG are perpendicular to BD at R and S, respectively, such that $DR = SB = (L/\sqrt{3} - L/2)/2$. Assume α is an arc of length L with midpoint O and subarcs β, γ. By Theorem 1, α can be positioned so that α lies within the rhombus $ABCD$. Assume also that α intersects both ΔDEH and ΔBGF (for otherwise either ΔDEH or ΔBGF may be truncated to yield the desired result).

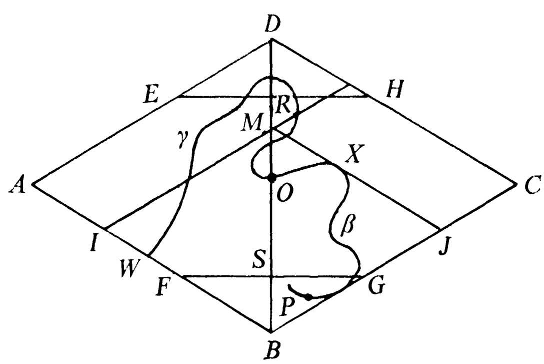


FIG. 2

First, suppose α is positioned as in the first part of the proof of Theorem 1. That is, both β and γ are contiguous with angle ABC. Suppose β has a point in ΔBGF

and let P be a point of β (in ΔBGF) for which the vertical distance ε from P to S is maximal and W is a point of γ on angle ABC (γ has no points in ΔBGF for if there were, neither β nor γ could have points in ΔDEH). Also, O is on or below M where M denotes the point on DB such that $RM = \varepsilon$. If IM and JM are parallel to AD and CD, respectively, then β has no points above angle IMJ. To see this, first, suppose β has a point X on IM (or (JM) between O and P. Then PX is greater than or equal to the vertical distance from P to X, and XO is greater than or equal to the vertical distance from X to M and, hence, $PX + XO \geqq SR = L/2$. Now suppose β has a point X on IM (or JM) such that P lies between O and X. Observe that $OD \leqq OB$ since γ has a point in ΔDEH and a point W on angle ABC. Therefore, $OP > L/4$ and $PX > L/4$ so that $\beta > L/2$. This shows that no point of β lies above angle IMJ.

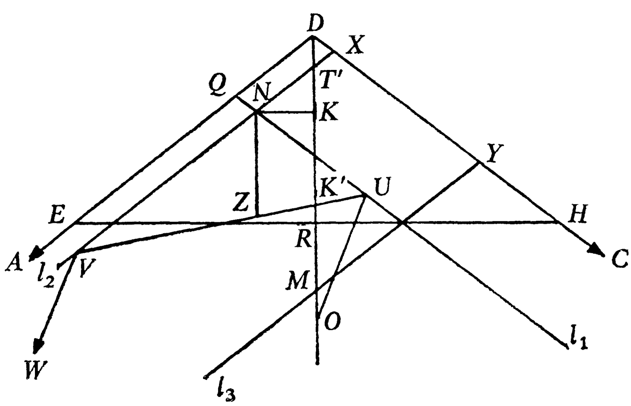


FIG. 3

Let l_1 denote the support line to α (from upper right) which is parallel to CD; l_1 lies above M (for otherwise the translation of α through the vector **MY** raises α by the distance $DM/2 = (\varepsilon + DR)/2 \geqq \varepsilon$ so that α is covered by $AFGCD$) and, hence, β cannot intersect l_1. Therefore l_1 intersects γ, say at U. Let K and K' be on DB with $DK = \varepsilon$ and $DK' = 2\varepsilon$. N denotes the point on l_1 such that NK is perpendicular to DK. Construct l_2 through N and parallel to AD.

Now suppose γ has a point V on l_2 in addition to the point U on l_1. Assume the order of these points along γ from O is $OUVW$. Z denotes the point on UV such that NZ is parallel to DB and l_3 is the line through M and parallel to AD. Clearly U and V are below N so that Z lies between U and V. If d_0 is the distance from O to l_2 and d_1 is the distance between l_2 and l_3, then $ZU + UO \geqq d_0 \geqq d_1$ (using a similar argument to the one in the proof of Theorem 1 on the appropriate rhombus). Similarly, $WV + VZ \geqq d_2$, the distance between AB and l_1. We may assume l_1 lies above K' for otherwise the translation **NX** of α places α within $AFGCD$. If $T = l_1 \cdot DB$ and $T' = l_2 \cdot DB$, then $DT = \varepsilon + KT = \varepsilon + KT' = K'T'$ so that the distance between CD and l_1 is the perpendicular distance d_3 from K' to l_2 which satisfies $d_3 \leqq d_1$. Therefore the length of

$$\gamma \geqq WV + VZ + ZU + UO \geqq d_2 + d_1 \geqq d_2 + d_3 = L/2$$

and γ lies on or below l_2. β also lies below l_2 since it has no points above the parallel lines through M. It now follows that $AFGCD$ covers α when α is translated through $\mathbf{NX} + \mathbf{NQ}$. For any other order of the points O, U, V, W on γ, it is clear that the length of γ will exceed $L/2$.

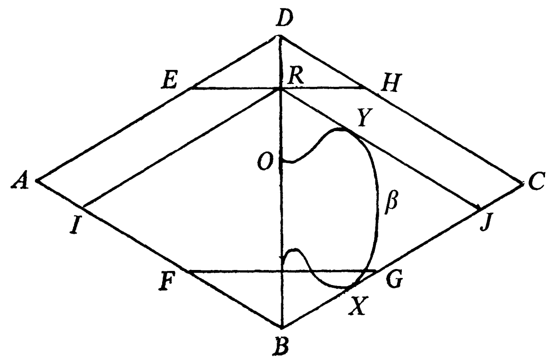


FIG. 4

Second, suppose α is positioned according to the second part of the proof of Theorem 1. That is, only one arc, say β, is contiguous with angle ABC. Since by the proof of Theorem 1 α is covered in any orientation (subject to the given conditions), select one for which the endpoint of β is on BD. Let RJ and RI denote line segments parallel to DC and DA, respectively. Let X be a point of β on angle ABC. Suppose β has a point Y on angle IRJ. Observe that (1) XY is greater than or equal to the vertical distance from X to Y, (2) the distance from X to BD is greater than the vertical distance from X to B, and (3) the distance from Y to BD is greater than the vertical distance from Y to R. It follows that the length of β is greater than BR, which is greater than $L/2$. Therefore β has no points on angle IRJ. Then γ has no points on angle IRJ, for (as in the second part of the proof of Theorem 1) if it did there would be an orientation of α such that points of γ lie on angle ABC. Therefore α is covered by $ABCHE$ and the proof is complete.

4. Conjectures on the worm problem. The truncation for the region described in Theorem 1 is not a minimal truncation. For example, we believe the other "tip" of the rhombus may be truncated. The figure in Conjecture 1 below is a member of a class of sectors that Wetzel has mentioned [**6**] but has no information on (see Section II).

CONJECTURE 1. *The sector $S(L, \pi/6)$ covers all arcs of length L.*

The area of the region above is $.261799L^2$. The following is a conjecture supported by the authors and John Wetzel.

CONJECTURE 2. *If a convex region contains all "two angle worms" of length L, then the region contains all worms of length L.*

By "two angle worms" we mean all arcs formed by joining consecutively, at any angles, three segments whose total length is L.

References

1. A. S. Besicovitch. On arcs that cannot be covered by an open equilateral triangle of side 1, Math. Gaz., 49 (1965) 286–288.

2. John Gerriets and George Poole, Covers accommodating curves of constant length, Unpublished solution.

3. Leo Moser, Poorly formulated unsolved problems of combinatorial geometry (mimeographed).

4. George Poole and John Gerriets, Minimum covers for arcs of constant length, To appear in the Bull. Amer. Math. Soc.

5. Jonathan Schaer and John E. Wetzel, Boxes for curves of constant length, Israel J. Math., 12 (1972) 257–265.

6. John E. Wetzel, Sectorial covers for curves of constant length, To appear in the Canadian Math. Bull.

DEPARTMENT OF MATHEMATICS, KANSAS STATE TEACHERS COLLEGE, EMPORIA, KANSAS 66801.

QUERIES

EDITED BY A. C. ZITRONENBAUM

This Department welcomes queries from readers about mathematics at the collegiate level, such as sources for exposition of a particular topic from a special point of view, references to vaguely remembered articles, descriptions of special kinds of courses or teaching methods, and methods for constructing illustrative examples for exercises of particular kinds (questions on research topics should, in general, be addressed to be the "Queries Department" of the Notices of the American Mathematical Society). Replies will be forwarded to the questioner and may also be edited into a composite answer for publication in this Department. Consequently all items submitted for consideration for possible publication should include the name and complete mailing address of the person who is to receive the reply. Queries and answers should be sent to A. C. Zitronenbaum, Department of Mathematics, Cornell University, Ithaca, NY 14850.

1. R. P. Boas, Jr. It is well known that it is possible to stack identical bricks into a cantilevered arch of arbitrarily large span (since the harmonic series diverges). A few years ago I saw a couple of articles that showed that with a given number of bricks one can get further by using some of the bricks as counterweights instead of using them to extend the arch directly. Does anybody know the references?

2. R. P. Boas, Jr. It is possible to give a proof that a connected graph is

unicursal if it contains at most two odd nodes, by using induction on the number of edges. Is this in print anywhere?

MATHEMATICAL NOTES

EDITED BY DAVID ROSELLE

Material for this Department should be sent to David Roselle, Department of Mathematics, Louisiana State University, Baton Rouge, LA 70803.

DISJOINT COVERING SYSTEMS

BŘETISLAV NOVÁK and ŠTEFAN ZNÁM

For an integer a and positive integer n the symbol $a(n)$ means the arithmetical sequence

$$\cdots,\ a-2n,\ a-n,\ a,\ a+n,\cdots.$$

A **disjoint covering system** (DCS) is such a system of arithmetic sequences

$$a_1(n_1), a_2(n_2), \cdots, a_k(n_k) \tag{1}$$

that every integer belongs to exactly one of them (see [**1**]). Without loss of generality we can suppose

$$1 < n_1 \leqq n_2 \leqq \cdots \leqq n_k; \tag{2}$$

$$0 \leqq a_t < n_t \text{ for } t = 1, 2, \cdots, k; \tag{3}$$

$$\text{if } n_t = n_{t+1} \text{ then } a_t < a_{t+1}\,. \tag{4}$$

We shall prove here a theorem giving a necessary and sufficient condition for a system to be disjoint covering.

From this theorem we can get very simply a lot of known assertions on disjoint covering systems.

LEMMA 1. *A system* (1) *is a DCS if and only if for any complex* z

$$\sum_{t=1}^{k} \frac{z^{a_t}}{1-z^{n_t}} = \frac{1}{1-z} \tag{5}$$

holds.

The proof follows immediately from the expansion of both sides in power series with the centre 0 or ∞, and from the uniqueness theorem.

THEOREM 1. (1) *is a DCS if and only if the relations*

$$\sum_{t=1}^{k} \frac{1}{n_t} \exp\left[2\pi i \frac{sa_t}{n_j}\right] = \begin{matrix} 0 \text{ for } s = 1,2,\cdots,n_j-1 \\ 1 \text{ for } s = n_j \end{matrix} \tag{6}$$

holds for all $j = 1,2,\cdots,k$.

Proof. If (1) is a DCS then by Lemma 1 the left and the right-hand sides of (5) have the same residues at the points

$$\exp\left[2\pi i \frac{s}{n_j}\right], \quad s = 1,2,\cdots,n_j; \ j = 1,2,\cdots,k.$$

Hence (6) holds. If (6) holds then the difference

$$F(z) = \sum_{t=1}^{k} \frac{z^{a_t}}{1 - z^{n_t}} - \frac{1}{1-z}$$

is a meromorphic function with (at most simple) poles at the points $\exp[2\pi i(s/n_j)]$ $s = 1,2,\cdots,n_j$, $j = 1,2,\cdots,k$; at these points the residues are zero (by (6)). Hence $F(z)$ is an entire function and $\lim_{z\to\infty} F(z) = 0$. By the Liouville theorem $F(z) \equiv 0$. Now, we can use Lemma 1.

In the sequel we shall suppose that (1) is a DCS.

THEOREM 2 (Erdös [1]). $\sum_{t=1}^{k} 1/n_t = 1$.

Proof. Put $s = n_j$ in (6).

THEOREM 3 (Davenport, Mirsky, Newman, Radó—see [1]). $n_{k-1} = n_k$.

Proof. Put $s = 1$, $j = k$ in (6).

If we put $s = 1$ in (6) we get the following generalization:

THEOREM 4. *For any* $j = 1,2,\cdots,k$ *there exists* $t \neq j$ $(t = 1,2,\cdots,k)$ *so that* $n_j \mid n_t$.

THEOREM 5 (Znám [6]). *Let* $n_1 \leqq n_2 \leqq \cdots \leqq n_{k-m} < n_{k-m+1} = n_{k-m+2} = \cdots = n_k$. *Let p be the smallest prime factor of* n_k. *Then* $m \geqq p$.

Proof. Putting $j = k$, $s = 1,2,\cdots,p-1$ in (6) we get the following system of equalities

$$\sum_{r=0}^{m-1} \exp\left[2\pi i \frac{sa_{k-r}}{n_k}\right] = 0.$$

Suppose $m < p$; then the system of equations

$$\sum_{r=0}^{m-1} x_r \exp\left[2\pi i \frac{sa_{k-r}}{n_k}\right] = 0, \qquad s = 1,2,\cdots,m$$

has a solution $x_0 = x_1 = \cdots = x_{m-1} = 1$. This is a contradiction because the determinant of this system is not equal to 0 (Vandermonde).

REMARK 1. Theorem 5 is a generalization of Theorem 3.

REMARK 2. We can similarly show that $n_k \nmid sn_t$ cannot hold for all $s = 1,2,\cdots,m$, $t < k - m$.

Our method is useful for the solving of other problems, too. Namely, if we suppose

$$n_1 < \cdots < n_{k-m} < n_{k-m+1} = \cdots = n_k,$$

then putting $s = 1$ and $j = k$ in (6) we get

$$\exp\left[2\pi i \frac{a_k}{n_k}\right] + \exp\left[2\pi i \frac{a_{k-1}}{n_k}\right] + \cdots + \exp\left[2\pi i \frac{a_{k-m+1}}{n_k}\right] = 0. \tag{7}$$

For the case $m = 2$ by elementary considerations (see [5]), we get from (7) that the system

$$a_1(n_1), a_2(n_2), \cdots, a_{k-2}(n_{k-2}), a_{k-1}(n_{k-1}/2) \tag{8}$$

is a DCS. Similarly, for $m = 3$ the system

$$a_1(n_1), a_2(n_2), \cdots, a_{k-3}(n_{k-3}), a_{k-2}(n_{k-2}/3) \tag{9}$$

is a DCS.

Now, according to Theorem 3 and using (8) step by step, we have $n_{k-1} = n_k$, $n_j/2 = n_{j-1}$ for $j = k-1,\cdots,2$. Finally, the pair $a_1(n_1)$, $a_2(n_2/2)$ is a DCS, hence $n_1 = 2$. So we have the proof of

THEOREM 6 (Stein [4]). *If $n_1 < n_2 < \cdots < n_{k-2} < n_{k-1} = n_k$, then $n_i = 2^i$ for $i = 1,2,\cdots,k-2$ and $n_k = n_{k-1} = 2^{k-1}$.*

Using (9) and similar considerations as above we have

THEOREM 7 (Znám [5]). *If $n_1 < n_2 < \cdots < n_{k-3} < n_{k-2} = n_{k-1} = n_k$ then $n_i = 2^i$ for $i = 1,2,\cdots,k-3$, $n_{k-2} = n_{k-1} = n_k = 3 \cdot 2^{k-3}$.*

The general case can be considered similarly. The main difficulty lies in the above mentioned "elementary considerations" The result is not always simple as is shown by the following theorem.

THEOREM 8 (Porubský [3]). *If $n_1 < n_2 < \cdots < n_{k-4} < n_{k-3} = n_{k-2} = n_{k-1} = n_k$, then there are two possibilities*:

(a) $n_i = 2^i$ *for* $i = 1,2,\cdots,k-4$, $n_{k-3} = n_{k-2} = n_{k-1} = n_k = 2^{k-2}$,

(b) $n_i = 2^i$ *for* $i = 1,2,\cdots,k-5$, $n_{k-4} = 3 \cdot 2^{k-5}$, $n_{k-3} = n_{k-2} = n_{k-1} = n_k = 3 \cdot 2^{k-4}$.

REMARK. From (6) some necessary conditions for (1) to be a DCS can be developed. For example: if for fixed j we add the equalities (6) for $s = 1,2,\cdots,n_j$, then we obtain

$$\sum_{t=1}^{k} \frac{1}{n_t} \left(\sum_{\substack{s=1 \\ n_j | sn_t}}^{n_j} \exp\left[2\pi i \frac{sa_t}{n_j}\right]\right) = 1, \quad j = 1,2,\cdots,k. \tag{10}$$

In the inner sum s runs over all multiples of $n_j/(n_j, n_t)$, hence

$$\sum_{\substack{s=1 \\ n_j | sn_t}}^{n_j} \exp\left[2\pi i \frac{sa_t}{n_j}\right] = \sum_{r=1}^{(n_t,n_j)} \exp\left[2\pi i \frac{a_t}{(n_t,n_j)} r\right] = \begin{matrix} 0 \text{ for } (n_t,n_j) \nmid a_t \\ 1 \text{ for } (n_t,n_j) \mid a_t. \end{matrix}$$

Afterwards, by (10) the expressions

$$\sum_{\substack{t=1 \\ (n_t,n_j)|a_t}}^{k} \frac{(n_t,n_j)}{n_t} = 1, \quad j = 1,2,\cdots,k$$

hold.

Theorem 5 has been proved also by M. Newman in the article "Roots of unity and covering sets," Math. Ann., 191 (1971) 279–282.

References

1. P. Erdös, Számelméleti megjegyzések, Mat. Lapok, 17 (1962) 228–255.

2. ———, Egy kongruenciarendszerekröl szóló problémaról, Mat. Lapok, 4 (1952) 122–128.

3. Š. Porubský, Generalization of some results for exactly covering systems, Mat. Časop., 22 (1972) 208–214.

4. S. K. Stein, Unions of arithmetic sequences, Math. Ann., 134 (1957–58) 289–294.

5. Š. Znám, On exactly covering systems of arithmetic sequences, Math. Ann., 180 (1969) 227–232.

6. ———, On exactly covering systems of arithmetic sequences, Coll. Math. Soc., János Bólyai, Number Theory 2, Debrecen, 1968.

BRETISLAV NOVÁK: KATEDRA MATEMATICKE ANALYSY MFFKU, PRAHA, SOKOLOVSKÁ 83, CZECHOSLOVAKIA.

ŠTEFAN ZNÁM: KATEDRA ALGEBRY A TEORIE ČISEL PFUK, BRATISLAVA, MLYNSKA DOLINA, CZECHOSLOVAKIA.

THE INDEPENDENCE OF SOME EXPONENTIAL VALUES

EUGENE SCHENKMAN

The object of this note is to show how ideas from a paper by Veblen [5] may be used to obtain an interesting and relatively simple proof of Lindemann's theorem (cf. [1] or [4]) from which it follows immediately that e and π are transcendental.

LINDEMANN'S THEOREM. *If for $i = 1, \cdots, n$, A_i and a_i are algebraic numbers with $a_i \neq a_j$ for $i \neq j$, and $A_i \neq 0$ for all i then*

$$(1) \qquad \sum_{i=1}^{n} A_i e^{a_i} \neq 0.$$

COROLLARY 1. *e is transcendental.*

Proof. For integers A_i, $i = 1, \cdots n$, $\sum_{i=1}^{n} A_i e^i \neq 0$.

Corollary 2. *π is transcendental.*

Proof. If π were algebraic so would be πi and then $e^{\pi i} + e^0$ would not be zero; but $e^{\pi i} = -1$.

We use Veblen's argument to prove the following lemma, which we use in the proof of the theorem.

LEMMA. *Let*

$$(2) \qquad F(x) = \sum_{m=0}^{\infty} C_m \frac{x^m}{m!} = \sum_{j=1}^{t} B_j e^{b_j x} \text{ for all } x$$

with the C_m rational numbers and the B_j and b_j non-zero algebraic numbers. Then $F(1)$ is not a non-zero rational integer.

Proof. Observe first that

$$(3) \qquad C_m = \sum_{j=1}^{t} B_j b_j^m \text{ for } m = 0, 1, \cdots$$

and hence there are natural numbers W and Z, depending only on F such that $W^m C_m$ is a rational integer for each m and

$$(4) \qquad |C_m| < Z^m \text{ for each } m.$$

Now let $f(x) = \sum_{i=0}^{s} q_i x^i$ with the q_i rational integers, $q_0 \neq 0$, such that each b_j is a zero of $f(x)$. For each prime p, let

$$(5) \qquad g_p(x) = x^{p-1}(f(x))^p = \sum_{h=0}^{sp} d_h x^{p-1+h}.$$

We note that $d_0 = q_0^p$ and that all the d_h are rational integers. Also if $M = \sum_{i=0}^{s} |q_i|$

then

$$|d_h| \leqq M^p. \tag{6}$$

Let

$$N_p = \frac{W^{sp}}{(p-1)!} \sum_{h=0}^{sp} d_h(p-1+h)! \tag{7}$$

and note that N_p is an integer satisfying

$$N_p \equiv W^{sp} d_0 \equiv W^{sp} q_0^p (\operatorname{mod} p). \tag{8}$$

We are going to show that there exists a prime p such that $N_pF(1)$ differs from a multiple of p by less than 1. Now

$$\begin{aligned}(p-1)!N_pF(x) &= W^{sp} \sum_{h=0}^{sp} d_h(p-1+h)! \sum_{m=0}^{\infty} C_m \frac{x^m}{m!} \\ &= \sum_{r=0}^{\infty} \sum_{h=v}^{sp} W^{sp} d_h(p-1+h)!\, C_{r+h-sp} \frac{x^{r+h-sp}}{(r+h-sp)!} \\ &= \sum_{r=0}^{\infty} U_r(x) = \sum_{r=0}^{sp-1} + \sum_{r=sp}^{sp+p-1} + \sum_{r=sp+p}^{\infty} .\end{aligned} \tag{9}$$

with $r = sp + m - h$, $v = \max(sp - r, 0)$ and

$$U_r(x) = \sum_{h=v}^{sp} d_h(p-1+h)!\, W^{sp} C_{r+h-sp} \frac{x^{r+h-sp}}{(r+h-sp)!} . \tag{10}$$

We shall consider separately the three sums in (9). We shall show that when $x = 1$, the first is divisible by $p!$, the second is zero, and the third is small.

Since $[(p-1+h)!]/[(h-1)!p!]$ is a binomial coefficient, it follows that for $r = 0, 1, \cdots, sp-1$ each $U_r(x)$ is a polynomial with coefficients that are integral multiples of $p!$. Hence

$$\sum_{r=0}^{sp-1} U_r(1) \equiv 0 \operatorname{mod} p!$$

From (10), (3) and (5) we see that

$$U_{sp+p-1}(x) = W^{sp} \sum_{h=0}^{sp} d_h C_{p-1+h} x^{p-1+h} = W^{sp} \sum_{j=1}^{t} B_j g_p(b_j x) \tag{11}$$

and

$$U_{sp+p-1-i}(x) = W^{sp} \sum_{j=1}^{t} B_j g_p^{(i)}(b_j x) \text{ for } i = 1, \cdots, p-1,$$

where $g_p^{(i)}(x)$ is the ith derivative of $g_p(x)$.

Since each b_j is a zero of $g_p^{(i)}(x)$ for $i = 0, 1, \cdots, p-1$, by (5) it follows that

$$\sum_{i=0}^{p-1} U_{sp+p-1-i}(1) = 0. \tag{12}$$

Finally $U_{sp+p}(1) = W^{sp} \sum_{h=0}^{sp} d_h C_{p+h}(1/p+h)$ so that from (6) and (4)

$$\left| U_{sp+p}(1) \right| \leqq W^{sp} \sum_{h=0}^{sp} M^p \frac{Z^{p+h}}{p} = \frac{(W^s MZ)^p}{p}(sp+1)Z^{sp} \leqq 2S(W^s MZ^{s+1})^p.$$

Similarly for $r = 1, 2, \cdots$

$$\begin{aligned} \left| U_{sp+p+r}(1) \right| &\leqq W^{sp} M^p \sum_{h=0}^{sp} Z^{p+r+h} \frac{(p-1+h)!}{(p+r+h)!} \\ &\leqq \frac{W^{sp} M^p Z^{p+r}}{(p+r)(p+r-1)\cdots(p+1)p}(sp+1)Z^{sp} \leqq 2S(W^s MZ^{s+1})^p \left(\frac{Z}{p}\right)^r. \end{aligned} \tag{13}$$

Thus if $p > 2Z$,

$$\left| \sum_{r=0}^{\infty} U_{sp+p+r}(1) \right| < 2S(W^s MZ^{s+1})^p \sum_{r=0}^{\infty} \frac{1}{2^r} = 4S(W^s MZ^{s+1})^p. \tag{14}$$

Now let p be chosen to satisfy (13) and also so that

$$4S(WMZ^{s+1}) < (p-1)! \tag{15}$$

Then from (9) with (11), (12), and (14) it follows that $N_p F(1)$ differs from a multiple of p by less than 1.

On the other hand if contrary to the Lemma, $F(1) = B_0$, a non-zero rational integer, then $N_p F(1) = N_p B_0$. But if p is chosen to satisfy (13), (15) and also so that $p > \max(W, q_0, B_0)$, then p is prime to B_0 and also to N_p by (8). Hence $N_p B_0$ differs from a multiple of p by at least 1, contrary to what was shown in the previous paragraph. This proves the Lemma.

Proof of the theorem. Let K be the normal closure of $Q[A_i, a_i;\ i = 1, \cdots, n]$, Q the field of rational numbers; and let Φ be the galois group of K over Q. We recall (cf. [2] or [3]) from Galois theory that Φ is finite and that Q is the fixed field of K under Φ; that is, if for $k \in K$, $k\phi = k$ for each $\phi \in \Phi$, then $k \in Q$. It follows that if $\sum a_i x^i$ is a polynomial (or power series) with the $a_i \in K$, then both $\prod_{\phi \in \Phi} \sum (a_i \phi) x^i$ and $\sum_{\phi \in \Phi} \sum (a_i \phi) x^i$ have coefficients in Q.

Now let D be a natural number and consider the function

$$H(x) = D \prod_{\phi \in \Phi} \sum_{i=1}^{n} (A_i \phi) e^{(a_i \phi)x}.$$

Then

$$H(x) = D \prod_{\phi \in \Phi} \sum_{i=1}^{n} (A_i \phi) \sum_{k=0}^{\infty} (a_i \phi)^k \frac{x^k}{k!} = \sum_{n=0}^{\infty} q_m \frac{x^m}{m!} \tag{16}$$

with $q_m \in Q$ and with D chosen so that q_0 is an integer. Furthermore $H(x) = \sum_{j=1}^{t} B_j e^{h_j x}$ with $h_i \neq h_j$ for $i \neq j$ and $B_j \neq 0$ for all j. We partition the above set of h_j into distinct conjugate classes under Φ; that is,

$$\{h_1, \cdots, h_t\} = \{b_1\phi, \phi \in \Psi_1\} \cup \cdots \cup \{b_c\phi, \phi \in \Psi_c\},$$

where $\Psi_1, \cdots, \Psi_c$ are subsets of Φ each containing the 1 of Φ and $b_1, \cdots, b_c$ are pairwise non-conjugate. Then

$$H(x) = \sum_{r=1}^{c} \sum_{\phi \in \Psi_r} B_{r,\phi} e^{(b_r\phi)x} \tag{17}$$

with each $B_{r,\phi} \neq 0$ since $B_j \neq 0$.

Since b_r is a sum of terms of the form $\sum_{\phi \in \Phi}(a_{i_\phi}\phi)$ and $B_{r,1}$ is a sum of products of the form $\prod_{\phi \in \Phi}(A_{i_\phi}\phi)$, and since ϕ is an additive as well as a multiplicative homomorphism, it follows that $B_{r,1}\phi = B_{r,\phi}$; therefore

$$H(x) = \sum_{r=1}^{c} \sum_{\phi \in \Psi_r} (B_{r,1}\phi) e^{(b_r\phi)x}. \tag{18}$$

From (16) and (18) we get by setting $x = 0$, $q_0 = \sum_{r=1}^{c} \sum_{\phi \in \Psi_r} B_{r,\phi}$ and we also have $t = |\Psi_1| + \cdots + |\Psi_c|$, where $|\Psi_r|$ is the cardinality of Ψ_r.

If $q_0 \neq 0$ we let $G(x)$ denote $\sum_{r=1}^{c} \sum_{\phi \in \Phi} e^{-(b_r\phi)x}$; while if $q_0 = 0$ we let $G(x)$ denote $\sum_{r=1}^{c} \sum_{\phi \in \Phi} (1 + B_{r,1}^{-1}\phi) e^{-(b_r\phi)x}$ noting that the power series expansion of $G(x)$ has rational coefficients in either case. Then $H(x) \cdot G(x)$ also has a power series expansion with rational coefficients and is a sum of exponentials;

$$H(x)G(x) = \sum_{m=0}^{\infty} C_m x^m = A + \sum_{j=1}^{s} D_j e^{d_j x} \text{ with } d_j \neq 0 \tag{19}$$

for each j and $C_m \in Q$ for each m. Furthermore A is a nonzero integer; for with the first definition of $G(x)$, $A = q_0$ (when $q_0 \neq 0$) while with the second definition of $G(x)$, $A = t$.

If now the theorem were not true and $\sum_{i=1}^{n} A_i e^{a_i}$ were zero, then $H(1)$ $G(1)$ would be zero. Then (19) would give that $\sum_{j=1}^{s} D_j e^{d_j x}$ equals the non-zero integer A contrary to the lemma. This proves the theorem.

References

1. A. O. Gelfond, Transcendental and Algebraic Numbers, translated, Dover, New York, 1960.
2. N. Jacobson, Lectures in Abstract Algebra, vol. III, Van Nostrand, Princeton, N. J., 1964.
3. S. Lang, Algebra, Addison-Wesley, Reading, Mass., 1965.
4. I. M. Niven, Irrational numbers, MAA Carus Math. Monograph No. 11, 1956.
5. O. Veblen, The transcendence of π and e, this MONTHLY, 11 (1904) 219–223.

DIVISION OF MATHEMATICAL SCIENCES, PURDUE UNIVERSITY, LAFAYETTE, INDIANA 47907.

Finally we note that a group can satisfy both chain conditions on normal subgroups and have infinitely many normal subgroups.

THEOREM 4. *A group G satisfying both chain conditions on normal subgroups has infinitely many normal subgroups if and only if there is a normal subgroup N of G, a group H and a simple H-module A with* $\mathrm{Aut}_H(A)$ *infinite such that G/N is an extension of $A \times A$ by H inducing the given operation of H on A.*

An example of such a group is provided by the semidirect product of $R^3 \oplus R^3$ by SO_3 where R^3 is Euclidean 3-space and SO_3 is its group of rotations.

A. W. HALES: DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA, LOS ANGELES, CALIFORNIA 90024.

R. J. NUNKE: DEPARTMENT OF MATHEMATICS, UNIVERSITY OF WASHINGTON, SEATTLE, WASHINGTON 98109.

AUTOMORPHISMS OF p-ADIC NUMBER FIELDS

C. G. WAGNER

In 1933 F. K. Schmidt [**4**, p. 3] proved a theorem which characterizes those fields which are complete with respect to at least two non-trivial inequivalent absolute values. A corollary of Schmidt's theorem states that a field complete with respect to a discrete absolute value is not complete with respect to an absolute value inequivalent to the original. An application of this corollary is the following standard proof of the fact that the identity map is the only automorphism of the field Q_p of p-adic numbers: Let g be an automorphism of Q_p with p-adic absolute value $|\ |_p$, and define an absolute value $|\ |_g$ on Q_p by $|\alpha|_g = |g^{-1}(\alpha)|_p$ for all $\alpha \in Q_p$. Then Q_p is complete with respect to $|\ |_g$, $|\ |_p$ and $|\ |_g$ agree on Q, and $|\alpha|_p = |g(\alpha)|_g$ for all $\alpha \in Q_p$. Since $|\ |_p$ and $|\ |_g$ are equivalent, they agree on Q_p, and so $|\alpha|_p = |g(\alpha)|_p$ for all $\alpha \in Q_p$. Hence g is continuous and is, thus, the identity map.

It may be of interest to note that there is an alternative "elementary" proof that any automorphism of Q_p is continuous, based on the fact that such an automorphism must preserve the units of Q_p. That this is the case is an obvious consequence of the following algebraic characterization of the units.

THEOREM. *Let $\alpha \in Q_p$. Then $|\alpha|_p = 1$ if and only if α has an m-th root in Q_p for all positive integers m prime to $p(p-1)$.*

Proof. Sufficiency. Immediate from the fact that the range of $|\ |_p$ is the discrete set $\{0, p^n : n \in Z\}$.

Necessity. Since $|\alpha|_p = 1$, $\alpha = a_0 + a_1 p + a_2 p^2 + \cdots$, where $0 \leqq a_i < p$ and $a_0 \neq 0$. Consider the polynomial $f(x) = x^m - \alpha$. Since $(m, p-1) = 1$, it follows

from a well-known fact about power congruences [3, p. 95] that there exists a natural number a such that $0 < a < p$ and $a^m \equiv a_0 \pmod p$. Hence $|f(a)|_p < 1$ and, since $(m, p) = 1$ and $0 < a < p$, $|f'(a)|_p = |ma^{m-1}|_p = 1$. Thus by Newton's method [2, p. 52] we may construct a sequence in Q_p which converges to a root of $f(x)$.

We remark that Ax and Kochen have proved a more general version of the preceding theorem as part of their study of formally p-adic fields [1, p. 633].

I wish to thank the referee for calling Schmidt's paper to my attention.

References

1. J. Ax and S. Kochen, Diophantine problems over local fields, II, Amer. J. Math., 87 (1965) 631–648.

2. G. Bachman, Introduction to p-adic Numbers and Valuation Theory, Academic Press, New York, 1964.

3. C. Long, Elementary Introduction to Number Theory, Heath, Boston, 1965.

4. F. K. Schmidt, Mehrfach perfekte Körper, Math. Ann., 108 (1933) 1–25.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF TENNESSEE, KNOXVILLE, TENNESSEE 37916.

A POINCARÉ TYPE COINCIDENCE THEOREM

SIMEON REICH

Let B denote the unit ball of a finite-dimensional Euclidean space E. According to Brouwer's fixed point theorem a continuous $g: B \to B$ has a fixed point. Let S denote the boundary of B. Recall that two functions f and g which map a set X into another set Y are said to have a coincidence if there exists a point $x \in X$ such that $f(x) = g(x)$. Schirmer [3] has established the following interesting coincidence theorem:

THEOREM 1. *Let f and g map B continuously into itself, and suppose that $f(S) \subset S$. If $f \mid S: S \to S$ is not nullhomotopic, then f and g have a coincidence.*

This proposition formally includes Brouwer's theorem because the identity map on S is not nullhomotopic. Of course, Brouwer's theorem is an immediate consequence of this (highly non-trivial) fact [1, p. 341].

Schirmer's proof is somewhat complicated. In this note we present a very simple proof of an extension of Theorem 1. It seems to be difficult to obtain this extension by adapting Schirmer's arguments. In the sequel, if $0 \neq x \in E$, then the point $x/|x|$, which belongs to S, will be denoted by $p(x)$.

THEOREM 2. *Let f and g map B continuously into E, and suppose that $f(S) \subset S$. If $f \mid S: S \to S$ is not nullhomotopic and $g(y) \neq mf(y)$ for all $y \in S$ and $m > 1$, then f and g have a coincidence.*

Proof. If $f(x) \neq g(x)$ for all $x \in B$, one can define $r\colon B \to S$ by $r(x) = p(f(x) - g(x))$. $r \mid S\colon S \to S$ is nullhomotopic [1, p. 316]. On the other hand, $H\colon S \times [0,1] \to S$ defined by $H(y,t) = p(f(y) - tg(y))$ establishes a homotopy of $f|S$ to $r|S$. This contradiction completes the proof.

COROLLARY. *If a continuous $g\colon B \to E$ satisfies $g(y) \neq my$ for all $y \in S$ and $m > 1$, then it has a fixed point.*

This result has been rediscovered many times. Actually it is due to Poincaré [2, p. 259].

References

1. James Dugundji, Topology, Allyn and Bacon, Boston, 1966.
2. Shmuel Kaniel, Quasi-compact non-linear operators in Banach space and applications, Arch. Rational Mech. Anal., 20 (1965) 259–278.
3. Helga Schirmer, A Brouwer type coincidence theorem, Canad. Math. Bull., 9 (1966) 443–446.

DEPARTMENT OF MATHEMATICS, THE TECHNION—ISRAEL INSTITUTE OF TECHNOLOGY, HAIFA, ISRAEL.

RESEARCH PROBLEMS

EDITED BY RICHARD GUY

In this Department the Monthly presents easily stated research problems dealing with notions ordinarily encountered in undergraduate mathematics. Each problem should be accompanied by relevant references (if any are known to the author) and by a brief description of known partial results. Manuscripts should be sent to Richard Guy, Department of Mathematics, Statistics, and Computing Science, The University of Calgary, Calgary 44, Alberta, Canada.

WHICH GRAPHS HAVE UNIQUE DISTANCE TREES?

GARY CHARTRAND and SEYMOUR SCHUSTER

A graph G is **connected** if there exists a path between every two of its vertices. The **distance** $d_G(u,v)$ between vertices u and v of G is the minimum length (number of edges) among all $u-v$ paths in G. The vertex set $V(G)$ of G is a metric space with this distance function.

If G and H are connected, isomorphic graphs, then it is a trivial observation that there exists a one-to-one mapping from $V(G)$ onto $V(H)$ which preserves distance; indeed, any isomorphism from G to H is such a mapping. It seems reasonable to ask the converse question: For a connected graph G, which graphs H have the

property that there exists a one-to-one mapping from $V(G)$ onto $V(H)$ which preserves distance? However, each such graph H is necessarily isomorphic to G. It is natural, therefore, to relax this restriction from one which involves a global distance property to one with a local distance property.

A **tree** is a connected graph with no cycles. If G is a connected graph, then a **spanning tree** of G is a subgraph of G which is a tree having all the vertices of G. For a vertex v of a connected graph G, a spanning tree T of G is **distance-preserving** from v if $d_T(v,u) = d_G(v,u)$ for every vertex u in G. Ore proved [**4**, p. 103] that for a given connected graph G and vertex v of G, there exists a spanning tree of G (perhaps several) which is distance-preserving from v. We refer to such spanning trees as **distance trees** (from v). For example, there are two (non-isomorphic) distance trees, T and T', from v in the graph G of Figure 1.

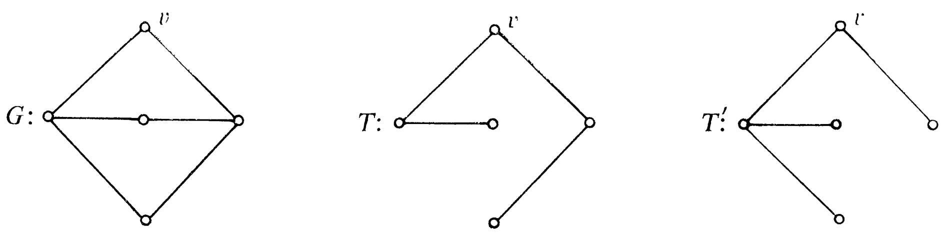


FIG. 1

Ore [**4**, p. 103] proposed and Baron [**1**] solved the problem of determining all graphs each of whose spanning trees is a distance tree.

If G is a connected graph such that all distance trees from all vertices are isomorphic to one another, then G is said to have a **unique distance tree**. We may now state our main problem.

PROBLEM 1. *Which graphs have unique distance trees?*

There are several classes of graphs which have unique distance trees, among which are the complete graphs, the graphs with $2n$ vertices, $n \geqq 2$, in which each vertex has degree $2n-2$, the Moore graphs (see [3]), and, of course, the trees. One Moore graph is the Petersen graph, shown in Figure 2 together with its (unique) distance tree.

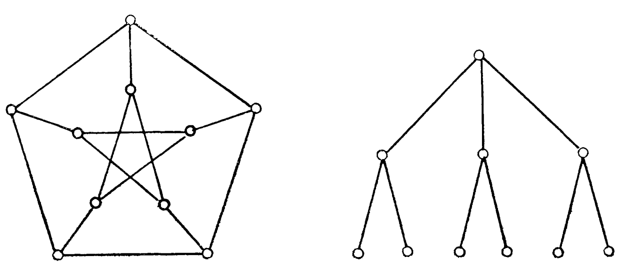


FIG. 2

A **cut-vertex** in a connected graph G is a vertex whose removal (together with all incident edges) results in a graph which is not connected. We have considered several examples of connected graphs having no cut-vertices and containing unique distance trees. Each of these graphs is regular (i.e., all vertices in the graph have the same degree).

CONJECTURE 1. *Every connected graph without cut-vertices having a unique distance tree is regular.*

Several examples of graphs with unique distance trees which contain cut-vertices exist (in addition to trees). Let G be a connected graph without cut-vertices having a unique distance tree such that $V(G) = \{v_1, v_2, \cdots, v_n\}$. Suppose T is an arbitrary tree (with possibly only one vertex) with a specified vertex v, and consider n copies of T denoted $T_1, T_2, \cdots, T_n$, where the vertex v is labeled v_i in T_i, $i = 1, 2, \cdots, n$. For each such i, all vertices different from v_i in T_i are distinct from the vertices of G and the vertices of each T_j, $j \neq i$. The graph H with vertex set $V(H) = [\bigcup_i V(T_i)] \cup V(G)$ and edge set $E(H) = [\bigcup_i E(T_i)] \cup E(G)$ possesses a unique distance tree.

Other examples of graphs with unique distance trees containing cut-vertices can be given. Suppose G is a cycle of even length n whose vertices are (in cyclic order) $v_1, v_2, \cdots, v_n$. As before, we let T be a tree (possibly having one vertex) with some specified vertex v and consider $n/2$ copies of T, denoted T_i for i odd and $1 \leqq i < n$; let v be labeled v_i in each such T_i. Also, for each odd i, $1 \leqq i < n$, the vertices different from v_i in T_i are distinct from the vertices of G and the vertices of T_j, j odd and $j \neq i$. Again, if we define H by letting $V(H) = [\bigcup_i V(T_i)] \cup V(G)$ and $E(H) = [\bigcup_i E(T_i)] \cup E(G)$, then H has a unique distance tree.

We propose the following.

CONJECTURE 2. *Every graph H (which is not a tree) having a unique distance tree is obtainable from a connected graph G without cut-vertices possessing a unique distance tree in one of two manners described above.*

In [2] it was shown that every graph (other than a tree) which possesses a unique distance tree has at least two vertices of maximum degree. Indeed, if Conjecture 2 is correct, then the only method for producing graphs H (not themselves trees) with unique distance trees and exactly two vertices of maximum degree is that described above where G is a cycle of length four.

In conclusion, we note that analogous problems exist for directed graphs (in which two vertices may be joined by two edges, one in each direction).

PROBLEM 2. *Which directed graphs have unique distance trees?*

G. Chartrand's work was supported in part by a grant from the office of Naval Research.
S. Schuster's work was supported by a grant from the National Science Foundation.

References

1. G. Baron, Charakterisierung der Graphen, deren spannende Bäume Distanzbäume resp. normale Bäume sind, Arch. Math., 20 (1969) 668–672.

2. G. Chartrand and M. J. Stewart, Isometric graphs, Recent Trends in Graph Theory (M. Capobianco, J. B. Frechen, and M. Krolik, eds.) Springer-Verlag, Berlin, 1971, pp. 63–67.

3. A. J. Hoffman and R. R. Singleton, On Moore graphs with diameters two and three, IBM J. Res. Develop., 4 (1960) 497–504.

4. O. Ore, Theory of Graphs. Amer. Math. Soc. Colloq. Publ., 38 (1962).

GARY CHARTRAND: DEPARTMENT OF MATHEMATICS, WESTERN MICHIGAN UNIVERSITY, KALAMAZOO, MICHIGAN 49001.

SEYMOUR SCHUSTER: DEPARTMENT OF MATHEMATICS, CARLETON COLLEGE, NORTHFIELD, MINNESOTA 55057.

A CONJECTURE ARISING FROM THEORETICAL GENETICS

M. A. B. DEAKIN

The "two-locus" model was introduced into mathematical genetics by Kimura [1] and by Lewontin and Kojima [2]. The model has recently been extensively discussed by Arunachalam and Owen [3]. It is the simplest of the genetic models to take into account processes such as recombination and epistasis which arise from the physical arrangement of genetic material into linear arrays on the chromosomes.

Mathematically, the model is specified by nine parameters R, w_{ij} $(1 \leqq i \leqq 4, 1 \leqq j \leqq 4)$ for which $w_{ij} = w_{ji}$ and $w_{14} = w_{23} = w$, $w_{ij} \geqq 0$. Without loss of generality w may be set equal to unity, as the only case for which this is invalid $(w = 0)$ is trivial. The ninth parameter R, which measures the "recombination" of the system, is subject to the constraint $0 \leqq R \leqq 0.5$.

Let $x_i (1 \leqq i \leqq 4)$ be variables subject to the restrictions $x_i \geqq 0$, $\Sigma_{i=1}^4 x_i = 1$. The governing equations of the system are then

$$\bar{w}\Delta x_i = x_i(w_i - \bar{w}) + \delta_i RD, \tag{1}$$

where

$$w_i = \sum_{j=1}^{4} w_{ij}x_j, \tag{2}$$

$$\bar{w} = \sum_{i=1}^{4} w_i x_i, \tag{3}$$

$$D = x_1x_4 - x_2x_3, \tag{4}$$

and

$$\delta_i = \begin{cases} +1 & \text{if } i = 2 \text{ or } 3 \\ -1 & \text{if } i = 1 \text{ or } 4. \end{cases} \tag{5}$$

Considerable attention has focussed on the determination of the equilibria of equations (1). Moran [4] was the first to point out explicitly that stable equilibria of (1) do not necessarily maximize $\bar{w}$, although this "result" was implicit in earlier formulations. Ewens [5] has subsequently provided plausible restraints on the quantities w_{ij} so that Moran's result does not apply. These relate to models in which the eight parameters w_{ij} may be reduced to four, the resulting model being termed "non-epistatic."

Scheuer and Mandel [6] and others have shown that $\bar{w}$ is maximised at equilibrium in the special case $R = 0$. This result is in accord with the general belief among geneticists that "tighter linkage" (i.e., lower values of R) leads to the preservation of "favourable gametotypes"—i.e., higher values of $\bar{w}_{eq}$, the equilibrium value of $\bar{w}$.

This may be formulated as a precise conjecture:

For fixed w_{ij}, $\bar{w}_{eq}$ *is a monotonic decreasing function of* R *under the constraints stated.* It is to be understood that this formulation applies to a given equilibrium as long as it can be followed as a function of R.

This formulation may be checked against the numerical results of Lewontin and Kojima [2] and Lewontin [7]. These authors consider certain numerically specialised cases which they submit to computer analysis. In all cases, the results are in accord with the conjecture.

A proof of the conjecture would aid considerably in elucidating the role played by fitness maximisation in genetics. A disproof would constitute another "disturbing result" [5] in that it would add one more paradox to the already disquieting number available under the two-locus model.

References

1. M. Kimura, A model of a genetic system which leads to closer linkage by natural selection, Evolution, 10 (1956) 278–287.

2. R. C. Lewontin and K. I. Kojima, The evolutionary dynamics of complex polymorphisms, Evolution, 14 (1960) 458–472.

3. V. Arunachalam and A. R. Owen, Polymorphisms with linked loci, Chapman and Hall, London, 1972.

4. P. A. P. Moran, On the nonexistence of adaptive topographies, Ann. Human Genet., 27 (1964) 383–393.

5. W. Ewens, Mean fitness increases when fitnesses are additive, Nature, 221 (1969) 1076.

6. P. A. G. Scheuer and S. P. H. Mandel, An inequality in population genetics, Heredity, 13 (1959) 519–524.

7. R. C. Lewontin, The interaction of selection and linkage, I. General considerations; heterotic models. Genetics, 49 (1964) 49–67.

Department of Mathematics, Monash University, Clayton, Australia.

CLASSROOM NOTES

EDITED BY DAVID ROSELLE

Material for this Department should be sent to David Roselle, Department of Mathematics, Louisiana State University, Baton Rouge, LA 70803.

AN ISOPERIMETRIC INEQUALITY FOR POLYHEDRA

H. H. JOHNSON AND J. OSAKA

There are many formulas and inequalities from differential geometry which can be extended to polygons and polyhedra when the basic notions of curvature, mean curvature, etc., are suitably defined. Moreover, these concepts are usually very intuitive and geometrically evident when defined on polyhedral surfaces, since these are discrete. Proofs which are quite technical for surface theory often have elementary analogues on polyhedra. Such material could well be introduced to students at a much earlier level than is possible in differential geometry.

In this note we define total and mean curvature for polyhedra in such a way as to extend an isoperimetric inequality of W. T. Reid [3] to these surfaces. Reid's proof used delicate analytic inequalities. Our proof is entirely elementary, using only vector algebra and the classical isoperimetric inequality for plane curves.

Reid's theorem concerns a surface S of area A bounded by a simple closed curve of length L in 3-space R^3. If the origin is assumed to be any point on the boundary and if X is the position vector to a point on S having mean curvature vector H, let $B = \int_S X \cdot H dA$. Then Reid's inequality is $L^2 \geqq 4\pi(A + B)$. When S is in a plane, $B = 0$ and Reid's inequality reduces to the classical isoperimetric inequality for plane areas.

Definitions. A **face** is a plane region bounded by a simple closed polygon $P = A_1A_2 \cdots A_n$ where the A_i are the vertices, the line segments A_iA_{i+1} and A_nA_1 are the edges. **Simple** means that edges meet only if they share a common vertex.

A **polyhedron** Π is a finite collection of faces satisfying: (1) two faces meet in at most a common edge or vertex, (2) no edge belongs to more than two faces, (3) if Q and Q' are faces which meet at a vertex A, it is possible to connect Q and Q' by a sequence $Q = Q_1, Q_2, \cdots, Q_r = Q'$ of faces having A as a vertex, in which Q_i and Q_{i+1} share a common edge, and (4) any two faces Q and Q' of Π can be connected by a sequence of faces where adjacent faces in the sequence share a common edge. The **boundary** of Π is the collection of all edges which belong to only one face. We also assume that Π is **orientable**. That is, it is possible to choose a "top side" to each of its polygons by means of a unit normal vector, so that along each common edge these top sides agree. We also assume that the origin is a vertex on the boundary. The right-hand rule determines the orientation of the boundary.

Let X_1X_2 be an edge common to two faces which have unit normals Y_1 and Y_2. Choose Y_1 and Y_2 so that Y_1, Y_2 and $X_2 - X_1$ forms a right-handed coordinate system. We define $\frac{1}{2}(X_2 - X_1) \times (Y_2 - Y_1)$ to be the mean curvature vector $H(X_1, X_2)$ along the edge X_1X_2. This measures the bending of Π along X_1X_2. When Y_1 and Y_2 are parallel, so that no bending occurs, $H(X_1, X_2) = 0$. It is easy to check that $H(X_1, X_2)$ does not depend on the accident of choosing Y_1 first.

In order to interpret Reid's inequality we must examine $H(X_1, X_2) \cdot X$, where X is a position vector to some point of the line segment X_1X_2.

LEMMA 1. *If X is any vector on the line segment X_1X_2, then*

$$H(X_1, X_2) \cdot X = \tfrac{1}{2}[X_2, Y_2 - Y_1, X_1] = \tfrac{1}{2}\{X_2 \times (Y_2 - Y_1)\} \cdot X_1 .$$

Proof. Let $X = tX_1 + (1 - t)X_2, 0 \leqq t \leqq 1$. Then

$$H(X_1, X_2) \cdot X = \tfrac{1}{2}(X_2 - X_1) \times (Y_2 - Y_1) \cdot (tX_1 + (1 - t)X_2),$$

and the result follows by elementary vector algebra.

In particular, $H(X_1, X_2) \cdot X$ is independent of X.

Let now B be the sum of these numbers $H(X_1, X_2) \cdot X$, where X is any vector on X_1X_2, summed over all edges X_1X_2 of the polyhedron Π. We shall prove that if Π has a simple boundary of length L, and if the area of Π is A, then $L^2 \geqq 4\pi(A + B)$.

First, we can subdivide the polygons of Π by adding extra edges so that each new polygon is a triangle. See Benson [1, p. 9]. This certainly does not change L or A. Also, B is unchanged, since $H(X_1, X_2) = 0$ if X_1X_2 is one of the new edges, for the unit vectors Y_1 and Y_2 to the polygons on each side of the edge are equal. Hence, we may assume that all polygons in Π are triangles.

PROPOSITION. *The area of triangle $X_1X_2X_3$ equals $\frac{1}{2}[(X_1 - X_3), (X_2 - X_3), Y]$ where Y is the unit normal to the triangle.*

This follows from the definition of cross product and its geometric interpretation as the area of a parallelogram formed by $X_1 - X_3$ and $X_2 - X_3$.

THEOREM 1. *$A + B = C$, where C is the sum, over all edges $X'X''$ along the boundary of Π, of the triple product $\frac{1}{2}[X', X'', Y]$, where Y is the orienting unit vector on the face having $X'X''$ as boundary, and the boundary is oriented from X' to X''.* (This theorem is true even if the boundary is not simple. A simple boundary is one consisting of a single polygon without self-intersections.)

Proof. Proof by induction on the number n of triangles in Π. For $n = 1$, we have only one triangle $X_1X_2X_3$, and

$$A + B = A = \tfrac{1}{2}[X_1 - X_3, X_2 - X_3, Y]$$
$$= \tfrac{1}{2}([X_1, X_2, Y] + [X_2, X_3, Y] + [X_3, X_1, Y]) = C.$$

Now assume the theorem true for all polyhedra which are decomposable into k triangles, and suppose Π has $k+1$ triangles. Let Π' be obtained by removing from Π one triangle, say $X_1X_2X_3$. Then if A', B', C' are the corresponding quantities for Π', $A'+B'=C'$ by induction.

Case 1. Suppose triangle $X_1X_2X_3$ has only one edge in common with Π', say X_1X_3. Suppose X_1X_3 is oriented in Π' from X_1 to X_3. Then Y, the unit normal to triangle $X_1X_2X_3$, orients this triangle in the order X_1 to X_2 to X_3. Hence,

$$\begin{aligned} A &= A' + \tfrac{1}{2}[X_2 - X_1, X_3 - X_1, Y] \\ &= A' + \tfrac{1}{2}([X_2, X_3, Y] + [X_3, X_1, Y] + [X_1, X_2, Y]). \end{aligned}$$

Since Π has one more interior edge, X_1X_3, than Π', let Y' be the unit normal to the triangle of Π' which adjoins X_1X_3. Then $B = B' + H(X_1, X_3)\cdot X_1 = B' + \tfrac{1}{2}[X_3, Y - Y', X_1]$.

Finally, $C = C' + \tfrac{1}{2}[X_1, X_2, Y] + \tfrac{1}{2}[X_2, X_3, Y] - \tfrac{1}{2}[X_1, X_3, Y']$. Hence,

$$\begin{aligned} A + B &= A' + \tfrac{1}{2}([X_2, X_3, Y] + [X_3, X_1, Y] + [X_1, X_2, Y] + B' + \tfrac{1}{2}[X_3, Y, X_1] \\ &\qquad - \tfrac{1}{2}[X_3, Y', X_1] \\ &= C' + \tfrac{1}{2}[X_2, X_3, Y] + \tfrac{1}{2}[X_1, X_2, Y] - \tfrac{1}{2}[X_1, X_3, Y'] = C. \end{aligned}$$

The other two cases, where triangle $X_1X_2X_3$ has two and then three edges in common with Π', are resolved similarly.

THEOREM 2. *If the boundary of* Π *is simple, and if* 0 *is a vertex on the boundary, then* $L^2 \geqq 4\pi(A+B)$.

Proof. Let the boundary be the polygon in space, $O, X_1, X_2, \cdots, X_n$, oriented in this order. Then

$$A + B = C = \tfrac{1}{2}([X_1, X_2, Y_1] + [X_2, X_3, Y_2] + \cdots + [X_n, O, Y_n]).$$

Now, $[X_i, X_{i+1}, Y_i] \leqq |X_i \times X_{i+1}|$, and $|X_i \times X_{i+1}|$ is twice the area of triangle OX_iX_{i+1}. Hence, $A + B \leqq S$, where S is the sum of the areas of the triangles OX_iX_{i+1}.

On the other hand, L is the sum of the lengths of sides $|X_1| + |X_2 - X_1| + |X_3 - X_2| + \cdots + |X_n|$. We can compare these quantities by supposing the triangles OX_iX_{i-1} all lie in the same plane, as a series of adjacent triangles sharing adjacent edges OX_i and the common vertex O. When the sum of the angles X_iOX_{i-1} is less than 2π, S is the area and L is the length of the boundary of this region. But then $L^2 \geqq 4\pi S$ by the classical isoperimetric inequality. All that remains is to consider what occurs if the triangles overlap, which happens if the sum of the angles at 0 is greater than 2π.

But then one obtains a plane figure consisting of an outer closed polygon with its area S_1 and length L_1 together with one or more inner polygons and their areas.

For each of these the isoperimetric inequality holds: $L_i^2 \geqq 4\pi S_i$. Hence,

$$(L_1 + \cdots + L_r)^2 \geqq L_1^2 + \cdots + L_r^2 \geqq 4\pi(S_1 + \cdots + S_r).$$

Hence the theorem is proved.

It may be noted that results of this kind are technically new. They are not included in Reid's theorem where differentiability of the surface is assumed.

Finally, Reid's theorem has been generalized to hypersurfaces in R^n by K. Hanes [2]. It would be illuminating and instructive to extend this result to polyhedra in R^n.

References

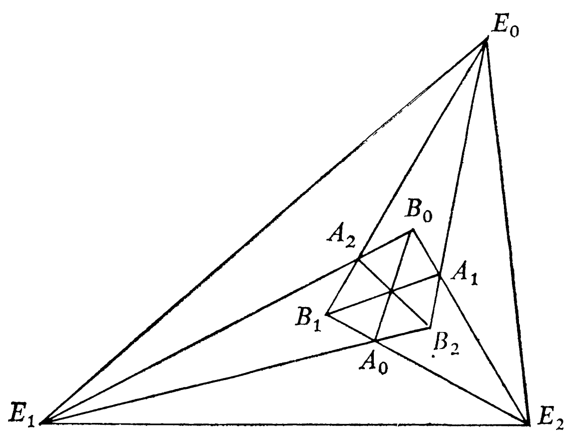
1. R. V. Benson, Euclidean Geometry and Convexity, McGraw-Hill, New York, 1966.
2. K. Hanes, Ph.D. Thesis, University of Washington, Seattle, Wash., 1970.
3. W. T. Reid, J. Math. Mech., 8 (1959) 897–906.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF WASHINGTON, SEATTLE, WASHINGTON 98195.

A GENERALIZATION OF MORLEY'S THEOREM

JOHN STRANGE

Morley's Theorem states that appropriately chosen pairs of the internal trisectors of the angles of a triangle meet in the vertices of an equilateral triangle. More precisely, with reference to the figure, $A_0A_1A_2$ *is an equilateral triangle.*



The line containing the distinct points X, Y will be denoted by XY. Furthermore a point of a projective space will, as usual, be identified with any one of its systems of homogeneous coordinates.

It is an easy exercise (Coxeter and Greitzer [**1**, pp. 49, 163]) in what is often called "elementary geometry" to deduce Morley's Theorem from the following

result: in the figure, *the lines A_0B_0, A_1B_1 and A_2B_2 are concurrent.* The theorem of this article is a generalization of this last result.

Let K be a commutative field having at least four elements, and let E_0, E_1, E_2 and E be four points of a projective plane on K, no three of which are collinear.

We define a projective involution on the set of lines containing the point E_0 by the following conditions:

The line E_0E is a self-corresponding element; the line E_0E_1 corresponds to the line E_0E_2.

Projective involutions are defined in a similar way on the set of lines containing E_1 and on the set of lines containing E_2.

Let l_0 be a line containing E_0. We suppose that l_0 is different from E_0E_1 and E_0E_2, and also that it is not a self-corresponding element. (Such a line l_0 exists since, by hypothesis, K has at least four elements). Let l'_0 be the line corresponding to l_0. Similarly, let l_1, l'_1 and l_2, l'_2 be pairs in the involutions on the sets of lines containing E_1 and E_2 respectively.

Let A_0 be the point of intersection of l_1 and l'_2, and let B_0 be the point of intersection of l'_1 and l_2. Let the points A_1, B_1, A_2, B_2 be defined similarly. We shall prove the following proposition:

The lines A_0B_0, A_1B_1, A_2B_2 are concurrent.

We choose the coordinate system for which

$$E_0 = (1,0,0),\ E_1 = (0,1,0),\ E_2 = (0,0,1) \text{ and } E = (1,1,1).$$

Since l_0 is not E_0E_1, let its equation be $\lambda_0x_1 = x_2$. Similarly, let the equations of l_1 and l_2 be $\lambda_1x_2 = x_0$ and $\lambda_2x_0 = x_1$ respectively. It follows that the equations of l'_0, l'_1, l'_2 are $x_1 = \lambda_0x_2$, $x_2 = \lambda_1x_0$, $x_0 = \lambda_2x_1$, respectively. Let

$$\mu_1(x_0 - \lambda_2x_1) + \mu_2(\lambda_1x_2 - x_0) = 0 \text{ and } \mu'_1(\lambda_2x_0 - x_1) + \mu'_2(x_2 - \lambda_1x_0) = 0$$

be equations of A_0B_0. Comparing coefficients in the usual way we find

$$\mu_1 - \mu_2 = \mu'_1\lambda_2 - \mu'_2\lambda_1,\ \mu_1\lambda_2 = \mu'_1,\ \mu_2\lambda_1 = \mu'_2$$

whence $\mu_1 - \mu_2 = \mu_1\lambda_2^2 - \mu_2\lambda_1^2$ so that $\mu_1(1 - \lambda_2^2) = \mu_2(1 - \lambda_1^2)$. We thus choose $\mu_1 = 1 - \lambda_1^2$ and $\mu_2 = 1 - \lambda_2^2$.

Now the equation of A_0B_0 is

$$(\mu_1 - \mu_2)x_0 - \lambda_2\mu_1x_1 + \lambda_1\mu_2x_2 = 0.$$

Similarly, the equations of A_1B_1 and A_2B_2 are

$$\lambda_2\mu_0x_0 + (\mu_2 - \mu_0)x_1 - \lambda_0\mu_2x_2 = 0$$

and

$$-\lambda_1\mu_0x_0 + \lambda_0\mu_1x_1 + (\mu_0 - \mu_1)x_2 = 0$$

respectively, where $\mu_0 = 1 - \lambda_0^2$. To show that these three lines are concurrent, it is sufficient to show that the determinant of coefficients is 0.

$$\begin{vmatrix} \mu_1 - \mu_2 & -\lambda_2\mu_1 & \lambda_1\mu_2 \\ \lambda_2\mu_0 & \mu_2 - \mu_0 & -\lambda_0\mu_2 \\ -\lambda_1\mu_0 & \lambda_0\mu_1 & \mu_0 - \mu_1 \end{vmatrix}$$

$= (\mu_1-\mu_2)(\mu_2-\mu_0)(\mu_0-\mu_1) + \lambda_0^2\mu_1\mu_2(\mu_1-\mu_2) + \lambda_1^2\mu_2\mu_0(\mu_2-\mu_0) + \lambda_2^2\mu_0\mu_1(\mu_0-\mu_1)$.
Replacing λ_0^2 by $1-\mu_0$, λ_1^2 by $1-\mu_1$ and λ_2^2 by $1-\mu_2$, we get

$$(\mu_1 - \mu_2)(\mu_2 - \mu_0)(\mu_0 - \mu_1) - \mu_0\mu_1\mu_2(\mu_1 - \mu_2 + \mu_2 - \mu_0 + \mu_0 - \mu_1) + \mu_1\mu_2(\mu_1 - \mu_2)$$
$$+ \mu_2\mu_0(\mu_2 - \mu_0) + \mu_0\mu_1(\mu_0 - \mu_1)$$
$$= (\mu_1 - \mu_2)(\mu_2 - \mu_0)(\mu_0 - \mu_1) - (\mu_1 - \mu_2)(\mu_2 - \mu_0)(\mu_0 - \mu_1) = 0.$$

The Euclidean result is obtained by taking E to be a point equidistant from the three lines E_1E_2, E_2E_0, E_0E_1.

I wish to express my thanks to my colleague Dr. R. V. Nillsen for his help in the preparation of this paper.

References

1. H. S. M. Coxeter and S. L. Greitzer, Geometry Revisited, Random House and Singer, New York, 1967.
2. C. Lubin, A proof of Morley's theorem, this MONTHLY, 62 (1955) 110–111.
3. K. Venkatachalienger, An elementary proof of Morley's theorem, this MONTHLY, 65 (1958) 612.

MATHEMATICS DEPARTMENT, THE ROYAL UNIVERSITY OF MALTA, MSIDA, MALTA.

CONTINUOUS LINEAR FUNCTIONALS ARISING FROM CONVERGENCE IN MEASURE

T. K. MUKHERJEE AND W. H. SUMMERS

The Hahn-Banach theorem guarantees that the continuous linear functionals on a separated locally convex topological vector space E can distinguish the points of E. There are several examples which dramatize the role of local convexity in this fundamental result, and in this note we consider perhaps the most intuitive of these; namely, the space of measurable functions with the convergence in measure topology. This example is often used to exhibit a trivial dual ([**2**, p. 55], for instance), but its scope does not appear to be widely known.

Let $(X, \mathscr{S}, \mu)$ be a finite measure space, and let $S(X, \mu)$ be the associated vector space of all complex valued measurable functions. (For measure theory, we primarily

follow the terminology and notation used in [1]. In particular, we adopt the convention which regards two functions in $S(X,\mu)$ as equal if they agree almost everywhere.) The topology of convergence in measure is induced on $S(X,\mu)$ by the metric

$$\rho(f,g) = \int \frac{|f-g|}{1+|f-g|} d\mu$$

[1, p. 178]. Hereafter, $S(X,\mu)$ will be assumed to have this topology, and hence $S(X,\mu)$ is a metrizable topological vector space [2, p. 55]. We omit the proofs of these well-known facts, but they require no greater sophistication than does the following development.

A measurable subset A of X is called an **atom** if $0 < \mu(A)$ and, for every measurable set B contained in A, either $\mu(B) = 0$ or $\mu(B) = \mu(A)$.

LEMMA 1. *If A is an atom, then there is a (unique) complex number $c(f,A)$ corresponding to each $f \in S(X,\mu)$ such that*

$$\mu(\{x \in A : f(x) \neq c(f,A)\}) = 0.$$

Proof. Let $\{z_k\}_{k=1}^{\infty}$ be a countable dense set in $\mathbb{C}$ (the complex number field with the usual Euclidean topology). For each ordered pair (k,n) of natural numbers, let $S(k,n) = \{z \in \mathbb{C} : |z - z_k| \leqq 1/n\}$, and let $A(k,n)$ denote the corresponding measurable set $A \cap f^{-1}[S(k,n)]$, where $f \in S(X,\mu)$. Since μ is subadditive and

$$A = \bigcup_{k=1}^{\infty} A(k,n),$$

it follows that, for each n, we can select a pair (k_n, n) such that $\mu[A(k_n,n)] = \mu(A)$. Setting $B = \bigcap_{n=1}^{\infty} A(k_n,n)$, we have from the continuity of μ that $\mu(B) = \mu(A)$. Since $f(B) \subseteq \bigcap_{n=1}^{\infty} S(k_n,n)$, then $f(B)$ is a singleton, and this completes the proof.

Given an atom A and $f \in S(X,\mu)$, it now follows that $\int_A f d\mu = c(f,A)\mu(A)$. Consequently, corresponding to each atom A, there is a linear functional ϕ_A defined on $S(X,\mu)$ by

$$\phi_A(f) = \int_A f d\mu.$$

The vector space of all *continuous* linear functionals on $S(X,\mu)$ will be denoted by $S(X,\mu)'$.

LEMMA 2. *If A is an atom, then $\phi_A \in S(X,\mu)'$.*

Proof. Take ε to be positive, and choose δ to be the smaller of $\mu(A)/2$ and $\varepsilon/2$. If $f \in S(X,\mu)$ with $\rho(f,0) < \delta$, then

$$|c(f,A)|(1 + |c(f,A)|)^{-1}\mu(A) < \delta,$$

whence $|c(f,A)|\mu(A) < \varepsilon$; i.e., $|\phi_A(f)| < \varepsilon$.

If both A and B are atoms, then $\phi_A = \phi_B$ if and only if the symmetric difference of A and B has measure zero. This relationship will be indicated by writing $A = B[\mu]$.

LEMMA 3. *If $(X,\mathscr{S},\mu)$ contains an atom, then there exists a countable collection $\mathscr{A}$ of pairwise disjoint atoms such that, if B is any atom, $B = A[\mu]$ for some $A \in \mathscr{A}$. Furthermore, if $\mathscr{B}$ is any other such collection, then $\mathscr{B} = \mathscr{A}[\mu]$.*

Proof. (This lemma is essentially given in [**1**, p. 169]. However, for completeness, we outline a proof.) The relation $A = B[\mu]$ is an equivalence relation on the set $\mathfrak{A}$ of all atoms in $(X,\mathscr{S},\mu)$, and if $[A]$ denotes the corresponding equivalence class for each $A \in \mathfrak{A}$, then $\{[A] : \mu(A) \geqq 1/n\}$ is finite. Since this implies $\{[A] : A \in \mathfrak{A}\}$ is countable, it is possible to select, one from each $[A]$, a countable collection $\mathscr{A}$ of pairwise disjoint atoms.

A subset B of X is called **locally measurable** if $B \cap E$ is measurable for each $E \in \mathscr{S}$, and it is easy to see that if $f \in S(X,\mu)$ and B is locally measurable, then $f\chi_B$ is measurable, where χ_B denotes the characteristic function of B. Therefore, for any locally measurable set B and any linear functional ψ on $S(X,\mu)$, there is an associated linear functional ψ_B defined by $\psi_B(f) = \psi(f\chi_B)$.

LEMMA 4. *Assume B is a locally measurable subset of X. If B contains no atoms and $\psi \in S(X,\mu)'$, then $\psi_B = 0$.*

Proof. Suppose there is an $f \in S(X,\mu)$ such that $\psi_B(f) = 1$. Setting $g = f\chi_B$ and $A = \{x \in X : |g(x)| > 0\}$, we have that A is a measurable set which contains no atoms. A standard argument ([**2**, p. 56], for example) now yields a sequence $\{A_k\}_{k=1}^{\infty}$ of measurable sets contained in A such that $\mu(A_k) = 2^{-k}\mu(A)$ and $|\psi(g_k)| \geqq 1$, where $g_k = 2^k g\chi_{A_k}$ and $k = 1,2,\cdots$. For each k, however, $\rho(g_k,0) \leqq \mu(A_k)$, and so the sequence $\{g_k\}$ converges in measure to zero. Since this contradicts the continuity of ψ, the proof is complete.

THEOREM 1. *The space $S(X,\mu)'$ contains a nonzero element if and only if $(X,\mathscr{S},\mu)$ contains an atom.*

Proof. If A is an atom in $(X,\mathscr{S},\mu)$, then $\phi_A \in S(X,\mu)'$ by Lemma 2 and $\phi_A(\chi_A) = \mu(A) \neq 0$. Since X is locally measurable, the converse follows from Lemma 4.

We can now define our candidate for $S(X,\mu)'$. Either $(X,\mathscr{S},\mu)$ is *nonatomic* (contains no atoms), in which case let L be the set consisting of the zero functional on $S(X,\mu)$, or it contains a set $\mathscr{A}$ of atoms with the properties promised by Lemma 3, and in this case let $L = \{\phi_A : A \in \mathscr{A}\}$. Combining Lemmas 2 and 3, we see that L is a uniquely determined subset of $S(X,\mu)'$. The linear span of L, call it $M(X,\mu)$, is thus a well defined linear subspace of $S(X,\mu)'$.

THEOREM 2. *The space $M(X,\mu) = S(X,\mu)'$.*

Proof. In view of Theorem 1, it suffices to assume $(X, \mathscr{S}, \mu)$ contains an atom, and hence, by Lemma 3, there is a countable set $\mathscr{A}$ of pairwise disjoint atoms such that, if C is an atom, $C = A[\mu]$ for some $A \in \mathscr{A}$. Setting $B = X \setminus \cup \{A : A \in \mathscr{A}\}$, we have that B is a locally measurable set which contains no atoms. Now let $\psi \in S(X, \mu)'$. If $\mathscr{A}$ is finite, say $\mathscr{A} = \{A_k\}_{k=1}^n$, then define $\alpha_k = \psi(\chi_{A_k})\mu(A_k)^{-1}$, where $k = 1, \cdots, n$. In this case, for $f \in S(X, \mu)$, Lemmas 1 and 4 show

$$\psi(f) = \sum_{k=1}^{n} c(f, A_k)\psi(\chi_{A_k}) = \sum_{k=1}^{n} \alpha_k \phi_{A_k}(f).$$

On the other hand, if $\mathscr{A} = \{A_k\}_{k=1}^{\infty}$, then for $f \in S(X, \mu)$, define $f_n = \sum_{k=1}^{n} f\chi_{A_k}$, where $n = 1, 2, \cdots$. Since $\sum_{k=1}^{\infty} \mu(A_k)$ converges, the sequence $\{f_n\}$ converges in measure to $f - f\chi_B$, and so, using Lemmas 1 and 4 once again,

$$\psi(f) = \sum_{k=1}^{\infty} c(f, A_k)\psi(\chi_{A_k}).$$

Define the sequence $\{g_k\}_{k=1}^{\infty}$ in $S(X, \mu)$ as follows: for each k such that $\psi(\chi_{A_k}) \neq 0$, set $g_k = \psi(\chi_{A_k})^{-1}\chi_{A_k}$; otherwise, let $g_k = 0$. This sequence converges in measure to zero also since $\sum_{k=1}^{\infty} \mu(A_k)$ converges, whereas $\psi(g_k) = 1$ if $\psi(\chi_{A_k}) \neq 0$. Consequently, the set of natural numbers for which $\psi(\chi_{A_k}) \neq 0$ is (at most) finite, and this concludes the proof.

If the measurable sets containing no atoms all have measure zero, then $(X, \mathscr{S}, \mu)$ is called *purely atomic*, while $S(X, \mu)'$ *separates the points* of $S(X, \mu)$ if, given $f \in S(X, \mu)$, either $f = 0$ or there is a $\psi \in S(X, \mu)'$ such that $\psi(f) \neq 0$.

COROLLARY 1. *The space* $S(X, \mu)'$ *separates the points of* $S(X, \mu)$ *if and only if* $(X, \mathscr{S}, \mu)$ *is purely atomic.*

When is $S(X, \mu)$ locally convex? A necessary condition has already been noted; namely, $S(X, \mu)'$ separates the points of $S(X, \mu)$. In closing, we mention that this condition is also sufficient. Indeed, if $(X, \mathscr{S}, \mu)$ is purely atomic, then $S(X, \mu)$ is topologically isomorphic to a product of countably many copies of $\mathbb{C}$.

References

1. P. R. Halmos, Measure Theory, Van Nostrand, Princeton, N. J., 1950.

2. J. L. Kelly, I. Namioka, and co-authors, Linear Topological Spaces, Van Nostrand, Princeton, N. J., 1963.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ARKANSAS, FAYETTEVILLE, ARKANSAS 72701.

THE TONELLI INTEGRAL

R. E. Dressler and K. R. Stromberg

In 1923, Tonelli [2] gave an alternate exposition of the Lebesgue integral, employing the notion of quasi-continuity. We find that this material is not readily available in textbooks and we feel that a proof (not given by Tonelli) of the equivalence of the Tonelli integral and the Lebesgue integral offers a valuable source for exercises for ancillary classroom material.

We fix a closed interval $[a,b]$ and we write $R(f)$, $L(f)$, and $T(f)$ for the Riemann integral, Lebesgue integral and Tonelli integral (respectively) of a function f defined on $[a,b]$. We need the following definition.

Definition. A real-valued function f defined on $[a,b]$ is called **quasi-continuous** (q.c.) if there exists a sequence $(K_n)_{n=1}^{\infty}$ of compact subsets of $[a,b]$ and a sequence $(f_n)_{n=1}^{\infty}$ of continuous real-valued functions on $[a,b]$ such that for each n, $f(x)=f_n(x)$ for $x \in K_n$ and $\lim_{n\to\infty}\lambda(K_n) = b-a$ (where λ denotes Lebesgue measure).

We observe that if $|f| \leqq M < \infty$ on $[a,b]$ then we can choose the f_n's so that $|f_n| \leqq M < \infty$ on $[a,b]$ for all n.

The following theorem is needed to define the Tonelli integral.

Theorem 1. *Let f be bounded and q.c. on $[a,b]$. Then there exists a real number $T(f)$ such that $T(f) = \lim_{n\to\infty} R(f_n)$ for any uniformly bounded sequence of f_n's as in the definition.*

Proof. First we note that if $(f_n)_{n=1}^{\infty}$ is given and M is a uniform bound on $[a,b]$ for f and each of the f_n's, then

$$|R(f_m)-R(f_n)| \leqq R(|f_m-f_n|) \leqq 2M(\lambda(K_m')+\lambda(K_n')) \to 0 \quad \text{as} \quad m,n\to\infty.$$

Hence, $(R(f_n))_{n=1}^{\infty}$ is a Cauchy sequence and we denote its limit by $T(f)$. A standard "interlacing" argument suffices to show that the value $T(f)$ is independent of the choice of the sequence $(f_n)_{n=1}^{\infty}$.

If f is as in Theorem 1, we call $T(f)$ the **Tonelli integral** of f.

Theorem 2. *A function f is q.c. on $[a,b]$ if and only if f is Lebesgue measurable on $[a,b]$.*

Proof. Let f be q.c. on $[a,b]$ and let $(f_n)_{n=1}^{\infty}$ and $(K_n)_{n=1}^{\infty}$ be as in the definition. Then for any real number α, we have

$$B_\alpha = \{x \in [a,b] \mid f(x) > \alpha\} = A_\alpha \cup \bigcup_{n=1}^{\infty} \{x \in K_n \mid f_n(x) > \alpha\}$$

for some subset A_α of $\bigcap_{n=1}^{\infty} K_n'$. Since $\lambda(\bigcap_{n=1}^{\infty} K_n') = 0$, we see that B_α is the union of a Borel set and a set of Lebesgue measure zero.

That every Lebesgue measurable function on $[a, b]$ is q.c. on $[a, b]$ is a restatement of a well-known theorem of Luzin ([**1**], (11.36)).

THEOREM 3. *If f is bounded and q.c. on $[a,b]$, then $T(f) = L(f)$.*

Proof. Let $(f_n)_{n=1}^{\infty}$ and $(K_n)_{n=1}^{\infty}$ be as in the definition and let M be a uniform bound on $[a, b]$ for f and each of the f_n's. Then,

$$|L(f) - L(f_n)| \leqq L(|f - f_n|) \leqq 2M\lambda(K_n') \to 0 \quad \text{as } n \to \infty .$$

So

$$L(f) = \lim_{n\to\infty} L(f_n) = \lim_{n\to\infty} R(f_n) = T(f).$$

Tonelli calls a non-negative q.c. function f (defined on $[a, b]$) **integrable** if its sequence of upper truncates, $(g_n)_{n=1}^{\infty}$, satisfies $\lim_{n\to\infty} T(g_n) < \infty$ and denotes this limit by $T(f)$. An application of the Monotone Convergence Theorem shows that, for such an f, $T(f) = L(f)$. By separation into positive and negative parts, one sees that the class of Tonelli integrable functions on $[a, b]$ is precisely $L_1([a, b])$ and furthermore $T = L$.

REMARK 1. We observe that quasi-continuity can be defined on any locally compact Hausdorff space with respect to a given regular measure. In this setting a proof similar to that of Theorem 2 shows that the converse to Luzin's Theorem obtains.

REMARK 2. The sequence $(K_n)_{n=1}^{\infty}$ in the above definition need not be nested. Thus, the corresponding sequence $(f_n)_{n=1}^{\infty}$ need not converge a.e. to f. However, once Theorem 2 is known, a theorem of F. Riesz ([**1**], (11.26)) provides a subsequence which does so.

REMARK 3. The definition of Tonelli's integral is "elementary." It reqires no knowledge of Lebesgue measure. Since the K_n's that appear above are always compact, each occurrence of $\lambda(K_n)$ or $\lambda(K_n')$ can be replaced by a statement about the sum of the lengths of the intervals of $[a, b]$ complementary to K_n. Tonelli did so in [**2**].

We are grateful to Professor H. H. Snyder for directing our interest to the Tonelli integral.

References

1. E. Hewitt, and K. Stromberg, Real and Abstract Analysis, Springer Verlag, New York, 1965.
2. L. Tonelli, Sulla Nozione di Integrale, Annali di Mat., Ser. IV, 1 (1923) 105–145.

DEPARTMENT OF MATHEMATICS, KANSAS STATE UNIVERSITY, MANHATTAN, KANSAS 66502.

THE WEAK CLOSURE OF A CERTAIN SET IN l^1

J. V. RYFF

Among the early examples a student encounters in a first course in functional analysis is one which illustrates the inadequacy of sequences in the weak topology.

If H is a separable Hilbert space with orthonormal basis $\{\phi_n\}$, then von Neumann noted that the set $\{\phi_m + m\phi_n\}$, $m, n = 1, 2, 3, \cdots$ has the origin as a weak limit point, yet no subsequence converges to that point. As norm bounded sets in H are metrizable in the weak topology, any such example must necessarily be unbounded. This will always be the case where a normed linear space has a separable dual. Certainly there must exist bounded, weakly sequentially closed, subsets of normed linear spaces which are not weakly closed (they cannot be convex, however). The purpose of this note is to set down one such example. Perhaps my real motive is to apply the delightful lemma of Erdös [**1**] (see also [**2**]):

From each finite sequence of $n^2 + 1$ real numbers one may select an increasing (or decreasing) subsequence containing at least $n + 1$ of these numbers.

If $(a_1, \cdots, a_k)$ is a sequence, we shall call k the *length* of the sequence. Then one can say that sequences of length $n^2 + 1$ contain monotonic subsequences of length $n + 1$. Evidently, sequences of length n^2 contain monotonic subsequences of length n.

Our example involves a subset of l^1, the space of infinite sequences of real numbers $x = \{\xi_n\}$ for which $\|x\|_1 = \Sigma \; |\xi_n|$ converges. The conjugate space l^∞ of bounded sequences induces the weak topology of l^1 in which neighborhoods of the origin θ are given by finite intersections of the sets

$$U(\theta, \alpha; \varepsilon) = \{x : |\Sigma \;\; \xi_n \alpha_n| < \varepsilon\},$$

where $x = (\xi_1, \cdots, \xi_k, \cdots)$ is in l^1, $\alpha = (\alpha_1, \cdots, \alpha_k, \cdots)$ is a fixed element of l^∞ and $\varepsilon > 0$ is arbitrary.

The set in which we shall be interested consists of those vectors on the unit sphere of l^1 which can be expressed typically by

$$x = \left(0, \cdots, \frac{1}{n}, 0, \cdots, -\frac{1}{n}, 0, \cdots\right),$$

where exactly n entries are nonzero and these entries are alternately $\pm 1/n$. We take all possible vectors of this type which can be formed for $n = 1, 2, \cdots$. This will be our set W. Clearly, each $x \in W$ has unit norm. Nevertheless, *the origin lies in the weak closure of W*. Intuitively, this is not unreasonable, and it only remains for one to supply a proof. To do so, we extend the Erdös lemma slightly:

Given any n infinite sequences of real numbers $x^{(k)} = \{\xi_m^{(k)}\}$ $k = 1, \cdots, n$, and any natural number N, it is possible to select indices $m_1, m_2, \cdots, m_N$ such that each finite subsequence $(\xi_{m_1}^{(k)}, \cdots, \xi_{m_N}^{(k)})$ is monotonic.

The proof is easy and we will only do it for $n = 2$. Choose a monotonic subsequence from $x^{(2)}$ of length N^2, say, $\xi^{(2)}_{m_1}, \cdots, \xi^{(2)}_{N^2}$. Then choose a monotonic subsequence of length N from $\xi^{(1)}_{m_1}, \cdots, \xi^{(1)}_{N^2}$. The resulting set of indices also determines a monotonic subsequence from $x^{(2)}$.

Now we are ready. Let

$$U = U(\theta, \alpha^{(1)}, \cdots, \alpha^{(m)}; \varepsilon) = \{x: \left| \Sigma\, \alpha^{(k)}_j \xi_j \right| < \varepsilon\}$$

be a basic weak neighborhood of the origin in l^1 $(\alpha^{(k)} \in l^\infty, k = 1, \cdots, m)$. Then $U \cap W \neq \phi$. To see this, first take $M = \max \| \alpha^{(k)} \|_\infty$, $k = 1, \cdots, m$. Then choose n so large that $1/n < \varepsilon/2M$. Employing our lemma, we select simultaneously m monotonic finite subsequences from the $\alpha^{(k)}$ of length n. Let their indices be $m_1, m_2, \cdots, m_n$. The sequence x consisting of 0's except for the entry $(-1)^{j+1}(1/n)$ in position m_j $(j = 1, \cdots, n)$ is an element of W. Furthermore,

$$\begin{aligned} \left| \langle x, \alpha^{(k)} \rangle \right| &= \frac{1}{n} \left| \Sigma\, (-1)^{j+1} \alpha^{(k)}_{m_j} \right| \leqq \frac{1}{n}\left(\left| \alpha^{(k)}_{m_1} \right| + \left| \alpha^{(k)}_{m_n} \right| \right) \\ &\leqq \frac{2}{n} \| \alpha^{(k)} \|_\infty < \varepsilon. \end{aligned}$$

Although the origin lies in the weak closure of W, no sequence from W can converge weakly to this point, for a sequence in l^1 converges weakly if and only if it converges in the norm. But each element of W has unit norm.

Two final remarks about the Erdös lemma are in order here. First, the lemma does not provide information regarding the length of maximal monotonic subsequences contained in sequences of arbitrary length. However, a simple canonical example illustrates that for $(n-1)^2 + 1 \leqq k < n^2 + 1$, a sequence of length k has at least one monotonic subsequence of length n, and this is the best possible. An example for $k = 15$ $(n = 4)$ is $(12, 13, 14, 15, 8, 9, 10, 11, 4, 5, 6, 7, 1 2, 3,)$. Second, modifications of this prototype can be used to show that the extended lemma does not extend to infinitely many sequences. That is, there exist infinite collections of sequences with common monotonic subsequences of length not greater than 2.

This example (as well as von Neumann's) illustrates a norm closed set whose weak closure is obtained by adding a single point—the origin.

References

1. P. Erdös and G. Szekeres, A combinatorial problem in geometry, Compositio Math., 2 (1935) 463–470.

2. A. Seidenberg, A simple proof of a theorem of Erdös and Szekeres, J. London Math. Soc., 34 (1959) 352.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CONNECTICUT, STORRS, CONNECTICUT 06268.

MATHEMATICAL EDUCATION

EDITED BY SHIRLEY HILL AND PAUL MIELKE

Material for this Department should be sent to Shirley Hill, Department of Mathematics, University of Missouri, Kansas City, MO 64110, or to Paul Mielke, Department of Mathematics, Wabash College, Crawfordsville, IN 47933.

THE OPERATION AND EVALUATION OF A PROCTORIAL SYSTEM OF INSTRUCTION IN MATHEMATICS

ARMSTRONG MALTBIE, R. G. SAVAGE and J. L. WASIK

Individuals with a responsibility for teaching courses with large enrollments are continually looking for new methods of presenting material which will hopefully optimize both learning and motivation of students. Nowhere is this need for developing new procedures more important than the first year undergraduate mathematics course where students come into the course with widely varying levels of mathematical ability. Use of a self-pacing instructional format has often been suggested as an alternative for optimizing student learning, and particularly for handling the problem of wide ranges of student ability within classes. The Mathematics Department at North Carolina State University (NCSU) decided to determine if this general approach to instruction would be an effective method of teaching algebra and trigonometry in a basic first year mathematics course (MA 111). The Proctorial System of Instruction (PSI) was selected as an appropriate model for the envisioned instructional system. Utilizing the PSI principles presented by F. S. Keller (1968), a set of 20 instructional units were developed and utilized first during the Fall, 1971 Semester at North Carolina State University for teaching two sections of the algebra and trigonometry course. The following is a description of the operation of the resultant system and an evaluation of its effectiveness.

Features of PSI. The Proctorial System of Instruction as described by Keller has the following operational features:

1. The student moves through the course at his own pace.
2. Complete mastery of each unit is expected before the student moves on to the next unit.
3. Each student is assisted, as needed, by the instructor and student proctors.

It should be noted that the use of proctors and the instructors to provide explanations and to grade tests with the student looking on, ensures a high degree of personal contact both with an instructor and with the student proctors. This feature, which is not found in programmed instruction and computer assisted instruction approaches, shows the student that someone is still concerned with his progress.

Carroll (1963) and Bloom (1971) have proposed a model for learning which suggests that intellectual ability is an indication of the amount of time required to learn a particular instructional topic. It is their thesis that given enough time any student with a minimum amount of aptitude for a subject will be able to demonstrate mastery of a subject. The self-pacing and complete subject matter mastery requirement features of PSI can be seen to be consistent with the principles of mastery learning as proposed by Carroll and Bloom.

Operation of NCSU Algebra and Trigonometry PSI Course. Twenty instructional units presently comprise the NCSU PSI course in Algebra and Trigonometry. The topics covered by the units range from simple Arithmetic Operation of Algebraic Expressions—Solutions of Equations of First Degree in Unit I to Inverse Trigonometric Functions in Unit XX. Since it is assumed that each unit requires knowledge gained through the study of the previous units, the student moves through each of the twenty units in the instructional process until he has finished the 20 units. Each unit is composed of the following instructional elements:

1. Specific objectives stated in a behavioral form.
2. Explanations of the material associated with identified behavioral objectives.
3. Study references from the course textbook.
4. Suggested problem assignment.
5. Collateral references where additional explanations and examples may be found.

Beginning with the first unit the student studies assigned references and works problems associated with unit objectives. The student then takes a readiness test which is administered by a designated proctor. If the student is able to respond correctly to all questions on the readiness test, the student is certified as having demonstrated a thorough knowledge and understanding of the unit content. The student then proceeds to the second unit and continues in a like manner. If a student is unable to demonstrate competency, he is required to restudy the unit and then take an alternate form of the readiness test. The student can take as many re-tests as are necessary without penalty. Four or more alternate forms of the readiness test are available for each unit. Since all students are required to show complete mastery on the readiness tests, only the fact that a student has passed a unit readiness test is recorded; partial learning of the subject matter is not acceptable in a mastery learning approach to instruction.

Upon successful completion of the 20 instructional units, the student is required to take a comprehensive final examination. A passing grade of C is guaranteed all students who successfully complete the 20 units irrespective of the grade obtained on the final examination. Above average and substantially above average performance on the final examination is required for the awarding of a B or A grade, respectively, in the course.

If, at the end of the semester, a student has not completed the course, an Incomplete (I) grade is assigned conditional upon evidence that the student is still attempting to complete the course. If a student is observed to be falling substantially behind his classmates, in terms of successful completion of units, he is counseled to drop the course and to sign up for a section whereby instruction is in a lecture mode. This is done under the assumption that the student requires the forced pacing of a lecture approach.

Maintenance of the PSI course at North Carolina State University requires two or three proctors and an instructor per section enrollment of 20–30 students. Proctors used in the Fall, 1971 semester were upper class undergraduate students enrolled in the Science and Mathematics Education Curriculum at North Carolina State University. The role of proctors may vary depending upon the instructional responsibility assigned to them. In the instructional system used at North Carolina State University, proctors and the instructor are present in class to answer questions raised by students; proctors also administer and grade unit readiness tests. The instructor has overall responsibility for development of new instructional materials where needed and also serves in an instructional role by answering questions raised by students.

Determination of the effectiveness of PSI. In order to demonstrate that the PSI method was an effective approach to instruction in algebra and trigonometry at North Carolina State University, it was considered imperative that a study be carried out which could evaluate the PSI approach in comparison to the traditional method of MA 111 instruction. The following sections report the results of this evaluation.

Evaluation Design: A four group Solomon (1949) experimental design was used in this study to ascertain if there were differences in mathematics performance at the end of the course between the groups of students who received the PSI experience and groups of students receiving the typical MA 111 lecture and discussion instruction. The Solomon design enables the evaluator to determine if observed results were due to the different instructional approaches or to other possible explanations such as the novelty of new instructional approach — i.e., "the hawthorne effect." A schematic of the design is given below:

	Instructional Method	
Testing Experience	PSI	Lecture and Discussion
Post Test Only	G_1	G_2
Pre and Post Tests	G_3	G_4

It should be noted G_1 and G_3 will be hereafter referred to as the PSI groups while G_2 and G_4 represent control groups.

As can also be noted by reference to the study design schematic, one PSI group and control group were tested at the beginning of the semester while all four groups were tested at the conclusion of the course.

Subjects. Students are computer assigned to sections of MA 111 and it was assumed that students enrolled in different sections at the same hour represent random assignment of subjects to groups. Four of the sections meeting at the 10:00 AM class hour were randomly selected to provide the subjects for this study. Two of the sections were then randomly selected to receive the PSI experience. Students enrolled in the two PSI sections were informed of the nature of the evaluation study and given the option to transfer to other MA 111 sections.[1] A total of 50 students were enrolled in the PSI sections while 49 students were taught by the traditional format.

Criterion measures. Forms A and B of the Algebra and Trigonometry Cooperative Mathematics Tests, published by Educational Testing Service of Princeton, New Jersey, were used to collect pre and post test data and served as the criterion measures of student performance. Each test form contains 40 items which are considered to provide an appropriate sampling of algebra or trigonometry content. It should be noted that the final testing for the control groups occurred in December, 1971, just before their final examination. For the two PSI sections, students took the criterion test after they had finished their 20 units and this ranged from November, 1971 to May, 1972.

Results. The performance of the students in the PSI and control lecture and discussion classes on the pre and post test administration of the algebra and trigonometry criterion tests are summarized in Table 1.

A comparison of the mean total correct test performance for the PSI and control classes (i.e., G_1 and G_3) on the Pretests administration showed no significant differences on either the algebra ($F = 3.41$; $df = 1, 45$; $p > .05$) or the trigonometry ($F = 0.87$, $df = 1, 45$; $p > .05$) measures. This comparison suggests that the PSI and control groups were comparable in terms of algebra and trigonometry knowledge at the beginning of the course. In other words, there seems to have been no factor operating which systematically assigned more capable students to PSI or control groups.

Since two of the groups were observed to have means on both the algebra and trigonometry tests which were similar, it was assumed that all four groups were roughly comparable in terms of mathematics performance at the beginning of the semester. For this reason, it was decided to use only the post test data in making decisions about the effectiveness of the PSI approach.

TABLE 1. Means, Standard Deviation, and Sample Sizes for Pre and Post Cooperative Algebra and Trigonometry Test Performance by Group.

Measure		PSI			Control		
		G_1	G_3	Total	G_2	G_4	Total
Algebra	Pre						
	M	13.00			10.68		
	SD	5.20			2.92		
	N	25			22		
	Post						
	M	20.91	20.74	20.80	17.21	19.10	17.85
	SD	5.09	2.73	4.12	4.76	4.91	4.82
	N	22	19	41	19	20	39
Trigon.	Pre						
	M	8.80			7.86		
	SD	3.92			3.01		
	N	25			22		
	Post						
	M	16.45	17.26	16.83	13.32	12.65	12.97
	SD	4.70	3.83	4.28	4.42	3.60	3.98
	N	22	19	41	19	20	39

Data was analyzed according to a 2 × 2 factorial design with the factors representing the PSI versus control and pre-test versus no-pre-test experience contrasts. A least squares analysis of the data for the two tests resulted in the same conclusion; namely that the two PSI groups performed significantly better, on both the algebra and trigonometry tests, than did the two control groups (Algebra: $F = 7.92$, $df = 1, 76$; $p < .01$. Trigonometry: $F = 17.16$, $df = 1, 76$; $p < .01$).

Differences between classes, with and without the pre-course testing experience, was non-significant thus indicating that being exposed to a pre-experimental condition — being tested — did not result in experimental bias. Similarly, the non-significant results obtained from the two tests of interaction suggested no differential bias was introduced to either the control or PSI groups which had taken the pre-test. The results of the analyses provide strong support for the conclusion that participation in the PSI approach in a first year college mathematics course resulted in a higher level of performance than did exposure to a traditional classroom presentation of the same content.

Additional Evidence of Effectiveness of PSI. The instructors responsible for the instruction of the two PSI MA 111 sections also collected other objective and sub-

jective information which was used as evidence of the effectiveness of the PSI approach.

As noted previously, a common final examination was administered to all sections electing MA 111 and was used for determining the grades received by the PSI students. The PSI students performed, on the average, at a significantly higher level than did the students in the non-PSI sections (PSI: $M = 68.12$; Non-PSI: $M = 59.00$; $t = 3.64$, $df = 41$, $p < .05$).

Student reaction to the PSI instructional approach was sampled through the administration of a brief course evaluation form at the completion of the 1971 Fall semester. An inspection of the distribution of responses given by the PSI students to questions relating to perceived course effectiveness indicated that out of 47 students, 43 did *not* regret their decision to remain in the PSI classes while 42 reported feeling they had learned more under the PSI mode than they would have under a lecture approach. Approximately sixty percent of the students (29 of 48) felt they worked harder under the PSI mode than they would have under a lecture mode while 44 of 48 would recommend taking the MA 111 PSI course to classmates with similar mathematical backgrounds. Student perception concerning helpfulness of the supplementary materials and proctors was highly favorable. A total of 37 or 48 students reported a willingness to take other courses under the Proctorial System of Instruction.

TABLE 2. Frequency Distribution of Grades Achieved in Spring, 1972 Mathematics Courses by PSI and Control Group Students.[a]

Group	Grade						
	A	B	C	D	F	CR[b]	W[c]
Control	7	7	9	7	2	0	0
PSI	7	10	9	2	0	1	3

[a] Frequencies for grades D and F were combined in X^2 analysis, CR and W were not included in X^2 analysis.
[b] CR = credit for course granted on basis of Pass/Fail.
[c] W = withdraw from course.

The instructors of the PSI students felt a third form of evidence of the course effectiveness would be a comparison of grade achieved by both PSI and Control group students in the mathematics course elected beyond MA 111. Of the original 50 PSI students, 32 students took a subsequent mathematics course while 32 of 49 control students elected a mathematics course in the Spring, 1972 semester. Frequency of grades earned for the experimental and control groups in their second mathematics course is presented in Table 2. While the PSI students had slightly better grades, a comparison of the grade distributions of A — F for the two methods of instruction showed differences to be statistically non-significant ($x^2 = 4.73$, $df = 3$, $p > .05$).

Discussion. The results of the evaluation of the PSI approach to teaching appears to provide support for the effectiveness of this method in presenting a first year university mathematics course in algebra and trigonometry. Using both the ETS Cooperative Algebra and Trigonometry and the Departmental Common Final Examination as evidence, the PSI students as a group performed significantly better than did students taught under traditional lecture-discussion methodology. Further, the positive responses to the course evaluation form suggested the students were also supportive of the PSI method of instruction. While the subsequent mathematics course performance after leaving MA 111 was not significantly different from the students identified as experimental and control, it was noted that more PSI than control students had taken a calculus course as their second course. Thus, the greater difficulty of mathematics courses elected by the PSI students may indicate why their grade distribution did not follow a significantly different pattern from the grade distribution of control students.

While the results are unequivocal in the interpretation that the experimentally treated student did perform better, it is not possible to conclude definitely whether the superior performance of these students was a function of student paced instructional method or the fact that they were allowed to take the final examination when they were ready to do so. Students in the control classes were tested at the end of the Fall, 1971 semester, however, the experimental group students were given until the end of the Spring, 1972 semester to complete the requirements for the course and to take the final. In other words, it may have simply been the fact that the experimental students "matured" in terms of mathematics knowledge.

In comparison of any two methods of instruction, the end product should be the ultimate measure of effectiveness. Thus, whether or not PSI students used more time to complete the course than did the control group students is not of concern in this evaluation since self-pacing is one of the features of PSI which distinguishes it from other instructional procedures.

In the final analysis, the results of these analyses suggest that exposure to the PSI approach in mathematics will result in superior overall group performance relative to groups instructed in a traditional manner. The Mathematics Department of the North Carolina State University intends to implement PSI as an option in the calculus courses as a result of this evidence of the success of PSI in algebra and trigonometry.

Footnote

1. Data concerning the number of students who elected to transfer out is not available; however, the instructors of the PSI sections estimated that less than 10 percent of the two classes utilized this option.

References

1. B. B. Bloom, Learning for mastery, In B. B. Bloom, T. R. Hastings, and G. F. Madaus, Formative and Summative Evaluation of Student Learning, McGraw-Hill, New York, 1971, 43–57.

2. J. Carroll, A model of school learning, Teachers College Record, 1963–64, 723–733.
3. F. S. Keller, Goodbye, teacher..., J. Appl. Behavior Analysis, 1, (1968) 79–80.
4. R. W. Solomon, An extension of control group design, Psychological Bulletin, 46, (1949) 137–150.

DEPARTMENT OF MATHEMATICS, NORTH CAROLINA STATE UNIVERSITY, RALEIGH, N. C. 27607.

FIRST HAND EXPERIENCE WITH INDEPENDENT STUDY OF MATHEMATICS FOR PROSPECTIVE SECONDARY MATHEMATICS TEACHERS

BEN V. FLORA, JR.

Introduction. Individualized instruction in mathematics has received much attention in recent years. One indicator of this is the fact that recent issues of both *The Arithmetic Teacher* (January, 1972) and *The Mathematics Teacher* (May, 1972) have been devoted almost entirely to articles about individualized instruction. Although the many individualized programs available differ substantially from one another in many respects, there are several characteristics common to most of them. These include: (1) The presence of different topics prescribed or selected for study by individuals, (2) Use of lists of behavioral objectives to be accomplished for each topic studied, (3) Flexibility in the amount of time the student may spend and/or the number of objectives a given individual must satisfy for a given topic of study, (4) Utilization of independent and/or small group study. Since so many schools today utilize some individualized programs of study which require independent study by students, it is important that students preparing to be teachers have some experience as a learner with independent study. With this thought in mind, it was decided to reconstruct the content and approach used in a mathematics course required by all mathematics students working on a B.S. in Education degree at Northern Illinois University, so that it possessed many of the characteristics of individualized programs. It was anticipated that such an experience would provide better insight into the problems encountered by students who are asked to learn independently as well as suggest procedures for structuring such a program. The purpose of this paper is to describe this course with the hope that others will consider developing courses along these lines.

Special topics in mathematics. The course selected for individualization was *Special Topics in Mathematics* (Math 416), a nine week course which is available to any student in mathematics during the half semester in which he is not involved in

student teaching. Generally, students enrolling in Math 416 have completed 24–30 hours of mathematics including 14 hours of calculus and linear algebra, 3–6 hours of abstract algebra, 3–6 hours of college geometry, and 3 hours of probability and statistics. The course and its broad aim are described by the following statements which are provided to the students enrolled in Math 416 at the first class meeting: "*Special Topics in Mathematics* (Math 416) is designed to provide the prospective teacher of secondary school mathematics with an opportunity to examine closely a number of selected topics in mathematics which are significantly related to the mathematics he will be teaching in the secondary school and which might have received limited attention during the course of his college mathematics studies. Since the majority of present day programs in mathematics at the junior and senior high levels primarily concern the mathematics of the real number system (with emphasis on understanding the structure of the system and its subsystems), Euclidean geometry (with special attention to mathematical reasoning and the nature of axiomatics), and the development of problem solving ability; the major topics examined in this course are addressed to these areas."

TABLE 1 — UNITS OF STUDY FOR MATH 416.

Unit one: *Formal axiomatics.*
Unit Two: *Euclid's contribution to mathematical thought.*
*Unit Three: *The nature of mathematics extended.*
*Unit Four: *The real number system.*
Units Five and Six: Select two of topics below.

A. *Arithmetic of infinites.*
B. *The theory of numbers.*
C. *Continued fractions.*
D. *Postulate sets in geometry.*
E. *Problems of antiquity and Euclidean constructions.*
F. *Inversion and ruler or compass alone constructions.*
G. *Graph theory.*
H. *Transformation geometry.*
I. *Basic combinatorics.*
J. *Game theory*

* If evidence suggests that this unit has already been sufficiently mastered, a student may, with the instructor's consent, elect one or more of the optional units in its stead.

Procedures for conducting Math 416. The adopted text presently used for Math 416 (*An Introduction to the Foundations and Fundamental Concepts of Mathematics* by Howard Eves and Carroll V. Newsom (Revised Edition, 1965), was chosen primarily because it provides a good survey of topics relevant to the prospective secondary school mathematics teacher, is written with an excellent historical perspective, and possesses excellent problem sets which not only provide one with a chance to apply some of the ideas presented in the chapters but are of such a variety that they provide the student with realistic problem solving experience.

With the above mentioned book as the basic reference source for the course, text reading and solutions of assigned exercises are to be accomplished by the student on his own. At present, students in Math 416 are required to complete six units of work. Four of these six are required of all students and the two additional units are selected from a list of topics provided in the course syllabus. Titles of these units are listed in Table 1. Selected excerpts from the defining information for Unit One are given in Table 2.

TABLE 2 — DEFINITION OF UNIT ONE: *Formal axiomatics*

I. **Objectives.** You will be able:

A. To define and give examples or non-examples of:
   1. a propositional function,
   4. equivalent postulate sets,
   5. a consistent postulate set,
   7. an independent postulate set,
   9. a categorical postulate set.

B. To explain the role of primitive terms, postulates, and theorems in axiomatics, and to explain the necessity of having primitive terms and postulates.

D. To state a technique for testing a postulate set for consistency and independence and to to be able to carry out the technique for simple examples.

F. To state Gödel's First Theorem (p. 335) and its significance to questions of completeness.

II. **Prerequisite:**

III. **Reading:** Basic: Eves & Newsom — Chapter 6, pp. 335–336.

IV. **Exercises:**

V. **Estimated Time:** Five days, plus quiz.

As a student completes a unit, he makes arrangements to take a test over the material studied. This unit test is taken by the student during out-of-class time and whenever he feels he is ready for the test. Since the student has a list of behavioral objectives for which tests of the units are constructed, the student should learn to judge when he has mastered the unit.

Use of class time. Very little class time is devoted to work on the six units of study. Generally one period is devoted to summarizing the unit ideas and one more period to answering questions on the reading material and exercises (for the first four units only). However, class meetings are held on a regular basis. Most of this class time is devoted to developing problem solving ability. This activity provides an aspect of communication and discussion of mathematics which would be lacking in a completely independent study setting. Challenging and interesting problems, generally unrelated to the units being studied and unrelated to one another, are attacked by the group. The emphasis is on employing *Pólya* techniques. The interesting *How to Solve It* by G. Pólya is a major reference for this activity. The pattern of such classes is generally as follows:

A. A problem is presented by the instructor.

B. The nature of the problem is briefly discussed.

C. Students spend 10–15 minutes individually or in small groups analyzing and trying to establish some results with the problem.

D. The students discuss their progress and by responding to leading questions, draw some conclusions and eventually a solution to the problem,

E. Once a solution is obtained, the problem is re-analyzed to determine characteristics of the problem which might have suggested the method of solution.

F. Some generalizations of the problem are stated.

G. Solutions of the generalizations are discussed. It is determined whether the technique of solving the original problem generalizes to these new problems.

Course evaluation. Although no carefully designed and controlled research was planned for evaluating the success of the new *Special Topics in Mathematics*, as a result of pupil and instructor judgments, some conclusions concerning the outcomes of the course seem justified. The course has now been taught three times (two different instructors and 8–12 students each time) using the approach described. Written evaluations by students indicate general satisfaction with the course and classify it as one of the most valuable courses they have had. They suggest they have learned at least as much mathematics through the independent study approach as they would through more conventional approaches and in addition, they feel they have acquired significant information about the process of learning. The text is classified by them as an interesting and readable one and the definitions or outlines for each unit are quite valuable. The problem solving experience is classified as the best aspect. They *do* find it difficult to provide sufficient intrinsic motivation to pace themselves but feel this is a result of past learning habits and suggest that such a learning experience should come much earlier. They indicate that they intend to provide such experiences for their students.

As additional evidence of success for this approach it was found that each student has been able to study more topics under this program than were studied through the more conventional approaches by the students before. A final exam given to one of the classes also suggests that the level of retention of ideas from the units studied by all of the students is good. Finally, improved class attitude and enthusiasm for learning mathematics were witnessed, and it was felt that these prospective teachers will now be able to work better with individuals and will have a better perception of individual learning problems.

Recommendations for others wishing to develop courses on an independent study basis. At the outset, it should be recognized that the nature of the *Special Topics in Mathematics* course lends itself nicely to an independent study approach. However, it is perceived that many other courses might also successfully employ at least some

of the characteristics of our program. The success with independent study in Math 416 has sufficiently encouraged several members of the Department of Mathematics at Northern Illinois University to give special consideration to the prospects of employing this approach to learning in several courses. At present, one other course (a geometry course designed for teachers of junior high mathematics) is being taught through similar techniques. Success there also stimulates continued course development along these lines. To others who wish to explore such techniques, the following suggestions are offered.

1. Team with one or two staff members in planning and developing the course.

2. Recognize that the course materials must all be developed before the class begins (or at least very early in the semester).

3. Anticipate that the instructor will need to be available regularly for out-of-class conferences with students.

4. Begin development of course materials by writing down broad goals for the course. Decide whether all of the mathematics content must be the same for all students.

5. Give careful thought to the background of students typically enrolled in the course. (Perhaps an independent study program should be constructed for only a selected subset of students who enroll in the course.)

6. Develop a topical outline for the course, including options where appropriate.

7. Before the course begins, develop materials (to be given to the students) which specifically define each unit of study. (We found the statement of behavioral objectives to be a great asset to the students and to us. We also found having answers available for most assigned exercises to be a valuable aid which significantly reduced the demand for use of class and out-of-class time of the instructor on such problems.)

8. Since students will complete units at various times, decide how evaluation will be accomplished and how students may obtain feed-back. (Communicate these decisions and all others to the students.)

9. Expect that some students will need to be continuously encouraged to move along more rapidly. Completion of a minimal course requirement can be a major problem. If course credit can be given in varied units (1, 2, or 3 semester hours, for example) or grades of incomplete can be assigned and work then completed during the following semester, this problem can be handled effectively. (Variability of time for completing course requirements is one of the main reasons for developing an independent study program and thus procedures for effectively allowing more time for slower students must be developed.)

10. Be prepared to adjust the original content or schedule for the course to fit the needs of the students. (As reported, we found that our students were able to complete more topics under this approach than under the more conventional one. However, the reverse might be the case for other courses.)

11. Be prepared to accept set-backs. Do not abandon a new approach after just one attempt. Work to correct and improve the original program. Remember, first attempts with any new procedure often do not measure up to expectations.

Acknowledgement. The author is grateful to Larry K. Sowder of the Department of Mathematics at Northern Illinois University for his principal role in the development of the Math 416 course syllabus and his helpful suggestions concerning this article.

References

1. R. Courant, H. Robbins, What is Mathematics?, Oxford University Press, London, 11th printing, 1961.

2. H. Eves, An Introduction to the History of Mathematics, Holt, Rinehart, and Winston, New York, 1964.

3. H. Eves, C. V. Newsom, An Introduction to the Foundations and Fundamental Concepts of Mathematics (Revised Edition), Holt, Rinehart, and Winston, New York, 1965.

4. M. Jeger, Transformation Geometry (Mathematical Studies: 1), George Allen and Unwin London, 1964.

5. J. G. Kemeny, J. L. Snell, G. L. Thompson, Introduction to Finite Mathematics, Prentice-Hall, Englewood Cliffs, N. J., 3rd printing, 1958.

6. A. N. Kostovskii, Geometrical Constructions Using Compasses Only (Popular Lectures in Mathematics Series), Translated from the Russian by H. Moss, Blaisdell, Waltham, Mass., 1961.

7. National Council of Teachers of Mathematics, The Arithmetic Teacher, NCTM, Number 1, 19 (1972).

8. ———, The Mathematics Teacher, NCTM, Number 5, 65 (1972).

9. I. Niven, Mathematics of Choice or How to Count Without Counting (New Mathematical Library), Random House, New York, 1965.

10. C. D. Olds, Continued Fractions (New Mathematical Library), Random House, New New, 1963.

11. O. Ore, Graphs and Their Uses (New Mathematical Library), Random House, New York, 1963.

12. A. L. Peressini, D. R. Sherbert, Topics in Modern Mathematics for Teachers, Holt, Rinehart and Winston, New York, 1971.

13. G. Pólya, How To Solve It, Doubleday Anchor Books, Garden City, New York, 2nd Edition, 1957.

14. A. S. Smogorzhevskii, The Ruler in Geometrical Constructions (Popular Lectures in Mathematics Series), Translated from the Russian by H. Moss, Blaisdell, Waltham, Mass., 1961.

DEPARTMENT OF MATHEMATICS, MOREHEAD STATE UNIVERSITY, MOREHEAD, KENTUCKY 40351.

PROBLEMS AND SOLUTIONS

EDITED BY EMORY P. STARKE

All problems (both elementary and advanced) proposed for inclusion in this Department should be sent to E. P. Starke, 1000 Kensington Ave., Plainfield, NJ 07060. Proposers of problems are urged to enclose any solutions or information that will assist the editors. Ordinarily, problems in well-known textbooks and results in generally accessible sources are not appropriate for this Department. No solutions (except those accompanying proposals) should be sent to Professor Starke.

ELEMENTARY PROBLEMS

Solutions of Elementary Problems should be sent to Problems Group, Mathematics Department, University of Maine, Orono, ME 04473. To facilitate their consideration, solutions of elementary Problems in this issue should be typed (with double spacing) and should be mailed before April 30, 1974.

An asterisk (*) means neither the proposer nor the editors supplied a solution.

E 2450. *Proposed by S. R. Conrad, B. N. Cardozo High School, Bayside, New York*

Prove that $x^{4n}+x^{3n}+x^{2n}+x^n+1$ is divisible by $x^4+x^3+x^2+x+1$ whenever n is a positive integer which is not a multiple of 5.

E 2451. *Proposed by F. W. Hartmann, Villanova University*

Let $a_0 = 1$, $a_1 = \sin 1$, $a_2 = \sin a_1, \cdots, a_{n+1} = \sin a_n$. Show that the sequence $\{n\, a_n^2\}$ has a limit and find that limit. What can be said for arbitrary a_0?

E 2452*. *Proposed by Joel Anderson, California Institute of Technology*

Starting with an arbitrary convex polygon P_1, a sequence of polygons is generated by successively "chopping off corners"; thus if P_i is a k-gon, then P_{i+1} is a $(k+1)$-gon. At the jth step let d_j be the altitude of the cut-off triangle, measured from the cut-off vertex. Prove or disprove: The series $\sum d_j$ converges.

E 2453. *Proposed by H. G. Niederreiter, Southern Illinois University*

Determine all rational numbers r for which 1, $\cos 2\pi r$ and $\sin 2\pi r$ are linearly dependent over the rationals.

E 2454. *Proposed by Doug Hensley, Graduate Student, University of Minnesota*

Suppose that $m \leqq n$ are positive integers. Show that

$$\Sigma \prod_{i=1}^{m} a_i^{-1} = m!\ \Sigma \prod_{i=1}^{m} b_i^{-1},$$

where the summation on the left hand side is taken over all ordered partitions of n into m positive integers, and where the summation on the right hand side is over all m-element subsets of $\{1, 2, \cdots, n\}$ which contain n.

E 2455. *Proposed by S. Audinarayana Moorthy, University of Bombay, India*

If n is a natural number let $g(n)$ denote the number of 1's in the representation of n in the base 2, and let $h(n)$ denote the highest power of 2 which divides $n!$. It is a known result due to Legendre that $g(n) + h(n) = n$.

(1) Let $F_n = 2^{2^n} + 1$ denote the nth Fermat number. Show that

$$\sum_{n=1}^{\infty} 2^{-n} g(n) = 2 \sum_{n=0}^{\infty} F_n^{-1}.$$

(2) Let $h(0) = 0$. Show that

$$\prod_{n=0}^{\infty} (1 + x^{2^n}) = \sum_{n=0}^{\infty} x^{h(n)}.$$

SOLUTIONS OF ELEMENTARY PROBLEMS

Almost Associative Systems

E 2385 [1972, 1134]. *Proposed by Burnett Meyer, University of Colorado*

Show that there exists no binary system with two-sided identity and two-sided inverses such that the associative law fails for exactly one ordered triple of elements.

Solution by D. P. Sumner, University of South Carolina. Let S be a binary system having a two-sided identity e and two-sided inverses. Suppose that S has at most one nonassociative triple. We will show that S is associative.

First observe that inverses are unique. For if a^* and a' are distinct inverses for the element a, then since not both (a^*, a, a') and (a', a, a^*) can be nonassociative triples, we can assume that (a^*, a, a') is an associative triple so that $a^* = a^*e = a^*(aa') = (a^*a)a' = ea' = a'$. We will henceforth denote the inverse of $a \in R$ by a^{-1}.

Let $a, b \in S$. We will show next that $a^{-1}(ab) = b$. For if this does not hold, then $T = (a^{-1}, a, b)$ must be the unique non-associative triple in S. But in this case, the following are associative triples:

(1) (ab, b^{-1}, a^{-1}) (4) (b^{-1}, a^{-1}, a)

(2) (a, b, b^{-1}) (5) $(b, (ab)^{-1}, ab)$

(3) (b, b^{-1}, a^{-1}) (6) $(b^{-1}a^{-1}, a, b)$.

If any of the triples (1) through (5) were identical to T, then we would have $a = b^{-1}$ and so $ab = e$ and $b = a^{-1} = a^{-1}e = a^{-1}(ab)$, contrary to assumption. If (6) were identical to T, then $b^{-1}a^{-1} = a^{-1}$, so using the fact that (4) is an associative triple, we would have $e = a^{-1}a = (b^{-1}a^{-1})a = b^{-1}(a^{-1}a) = b^{-1}e = b^{-1}$, implying $b = e$, and thus $a^{-1}(ab) = a^{-1}(ae) = a^{-1}a = e = b$, again contrary to assumption. Thus (6) also must associate.

Since the above six triples are associative, we have $(ab)(b^{-1}a^{-1}) = [(ab)b^{-1}]a^{-1} = [a(bb^{-1})]a^{-1} = aa^{-1} = e$ and also $(b^{-1}a^{-1})(ab) = [(b^{-1}a^{-1})a]b = [b^{-1}(a^{-1}a)]b = b^{-1}b = e$ so that $(ab)^{-1} = b^{-1}a^{-1}$. We have then $a^{-1} = (bb^{-1})a^{-1} = b(b^{-1}a^{-1}) = b(ab)^{-1}$ and therefore $a^{-1}(ab) = [b(ab)^{-1}](ab) = b[(ab)^{-1}(ab)] = b$. This contradiction establishes that $a^{-1}(ab) = b$ for every $a, b \in S$. In a similar way, we can show the dual result that $(ab)b^{-1} = a$ for every $a, b \in S$.

Now we show that S must be associative, for suppose $a(bc) \neq (ab)c$ for some $a, b, c \in S$. Then (a, b, c) is the unique non-associative triple in S. If (ab, b^{-1}, bc) were different from (a, b, c), then $a(bc) = [(ab)b^{-1}](bc) = (ab)[b^{-1}(bc)] = (ab)c$, a contradiction; thus the two triples must be identical. This implies that $ab = a$ and $bc = c$ so that $a(bc) = ac = (ab)c$, again a contradiction. We conclude that S must be associative.

Also solved by W. E. Bodden, Neal Brand, Gary McDonald & Merry McDonald, Lothar Redlin, Phil Tracy, and Qazi Zameeruddin (India).

Editor's comment. Notice that the bulk of the published solution goes to show that $a^{-1}(ab) = b$. Several solvers assumed this without proof, thus simplifying their solutions considerably. Other oversights were the assumption without proof of the cancellation laws, or the assumption without proof that if $xy = x$ then necessarily $y = e$ (a consequence of the cancellation law).

The proposer presents the following example which has exactly two nonassociative triples: Let $S = \{0, 1, 2\}$ and define $x * y = |x - y|$. Then 0 is the identity, each element is its own inverse and (2, 1, 1) and (1, 1, 2) are the only nonassociative triples. J. Conway asks for what k do there exist systems of the required type with precisely k nonassociative triples?

Conditions for a Regular Ring to be Boolean

E 2387 [1972, 1134]. *Proposed by David Jacobson, Rutgers University*

It is well known that a Boolean ring with identity 1 is (von Neumann) regular and 1 is the only unit in the ring. Conversely, show that if R is a commutative regular ring and 1 is the only unit in R, then R is a Boolean ring.

Solution by E. T. Wong, Oberlin College. The problem will follow from the known result below:

LEMMA. *Let R be a regular ring with identity* 1 (*not necessarily commutative*) *with the property that $x^2 = 0$ only for $x = 0$. Then, given any $a \in R$, there exists a unit $u \in R$ such that $au = ua$ and $aua = a$.*

Proof. Suppose that $e \in R$ is an idempotent. An easy calculation shows that for every $x \in R$, $(ex - exe)^2 = (xe - exe)^2 = 0$, so that $ex = exe = xe$ and thus e lies in the center $Z(R)$. Let $a \in R$ be arbitrary. Then $aba = a$ for some $b \in R$ so that $(ab)^2 = ab$ and $(ba)^2 = ba$; therefore $ab, ba \in Z(R)$ and thus $ab = abab = a(ba)b = (ba)ab = ba(ab) = b(ab)a = baba = ba$. Let $u = 1 - ab + bab$. It is now routine to verify that $ua = ba = ab = au$, that $aua = a$, and that $u' = 1 - ab + aba$ serves as an inverse for u so that u is a unit. This completes the proof of the lemma.

Suppose now that R is a regular ring with identity 1 with the property that 1 is the only unit in R. (We do not assume commutativity.) Then R satisfies the hypotheses of the lemma for if $x^2 = 0$, then $1 = 1 - x^2 = (1 + x)(1 - x)$ so that $1 + x$ is a unit; this implies that $1 = 1 + x$ so that necessarily $x = 0$. (Note that R can have no nonzero nilpotent elements at all.) Applying the lemma, we see that for any $a \in R$, $aua = a$ for some unit u. But necessarily $u = 1$ so that $a^2 = a$; that is, R is a Boolean ring.

Also solved by Donald Adams & Helen Adams, S. Alamelu (India), J. T. Arnold, W. E. Bodden, G. Conn & F. Tessier, S. C. Currier, Jr., Gertrude Ehrlich, B. M. Green, Melvin Henriksen, E. J. Howard, A. A. Jagers (Netherlands), S. J. Jain, Susanne Johannsen (Switzerland), K. Koh, Harbans Lal & Qazi Zameeruddin (India), Gary McDonald & Merry McDonald, P. V. R. Murty (India), W. K. Nicholson, John O'Neill, Henryk Pierog (Denmark), David Promislow, S. K. Sim, Art Steger, D. P. Sumner, R. G. Symonds, R. C. Wagner, E. T. H. Wang, G. P. Wene, and the proposer.

Editorial comment. Jain shows that if R is a regular ring (not necessarily commutative) with identity, if the group of units R^* is of order n, and if $na = 0$, $a \in R$, implies $a = 0$ then for all $x \in R$, $x^{n+1} = x$.

The Symmetric Derivative is Aull Done

E 2390 [1972, 1135]. *Proposed by Anon, Erewhon-upon-Yarkon*

Let $f(x)$ be continuous on (a, b) and suppose

$$D_s f(x) = \lim_{h \to 0} \frac{f(x + h) - f(x - h)}{2h} = 0$$

for all x in (a, b). Prove that $f(x)$ is constant.

Comments by George Crofts, B. F. Dunn, Alex Ferrer (Mexico), E. M. Klein,

J. F. Leetch, P. A. Lindstrom, E. J. Moulis, Jr., Phil Tracy, and W. T. Whitley (independently). This problem is an immediate consequence of the Quasi-Mean Value Theorem of C. E. Aull. See Theorem 1 (p. 709) of C. E. Aull, *The first symmetric derivative*, this MONTHLY 74 (1967), 708–711.

Also solved by Fred Barber, D. Borwein & A. Meir, W. O. Egerland, C. Gardner, M. G. Greening (Australia), Rev. William Habakkuk (Erewhon), A. Jakimowski (Israel), M. S. Klamkin, Charlotte Krauthamer, Felix Magnotta, R. Opp, D. B. Price, Kenneth Schilling, R. M. Warten, and R. A. Zalik.

Editor's comment. Most solvers solve the problem directly, using standard $\varepsilon - \delta$ techniques. Several solvers show that if F is an antiderivative of f, then the second symmetric (Schwarz) derivative of F vanishes, implying that F is linear by a known theorem of H. A, Schwarz (see I. P. Natanson, *Theory of Functions of a Real Variable*, Vol. II, Ungar, New York, 1960, pp. 36–37). Price uses the techniques of the theory of distributions of Laurent Schwartz. Klamkin comments that the result is known and appears with proof in T. Chaundy, *The Differential Calculus*, Oxford University Press, London, 1935, pp. 95–98. Klamkin also notes that the result still holds for the more general ratio

$$\frac{f(x+ph)-f(x-qh)}{h},$$

where p and q are fixed positive numbers independent of x and h. C. J. Neugebauer refers to A. Khintchine, Fund. Math. 9 (1923), pp. 212–279.

A Property Shared by Ellipses and Other Curves

E 2397 [1973, 202]. *Proposed by K. A. Brons, Cherry Hill, New Jersey*

The ellipse has the property that corresponding to any point on it there exist two other points on it such that the tangent to the curve at any of the three points is parallel to the chord joining the other two. Do any other simple closed convex planar curves enjoy this property?

I. *Solution by Erhard Heil, Technische Hochschule, Darmstadt, Germany.* The answer is affirmative. This has been shown by R. Inzinger, *Über konvexe ebene Bereiche die eine einparametrige Schar von Grösstdreiecken besitzen*, Österreich. Akad. Wiss., Math.-Nat. Kl., Sitzungsber., IIa 156, 263–285 (1948). The paper is reviewed by Busemann (Math. Reviews 10, 205) and by Hadwiger (*Zentralblatt* 37, 252).

II. *Solution by G. D. Chakerian, University of California, Davis.* Smooth convex curves C having the required property and different from ellipses have been constructed by J. Sancho San Roman [*Sobre un nuevo conceptode anchura de ovalos, invariante afin*, Rev. Mat. Hispano-Amer. (4) 16(1956), 151–171; Math. Reviews 18(1957), p. 505]. For each point $p \in C$, let $T(p)$ be a triangle of maximum area inscribed in C with one vertex at p. The required curves are characterized by the property that the area of $T(p)$ is the same for all $p \in C$.

A Result Known to Johnson

E 2398 [1973, 202]. *Proposed by C. W. Dodge, University of Maine at Orono*

Prove that the point of intersection of the diagonals of a parallelogram lies on the pedal circle for any vertex with respect to the triangle formed by the other three vertices.

I. *Solution by Huseyin Demir, Middle East Technical University, Ankara, Turkey.* Let $ABCD$ be a given parallelogram with I as center. Let the projections of D on sides BC, CA, AB of triangle ABC be A', B', C', respectively. If $\measuredangle D = \pi/2$, the pedal triangle degenerates into the Simson line AC containing the point I.

We give the proof in the case where $\measuredangle D > \pi/2$ and A' is on the segment BC and C' is on the segment AB. Similar proofs may be given in other cases. We need only show that $\measuredangle C'A'I = \measuredangle AB'C'$. In obtaining this equality we use the properties that $AC'B'D$ and $DC'BA'$ are cyclic and triangle DIA' is isosceles. We have

$$\measuredangle C'A'I = \measuredangle C'A'D - \measuredangle IA'D = \measuredangle C'BD - \measuredangle IDA' = \measuredangle A'DC$$
$$= \pi/2 - \measuredangle C = \pi/2 - \measuredangle A = \measuredangle ADC' = \measuredangle AB'C'.$$

II. *Solution by A. W. Walker, Toronto, Canada.* Let D be the reflection of the vertex A of triangle ABC in the midpoint M of the side BC. If BAC is a right triangle, the pedal "circle" of D for triangle ABC is the line BC; if not, let E be the meet of the lines tangent to circle ABC at B and C. Then BD and BE are isogonal conjugate lines in the angle ABC, and likewise for CD and CE in angle BCA, so D and E are isogonal conjugate points in triangle ABC and therefore (R. A. Johnson, *Modern Geometry*, p. 155) have a common pedal circle passing through the projection M of E on BC.

REMARK. E 2398 is a special case of the theorem: *For a plane non-orthocentric quadrangle ABCD there are four pedal circles (and four nine-point circles) passing through the center of the rectangular hyperbola ABCD* (Johnson, p. 242).

Also solved by Günter Bach (Germany), Leon Bankoff, Howard Eves, Michael Goldberg, M. G. Greening (Australia), Lew Kowarski, L. Kuipers, and the proposer.

ADVANCED PROBLEMS

All solutions of Advanced Problems should be sent to J. Barlaz, Rutgers — The State University, New Brunswick, N. J. 08903. Solutions of Advanced Problems in this issue should be typed (with double spacing) on separate, signed sheets and should be mailed before April 30, 1974.

An asterisk (*) means neither the proposer nor the editors supplied a solution.

5946. *Proposed by Andrzej Ehrenfeucht, University of Colorado*

Find a sequence of groups $G_1 \supseteq G_2 \supseteq G_3 \supseteq \cdots$ such that each G_n is a finitely generated group but $\bigcap_{n=1}^{\infty} G_n$ is not a finitely generated group.

5947. *Proposed by C. H. Kimberling, University of Evansville*

If $a(x) = x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n = \prod_{i=1}^{n}(x - \alpha_i)$ and $s_m = s_m[a(x)] = \sum_{i=1}^{n}\alpha_i^m$, then (1) find

$$A(x) = x^n + A_1x^{n-1} + \cdots + A_{n-1}x + A_n$$

satisfying $s_m[A(x)] = -s_m[a(x)]$ for $m = 1,2,\cdots,n$; and (2) find the characteristic polynomial $f(x)$ of

$$\begin{bmatrix} s_1 & -1 & 0 & \cdots & & 0 \\ s_2 & s_1 & -2 & \cdots & & 0 \\ s_3 & s_2 & s_1 & \cdots & & 0 \\ \vdots & & & & s_1 & 1-n \\ s_n & s_{n-1} & s_{n-2} & \cdots & s_2 & s_1 \end{bmatrix}.$$

5948*. *Proposed by A. A. Mullin, U.S.A. Research Office, Arlington, Virginia*

Does there exist an integer-valued sequence g which is both additive and relatively prime? I.e., does there exist a g such that $g(a \cdot b) = g(a) + g(b)$, if $(a,b) = 1$ and $(g(a), g(b)) = 1$, if $a \neq b$, where (a,b) is the greatest common divisor of a and b? Clearly, there exists a nonconstant integer-valued sequence which is both multiplicative and relatively prime.

5949*. *Proposed by C. W. Anderson, University of California, Berkeley*

Let $\Sigma(n) = \sigma(n)/n$, where $\sigma(n)$ is the sum of the divisors of n. It is known that $\Sigma : N \to [1,\infty)$ is a dense map. It is also known that there exists p/q, $(p,q) = 1$, $p > q$, such that $\Sigma(n) = p/q$ has no solutions, or $\Sigma(n) = p/q$ has exactly one solution.

(1) Show there exists a dense infinity of rationals p/q in $[1, \infty)$ for which $\Sigma(n) = p/q$ has no solution. (2) Also show there exists a dense infinity of such rationals for which $\Sigma(n) = p/q$ has only one solution. For example, $\Sigma(n) = 5/4$ has no solution; $\Sigma(n) = 3/2$ has exactly one solution.

5950. *Proposed by M. G. Glasser, Battelle Memorial Institute*

Sum the series

$$S = \frac{1}{ab} - \frac{(a+b+2)}{(a+1)(b+1)} + \sum_{n=2}^{\infty}(-1)^n \frac{(a+b+1)\cdots(a+b+n-1)(a+b+2n)}{n!(a+n)(b+n)}.$$

5951*. *Proposed by J. Gilles, University of Charleroi, Belgium*

Show that

$$(1)\qquad \int_0^\infty \frac{x^3 \sin \frac{1}{2}\pi x}{e^{2\pi\sqrt{x}} - 1}\,dx = \frac{17}{16} - \frac{8}{3\pi} - \frac{7}{\pi^2} + \frac{35}{2\pi^3} - \frac{105}{16\pi^4},$$

$$(2)\qquad \int_0^\infty \frac{x^3 \cos \frac{1}{2}\pi x}{e^{2\pi\sqrt{x}} - 1}\,dx = \frac{-8}{3\pi} + \frac{7}{\pi^2} + \frac{35}{2\pi^3} + \frac{873}{16\pi^4}.$$

SOLUTIONS OF ADVANCED PROBLEMS

The Translates of a Linear Set

5866 [1972, 779]. *Proposed by Hal Forsey, San Francisco State College*

Let A be a nonempty proper subset of R, the real numbers. Show that $\{A + t\colon t \text{ in } R\}$ is infinite.

Solution by Douglas Costa, University of Kansas. The resolution of the problem is an immediate consequence of the following more general result: Let *A be a nonempty proper subset of a divisible group G. Then* $\{A + t\colon t \text{ in } G\}$ *is infinite.*

Let $T = \{t \text{ in } G\colon A + t = A\}$. T is a subgroup of G, and the hypotheses on A guarantee that it is proper. Thus G/T is a nontrivial divisible group and hence infinite.

Observe that if $A + t = A + s$, then $t - s$ is in T. Therefore $A + t \to t + T$ is a well-defined mapping of $\{A + t\colon t \text{ in } G\}$ onto G/T. It follows that $\{A + t\colon t \text{ in } G\}$ is infinite.

Also solved by R. B. Brook, Robert Brooks, J. H. Elton, W. R. Emerson, E. W. Ewing, G. F. Feissner, D. Fieldhouse & H. Pesotan, P. Fisher & J. Holbrook, D. P. Giesy, J. D. Gillam, Robert Gilmer, S. M. Ginn, Donald Girod, P. D. Humke, A. A. Jagers (Netherlands), K. O. Leland, O. P. Lossers (Netherlands), Milan Lustig (Czechosolvakia), J. G. Mauldon, Ka Menehune, L. F. Meyers, Warren Page, Tom Parker, J. Pasciak, B. R. Toskey, Phil Tracy, Charles Tucker, M. H. Yu, and the proposer.

Notes. Humke notes that the result is best possible in that there is a set A having enumerably many distinct translates. Let H be a Hamel basis for R, and let $h \in H$. Let A be the set of all real numbers which do not have h in their expansion over H. The set of translates of A is $\{A + qh\colon q$ is rational$\}$ and hence is enumerable. Fisher and Holbrook (using a comment by Professor W. Fraser), and the proposer note that such a set must be non-measurable.

Compact Subsets in a T_0-Space

5874 [1972, 913]. *Proposed by T. E. Elsner, Michigan State University*

Let X be a compact T_0-space and let A be the set of closed singletons in X. Show that every subset containing A is compact.

Solution by A. A. Jagers, Technische Hogeschool Twente, Enschede, Netherlands. We first show that any nonempty closed subset C of X intersects A. An ap-

plication of Zorn's lemma and the compactness of X imply the existence of a minimal nonempty closed subset M of C. Being minimal, M is equal to the closure $\bar{x}$ of $\{x\}$ for all $x \in M$, whereas, since X is a T_0-space, $\bar{p} \neq \bar{q}$ if (and only if) $p, q \in X$ and $p \neq q$. Hence $M = \{m\} = \bar{m}$ for some $m \in X$. Clearly $m \in A \cap C$.

Let Y be a subset of X containing A; let $\mathscr{F}$ be a collection of relatively closed subsets of Y with the finite intersection property. To each $F \in \mathscr{F}$ associate a closed subset F_0 of X with $F = Y \cap F_0$. Set $D = \cap \{F_0 \mid F \in \mathscr{F}\}$. Then D is closed and, since X is compact, D is nonvoid. Hence $D \cap Y \supset D \cap A$ is nonvoid. Thus the intersection taking all of the elements in $\mathscr{F}$ is nonvoid. This proves the compactness of Y.

Also solved by G. C. L. Brümer (South Africa), D. Ž. Djoković, J. W. Grossman, R. A. Herrmann, C. T. Kendrick, D. G. Lash, John MacBain, S. E. Minear, B. W. Miller, T. B. Muenzenberger & R. E. Smithson, Tom Parker, J. M. Rosenberg (England), Ira Rosenholtz & Bob Christiansen, W. J. Śanchez, J. R. Smith, Paul Strong, W. W. Williams, M. H. Yu, and the proposer.

Limit of Solutions of a Linear Differential Equation

5875 [1972, 913]. *Proposed by Anon, Erewhon-upon-Yarkon*

Suppose $f(t)$ is twice differentiable and

$$\lim_{t \to \infty} [f(t) + f'(t) + f''(t)] = L.$$

Prove $\lim_{t \to \infty} f(t) = L$. (Compare an exercise in Hardy, *Pure Mathematics*; $f + f'' \to L$.)

Solution by W. C. Waterhouse, Cornell University. A generalization of an exercise in G. H. Hardy, *Pure Mathematics* is: if f is differentiable, $f' + \alpha f \to 0$, and $a = \mathrm{Re}(\alpha) > 0$, then $f \to 0$. Indeed, given $\varepsilon > 0$ choose c with $|f' + \alpha f| \leqq a\varepsilon$ for $t \geqq c$. Then

$$|D(e^{\alpha t} f)| = |e^{\alpha t}(f' + \alpha f)| \leqq a\varepsilon e^{at},$$

and hence

$$|e^{\alpha t} f(t) - e^{\alpha c} f(c)| \leqq \varepsilon[e^{at} - e^{ac}].$$

This implies

$$|f(t)| \leqq e^{a(c-t)} |f(c)| + \varepsilon[1 - e^{a(c-t)}],$$

whence $|f(t)| \leqq 2\varepsilon$ for large t.

Now let $P(D)$ be a polynomial in D with constant coefficients, and factor it as a constant times $(D - \lambda_1) \cdots (D - \lambda_n)$. If all λ_i have negative real part, then the above argument applied inductively n times shows that $P(D)f \to 0$ implies $f \to 0$. This

is in particular the case for $P(D) = D^2 + D + 1$; the reduction to $L = 0$ is obtained by subtracting a constant from f.

The condition of negative real parts is necessary as well as sufficient, since $f(t) = e^{\lambda_i t}$ certainly satisfies $P(D)f \to 0$. Hence the analogous statement for $f + f' + \cdots + f^{(n)}$ is false for $n \geqq 3$.

Also solved by J. M. Ash & Stephen Vagi, D. G. Belanger, D. Borwein, T. A. Burton & L. Kuipers, R. W. Cheney (Sweden), Michael Golomb, M. G. Greening (Australia), A. A. Jagers (Netherlands), Sol Kaufman, C. E. Langenhop, O. P. Lossers (Netherlands), J. S. Muldowney, Frank Witte, P. H. Young, and the proposer.

A $(C, 1)$ Summable Sequence

5877 [1972, 914]. *Proposed by R. Shantaram, University of Michigan at Flin*

Let $\{a_n\}$ be a sequence of positive real numbers such that

$$\lim_{n\to\infty} (a_1 + a_2 + \cdots + a_n)/n = a, \quad 0 < a < \infty.$$

For $\alpha > 0$, find $\lim_{n\to\infty}(a_1^\alpha + a_2^\alpha \cdots + a_n^\alpha)/n^\alpha$. What if $a = 0$?

Solution by Ellen Hertz, Bronx, New York. We find

$$\lim_{n\to\infty} \frac{\sum_{k=1}^n a_k^\alpha}{n^\alpha} = \begin{cases} 0 \quad \text{if} \quad \alpha > 1,\ a \geqq 0 & (1) \\ \infty \quad \text{if} \quad \alpha < 1,\ a > 0 & (2) \\ \text{indeterminate if } \alpha < 1,\ a = 0. & (3) \end{cases}$$

Let $\alpha > 1$, $a \geqq 0$. Set $\alpha = 1 + \varepsilon$. Let $k(n)$ be the greatest index not exceeding n such that $a_{k(n)} = \max(a_1, \cdots, a_n)$. Then

$$\frac{\sum_{k=1}^n a_k^{1+\varepsilon}}{n^{1+\varepsilon}} \leqq \frac{a_{k(n)}^\varepsilon}{n^\varepsilon} \cdot \frac{\sum_{k=1}^n a_k}{n} \sim \left(\frac{a_{k(n)}}{n}\right)^\varepsilon \cdot a.$$

If $a_{k(n)}$ is bounded, then surely $a_{k(n)}/n \to 0$. If not, then $k(n) \to \infty$ and since

$$\frac{a_n}{n} = \frac{\sum_{k=1}^n a_k}{n} - \frac{n-1}{n}\,\frac{\sum_{k=1}^{n-1} a_k}{n-1} \to 0$$

it follows that $a_{k(n)}/n \leqq a_{k(n)}/k(n) \to 0$. This proves (1).

For (2), set $\alpha = 1 - \varepsilon$, $a > 0$. Then

$$\frac{\sum_{k=1}^n a_k^{1-\varepsilon}}{n^{1-\varepsilon}} \geqq \frac{\sum_{k=1}^n a_k}{a_{k(n)}^\varepsilon n^{1-\varepsilon}} = \left(\frac{n}{a_{k(n)}}\right)^\varepsilon \cdot \frac{\sum_{k=1}^n a_k}{n} \sim \frac{n}{a_{k(n)}} \cdot a \to \infty.$$

For (3), let $a_k = k^{-p}$, $\alpha p \neq 1$; then

$$n^{-\alpha}\sum_{k=1}^{n} a_k^{\alpha} = n^{-\alpha}\sum_{k=1}^{n} k^{-p\alpha} \sim C\cdot n^{-p\alpha+1-\alpha} \to \infty \text{ if } \alpha<(1+p)^{-1},$$
$$\to 0 \text{ if } \alpha>(1+p)^{-1},$$
$$\to C \text{ if } \alpha=(1+p)^{-1}.$$

Also solved by A. A. Jagers (Netherlands), O. P. Lossers (Netherlands), L. E. Mattics, T. Šalát (Czechoslovakia), Alan Stein, Aleksandras Zujus, and the proposer.

Editor's comment. The result is somewhat intuitive if we consider the case of the sequence being convergent. An interesting question which may also be raised concerns the limit of $n^{-q}\sum_{i=1}^{n}a_i^{\alpha}$ for general q, or even for the special case $q = 1$.

Characterizing Non-concave Functions on [0, 1]

5879 [1972, 1043]. *Proposed by Alexandru Lupas, Institutul de Calcul, Cluj, Romania, and the University of Stuttgart, Germany*

Let L_n: $C(K) \to C(K)$, $n = 1,2,\cdots, K = [0,1]$, be a sequence of linear positive operators with the properties:

(1) $L_n e_0 = e_0$, $L_n e_1 = e_1$, $L_n e_2 = e_2 + a_n$, $(n = 1,2,\cdots)$, where $e_k(t) = t^k$ and $\{a_n\}$, $n = 1,2,\cdots$, is a sequence of nonnegative continuous functions, uniformly convergent to zero on K, and such that there is $x_0, x_0 \in K$, for which $a_n(x_0) > 0$.

(2) For every $g \in C(K)$ and $n = 1,2,\cdots$,

$$(L_n g)(0) = g(0), \ (L_n g)(1) = g(1).$$

Prove or disprove the following assertion: A function $f, f \in C(K)$, is non-concave on K if and only if $f(x) \leqq (L_n f)(x)$ for every $x \in K$. Eventually study the same problem without the second property of the operators.

Solution by S. P. Lloyd, Bell Laboratories, Murray Hill, New Jersey. The assertion does not hold; consider

$$(L_n f)(x) = (1-nx)f(0) + nxf(n^{-1}) \quad \text{if } 0 \leqq x \leqq n^{-1},$$
$$= f(x) \quad \text{if } n^{-1} \leqq x \leqq 1,$$

noting that $a_n(x) = (x/n)(1-nx)^+ \leqq 1/(2n)^2$, $x \in K$. The function $f(x) = x[-1 + \cos(2\pi/x)]$ if $0 < x \leqq 1$ $f(0) = 0$, satisfies $f \leqq L_n f$ for $n = 1,2,\cdots$, but is strictly concave on infinitely many disjoint subintervals of K.

Any $L: C(K) \to C(K)$ satisfying $L \geqq 0$, $Le_0 = e_0$, has the form $(Lf)(x) = \int_K f(\xi)p_x(d\xi)$, $x \in K$, where each p_x is a probability measure. If further $Le_1 = e_1$, then

$$a(x) = (Le_2 - e_2)(x) = \int_K (\xi - x)^2 p_x(d\xi)$$

is the variance of p_x. From $x = \int_K \xi p_x(d\xi)$ it is clear that $p_0 = \delta_0$, $p_1 = \delta_1$, where δ_x is the point measure at x, so that condition (2) is redundant.

REMARKS. With $[L_n]$ the given sequence, define $L = \sum_{n=1}^{\infty} 2^{-n} L_n$, so that $Le_0 = e_0$, $Le_1 = e_1$, $a = Le_2 - e_2 = \sum_{n=1}^{\infty} 2^{-n} a_n$. By the conditions stated, $a(x) > 0$ if $0 < x < 1$. Regard L as the Markov operator for a Feller process $[x_t(\omega), t = 0, 1, \cdots]$ in state space K. Since $[x_t(\omega)]$ is a bounded martingale, $x_\infty(\omega) = \lim_t x_t(\omega) \in K$ exists with probability 1. Thus the bounded submartingale $[x_t^2(\omega)]$ satisfies $E\{x_0^2(\omega)\} + \sum_{t=0}^{\infty} E\{a(x_t(\omega))\} = E\{x_\infty^2(\omega)\} < \infty$, and from $a(x) > 0$, $0 < x < 1$, it is clear that $x_\infty(\omega) = 0$ or 1 with probability 1.

Let $\mathscr{M} = \{f \in C(K): f = Lf)$ be the continuous regular functions for the process. If $f \in \mathscr{M}$, then

$$f_0 = f - [f(1)]e_1 - [f(0)] \quad e_0 - e_1) \in \mathscr{M}$$

has the property $f_0(x_\infty(\omega)) = 0$ with probability 1, whence $f_0 \equiv 0$. In other words, $\mathscr{M}$ consists of the linear functions on K.

Let $\mathscr{E} = \{f \in C(K): f \geqq Lf\}$ be a the continuous excessive functions for the process, and let $\mathscr{M}^\wedge \in \mathscr{E}$ be the $\mathscr{M}$-concave functions, i.e., the closure in $C(K)$ of

$$\bigcup_{m=1}^{\infty} \{f_1 \wedge \cdots \wedge f_m : f_i \in \mathscr{M} \text{ for } 1 \leqq i \leqq m\}.$$

By the above $\mathscr{M}^\wedge$ consists of the bounded functions on K which are concave in the usual sense. If $f \leqq L_n f$, $n = 1, 2, \cdots$, then $-f \in \xi$ but $-f \in \mathscr{M}^\wedge$ may fail, as the example shows: the inclusion $\mathscr{M}^\wedge \subset \mathscr{E}$ may be proper.

The result does not change if it is required that every $a_n(x_0) > 0$ for $0 < x_0 < 1$. Let $[0 < \alpha_n < 1]$ be any sequence converging to 0, and consider

$$(L_n f)(x) = (1 - \alpha_n) f(x) + \alpha_n [(1-x) f(0) + x f(1)], \quad x \in K,$$

for which $a_n(x) = \alpha_n x(1 - x)$, $\| a_n \| = \alpha_n / 4$. The following conditions on $f \in C(K)$ are seen to be equivalent:

(i) $L_n f \geqq f$ for $n = 1, 2, \cdots$: (ii) $Lf \geqq f$: (iii) $f(x) \leqq (1-x)f(0) + xf(1)$, $x \in K$. The failure of $\mathscr{E} \subset \mathscr{M}^\wedge$ is substantial, e.g., for any $f \in C(K)$ and any $\varepsilon > 0$ there exists $f_\varepsilon \in \mathscr{E}$ such that $f(x) = f_\varepsilon(x)$, $\varepsilon \leqq x \leqq 1 - \varepsilon$.

The problem of identifying $\mathscr{M}$ is related to questions in R. A. Hunt and R. L. Wheeden, *Positive harmonic functions on Lipschitz domains*, Trans. A.M.S. 147 (1970), 507–527, and in W. A. Veech, *A converse to Gauss' theorem*, Bull. A.M.S. 78 (1972), 444–446. Identifying $\mathscr{E}$ would seem to be more difficult.

Also solved by Charles Micchelli

REVIEWS

EDITED BY J. ARTHUR SEEBACH, JR. AND LYNN A. STEEN

with the assistance of the mathematics departments of St. Olaf and Carleton Colleges

COLLABORATING EDITOR FOR FILMS: SEYMOUR SCHUSTER, CARLETON COLLEGE

We invite readers to submit reviews of significant recent college-level mathematics books. We especially encourage reviews based on classroom use, or comparative reviews of several related books. Reviews should ordinarily not exceed two pages (per book) typed double spaced. Manuscripts of reviews as well as books submitted for review should be sent to: Book Review Editor, American Mathematical Monthly, St. Olaf College, Northfield MN 55057.

Calculus: *A Linear Approach*, *Volume* 1. By John B. Fraleigh. Addison-Wesley, Reading, Mass., 1971. vii + 663 pp. $11.95. (Telegraphic Review, January 1972.)

This book was used for two years at Bucknell University in several freshman sections consisting almost entirely of mathematics majors. It was well received by both students and instructors. The book is especially appropriate for use in a four semester "analysis" sequence which contains a formal study of linear algebra, since the author prepares the student for such a study from the outset. Throughout the book a high level of rigor and sophistication is maintained, yet our students seemed to find most of the material quite readable. The more difficult topics and proofs are put into starred sections which we felt desirable for mathematics majors, but which may be omitted without loss of continuity. Regarding the current enthusiasm for applications—the author includes many in physics, but applications in other disciplines are lacking.

The author avoids spending an inordinate amount of time on plane analytic geometry in Chapter 1 by proceeding directly to the geometry of $\mathbb{R}^n$. An excellent section is included about the "amount of room in $\mathbb{R}^4$", which will strengthen anyone's intuition regarding n-dimensional Euclidean space. The fundamental completeness property of the real numbers is also introduced in this chapter in the form of the Supremum Property. The notion of a linear mapping is introduced in Chapter 2 and the relationship between linear mappings from $\mathbb{R}^n$ to $\mathbb{R}$ and hyperplanes in $\mathbb{R}^{n+1}$ is discussed. As the title suggests, the language of linearity is used throughout the remainder of the book. After giving the standard definition of the derivative of a function in Chapter 3, the author proves the characterization which prepares the student for the generalization of differentiability to functions of several variables. The message which is repeatedly emphasized in this material is that a function's being differentiable at a point means it has a good linear approximation at the point.

Many of the deeper theorems on continuous and differentiable functions are presented in Chapter 4; accessible proofs are always provided along with comments and examples showing the role of "completeness" in the results. The notions of compactness and uniform continuity are also introduced in this chapter (in starred

sections). It is highly unlikely that even the honor student will develop a working knowledge of these concepts in a first course; however, as a result of this contact, he will find them much less imposing the next time he meets them. To quote the author, "having been 'exposed' to a concept ahead of time is often half the battle." Besides, the fundamental importance of these ideas is alone enough to justify their inclusion in an honors section.

The treatment of integration in Chapter 5 is excellent. The theory is clear and well-motivated, and numerous applications are provided. The author resists presenting a host of integration tricks for dealing with various kinds of problems. He does include a section on integration by parts and provides the student with plenty of opportunity in the exercises for developing integration techniques while encouraging the use of integral tables.

The introduction of inverse functions is delayed until Chapter 6. At that time the inverse trigonometric functions, the logarithm and exponential functions, and the hyperbolic functions are presented in standard fashion. Chapters 7 and 8 are devoted to sequences and series of real numbers and functions. Most of the familiar convergence tests are included and the student is encouraged to gain skill at determining convergence by inspection. Here again the author goes fairly deeply into the theory — for example, uniform convergence is introduced and theorems about term-by-term integration and differentiation of series of functions are given.

The differential calculus of several variables is taken up in the last two chapters. Matrices are introduced and applied to the solution of systems of linear equations and finally to the differential calculus from $\mathbb{R}^n$ into $\mathbb{R}^m$. A formal study of linear algebra and a treatment of multiple integration is given in Volume II. A word of caution regarding Volume II — the treatment of linear algebra is considerably abbreviated, with many standard theorems left to the exercises. Also, the treatment of determinants is non-standard, and many of their basic properties are not proved. Dissatisfaction with the quantity and treatment of linear algebra has led to our replacing Volume II after one year.

Several topics that are standard freshman fare are left to Volume II. The most notable example is the theory of plane curves (conic sections, parametric equations, arc length, polar coordinates); also omitted are numerical approximation of integrals and trigonometric substitution. These omissions may cause difficulty in choosing a second year text other than Volume II.

The author distributes sections on differential equations throughout the two volumes, treating the various types of equations as the needed tools for their solution are developed. For example, the equation $y' = ay$ is solved as an application of the exponential function, and linear differential equations are treated as an application of power series. The reviewer feels this is far preferable to including a separate chapter on differential equations which no one gets to.

A large number of exercises is included throughout the book to give the student plenty of practice in "drill" problems. There are also many problems of a more challenging nature, which require a deeper understanding of the underlying theory. The format of the book is pleasing and the text is remarkably error free. (One error of substance does occur in Theorem 6.4, p. 315, when the author claims to prove $(e^a)^b = e^{ab}$ for $a, b, \in \mathbb{R}$. At this point, the reader has no definition of the left hand side unless $a = 1$ or b is rational.)

This is an excellent book, and I enthusiastically recommend it for honors sections. With the omission of many of the starred sections, it might also be well-suited for general use.

PAUL STRONG, Bucknell University

Numerical Methods that Work. By Forman S. Acton. Harper and Row, New York, 1970. xviii + 532 pp. $13.95. (Telegraphic Review, January 1971.)

This is an unusual first text in numerical methods, written for bright engineering students. I used it jointly with Ralston's *First Course in Numerical Analysis* in two terms of advanced undergraduate numerical analysis for mathematics and physics students. I needed the additional text because Acton seldom derives or provides mathematical background for the methods he presents. His method is to teach by example. Acton is aware of this and frequently refers his reader to other texts for mathematical background. He especially avoids error analysis: his reader rarely sees a Taylor series error term. To me the course was an opportunity to solidify and enlarge upon my students' knowledge of linear mathematics, multivariable calculus, differential equations, and so on. If you share these goals, Acton will be a poor choice of sole text, in my opinion. But it contains so much mathematical and computational insight found nowhere else that I heartily recommend it as serious side reading for student and teacher.

This text explains what *not* to compute and in what order to compute. It is outstanding in applying mathematical effort to the part of the problem that needs it most. For example, Acton is not interested in super-high-order-of-accuracy quadrature formulas. From his many examples one draws the moral: your troubles in $\int f$ are with f, not with the integration. Then, work at replacing or approximating f with suitable mathematics: power series, series around ∞, change of variables, subtraction of singularities, etc. All these are used successfully. Methods are presented for conversion among various forms of a function, including rational approximations and continued fractions.

For such topics as eigenvalue calculation, solution of partial differential equations (Laplace and parabolic equations are treated starting with physical examples) and network analysis, Acton starts from scratch. But the reader must have preparation to read other sections. In the chapter on interpolation, Everett's and Bessel's formulas arrive out of the blue: the reader should have seen Lagrange interpolation and the

classical theory elsewhere. The reader also needs experience with singular integrals to read Acton.

The chapter on interpolation is one instance where a few more paragraphs placing the material in context are really needed. Also, the idea of orthogonality of functions with respect to a finite net of points is presented well, but the few paragraphs needed to explain its use in least square applications are missing.

These are small complaints. By leaving out the background material Acton has made space for an exhibition of the professional numerical analyst at work. The book is well written in a style that is casual, friendly, and genuinely entertaining. It deserves attention from numerical methods instructors.

PHILIP TUCHINSKY, Ohio Wesleyan University

Mathematics in the Time of the Pharaohs. By Richard J. Gillings. MIT Press, Cambridge, Massachusetts, 1972. xi + 286 pp. $25.00 (Telegraphic Review, October 1972.)

English publications in the history of mathematics have been rather consistently marked by a disconcerting degree of ignorance in their treatment of the mathematics of ancient Egypt. Standard criticisms have pointed to the methods used in solving problems as primitive and devoid of formal proof, and have perpetuated the view that the mathematics of that era remained on the level of practical arithmetic needed in daily life. Gillings' work, the first comprehensive study on the subject to be published in English, recommends itself by a high level of scholarship as well as the rigor with which it contests such misconceptions.

The presentation is based on facts given by mathematical papyri, specifically the *Rhind Papyrus*, the *Moscow Papyrus*, and the *Egyptian Mathematical Leather Roll*. The author demonstrates that from the development of the arithmetical calculations of finding two thirds of any number and the twice times table, the Egyptians were able to reach a high level of mathematical sophistication. Despite the fact that they had no mathematical tools to speak of, no notations or signs for the four operations of arithmetic, the accomplishments were significant. Gillings focuses in particular on the determination of the value of π with an accuracy of less than 0.6 of one percent error, the solution of first and second degree equations, the summing of geometric and arithmetic progressions, the formulation of the volume of the frustrum and of the surface area of a hemisphere.

One of the chief contributions of the text is the success with which the author demonstrates explicitly obvious mathematical relationships implicit in the ancient manuscripts. The chapter on the "G Rule" serves as a good example. The rule is stated as follows:

"If one unit fraction is double another then their sum is a different unit fraction if and only if the larger denominator is divisible by 3. The quotient of the division is the unit fraction of the sum."

Knowledge of the "G Rule" by the ancient Egyptians is inferred in a manner similar to that used by Neugebauer and Sachs. The method rests on the assumption that simple mathematical relationships must have suggested themselves visually from tabulated figures governed by rules expressing those relationships. Gillings' inference concerning the "G Rule" is substantiated by him in numerous examples taken from the papyri which indicate its application. These cases are then used to illustrate that specific mathematical methods, for example, the simplest decomposition of fractions, were not improved upon for centuries and in some instances remained unchallenged until the fifteenth century.

The text was most successfully used in a history of mathematics course for undergraduates. Favorable reception was due to its readability, ample illustrations, and imaginative line of argument. An appendix of problems would have been desirable for this purpose. Unfortunately, the prohibitive price relegates this excellent study to the reserve shelf.

JOHN NIMAN, Hunter College of the City University of New York

Introduction to Mathematical Theory of Control Processes. By Richard Bellman. Vol. I, Academic Press, New York, 1967. xvi + 245 pp. $11.50. (Telegraphic Review, April 1972.) Vol. II, 1971. xix + 306 pp. $16.00. (Telegraphic Review, August–September 1971.)

Control and Dynamic Systems. By Yasundo Takahashi, Michael J. Rabins, David M. Auslander. Addison-Wesley, Reading, Massachusetts, 1970. xiv + 800 pp. $17.50. (Telegraphic Review, April 1972.)

Mathematical Methods of Optimal Control. By V. G. Boltyanski. Transl: K. N. Trirogoff. Holt, Rinehart and Winston, New York, 1971. xiv + 272 pp. $9.95. (Telegraphic Review, April 1972.)

Because of a large and growing human population it is becoming ever more necessary to utilize efficiently the limited resources of our planet. Control theory provides a scientific approach to the decision making involved in the allocation of resources. In the two books authored by Bellman, control theory is defined as the care and feeding of systems — control theory being the technique whereby the best feeding plan is established. The four books under review are each devoted to the topic of control theory.

The control theory approach is to assume that the controlled system (be it an airplane, a human, or an economy) is fixed, usually unalterable, and equivalent from an input-output point of view to its mathematical model. The inputs can be thought of as the influence variables (the quantities which are subject to control) and the output can be thought of as the sensed variables. Control theory provides the connection

between the outputs and the inputs so that the controlled system behaves in a satisfactory (or perhaps even in a most efficient) manner. This connection between output and input is the concept of feedback. Models commonly used for systems are difference equations, differential equations (or the equivalent transfer functions), functional differential equations, partial differential equations or combinations of these.

The mathematical model for a system is obtained in a number of ways. In engineering it is common to build the model based on the known structure of the system; for example, it may be composed of electrical devices such as resistors and capacitors, which have a well established relationship between their inputs and outputs, and rules (Kirchhoff's current and voltage laws) for interconnecting which determine the model of the composed system. Mechanical and chemical systems can be modelled in a similar fashion. Another approach to modelling is to postulate a relationship between the input and output variables with unknown parameters which are determined by a comparison with experimental data from the controlled system.

To evaluate the performance of the controlled system a comparison is usually made between the performance of the actual controlled system and some ideal system in certain critical situations. In control engineering this comparison may be in terms of such quantities as the rise time of the output for a step type input, the peak overshoot, phase margin, gain margin, etc. The modern control approach makes this comparison by considering as the performance index the mean square error between the actual system response and the ideal system response (or some other mathematical expression of efficiency). Frequently the goals of the system are not clear and one chooses an index of performance for mathematical simplicity to carry out the controller synthesis and afterwards compares the controlled system with other norms.

Another part of the problem to consider is the constraints and restraints that the system may be subject to. For example, the controlled inputs may be restricted as to size or shape or to the total number available. Such constraints are frequently stated in mathematical terms by putting a limit on the absolute value of a control variable or by a limitation on the integral of the control variable squared.

The mathematical control problem is then to select the best relationship (in the sense of the performance criteria) between the outputs and the inputs subject to the mathematical model and constraints. This mathematical control problem is the central topic of the four books under review.

In the two books authored by Bellman, the mathematical control problem for systems modeled by ordinary differential equations is considered from two points of view: a classical variational one and the dynamic programming (feedback) approach. The control of processes with difference equations and partial differential equation models are briefly discussed.

Volume I begins with motivational material in the form of nine very short sections concerning mathematical modeling, uncertainty, questions of observing systems, the concept of feedback, and the like. Chapter 2 is devoted to second order linear differen-

tial and difference equations, while in Chapter 3 the concepts of stability and control are discussed and the mathematical control problem (linear model and quadratic performance index) is formulated. The remaining 6 chapters elucidate techniques for solving mathematical control problems when various constraints and generalizations are admitted. Such questions as the existence of an optimalizing control function, its uniqueness, and necessary and sufficient conditions that the optimalizing control function must satisfy are considered. Chapter 9 outlines briefly how the theory of functional analysis can be applied to the quadratic optimal control problem.

Volume II discusses the discrete version of the mathematical control problem, namely when the controlled system model is a nonlinear difference equation and the performance index is a finite sum of terms involving the output and input variables. In early chapters, the technique of dynamic programming is the tool used for determining the optimum control policy. The general mathematical control problem for the nonlinear differential control model and an integral type performance index is considered using results and computations from the calculus of variations. Middle chapters contain information about approximating a continuous process by a discrete process useful for controller synthesis, and give details of the theory when the time interval for control becomes unbounded (asymptotic control theory). The remaining chapters contain results for applying the theory to more general situations, such as allocation of resources and to problems with distributed parameters.

Bellman's stated goal was to provide a rigorous, yet elementary, account of control theory geared to the abilities of an undergraduate student with training in calculus, differential equations, and matrix theory (each of these topics is again discussed in the two volumes under review). In general, he has admirably achieved this goal. On the other hand, there are many topics of interest in control theory beyond the mathematical control problem. Such topics as modelling, the information set (observability–controllability) are treated only briefly, as are realistic applications of the theory. The books should be of interest to students in all fields which are concerned with allocation of resources and dynamic processes (including medicine, economics, engineering and computer science).

The senior-first year graduate level text, *Control and Dynamic Systems* by Takahashi, Rabins and Auslander, is excellent for its motivation, clarity, and coverage of topics. It has a remarkably good coverage of classical control topics such as Bode diagrams, Nyquist criterion and root locus. In addition to a lucid description of these techniques in the analysis and design of linear feedback control systems, the text is replete with well-chosen examples from a variety of fields. These are presented in sufficient detail to make the book suitable for self study.

Modern control topics such as state space description of systems, controllability and observability are discussed with an abundance of well-chosen examples. Optimization methods such as dynamic programming and maximum principle are also dis-

cussed in a manner appropriate to the level of the text. A chapter is devoted to linear stochastic systems where the principles of Kalman-Bucy filtering are discussed. This is one of the clearest discussions that exists at this level. The discussion of nonlinear systems includes the phase-plane, Lyapunov stability theory and the describing function method. A unique feature of the book is the inclusion of a chapter on switching systems where logic elements, switching algebra, combinatorial systems, sequential systems and finite state machines are discussed.

The chapters have at the end enough challenging problems to make it a good book for classroom use. One of the reviewers has used parts of this text at the senior level and has found the student response very favorable. The book can be recommended for classroom use and for self study by practicing engineers.

Mathematical Methods of Optimal Control by V. G. Boltyanski is a translation from the Russian original. As we have been accustomed to expect, the translation by K. N. Trirogoff is very smooth indeed.

The book deals mainly with the methods of Pontryagin's maximum principle and Bellman's dynamic programming. The maximum principle is discussed at a level so that first year graduate students can follow most of the technical discussion. The author has chosen two dimensional systems for illustrative purposes so that the phase plane can be exploited to great advantage. In this, he has succeeded well. Linear time-optimal processes are given considerable emphasis. Numerical algorithms of Neustadt and Eaton are discussed. Dynamic programming and its application to Optimal Controller synthesis is well done. Numerical aspects of this technique are not discussed. The reader should refer to the books of Bellman for this point. The last chapter introduces the reader to various generalizations of the optimal control problem.

The basic weakness of the book as a classroom text is the lack of problems. Even if the instructor should supply the problems, the scope is rather narrow for a quarter or a semester course. There are other textbooks that could well be used for classroom teaching. The book is well suited for self-study by practicing engineers and students.

In summary, the two books by Bellman can be read profitably by students oı engineering, economics and mathematics. They are good for getting an overall picture of control problems. The book by Takahashi, *et al*, is an excellent text which can be used in a variety of ways by selective omission of topics. The book by Boltyanski is suitable for self-study by students and practicing engineers. One gets an easier picture of the maximum principle. This is not suitable as a textbook.

E. B. LEE and K. S. P. KUMAR, University of Minnesota

TELEGRAPHIC REVIEWS

Telegraphic reviews are designed to give prompt notice of new books with sufficient information to assist our readers in deciding whether to order an examination copy or to suggest library purchase. Possible uses are indicated as follows:

T = textbook P = professional reading
S = supplementary reading L = undergraduate library purchase
13 to 18 = freshman to second year graduate level usage
1 to 4 = appropriate time in semesters to cover text

Asterisks (*) or question marks (?) denote special positive or negative emphasis, respectively. Publishers are denoted by standard abbreviations; complete addresses may be found in *Books in Print*.

GENERAL, S**, P**, L**, *Africa Counts: Number and Pattern in African Culture*. Claudia Zaslavsky. Prindle, 1973, x + 328 pp, $12.50. A rich and beautiful book! Full of pictures and figures illustrating the development of mathematics among the tribes of sub-Sahara Africa: numeration systems, cultural significance of numbers, mathematical games, geometric form and pattern in art and architecture. A rare interdisciplinary perspective reigns throughout, pervaded by a passion and love for the peoples whose accomplishments the author relates. PJC

GENERAL, S**, *Man and the Computer*. John G. Kemeny. Scribner, 1972, viii + 151 pp, $6.95. Personalized view, critical evaluation of present state of the art and various applications of computers, together with a description of the "predictable evolution of the computer species" (automated libraries, personalized newspaper service, shopping by computer). Emphasizes time-sharing and man-machine interaction: "My major concern is whether man can keep up with his symbiotic partner." PJC

GENERAL, S, L, *Secret Writing: The Craft of the Cryptographer*. James Raymond Wolfe. McGraw, 1970, 192 pp, $5.50. A popularized treatment, replete with cloak and dagger anecdotes, including examples of both modern and ancient historical interest. No mathematical, statistical or computer methods. Bibliography. LAS

BASIC, T(13), *Elementary Algebra for the College Student*. John C. Knutson, James V. Rogers. Rinehart Pr, 1972, xi + 641 pp, $10.95. Precollege algebra with 1100 classroom exercises, 3100 regular exercises, and 1300 review exercises. Verbal problems are an outstanding feature. LLK

PRECALCULUS, T***(13: 1), L, *The Elementary Functions: An Algorithmic Approach*. G. Albert Higgins, Jr. P-H, 1974, xiv + 335 pp, $10.50. An algorithmic approach to the elementary functions is indeed a step in the right direction especially where access to computing services is available. This presentation includes a wealth of programming suggestions and techniques. LLK

HISTORY, P, L*, *Origins of Modern Algebra*. Lubos Nový. Wolters-Noordhoff, 1973, viii + 252 pp, $24. A detailed, scholarly study of the evolution of algebra between 1770 and 1870--a period in which the fundamental structures of modern algebra emerged, and algebra itself developed a separate identity. Extensive bibliography, well referenced throughout the text and its hundreds of footnotes. LAS

HISTORY, S*, P*, L***, *Albert Einstein: Creator and Rebel.* Banesh Hoffmann. Viking Pr, 1972, xv + 272 pp, $8.95. This rich, illuminating "indication of the man" by his distinguished and unusually articulate former assistant balances personal anecdotes with extensive scientific exposition. A remarkable portrait of the person who more than any other made the general public aware of the scientific power of abstract mathematics. LAS

FOUNDATIONS, T(18: 1), P, L, *Normative Systems.* Carlos E. Alchourrón, Eugenio Bulygin. Lib. of Exact Phil., V. 5. Springer-Verlag, 1971, xviii + 208 pp, $21.50. An approach to traditional legal philosophy which assumes that legal science can be a pre-analytic ground for deontic logic and that deontic logic can be of value to legal science. Legal science is construed as a normative (rather than formal or empirical) science and approached in terms of the notion of a "normative system", defined as "a set of sentences that has (some) normative consequences." Part I develops a scheme for studying such metalogical properties as consistency, completeness, and independence. Part II applies the scheme to some specific problems of legal science. The text is relatively informal; an appendix gives definitions and theorems for a rigorously formalized treatment of the main lines of thought. A competent book (over-priced), with a brief but good bibliography. FS

FOUNDATIONS, T(17-18), S, P*, L, *The Axiom of Choice.* Thomas J. Jech. Stud. in Logic and Found. of Math., V. 75. North-Holland, 1973, xi + 202 pp, $14.95. An excellent survey of independence, consistency and equivalence results (largely from the past decade) in a straightforward notation easily accessible to the mathematician who is not a specialist in mathematical logic. Includes consequences and indispensibility of AC, consistency models, independence models, weaker versions of AC, consequences of ~AC, and axioms (e.g., determinateness) which contradict AC. LAS

COMBINATORICS, P, L, *Graphs and Hypergraphs.* Claude Berge. Transl: Edward Minieka. North-Holland, 1973, xiv + 528 pp, $24.95. Translation and revised edition of *Graphes et Hypergraphes* (Dunod, Paris 1970). Compendium of a still-young subject. Author remains convinced that network flow theory and theory of alternating chains should form the foundation of graph theory. PJC

COMBINATORICS, P, *Introduction à la Théorie des Hypergraphes.* Claude Berge. Pr U Montreal, 1973, 114 pp, $3.50 (P). If $X = \{x_1, \dots x_n\}$ is a finite set, then a family $\{E_i\}$ of non-empty subsets of X is a hypergraph iff $\cup E_i = X$. More succinctly: a hypergraph is a family of hyperedges, which in turn are sets of vertices of cardinality not necessarily 2 (as for graphs). First part, coloration problems, appears also in *Graphs and Hypergraphs* (but not in the earlier French edition). Second part, isomorphism problems, contains some newer results. PJC

LINEAR ALGEBRA, T(14: 2), *Linear Algebra with Differential Equations.* Donald L. Bentley, Kenneth L. Cooke. HR&W, 1973, xiv + 625 pp, $14.50. Material can be separated and taught as a traditional course in linear algebra (chapters 4-7, 10-11, 14), and a course in differential equations with linear algebra prerequisite. The authors feel there is efficiency in integrating the two. LLK

LINEAR ALGEBRA, T(15: 1), S. *Introduction to Matrices with Applications in Statistics*. Franklin A. Graybill. Wadsworth, 1969, 372 pp, $10.95. Prerequisite: elementary linear algebra. There are 3 chapters summarizing matrix theory; the remainder of the book is the development of topics in matrix algebra useful for the study of linear statistical models. Includes problems (no answers) and many references at the end of each chapter. LLK

CALCULUS, T(13-14: 1-3). *Calculus for College Students, Second Edition*. Murray H. Protter, Charles M. Morrey, Jr. A-W, 1973, xv + 932 pp, $13.95. This second edition closely resembles the first. There is some rearranging of topics and addition of problems but the lack of variety or challenge still exists. LLK

REAL ANALYSIS, T(15-17: 1), L. *Lebesgue Integration and Measure*. Alan J. Weir. Cambridge U Pr, 1973, xii + 281 pp, $16.50, $6.95 (P). Suitable for advanced undergraduates. Integral is defined on R and on R^n using null sets. Other measure theory is developed after proofs of the convergence theorems. Chapter on L^p spaces. Answers to exercises. RBK

REAL ANALYSIS, S*(14-16), P, L**, *A Primer of Real Functions*. Ralph P. Boas, Jr. Carus Math. Mono., No. 13. MAA, 1972, xi + 196 pp, $8. Second edition of a work first published in 1960; not a text, but an orderly composition of 24 partially independent elegant snapshots from the theory of sets and real functions. A gold mine of interesting, uncommon insight and examples. LAS

TOPOLOGY, T(16-18: 1, 2), P, *Introduction to Piecewise-Linear Topology*. C.P. Rourke, B.J. Sanderson. Ergebnisse der Math., B. 69. Springer-Verlag, 1972, viii + 123 pp, $13.40. Even though this book includes all necessary algebra and algebraic topology in appendices and formally requires only a good semester of point set topology as a prerequisite, it is a very substantial work including recent results concerning the Poincaré conjecture via the h-cobordism theorem. JAS

TOPOLOGY, T(18), P, L, *Topological Embeddings*. T. Benny Rushing. Pure and Appl. Math., V. 52. Acad Pr, 1973, xiii + 316 pp, $18.50. A comprehensive introduction, with no intention of being complete, to the currently lively subject of the title. The exercises scattered throughout would make this suitable for an advanced text (algebraic topology not required as a prerequisite) while the bibliography and presentation are focused on getting to the core problems of the field. The emphasis is on geometry with pictures. The index appears adequate. In summary: a nicely done volume with something new to say. JAS

TOPOLOGY, P, *Geometrical Combinatorial Topology, V. II*. Leslie C. Glaser. Van-N-Rein, 1972, v + 175 pp, $9.50 (P). Two chapters--one on applications of Stallings' Engulfing Lemma, the other on embeddings in Euclidean space--show the breadth and depth of geometrical results obtainable using the concepts of piecewise linear topology as introduced in V. I (TR May 1970). JAS

STATISTICS, T(15-17: 1, 2), L. *Stable Chaos: An Introduction to Statistical Control*. David Durand. GLP, 1971, xv + 582 pp, $14.50. Not a book on quality control, but a beginning statistics text,

oriented toward students in management, operations research or industrial engineering. Arrangement of topics is unique, following the central theme of statistical control in random processes. Calculus is assumed, but emphasis is on applications rather than mathematical proofs (e.g., no mention is made of moment generating functions). Does not include regression analysis. RSK

STATISTICS, T(13-14: 1), *Concepts of Statistical Inference, Second Edition*. William C. Guenther. McGraw, 1973, 11 + 561 pp, $11.50. Presupposes only high school algebra. This new edition has additional "realistic" exercises and a chapter on non-parametric statistics. Many tables and graphs. No Bayesian methods. FLW

STATISTICS, T(13-14: 1), *Elementary Probability and Statistics With Optional Computer Applications*. A. William Gray, Otis M. Ulm. Glencoe Pr, 1973, 272 pp, $10.95. Presupposes only high school algebra. Includes an optional chapter on BASIC and problems designed for its use. Standard topics. No Bayesian methods. FLW

APPLICATIONS (PHYSICS), S(16-18), P*, L*, *The Large Scale Structure of Space-Time*. S.W. Hawking, G.F.R. Ellis. Cambridge U Pr, 1973, xi + 391 pp, $28.50. A superb monograph which both develops and interprets general relativity in order to demonstrate that this theory entails the existence of black holes and a "creation point" for the universe: such objects are singularities (non-existent limit points) of certain space-time geodesics. Assuming only calculus and topology, it includes all necessary differential geometry in Chapter 2. An appendix contains a translation of a paper by Laplace in which he shows that gravity of a star may in some cases prevent light from leaving it. LAS

APPLICATIONS (PHYSICS), T(16-17: 1, 2), S, P, L, *Statistical Mechanics, Kinetic Theory, and Stochastic Processes*. C.V. Heer. Acad Pr, 1972, xvi + 602 pp, $18.50. Introduces statistical topics through kinetic theory. Many exercises interspersed in the text. Applications are plentiful. Theory is continually supported by experimental results from the recent literature. Considerable space is devoted to stochastic processes, noise and fluctuations. The book should have a strong appeal to experimentalists. DG

APPLICATIONS (PHYSICS), T*(17-18: 1, 2), S, P, L, *Statistical Mechanics*. R.K. Pathria. Pergamon Pr, 1972, xiii + 527 pp, $24.Strongly recommended as a text for graduate level course; also reads easily enough to be used for self study. Extensive bibliography. Primary criticism is its traditional manner. Ensemble theory is developed carefully followed by non-interacting gaseous systems, weakly interacting systems and finally phase transitions. A brief treatment of fluctuations concludes this superb book. DG

APPLICATIONS (PHYSICS), P, L, *Mathematics of Contemporary Physics*. Ed: R.F. Streater. Acad Pr, 1972, xi + 274 pp, $17.50. Expository papers from a 1971 "teaching conference" at Bedford College on quantum field theory, functional analysis, statistical mechanics, quantum scattering, linear fields, free Boson gas and current groups. LAS

Reviewers Whose Initials Appear Above

Paul J. Campbell, St. Olaf; David Grimsrud, St. Olaf; Lorraine L. Keller, St. Olaf; Richard B. Kirchner, Carleton; Richard S. Kleber, St. Olaf; J. Arthur Seebach, Jr., St. Olaf; Lynn A. Steen, St. Olaf; Frederick Stoutland, St. Olaf; Frank L. Wolf, Carleton.

NEWS AND NOTICES

EDITED BY RAOUL HAILPERN, SUNY at Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to Mathematical Association of America, 1225 Connecticut Avenue, NW, Washington, D. C. 20036. Items must be submitted at least two months before publication can take place.

PERSONAL ITEMS

University of Maryland, College Park: Assistant Professors J. C. Alexander and G. A. Helzer have been promoted to Associate Professors; Associate Professor S. Gulick has been promoted to Professor; Professors L. W. Cohen and J. L. Walsh retired on June 15, 1973, with the title of Professor Emeritus; Professor L. J. Goldstein received the Maryland Distinguished Young Scientist Award for 1972.

A. M. Gleason, Hollis Professor of Mathematicks (sic) and Natural Philosophy at Harvard University, has been named chairman of the Division of Mathematical Sciences of the National Research Council. He assumed the two-year position July 1, 1973.

Dr. D. R. Morrison, University of New Mexico, has been appointed Director of the newly formed Division of Computing and Information Science.

Professor Andrew Sterrett, Denison University, has been appointed Dean of the college.

Professor Alson H. Bailey, Georgia Institute of Technology, died on April 10, 1973, at the age of 64. He was a member of the Association for forty-two years.

Mr. Clarence A. Balof, Lincoln College, died on April 20, 1973. He was a member of the Association for forty-seven years.

INTERNATIONAL SYMPOSIA ON MATHEMATICAL EDUCATION

The Second International Congress on Mathematical Education (ICME) was held at Exeter University in England during late summer, 1972. That Congress exhibited a strong interest in discussion of aspects of mathematical education on a wide international scale. The International Commission on Mathematical Instruction (ICMI) responded by giving sponsorship to the following international symposia, which are listed here for the benefit of American mathematics educators. For information concerning particular symposia, please write directly to the person and addresses listed.

(1) *Poland.* Symposium at Warsaw; 1974. Main Subject: Mathematics in Primary Schools (children from 6 to 11 years of age). Professor Z. Semadeni, Institute of Mathematics, Polish Academy of Sciences, Ul. Sniadeckich 8, Warszawa 1, Poland.

(2) *Africa.* Regional Conference. Probably Nairobi, 1974. Main Subject: Interactions between mathematical education and linguistics. Dr. D. Saint-Rossy, Unesco House, Malik Street, PO Box 30592, Nairobi, Kenya.

(3) *Japan.* ICMI-JSME Tokyo Conference; 1974. Preliminary proposal: 5–9 November, 1974. Main Subject: Curriculum and teachers' training. Professor S. Iyanaga, 12–4, Otsuka 6-Chome, Bunkyo-Ku, Tokyo, Japan.

(4) *India.* Regional Conference; late 1974. Main Subject: The Development of an integrated curriculum in mathematics for the underdeveloped countries. Professor P. L. Bhat-

nagar, Dean of Studies, Department of Mathematics, Himachal Pradesh University, Simla 5, India.

(5) *Copenhagen*. Symposium; 1974 or 1975. Planning is at a preliminary stage. Main Subject: Aspects of geometry at school level. Professor H. G. Steiner, 8580 Bayreuth, Geschwister-Scholl-Platz 3, Germany GFR.

(6) *IFIP Conference*. Marseilles; August 1975. Plans are in hand for ICMI to be closely associated with this conference of the International Federation for Information Processing — the Second World Conference on Computer Education. Professor J. Hebenstreit, Ecole Supérieure D'Electricité, 10 Avenue Pierre Larousse, 92 Malakoff, Paris, France.

Two other symposia, already held, are listed here for general information. It may be possible to get post conference information (e.g., proceedings) by writing to the persons referred to.

(1) *Luxembourg*. 3rd Mathematical Seminar at Echternach; 4–9 June, 1973. Main Subject: New Aspects of mathematical applications at school level. Séminaire CIEM, c/o Mr. Jos. Hallé, Lycée Classique, Echternach, CCP 34 540 Luxembourg.

(2) *Hungary*. International Colloquium at Eger; 18–22 June, 1973. Main Subject: Theoretical problems of teaching mathematics in primary schools. Languages: English, French, Russian, German. A. Recski, Bolyai Janos Mathematical Society, Szabadsag Ter 17.II.203, Budapest V, Hungary.

JOURNALS IN MATHEMATICS EDUCATION

The External Affairs Committee of the National Council of Teachers of Mathematics has compiled a "Listing of Foreign and Domestic Journals in Mathematics Education." The "Listing" gives titles and addresses for various foreign and domestic journals as well as, where possible, an indication of education level (s) dealt with in journal articles. The journals are concerned with topics in mathematical education, as opposed to pure mathematics and as opposed to general education. Persons interested in having a copy of the "Listing" should request one (postcard or letter) by writing to: *Listing* Request, National Council of Teachers of Mathematics, 1906 Association Drive, Reston, Virginia 22091.

FELLOWSHIP AND RESEARCH OPPORTUNITIES IN THE MATHEMATICAL SCIENCES

In its annual brochure on Fellowship and Research Opportunities in the Mathematical Sciences, the Division of Mathematical Sciences of the National Research Council calls attention to a number of fellowships and other kinds of support for research in the mathematical sciences at both the predoctoral and postdoctoral levels to be awarded during the year 1973–74. Copies of this brochure are available from:

Division of Mathematical Sciences,
National Research Council,
2101 Constitution Avenue, N. W.,
Washington, D. C. 20418.

MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

CALENDAR OF FUTURE MEETINGS

Summer Meeting 1974: There will be no joint summer meeting in 1974, in order that mathematicians may attend the International Congress of Mathematicians to be held in Vancouver, British Columbia, August 21–29, 1974.

Fifty-eighth Annual Meeting, Washington, D. C., January 25–27, 1975.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Editorial Director.

ALLEGHENY MOUNTAIN, Allegheny College, Meadville, Pennsylvania, May 3–4, 1974.

FLORIDA, University of Florida, Gainesville, March 8–9, 1974.

ILLINOIS, Knox College, Galesburg, May 10–11, 1974.

INDIANA, Rose-Hulman Institute of Technology, Terre Haute, April 27, 1974.

IOWA, Upper Iowa College, Fayette, April 19, 1974.

KANSAS, Ottawa University, Ottawa, Spring 1974.

KENTUCKY

LOUISIANA-MISSISSIPPI

MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA

METROPOLITAN NEW YORK

MICHIGAN, Central Michigan University, Mount Pleasant, May 3–4, 1974.

MISSOURI, University of Missouri at Rolla, Rolla, March 29–30, 1974.

NEBRASKA, University of South Dakota, Vermillion, April 19–20, 1974.

NEW JERSEY

NORTH CENTRAL, South Dakota State University, Brookings, April 27, 1974.

NORTHEASTERN

NORTHERN CALIFORNIA, Chabot College, Hayward, February 1975.

OHIO, Muskingum College, New Concord, May 3–4, 1974.

OKLAHOMA-ARKANSAS, University of Arkansas, Little Rock, April 5–6, 1974.

PACIFIC NORTHWEST, University of British Columbia, Vancouver, August 21–24, 1974 (business meeting only — no general meeting).

PHILADELPHIA

ROCKY MOUNTAIN, Colorado School of Mines, Golden, April 26–27, 1974.

SEAWAY, Union College, Schenectady, April 27, 1974.

SOUTHEASTERN, University of Tennessee at Knoxville, March 29–30, 1974.

SOUTHERN CALIFORNIA, Harvey Mudd College, Claremont, March 2, 1974.

SOUTHWESTERN, New Mexico State University, Las Cruces, April 5–6, 1974.

TEXAS

WISCONSIN, Marquette University, Milwaukee, May 3–4, 1974.

FUTURE MEETINGS OF OTHER ORGANIZATIONS

AMERICAN ASSOCIATION FOR THE ADVANCEMENT OF SCIENCE, San Francisco, February 24–March 1, 1974.

AMERICAN MATHEMATICAL SOCIETY, Washington, D. C., January 23–26, 1975.

AMERICAN SOCIETY FOR ENGINEERING EDUCATION

ASSOCIATION FOR COMPUTING MACHINERY, San Diego, California, November 12–16, 1974.

ASSOCIATION FOR SYMBOLIC LOGIC

FIBONACCI ASSOCIATION

INSTITUTE OF MATHEMATICAL STATISTICS

MU ALPHA THETA, University of Arkansas, Fayetteville, August 4–7, 1974.

NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS, Atlantic City, New Jersey, April 17–20, 1974.

OPERATIONS RESEARCH SOCIETY OF AMERICA, Boston, April 19–20, 1974.

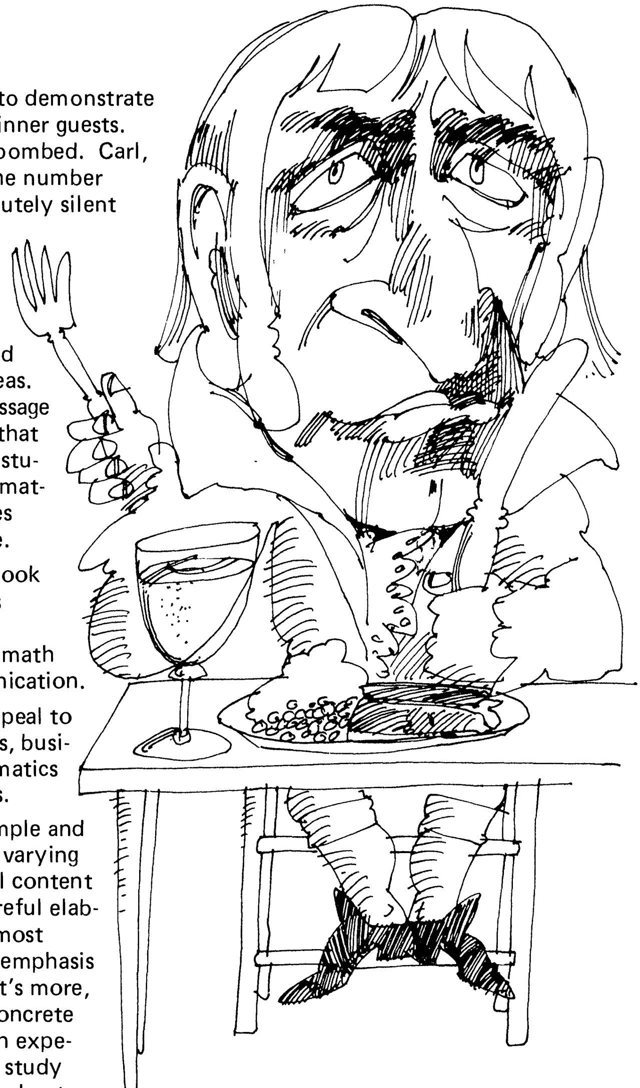
PI MU EPSILON, Western Michigan University, Kalamazoo, August 18–23, 1975.

SCHOOL SCIENCE AND MATHEMATICS ASSOCIATION

SOCIETY FOR INDUSTRIAL AND APPLIED MATHEMATICS

THE AMERICAN MATHEMATICAL MONTHLY, Vol. 81, No. 1, January 1974, pp. 1–110

Speak, Gauss!
C'mon boy, Speak!



As a child, Carl Gauss was made to demonstrate his mathematical precocity for dinner guests. On at least one occasion the act bombed. Carl, being nearsighted, couldn't see the number slate his father held and sat absolutely silent while the audience looked on.
And on. And on.

What's the point?

At Addison-Wesley we know math inside and out. Its ideas and the personalities behind those ideas. And we know how to get the message across. We publish college texts that blend theory and application, so students come to understand mathematics. Thoroughly. And that makes math a whole new numbers game.

An obvious example is the new book by Joseph Newmark and Frances Lake – **Mathematics as a Second Language.** It's designed to show math as an everyday means of communication.

In topic selection, the authors appeal to the diverse interests of liberal arts, business, social science, finite mathematics and elementary education majors.

Although the language used is simple and easily understood by students of varying reading abilities, its mathematical content is complete and correct. And careful elaboration is given to points which most frequently cause difficulty. The emphasis is on practical applications. What's more, each concept is motivated by a concrete example within the student's own experience. And as an additional aid, study guides appear at the end of every chapter.

Another outstanding feature of the book is its highly attractive, open format. Contemporary pictograms at the opening of chapters further enhance the book's overall appearance and illustrate difficult concepts within the text.

One more thing. The notes for this text were successfully class-tested over a two-year period.

So if you've been waiting for the math course that dissolves mental blocks, wait no more. Newmark and Lake are here.

College Division
ADDISON-WESLEY PUBLISHING COMPANY
Reading, Massachusetts 01867

WHERE MATH WARMS UP.

THE AMERICAN
MATHEMATICAL MONTHLY

(FOUNDED IN 1894 BY BENJAMIN F. FINKEL)

THE OFFICIAL JOURNAL OF

THE MATHEMATICAL ASSOCIATION OF AMERICA

VOLUME 81 **NUMBER 2**

CODEN: AMMYAE

CONTENTS

(*Continued on inside cover*)

FEBRUARY **1974**

NOTICE TO AUTHORS

Research papers per se are unsuitable; see statement of policy (Vol. 81, p. 1).

Please follow the format in current issues of the Monthly. Manuscripts must be legibly typewritten or reproduced from typewritten copy, double spaced with wide margins. Three copies should be submitted to the appropriate editor and one kept by the author as protection against loss. The author's full address *must* appear at the end of the manuscript.

Backlog: Main Articles 18 months, Math. Notes 27 months, Research Problems 24 months, Classroom Notes 24 months, Math. Education 19 months.

EDITORIAL CORRESPONDENCE AND MAIN ARTICLES: to Alex Rosenberg, Department of Mathematics, Cornell University, Ithaca, N.Y. 14850; NOTES, etc.: to the corresponding Associate Editor; ADVERTISING CORRESPONDENCE: to Raoul Hailpern, Mathematical Association of America, SUNY at Buffalo, Buffalo, N. Y. 14214; CHANGE OF ADDRESS and SUBSCRIPTIONS: to A. B. Willcox, Mathematical Association of America, 1225 Connecticut Ave., N.W., Washington, D.C. 20036.

Annual dues for members of the Association (including a subscription to the American Mathematical Monthly) are $14.00 for each of the first two years of membership and $18.00 thereafter. Student Membership is available with annual dues of $9.00. For nonmembers the subscription price is $18.00.

Published by the Association at Washington, D. C., and Menasha, Wisconsin, during the months of January, February, March, April, May, June–July, August–September, October, November, December.

Second-class postage paid at Washington, D. C., and additional mailing offices.

PRINTED IN THE UNITED STATES OF AMERICA

AWARD FOR DISTINGUISHED SERVICE TO PROFESSOR R. H. BING

In any assessment of contemporary mathematicians who have served and are serving the causes of mathematics, R. H. Bing is both an essential singularity and a pole. He is an essential singularity because "in the whole human race, there's nobody else quite like R. H. Bing" and because in any neighborhood of R. H. Bing there is activity of all the many facets of mathematics including inspirational teaching, ingenious research, dedicated leadership, and delightfully effective salesmanship for our subject. He is a pole because like a magnet he has attracted so many gifted young people to the field of mathematics and because in any neighborhood of R. H. Bing, the enthusiasm for mathematics is unbounded.

R. H. Bing, or R. H. as his family and friends call him, was born and bred a Texan and his speech still attests to that fact. He was one of those Texans with no given names, just initials. It is said that he was once called Ronly Honly Bing for R. (only) H. (only) Bing.

Along with L. B. J., R. H. was a graduate of San Marcos State Teachers College in Texas. It took him two and a half years to finish and then he did what graduates were trained to do — he taught. R. H. was a high school mathematics teacher and coach for six years before beginning his serious graduate work in mathematics under R. L. Moore. He not only coached football but basketball and track as well. He is one of the few high school coaches who made it to the National Academy of Sciences. It is reported that in 1938 he took a summer course in geometry under Moore while working on his Master of Education degree and he decided that if those other guys in the class could be mathematicians, then so could he. He was right, he could.

When he later took Moore's topology courses he was quickly identified as a most promising student and his later research achievements fully justified Moore's judgement of him. Since his graduate school days, he was the principal yardstick by which Moore measured his later students.

Two years after getting his degree in 1945, R. H. Bing moved to the University of Wisconsin and just this past summer returned from there to the University of Texas.

At Wisconsin, he had 30 Ph.D. students and his seminar produced more than a dozen others. Among his own students have been Morton Brown, James Kister, Russ McMillan, Les Glaser, David Henderson, John Hempel and Dallas Webster to name only a few. However, the seminar he organized and led at Wisconsin was not just for students but for Wisconsin's faculty and for many visiting faculty as well. It is fair to say that Wisconsin under R. H. Bing became perhaps the world's major center of point set topology. His students and his students' students and those influenced by him and his mathematical descendents have had great impact on much of modern topology.

R. H. BING

But his successes in teaching and in research have been just a part of his many contributions to mathematics. Perhaps it is his complete and delightfully contagious enthusiasm for mathematics and his desire and ability to communicate that enthusiasm that make him most distinctive. He has long been one of mathematics' best salesmen — a man always willing to give of his time and energy to talk in his folksy way to student groups at all levels, to teachers, to research seminars, to bankers, to chambers of commerce, to service clubs, to any group interested in how a mathematician views or does mathematics. And he was and is in demand as a speaker because he developed a richly deserved reputation as a person anxious to communicate at the level of his audience in showing that mathematics is an interesting, dynamic, and vital part of human experience.

It has been said that, to R. H. Bing mathematics is like a game — there is mathematics and there is R. H. Bing and the battle is to be joined. They were and are serious but friendly adversaries, and RH has won more than his share of the contests. Perhaps this analogy is too frivolous, but it has its element of truth in the joy and verve with which RH attacked his problems.

Throughout his mathematical career, his wife, Mary, has been a constant source of both inspiration and patient understanding. She has shared his burdens and fully deserves to share his recognitions.

Back in the mid-1950's he helped organize and run the NSF sponsored summer conference on topology at Madison. Even though he was deeply involved with much of the mathematics being discussed and done at the conference he was still able to handle the local arrangements with skill and aplomb. Many of us there could only stand in awe at his virtuoso performance. Later we discovered one of the secrets of his success. He got up at five o'clock to do mathematics. It gave him quite a head start.

In the early days of SMSG, RH was one of the active participants in the geometry writing group bringing to the effort his own previous high school experience, his knowledge of the power of geometric thought, and his training in that first graduate course he took from R. L. Moore.

R. H. has given long and distinguished service to the mathematical community in his many terms of duty in the MAA, in the AMS, on various national and international committees, and for the past six years as a member of the National Science Board. Recently he has been the only mathematician on the Board.

He has served as President of the MAA and as a long time member of the Board of Governors. In the AMS, he was twice an elected member-at-large of the Council, served on the Executive Committee, and more recently, served as Vice-President. His judgement of things mathematical is so widely respected that he has been on the editorial boards of the *Bulletin* and the *Proceedings* and has been on numerous nominating and selection committees for many organizations including a three-year term as a member of the nominating committee to the National Medal of Science.

One of RH's many services to this country's mathematics has been his frequent

involvement with international mathematical activities. He has been an invited speaker at various international symposia and conferences in topology; he gave an invited talk at the International Congress in Stockholm, and in 1966 he served on the committee of the International Mathematical Union concerned with the Moscow Congress. In the summer of 1973, he was an active participant in the Binational Conference in Mathematics Education and Research in Bangalore, India.

As a man of mathematics, R. H. Bing is one of the effective, dedicated and useful men of our time. As a human being he glows with a warmth, a love of fun, and a joyous spirit which translates itself into genuine concern for his fellow man. The happy mixture of the two is what we honor today.

R. D. ANDERSON

AWARD OF THE 1974 CHAUVENET PRIZE TO PROFESSOR PETER D. LAX

The Board of Governors of the Mathematical Association of America at its meeting on August 19, 1973, at Missoula, Montana, voted to award the 1974 Chauvenet Prize to Professor Peter D. Lax for his paper "The Formation and Decay of Shock Waves", which appeared in this MONTHLY, 79 (1972) 227–241.

A certificate and monetary award in the amount of five hundred dollars was presented to Professor Lax at the time of the Annual Business Meeting of the Association on January 18, 1974, in San Francisco.

The Chauvenet Prize is awarded for a noteworthy paper of an expository or survey nature published in English, which comes within the range of profitable reading for members of the Association. The purpose of the Prize is to stimulate the writing of expository and survey articles. The 1974 Prize, awarded for a paper published in the three-year period 1970-72, is the twenty-second award of the Chauvenet Prize since its institution by the MAA in 1925. For the list of names of previous winners, see this MONTHLY, 71 (1964) p. 589, 73 (1965) pp. 2–3, 74 (1967) p. 3, 75 (1968) pp. 3–4, 77 (1970) pp. 117–118, 78 (1971) pp. 112–113, 79 (1972) pp. 112–113, and 80,(1973) p. 120.

Professor Lax was born on May 1, 1926, in Hungary. He received his A.B. degree in 1947 and his Ph.D. degree in 1949 under Professor K. O. Friedrichs, both degrees at New York University. He joined the staff at New York University in 1949 as assistant professor, advancing rapidly through the ranks and becoming Director of the Courant Institute of Mathematical Sciences at New York University in 1972, after having served as a director of the A.E.C. Computing and Applied Mathematics Center at the Courant Institute of Mathematical Sciences.

He has been a frequent summer visitor at Stanford, a member of the staff of the Los Alamos Scientific Laboratories in 1950, a consultant to the Radiation Laboratory in California, and a Fulbright Lecturer in the summer of 1958.

In the summer of 1968, Professor Lax gave the SIAM von Neumann lecture; in the spring of 1972, he gave the Weyl Lectures at the Institute for Advanced Study at Princeton, and in the summer of 1973 he was the Hedrick Lecturer of the Association at the meeting at Dartmouth.

Professor Lax is a member of the National Academy of Sciences. He was Vice-President of the American Mathematical Society in 1969–71. He has served the Association in many capacities: as a member of the Joint Committee (with the AMS) on the Graduate Program in Mathematics; a member of the Panel of CUPM on Pregraduate Training in 1962–64, also its Subcommittee on Applied Mathematics, and a member of the CUPM Panel on Computing in 1968–69. He was an Associate Editor of this MONTHLY for the period 1969–73, a member of the Advisory Committee on Individual Lecture Films since 1964, and was appointed to the Committee on Earle Raymond Hedrick Lectures in 1973. The same year, he was elected a member of the Board of Governors.

He has twice received the Association's Lester R. Ford Award: in 1966 for his paper *Numerical Solution of Partial Differential Equations*, in this MONTHLY, 72 (1965) Part II (Slaught Paper No. 10) 74–84, and in 1973 for the paper for which he has now been awarded the Chauvenet Prize.

Professor Lax's interest in mathematics has centered on partial differential equations, linear and nonlinear functions of mathematical physics, computing, and functional analysis whenever needed. His many significant contributions to these areas of mathematics are contained in his nearly sixty publications, including the book published with R. Phillips *Scattering Theory*, Academic Press, 1967, and the monograph *Hyperbolic Systems of Conservation Laws and the Mathematical Theory of Shock Waves*, Conference Board of the Mathematical Sciences, Monograph No. 11, SIAM, 1973.

In his acceptance, Professor Lax expressed his gratitude for the honor of having been awarded the 1974 Chauvenet Prize.

He expressed his indebtedness to James Glimm for his fertile ideas on the decay of shock waves. He declared himself extremely fortunate in having teachers who were outstanding expositors: Miss R. Peter, author of the charming popular book, *Playing with Infinity*; Professors Pólya and Szegö of *Aufgaben und Lehrsätze* fame (now happily available in an English enlarged edition) and, most of all, to Richard Courant — a master expositor in both pure and applied mathematics.

REAL PROOFS OF COMPLEX THEOREMS (AND VICE VERSA)

LAWRENCE ZALCMAN

Introduction. It has become fashionable recently to argue that real and complex variables should be taught together as a unified curriculum in analysis. Now this is hardly a novel idea, as a quick perusal of Whittaker and Watson's *Course of Modern Analysis* or either Littlewood's or Titchmarsh's *Theory of Functions* (not to mention any number of *cours d'analyse* of the nineteenth or twentieth century) will indicate. And, while some persuasive arguments can be advanced in favor of this approach, it is by no means obvious that the advantages outweigh the disadvantages or, for that matter, that a unified treatment offers any substantial benefit to the student. What is obvious is that the two subjects do interact, and interact substantially, often in a surprising fashion. These points of tangency present an instructor the opportunity to pose (and answer) natural and important questions on basic material by applying real analysis to complex function theory, and vice versa. This article is devoted to several such applications.

My own experience in teaching suggests that the subject matter discussed below is particularly well-suited for presentation in a year-long first graduate course in complex analysis. While most of this material is (perhaps by definition) well known to the experts, it is not, unfortunately, a part of the common culture of professional mathematicians. In fact, several of the examples arose in response to questions from friends and colleagues. The mathematics involved is too pretty to be the private preserve of specialists. Publicizing it is the purpose of the present paper.

1. The Greening of Morera. One of the most useful theorems of basic complex analysis is the following result, first noted by Giacinto Morera.

MORERA'S THEOREM [37]. *Let $f(z)$ be a continuous function on the domain D. Suppose that*

$$\int_\gamma f(z)dz = 0 \tag{1}$$

for every rectifiable closed curve γ lying in D. Then f is holomorphic in D.

Morera's Theorem enables one to establish the analyticity of functions in situations where resort to the definition and the attendant calculation of difference quotients would lead to hopeless complications. Applications of this sort occur, for instance, in the proofs of the Schwarz Reflection Principle and other theorems on the extension of analytic functions. Nor is its usefulness limited to this circle of ideas; the important fact that the uniform limit of analytic functions is again analytic is an immediate consequence (observed already by Morera himself, as well as by Osgood [**39**], who had rediscovered Morera's theorem).

Perhaps surprisingly, the proofs of Morera's theorem found in complex analysis texts all follow a single pattern. The hypothesis on f insures the existence of a single-valued primitive F of f, defined by

$$F(z) = \int_{z_0}^{z} f(\zeta)d\zeta. \tag{2}$$

Here z_0 is some fixed point in D and the integral is taken over any rectifiable curve joining z_0 to z. The function F is easily seen to be holomorphic in D, with $F'(z) = f(z)$; since the derivative of a holomorphic function is again holomorphic, we are done.

Several remarks are in order concerning the proof sketched above. First of all, the assumption that (1) holds for all rectifiable closed curves in D is much too strong. It is enough, for instance, to assume that (1) holds for all closed curves consisting of a finite number of straight line segments parallel to the coordinate axes; the integration in (2) is then effected over a (nonclosed) curve composed of such segments, and the proof proceeds much as before. Second, since analyticity is a local property, condition (1) need hold only for an arbitrary neighborhood of each point of D; that is, (1) need hold only for *small* curves. Finally, the proof requires the fact that the derivative of an analytic function is again analytic. While this is a trivial consequence of the Cauchy integral formula, it can be argued that that is an inappropriate tool for the problem at hand; on the other hand, a proof of this fact without complex integration is genuinely difficult and was, in fact, only discovered (after many years of effort) in 1961 [**44**], [**10**], [**46**].

There is an additional defect to the proof, and that is that *it does not generalize*. Thus, it was more than thirty years after Morera discovered his theorem that Torsten Carleman realized the result remains valid if (1) is assumed to hold only for all (small) *circles* in D. It is an extremely instructive exercise to try to prove Carleman's version of Morera's theorem by mimicking the proof given above. The argument fails because it cannot even be started: the very existence of a single-valued primitive is in doubt. This leads one to try a different (and more fruitful) approach, which avoids the use of primitives altogether.

Suppose for the moment that f is a smooth function, say continuously differentiable. Fix $z_0 \in D$ and suppose (1) holds for the circle $\Gamma_r(z_0)$ of radius r, centered at z_0. Then, by the complex form of Green's theorem

$$0 = \int_{\Gamma_r(z_0)} f(z)dz = 2i \iint_{\Delta_r(z_0)} \frac{\partial f}{\partial \bar{z}} dxdy,$$

where $\Delta_r(z_0)$ is the disc bounded by $\Gamma_r(z_0)$ and $\partial f/\partial \bar{z} = \frac{1}{2}(\partial f/\partial x + i\, \partial f/\partial y)$. (There's no cause for panic if the $\partial/\partial \bar{z}$ operator makes you uneasy or you are not familiar with the complex form of Green's theorem; just write $f(z) = u(z) + iv(z)$, $dz = dx + idy$, and apply the usual version of Green's theorem to the real and imaginary parts of the integral on the left.) Dividing by an appropriate factor, we

have

$$\frac{1}{\pi r^2} \iint_{\Delta_r(z_0)} \frac{\partial f}{\partial \bar{z}} dxdy = 0;$$

i.e., the average of the continuous function $\partial f/\partial \bar{z}$ over the disc $\Delta_r(z_0)$ equals 0. Make $r \to 0$ to obtain $(\partial f/\partial \bar{z})(z_0) = 0$. Since this holds at each point $z_0 \in D$, $\partial f/\partial \bar{z} = 0$ identically in D. Writing this in real coordinates, we see that $u_x = v_y$, $u_y = -v_x$ in D; thus the Cauchy-Riemann equations are satisfied and f is analytic.

Notice that we did not need to assume that (1) holds for all circles in D or even all small circles; to pass to the limit it was enough to have, for each point of D, a sequence of circles shrinking to that point. Moreover, since f has been assumed to be continuously differentiable, it is sufficient to prove that $\partial f/\partial \bar{z}$ vanishes on a dense set. Finally, and most important, *the fact that our curves were circles was not used at all*! Squares, rectangles, pentagons, ovals could have been used just as well. To conclude that $(\partial f/\partial \bar{z})(z_0) = 0$, all we require is that (1) should hold for a sequence of simple closed curves γ that accumulate to z_0 (z_0 need not even lie inside or on the γ's) and that the curves involved allow application of Green's theorem. It is enough, for instance, to assume that the curves are piecewise continuously differentiable.

To summarize, we have shown that Green's theorem yields in a simple fashion a very general and particularly appealing version of Morera's theorem for C^1 functions. It may reasonably be asked at this point if the proof of Morera's theorem given above can be modified to work for functions which are assumed only to be continuous. That is the subject of the next section.

2. Smoothing. Let $\phi(z)$ be a real valued function defined on the entire complex plane which satisfies

(a) $\phi(z) \geqq 0$,

(b) $\iint \phi(z)dxdy = 1$,

(c) ϕ is continuously differentiable,

(d) $\phi(z) = 0$ for $|z| \geqq 1$.

It is trivial to construct such functions; we can even require ϕ to be infinitely differentiable and to depend only on $|z|$, but these properties will not be required in the sequel. Set, for $\varepsilon > 0$, $\phi_\varepsilon(z) = \varepsilon^{-2}\phi(z/\varepsilon)$. Then, clearly, ϕ_ε satisfies (a) through (c) above and ϕ_ε vanishes off $|z| < \varepsilon$. The family of functions $\{\phi_\varepsilon\}$ forms what is known in harmonic analysis as an approximate identity (a smooth approximation to the Dirac delta function); workers in the field of partial differential equations, where the smoothness properties of the ϕ_ε are emphasized, are accustomed to call similar functions (Friedrichs) mollifiers.

Suppose now that f is a continuous function on some domain D and set

$$f_\varepsilon(z) = \iint f(z-\zeta)\phi_\varepsilon(\zeta)d\xi d\eta \qquad \zeta = \xi + i\eta, \tag{3}$$

where the integral is extended over the whole complex plane. This integral exists and defines a continuous function for all points z whose distance from the boundary of D is greater than ε. Moreover, $f_\varepsilon(z)$ is continuously differentiable for such points. Indeed, changing variable in (3), we have

$$f_\varepsilon(z) = \iint \phi_\varepsilon(z-\zeta)f(\zeta)d\xi d\eta$$

and the x and y derivatives can be brought inside the integral since we have chosen ϕ_ε to be continuously differentiable. Finally, we note that for any compact subset K of D, $f_\varepsilon(z)$ converges uniformly to $f(z)$ on K as $\varepsilon \to 0$. This expresses the delta-function-like behavior of the family $\{\phi_\varepsilon\}$. Here is the simple proof. By (b),

$$f(z) - f_\varepsilon(z) = \iint_{|\zeta| \leqq \varepsilon} \{f(z) - f_\varepsilon(z-\zeta)\}\phi_\varepsilon(\zeta)d\xi d\eta,$$

whence by (a) and (b)

$$\begin{aligned} |f(z) - f_\varepsilon(z)| &\leqq \iint_{|\zeta| \leqq \varepsilon} |f(z) - f(z-\zeta)|\,\phi_\varepsilon(\zeta)d\xi d\eta \\ &\leqq \sup_{|\zeta| \leqq \varepsilon} |f(z) - f(z-\zeta)|. \end{aligned} \tag{4}$$

Since K is compact, f is uniformly continuous on K; so (4) shows that

$$\sup_{z \in K} |f(z) - f_\varepsilon(z)| \to 0 \qquad \text{as } \varepsilon \to 0.$$

The proof of Morera's theorem is now easily completed. Suppose, for instance, that f is continuous on D and that there exists a sequence of positive numbers $r_1 \geqq r_2 \geqq r_3 \geqq \cdots \to 0$ such that

$$\int_{\Gamma_{r_n}(z)} f(w)dw = 0 \tag{5}$$

for each $z \in D$ whenever the circle $\Gamma_{r_n}(z) = \{w: |w - z| = r_n\}$ lies in D. Fix a compact set $K \subset D$ and take $\varepsilon < \frac{1}{2}\operatorname{dist}(K, \partial D)$. Then for $r = r_n < \frac{1}{2}\operatorname{dist}(K, \partial D)$ and $z \in K$ we have

$$\begin{aligned} \int_{\Gamma_r(z)} f_\varepsilon(w)dw &= \int_{\Gamma_r(z)} \left\{\iint f(w-\zeta)\phi_\varepsilon(\zeta)d\xi d\eta\right\} dw \\ &= \iint \left\{\int_{\Gamma_r(z)} f(w-\zeta)dw\right\} \phi_\varepsilon(\zeta)d\xi d\eta \\ &= \iint \left\{\int_{\Gamma_r(z-\zeta)} f(w)dw\right\} \phi_\varepsilon(\zeta)d\xi d\eta \\ &= 0. \end{aligned}$$

Since f_ε is continuously differentiable, it is analytic on the interior of K; and since f_ε converges to f uniformly on K, f must be analytic there. Finally, because K is arbitrary, f is analytic on all of D.

Again, there is nothing particularly sacred about circles: if $\{\gamma_n\}$ is a sequence of simple closed piecewise continuously differentiable curves which shrink to the origin and $\gamma_n(z)$ is the image of γ_n under the map $w \mapsto w + z$, we may replace (5) by

$$\int_{\gamma_n(z)} f(w)dw = 0 \tag{6}$$

and the rest of the argument remains unchanged. Similarly, it is enough to assume that (5) or (6) hold only for a dense set of $z \in D$, since the full condition then follows from the continuity of f.

The result can be extended even further. The requirement that f be continuous may be relaxed to the assumption that f is measurable and integrable with respect to Lebesgue area measure on compact subsets of D. Of course, the conclusion now reads that f agrees almost everywhere with a function analytic on D. For a complete treatment, together with an historical discussion, see [**62**]. The use of smoothing operators is a standard tool among workers in partial differential equations and approximation theory; for a systematic exposition of its use in this last subject, see [**52**].

The success of the smoothing technique in dealing with Morera's theorem suggests using it to prove Cauchy's theorem. This is a good idea, but one which, unfortunately, simply does not work. Here's the rub. Suppose $f(z)$ is analytic in the disc D. We know (by Green's theorem) that $\int_T f(z)dz = 0$ for every triangle T in D *if* f is continuously differentiable. Of course, in general, f is *not* known *a priori* to be continuously differentiable; but we may construct $f_\varepsilon(z)$, as in (3), which is. *However, it is not clear that* $f_\varepsilon(z)$ *is holomorphic.* The problem is that while $f'(z)$ is known to exist for each $z \in D$, and is easily proved to be measurable, it is *not* known to be integrable; we cannot, therefore, differentiate f inside the integral sign of (3). (A similar difficulty arises in the proof of Hartogs' theorem: *If a function of two complex variables* $g(z_1, z_2)$ *is analytic in each variable separately, then it is analytic as a function of the joint variables* z_1, z_2). The argument does work if f' is assumed to be area integrable, but this assumption is (of course) unnecessary, and it seems best to base the proof of Cauchy's theorem on Pringsheim's device [**45**] of subdividing triangles. This is the pattern followed in most modern texts.

3. In circles. All the versions of Morera's theorem discussed up to now have depended in an essential fashion on the fact that (1) holds for a certain class of contours containing arbitrarily small curves. The obvious question to ask is what happens if (1) holds for circles which do *not* shrink in radius. In this situation, it is natural to assume that the function in question is defined on the entire complex plane. A satisfying answer is provided by the following result, proved in 1970 [**62**].

THEOREM. *Let f be a continuous function on the complex plane and suppose that there exist numbers $r_1, r_2 > 0$ such that*

$$\int_\Gamma f(z)dz = 0 \tag{7}$$

for every circle having radius r_1 or r_2 (and arbitrary center). Then f is an entire function unless r_1/r_2 is a quotient of zeroes of the Bessel function $J_1(z)$.

The hypothesis on f may be relaxed to the assumption of local integrability, and (7) need hold only for 'almost all' circles. The restriction on the pair r_1, r_2 is, however, essential: in case it is not satisfied, f may fail to be holomorphic anywhere.

The proof is considerably more involved than (and of an altogether different character from) the sort of argument we have seen in the preceding sections; essential ingredients include the harmonic analysis of an appropriate space of distributions and the Delsarte-Schwartz theory of mean-periodic functions. See [**62**], where related results are discussed, for details. One can also show that if f is continuous on the plane and (7) holds for every square (of arbitrary center and orientation) having side of fixed length, then f is entire. Again, a reference is [**62**]. Further perspectives on results of this sort will be found in [**63**].

4. Reflections on reflection. According to the Schwarz Reflection Principle, if $f(z)$ is analytic in $\Delta = \{z: |z| < 1\}$ and continuously extendible to an open arc γ of $\Gamma = \{z: |z| = 1\}$, and if the values of f corresponding to points of γ lie on a circular, or, more generally, an analytic arc γ^*, then f may be extended by 'reflection' to a function analytic in a domain containing $\Delta \cup \gamma$. The usefulness of this technique can hardly he overestimated: it provides an essential tool in problems involving the extension of conformal mappings and plays a traditional role in the 'slick' proof [**49**, pp. 322–325] of Picard's little theorem. Another application yields what is surely the simplest proof that a nonzero function analytic in Δ cannot vanish identically on an arc of Γ.

The question thus naturally arises whether an analogous result holds if γ^* is no longer analytic but simply smooth, C^∞ say. A negative answer is immediate. Indeed, let Γ^* be an infinitely differentiable, nowhere analytic, simple closed Jordan curve and let f map Δ conformally onto the interior D of Γ^*. The univalent function f, extends to a homeomorphism of $\Delta \cup \Gamma$ onto $D \cup \Gamma^*$ and induces a one-one correspondence between the points of Γ and those of Γ^*. However, f cannot be continued analytically across any subarc of Γ, for then f would establish an analytic correspondence between a subarc γ of Γ and a subarc γ^* of Γ^*. Thus γ^* would be analytic, contrary to hypothesis. This example is really quite striking, providing, as it does, an example of a (univalent!) function analytic on Δ and of class C^∞ on $\Delta \cup \Gamma$ which cannot be extended analytically across any arc of Γ.

What is not generally realized is that the example can be worked backward to provide an example of an infinitely differentiable, yet nowhere analytic, Jordan

curve. This approach avoids altogether reliance on the plausible (and true) but nonobvious facts concerning smoothness and univalence of the boundary function which we invoked so shamelessly above. The tools we need are two, the first of which is the following simple lemma.

LEMMA. *Let $f(z) = z + a_2 z + a_3 z^3 + \cdots$ be analytic in Δ. Suppose that $\sum_{n=2}^{\infty} n\,|\,a_n\,| < 1$. Then f is continuous on $\Delta \cup \Gamma$ and univalent there.*

Proof. Continuity is clear from the absolute convergence of the series. Let $z, \zeta \in \Delta \cup \Gamma$. Then

$$\frac{f(z)-f(\zeta)}{z-\zeta} = 1 + \sum_{n=2}^{\infty} a_n (z^{n-1} + z^{n-1}\zeta + \cdots + \zeta^{n-1}).$$

Thus

$$\left|\frac{f(z)-f(\zeta)}{z-\zeta}\right| \geqq 1 - \sum_{n=2}^{\infty} n\,|\,a_n\,| > 0,$$

so that f is univalent.

The second ingredient we need is the celebrated Hadamard gap theorem.

HADAMARD GAP THEOREM. *Let $f(z) = \sum_{k=0}^{\infty} a_k z^{n_k}$ have Δ as its disc of convergence. If $n_{k+1}/n_k \geqq q$ for some $q > 1$ and all large k, then f has Γ as its natural boundary; that is, f cannot be continued analytically across any subarc of Γ.*

The beautiful proof of this theorem due to L. J. Mordell ([**36**], cf. [**54**, p. 223]) should be standard fare in graduate courses in complex analysis.

The construction of the required function is now almost trivial. We choose the sequences $\{a_k\}$ and $\{n_k\}$ to satisfy

(a) $a_0 = n_0 = 1$,
(b) $\sum_{k=1}^{\infty} n_k\,|\,a_k\,| < 1$,
(c) $(a_k)^{1/n_k} \to 1$,
(d) $n_{k+1}/n_k \geqq 2$,
(e) $\sum_{k=0}^{\infty} n_k^j\,|\,a_k\,| < \infty \qquad j = 0, 1, 2, \cdots$.

A simple concrete example is provided by the function

$$f(z) = z + \sum_{n=5}^{\infty} z^{2^n}/n!.$$

By the lemma, f establishes a homeomorphism between Γ and a simple closed Jordan curve Γ^*. Since f satisfies the hypothesis of Hadamard's gap theorem, f cannot be extended analytically across any arc of Γ. Hence, Γ^* must be nowhere analytic since otherwise the Schwarz principle would apply. Finally, by (e), the series for $f^{(j)}(z)$ converges absolutely on $\{z : |\,z\,| \leqq 1\}$ for each j; thus f is infinitely differentiable on $\Delta \cup \Gamma$, so that Γ^* is a C^∞ curve.

Interestingly enough, one can trace the basic ideas of this section back to before the turn of the century, (see Osgood [**38**]). In particular, the lemma, which is usually attributed to the American topologist J. W. Alexander [**67**], was known to Fredholm as early as 1897 ([**38**, p. 17]).

5. Extensions. The reflection principle enables one (in certain circumstances) to extend a holomorphic function across an analytic arc to a somewhat larger domain. As we have seen, it is in general impossible to relax the condition of analyticity; nevertheless, the much weaker hypothesis of rectifiability suffices in case a continuous extension analytic in an abutting domain is already known. The precise result may be stated (somewhat informally) as follows.

THEOREM. *Let D be a domain and let J be a simple rectifiable Jordan arc dividing D into disjoint domains D_1 and D_2. Suppose f_j $(j = 1,2)$ is analytic in D_j and continuous on $D_j \cup J$ and that $f_1 = f_2$ on J. Then the function f obtained by setting $f(z) = f_j(z)$ for $z \in D_j \cup J$ is analytic in D.*

The proof is a standard application of Morera's theorem, with due care exercised in dealing with the assumption that J is merely rectifiable.

The precise nature of the hypothesis of rectifiability on J in the above theorem is by no means clear, and the proof (which we leave to the reader) does little to explicate it. My experience has been that students — especially good ones — generally guess that the result remains true if rectifiability is dispensed with. This, however, is *not* the case, as the following example shows.

Let K be a compact set of positive Lebesgue measure and set

$$f(z) = \iint_K \frac{d\xi d\eta}{\zeta - z} \qquad \zeta = \xi + i\eta. \tag{8}$$

The function $f(z)$ is obviously analytic off K and satisfies $f(\infty) = 0$; moreover since $\lim_{z \to \infty} zf(z) = -\iint_K d\xi d\eta \neq 0$, f is nonconstant on the unbounded component of K. We claim f is actually continuous on the complex sphere. Indeed, formula (8) exhibits f explicitly as the convolution of the locally (area) integrable function $1/\zeta$ with the bounded measurable function of compact support $\chi_K(\zeta)$, the characteristic function of K. Such a convolution is well known (and easily proved) to be continuous (see, for instance, [**5**, p. 154]).

Suppose now that $K = J$, a simple closed Jordan curve. The existence of such curves having positive area was first proved by Osgood [**41**] in 1902. (This is one of the relatively few examples in mathematics that retains its original vigor unimpaired: students today — even those who know about Peano curves — are as baffled and surprised by this fact as mathematicians were 70 years ago. The construction is not too complicated for presentation in class, and the example itself instills a healthy respect for the Jordan curve theorem.) One can actually construct J to have the

additional property that it has positive area everywhere, that is, if D is an open set and $D \cap J \neq \emptyset$ then $D \cap J$ has positive area. The function f defined by (8) with $J = K$ is continuous on $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ and analytic off J; thus, it is analytic in both components D_1, D_2 of $\hat{\mathbb{C}} \setminus J$. However, f is not analytic at any point of J. Indeed, suppose f analytic at $z_0 \in J$ and let D be a small open disc about z_0 lying in the domain of analyticity of f. Set $J_1 = D \cap J$. Then

$$f(z) = \int_{J \setminus J_1} \frac{d\xi d\eta}{\zeta - z} + \int_{J_1} \frac{d\xi d\eta}{\zeta - z} = g(z) + h(z)$$

for $z \notin J$, and $g(z)$ is clearly analytic in D. Thus $h(z)$ must be analytic in D as well. But h is obviously analytic off $\bar{D}$ and continuous on $\hat{\mathbb{C}}$. Thus, according to the theorem of the present section, h is analytic on all of $\hat{\mathbb{C}}$, hence a constant. But $J \cap D = J_1$ has positive area, so that $h(z)$ is nonconstant. We have reached the desired contradiction.

Thus f cannot be continued analytically across any arc of J. In particular, the restrictions f_1, f_2 of f to the components D_1, D_2 of $\hat{\mathbb{C}} \setminus J$ determine analytic functions *which are not analytic continuations of one another*; indeed, J forms a natural boundary for each of these functions.

Actually, the requirement that J have positive measure was used merely to insure the existence of nontrivial functions continuous on $\hat{\mathbb{C}}$ and analytic off J. The same result can be obtained (but with more work) if the set in question has positive Hausdorff $(1 + \varepsilon)$-measure for some $\varepsilon > 0$ [**61**]. Even this condition is not necessary; in fact, Denjoy [**11**] has constructed an arc which is the graph of a function and which has the required property.

6. Blowing up the boundary. Questions involving length and area arise in conformal mapping as well. A conformal map, being analytic, must map sets of zero area to sets of zero area; however, distortion at the boundary is an *a priori* possibility. Writing $\Delta \cup \Gamma = \{z : |z| \leqq 1\}$ as before, let us assume that the univalent function $f(z)$ maps Δ conformally onto the Jordan region D. According to the Osgood-Taylor-Carathéodory theorem, f extends to a homeomorphism of $\Delta \cup \Gamma$ onto $D \cup \partial D$. (Proofs of this important result, announced by Osgood [**65**] and proved independently by Osgood and Taylor [**66**, p. 294], and Carathéodory [**6**], [**7**] are available in [**9**, pp. 46–49] and [**24**, p. 129–134]. The reader will find a comparison of the treatments in these references particularly instructive in the matters of style of exposition and attention to detail.) In case ∂D is rectifiable, a theorem of the Riesz brothers [**47**] insures that f and f^{-1} preserve sets of zero length (= Hausdorff one-dimensional measure). When ∂D fails to be rectifiable, however, all hell breaks loose. In particular, a subset of ∂D having positive area may correspond to a subset of Γ having zero Lebesgue (linear) measure! For the construction, we need an important result from plane topology.

MOORE-KLINE EMBEDDING THEOREM [**35**]. *A necessary* and *sufficient condition*

that a compact set $K \subset \mathbb{C}$ should lie on a simple Jordan arc is that each closed connected subset of K should be either a point or a simple Jordan arc with the property that $K - \gamma$ does not accumulate at any point of γ, except (perhaps) the endpoints.

Now let K be a Cantor set having positive area; K may be realized, for instance as the product of two linear Cantor sets, each of which has positive linear measure. Construct countably many disjoint simple Jordan arcs $J_n \subset \mathbb{C} \setminus K$ such that the sequence $\{J_n\}$ accumulates at each point of K and at no other points of $\hat{\mathbb{C}}$ and with the additional property that if $z_0 \in \mathbb{C} \setminus (K \cup \{J_n\}) = R$ and $z \in K$, then any arc from z_0 to z which lies, except for its final endpoint, in R must have infinite length. By the Moore-Kline embedding theorem, we may pass a simple closed Jordan curve J through $K \cup \{J_n\}$. Let f be a conformal map from Δ to D, the domain bounded by J. Then f extends to a homeomorphism from Γ to J. Let $S = f^{-1}(K)$. That S has zero linear measure follows at once from the following theorem, due to Lavrentiev.

THEOREM. *Let f be a conformal homeomorphism of $\Delta \cup \Gamma$ onto the Jordan domain $D \cup J$. If $S \subset J$ is not rectifiably accessible from D then $f^{-1}(S) \subset \Gamma$ has zero measure.*

Proof. Since D is a bounded domain, its area, given by the expression $\iint_\Delta |f'(z)|^2 dxdy$, is finite. Thus

$$\int_0^{2\pi} \int_0^1 |f'(re^{i\theta})| \, r dr d\theta$$
$$\leqq \left(\int_0^{2\pi} \int_0^1 r dr d\theta \right)^{1/2} \left(\int_0^{2\pi} \int_0^1 |f'(re^{i\theta})|^2 r \, dr d\theta \right)^{1/2} < \infty.$$

It follows that $\int_0^1 |f'(re^{i\theta})| \, r \, dr < \infty$ for almost all θ or, what is the same, $l(\theta) = \int_0^1 |f'(re^{i\theta})| \, dr < \infty$ almost everywhere. But $l(\theta)$ is the length of the image of the radius from 0 to $e^{i\theta}$ under f. So almost every point of Γ corresponds to a rectifiably accessible point of J, and we are done.

Actually, much more is true. It follows from a result of Beurling [**3**] (cf. [**9**, p. 56]) that the set of points on J which are not rectifiably accessible from D must correspond to a set of logarithmic capacity 0 on the unit periphery. It would take us too far afield to enter into a detailed discussion of the capacity of plane sets here; for our purposes it is enough to know that sets of capacity zero are exceedingly small. For instance, such a set must have zero Hausdorff ε-measure for all $\varepsilon > 0$. The first person to show that a set of capacity zero on Γ could correspond under a conformal mapping to a set having positive area was Kikuji Matsumoto [**33**]. He actually proved (what is implicit in the above discussion) that for each totally disconnected compact subset K of the plane there exists a Jordan domain D with boundary $J \supset K$ such that K corresponds under conformal mapping to a set of capacity zero

on Γ. The discussion here (in particular, the ingenious proof of the central result) is based on an idea of Walter Schneider [**51**].

The *compression* of the boundary of the unit disc presents greater difficulties. Lavrentiev, however, has shown that a set of positive measure on Γ may be mapped onto a set of zero length under a conformal mapping of Jordan domains [**30**]. A more recent construction is due to McMillan and Piranian [**32**].

7. Absolute convergence and uniform convergence. Conformal mapping techniques are also useful in constructing examples concerning the convergence of power series and Fourier series. Below we offer some simple but instructive examples.

The first example of a power series which converges uniformly but not absolutely on the closed unit disc was given by Fejér [**15**], cf. [**25**, vol. 1, p. 122]. The following geometric example, due to Gaier [**68**] and rediscovered by Piranian (see [**1**, pp. 289, 314]), is particularly appealing. Let D be the region of figure 1, a triangle from

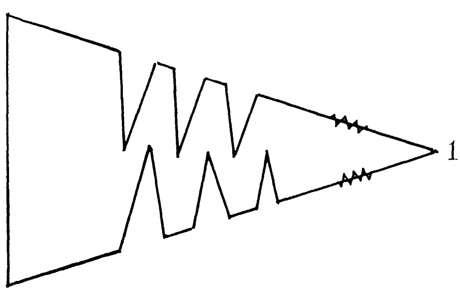


FIG. 1

which wedges have been removed in such a way that the vertex at $z = 1$ is not rectifiably accessible from the interior of D. Since D is a Jordan region, any conformal map of $\Delta = \{z : |z| < 1\}$ onto D extends to a homeomorphism of the closed regions. Suppose that $f(z) = \Sigma a_n z^n$ is such a homeomorphism satisfying $f(1) = 1$. Clearly,

$$(9)\qquad \begin{aligned} \int_0^1 |f'(r)|\,dr &= \int_0^1 \Big|\sum_{n=1}^{\infty} n a_n r^{n-1}\Big|\,dr \\ &\leqq \int_0^1 \left(\sum_{n=1}^{\infty} n\,|a_n|\,r^{n-1}\right)dr = \sum_{n=1}^{\infty} |a_n|. \end{aligned}$$

Since the length of the image of $[0,1]$ under f is infinite and is given by the extreme left member of (9), the series for f is not absolutely convergent. That the series is uniformly convergent on the closed disc follows from a result due to Fejér.

FEJÉR'S TAUBERIAN THEOREM [**16**], [**55**, p. 357]. *Let* $f(z) = \sum_{n=0}^{\infty} a_n z^n$ *and suppose* $\sum_{n=1}^{\infty} n\,|a_n|^2 < \infty$. *If* $\lim_{r\to 1-} f(re^{i\theta}) = f(e^{i\theta})$ *exists, then the sum* $\sum_{n=0}^{\infty} a_n e^{in\theta}$ *exists and is equal to* $f(e^{i\theta})$. *Moreover, if* $\lim_{r\to 1-} f(re^{i\theta}) = f(e^{i\theta})$ *uniformly for* $\theta_1 \leqq \theta \leqq \theta_2$, *then* $\sum_{n=0}^{\infty} a_n e^{in\theta} = f(e^{i\theta})$ *uniformly for* $\theta_1 \leqq \theta \leqq \theta_2$.

This is a typical theorem of Tauberian type, the Tauberian condition being, of course, $\Sigma_{n=1}^{\infty} n\,|\,a_n\,|^2 < \infty$.

Proof. Set $s_N(e^{i\theta}) = \Sigma_{n=0}^{N} a_n e^{in\theta}$. Then

$$|f(re^{i\theta}) - s_N(e^{i\theta})| \leqq \sum_{n=1}^{N} |\,a_n\,|(1 - r^n) + \sum_{n=N+1}^{\infty} |\,a_n\,| r^n = S_1 + S_2.$$

Now $1 - r^n \leqq n(1 - r)$ (divide both sides by $1 - r$). Thus,

$$S_1 \leqq (1 - r) \sum_{n=1}^{N} n\,|\,a_n\,| \leqq (1 - r)\left(\sum_{n=1}^{N} n\right)^{1/2}\left(\sum_{n=1}^{N} n\,|\,a_n\,|^2\right)^{1/2} \leqq KN(1 - r),$$

where $K = (\Sigma_{n=1}^{\infty} n\,|\,a_n\,|^2)^{1/2}$. Here we have used the Cauchy-Schwarz inequality and the fact that $\Sigma_{n=1}^{N} = N(N+1)/2 \leqq N^2$. Applying Cauchy-Schwarz to S yields

$$S_2 = \sum_{n=N+1}^{\infty} \sqrt{n}\,|\,a_n\,| \frac{r^n}{\sqrt{n}} \leqq \left(\sum_{n=N+1}^{\infty} \frac{r^{2n}}{n}\right)^{1/2}\left(\sum_{n=N+1}^{\infty} n\,|\,a_n\,|^2\right)^{1/2}$$

$$\leqq \left(\frac{1}{N(1 - r)} \sum_{n=N+1}^{\infty} n\,|\,a_n\,|^2\right)^{1/2}$$

since

$$\sum_{n=N+1}^{\infty} r^{2n}/n \leqq 1/N \sum_{n=0}^{\infty} r^n = 1/N(1 - r).$$

Having fixed N, we may, by the intermediate value theorem for continuous functions, choose $r = r_N$ such that $N(1 - r_N) = (\Sigma_{n=N+1}^{\infty} n\,|\,a_n\,|^2)^{1/2}$. Clearly, as $N \to \infty$, $r_N \to 1$. Thus

$$|f(r_N e^{i\theta}) - s_N(e^{i\theta})| \leqq K\left(\sum_{n=N+1}^{\infty} n\,|\,a_n\,|^2\right)^{1/2} + \left(\sum_{n=N+1}^{\infty} n\,|\,a_n\,|^2\right)^{1/4},$$

and the right hand side tends to 0 as $N \to \infty$ since $\Sigma_{n=1}^{\infty} n\,|\,a_n\,|^2 < \infty$. Since $f(re^{i\theta}) \to f(e^{i\theta})$, $s_N(e^{i\theta}) \to f(e^{i\theta})$; hence $\Sigma_{n=0}^{\infty} a_n e^{in\theta} = f(e^{i\theta})$. Finally, all our calculations are uniform in θ, so if $f(re^{i\theta}) \to f(e^{i\theta})$ uniformly on some arc, then $\Sigma_{n=0}^{\infty} a_n e^{in\theta} = f(e^{i\theta})$ uniformly on that arc.

To apply Fejér's theorem to the situation at hand, simply note that if f maps Δ conformally onto the Jordan region D then (by the Osgood-Taylor-Carathéodory theorem) f extends continuously to $\Delta \cup \Gamma$ so that $f(re^{i\theta}) \to f(e^{i\theta})$ uniformly for $0 \leqq \theta \leqq 2\pi$. Since

$$\pi \sum_{n=1}^{\infty} n\,|\,a_n\,|^2 = \int_0^1 \int_0^{2\pi} |f'(re^{i\theta})|^2 \, d\theta \, r \, dr = \text{area of } D < \infty,$$

the Taylor series for f converges uniformly on Γ.

We should observe that it is easy to modify the domain of figure 1 so that its boundary becomes analytic at every point except $z = 1$. The corresponding mapping function then extends (by Schwarz reflection) across $\Gamma \setminus \{1\}$ and yields a function univalent and analytic on a domain containing $(\Delta \cup \Gamma) \setminus \{1\}$ whose Taylor series converges uniformly but not absolutely on $\Delta \cup \Gamma$.

8. Fourier series. One of the loveliest applications of complex analysis to real variables occurs in the theory of Fourier series. The result in question is the so-called Pál-Bohr theorem, which may be stated as follows.

PÁL-BOHR THEOREM. *Let $f(e^{i\theta})$ be a continuous real-valued function on the unit circle Γ. There is a self-homeomorphism ϕ of Γ such that the Fourier series of $f \circ \phi$ converges uniformly.*

It is well known, of course, that the Fourier series of a continuous function may diverge on a dense subset of Γ [**28**, p. 58]; this gives the Pál-Bohr theorem added poignancy. On the other hand, a deep and famous result of Lennart Carleson [**64**] insures that the Fourier series of a continuous function converges *almost* everywhere in the sense of Lebesgue measure.

The Pál-Bohr theorem has an interesting history. It was first proved by Jules Pál in 1914 with the weaker conclusion that uniform convergence could be obtained on any *proper* closed subarc of Γ, however large. Bohr [**4**], in 1935, removed the restriction in Pál's theorem. Finally, in 1944, Salem [**50**] introduced a trick which yields the full strength of the result very quickly.

Proof of the Pál-Bohr Theorem. Regard f as a function on the interval $[-\pi, \pi]$ satisfying the periodicity condition $f(-\pi) = f(\pi)$. We rule out at the outset the trivial case in which f is identically constant. By adding, if necessary, a continuous periodic function of bounded variation, we may assume that $f(-\pi) = f(\pi) = f(x)$ for exactly one point $x \in (-\pi, \pi)$. (This is Salem's trick; see [**50**] for a complete verification.) Since the Fourier series of a continuous function of bounded variation converges uniformly, it is enough to prove the theorem under this additional assumption. Let g be a continuous periodic function on $[-\pi, \pi]$ which increases on $(-\pi, x)$ and decreases on (x, π). Then the image of $[-\pi, \pi]$ under the map $H(t) = g(t) + if(t)$ is a simple closed Jordan curve J in the plane.

Let $F(z) = \sum_{n=0}^{\infty} a_n z^n$ be a Riemann map of Δ onto the interior of J such that $F(-1) = H(-\pi)$. Then F extends to a homeomorphism of Γ onto J, and by the discussion following the proof of Fejér's theorem, the series $F(e^{i\theta}) = \sum_{n=0}^{\infty} a_n e^{in\theta}$ converges uniformly on Γ. The required homeomorphism of $[-\pi, \pi)$ is obtained by setting $\phi(t) = H^{-1}(F(e^{it}))$. Indeed, this is clearly a homeomorphism, and $f(\phi(t)) = f \circ H^{-1}(F(e^{it})) = \operatorname{Im} F(e^{it})$, which has a uniformly convergent Fourier series since $F(e^{it})$ does.

Perhaps surprisingly, the argument given above is (essentially) the *only* known proof of this theorem. Whether an analogous result holds for complex-valued

functions remains an open question; of course, this is equivalent to the question of whether, given *two* real-valued continuous functions f, g on Γ, one can find a single homeomorphism ϕ for which $f \circ \phi$ and $g \circ \phi$ both have uniformly convergent Fourier series.

We learned of the Pál-Bohr theorem from the interesting survey article of Goffman and Waterman [**20**], and our treatment parallels the discussion given there. The decision to reproduce the proof in some detail was based on our feeling that this beautiful result deserves a wider public.

9. Harmonic conjugates. A somewhat different application of conformal mapping to problems involving Fourier series involves the construction of functions having certain prescribed bad boundary behavior. Thus, one may ask (and Prof. A. Devinatz did) for an explicit example of a function harmonic on Δ and continuous on $\Delta \cup \Gamma$ whose harmonic conjugate is discontinuous but bounded. Although the problem has been framed (for simplicity) in terms of harmonic functions, it is actually a pure real variable question concerning the lack of smoothness of a certain singular integral operator.

For the solution, consider the simply connected domain D, indicated in Figure 2, bounded by an (open) analytic curve J together with its asymptote the segment

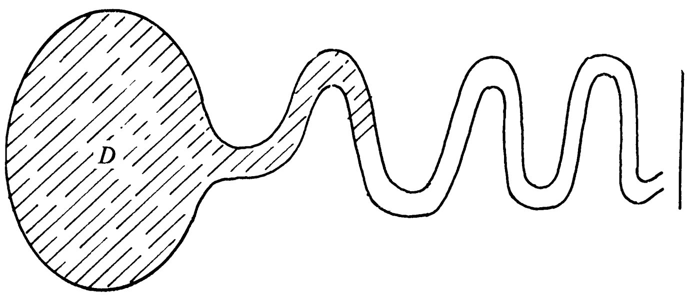


FIG. 2

$\{y: -1 \leqq y \leqq 1\}$ of the y-axis in the complex plane. Map Δ conformally onto D by the univalent function $f(z) = u(z) + iv(z)$. A standard result in conformal mapping [**55**, p. 353] insures that a single point of Γ, say 1, corresponds to the "bad" part of the boundary and that f establishes a homeomorphism between $\Gamma \setminus \{1\}$ and J. By the reflection principle, f actually extends analytically across $\Gamma \setminus \{1\}$. One proves that as $z \to 1$, $u(z) \to 0$; and it is now obvious that u is not only harmonic on Δ and harmonically extendible across $\Gamma \setminus \{1\}$ but also continuous on $\Delta \cup \Gamma$. On the other hand, the harmonic function v, which is clearly bounded, is *not* continuous at $z = 1$. The details of the proof will be easily supplied by anyone familiar with Carathéodory's important theory of prime ends [**7**], [**55**,

pp. 352–355], [9]. An obvious modification yields a bounded continuous function whose conjugate is unbounded.

10. Tauberian theorems. Tauberian theorems, such as Fejér's, have an intrinsic interest quite independent of applications. Of these, the most celebrated is certainly that due to Littlewood, which states that if $\lim_{r\to 1-} \sum_{n=0}^{\infty} a_n r^n = L$ exists and $a_n = O(1/n)$, then $\sum_{n=0}^{\infty} a_n = L$. This result resisted considerable efforts at proof for several years before it was finally settled by Littlewood [31], whose argument required six pages of ingenious and delicate analysis. Much later, Karamata [27] introduced a new technique, based on approximation theory, resulting in an enormous simplification of the proof. Much less well-known is Wielandt's modification [59] of Karamata's proof, which yields a simple and transparent proof of the theorem in question. Below, we present Wielandt's proof of a strengthened version (due to Hardy and Littlewood [22]) of the Littlewood Tauberian theorem.

THEOREM. *Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be analytic in $|z| < 1$ and suppose that the a_n are real and that $na_n \leqq K$ for some $K > 0$. If $\lim_{r\to 1-} f(r) = L$ exists as $r \to 1-$, then $\sum_{n=0}^{\infty} a_n = L$.*

The advantage of this result over the original Littlewood theorem lies, of course, in the fact that the order estimate on the coefficients is replaced by a *one-sided* bound.

Proof. Trivial normalizations allow us to assume that $L = 0$, $a_0 = 0$, $K = 1$. Consider the family $\mathscr{F}$ of real functions $\phi(x)$ on $(0,1)$ which satisfy

(a) $\sum_{n=1}^{\infty} a_n \phi(x^n)$ is convergent for $x \in (0,1)$,

(b) $\Phi(x) = \sum_{n=1}^{\infty} a_n \phi(x^n) \to 0$ as $x \to 1-$.

Clearly, if $\phi(x) \in \mathscr{F}$, $\phi(x^k) \in \mathscr{F}$ $(k = 1, 2, \cdots)$ and $\mathscr{F}$ is closed under linear combinations. Since (by hypothesis) $x \in \mathscr{F}$, each polynomial vanishing at the origin belongs to $\mathscr{F}$. The proof depends on a simple lemma concerning the approximation of functions.

LEMMA. *Let $\phi(x)$ satisfy* (a). *Suppose that for each $\varepsilon > 0$ there exist polynomials $p_1(x)$, $p_2(x)$ such that $p_i(0) = 0$, $p_i(1) = 1$ $(i = 1, 2)$ and*

$$p_1(x) \leqq \phi(x) \leqq p_2(x) \qquad \frac{p_2(x) - p_1(x)}{x(1-x)} = q(x) > 0,$$

where $\int_0^1 q(x)dx < \varepsilon$. Then $\phi(x)$ satisfies (b) *and hence belongs to $\mathscr{F}$.*

Proof of Lemma. Let $\Phi(x) = \sum_{n=1}^{\infty} a_n \phi(x^n)$, $q(x) = \sum_{k=0}^{r} b_k x^k$.
Then

$$\Phi(x) - \sum_{n=1}^{\infty} a_n p_1(x^n) = \sum_{n=1}^{\infty} a_n(\phi(x^n) - p_1(x^n)) \leqq \sum_{n=1}^{\infty} \frac{1}{n}(p_2(x^n) - p_1(x^n))$$

$$= \sum_{n=1}^{\infty} \frac{1}{n}(1 - x^n)x^n q(x^n) \leqq (1 - x) \sum_{n=1}^{\infty} x^n q(x^n)$$

$$= (1 - x) \sum_{k=0}^{r} b_k \sum_{n=1}^{\infty} x^{n(k+1)} = \sum_{k=0}^{r} \frac{b_k(1 - x)x^{k+1}}{1 - x^{k+1}} \to \sum_{k=0}^{r} \frac{b_k}{1 + k}$$

$$= \int_0^1 q(x)dx < \varepsilon \text{ as } x \to 1 - .$$

Here we have used the fact that $1 - x^n \leqq n(1 - x)$ and that $(1 - x^n)/(1 - x) \to n$ as $x \to 1$. Since $p_1(x) \in \mathscr{F}$, $\sum_{n=1}^{\infty} a_n p_1(x^n) \to 0$ as $x \to 1 -$ so that $\Phi(x) < \varepsilon$ for x near 1. Consideration of $p_2(x) - \phi(x)$ shows similarly that $\Phi(x) > -\varepsilon$ if x is near enough to 1. Thus $\Phi(x) \to 0$ as $x \to 1$, so that $\phi \in \mathscr{F}$.

Continuing with the proof of the theorem, let

$$\phi^*(x) = \begin{cases} 0 & 0 \leqq x < \frac{1}{2} \\ 1 & \frac{1}{2} \leqq x \leqq 1 \end{cases}$$

so that $\Phi(x) = \sum_{n=1}^{\infty} a_n \phi^*(x^n) = \sum_{2x^n \geqq 1} a_n = \sum_{n=1}^{N} a_n = s_N$, where

$$N = \left[\log 2/\log \frac{1}{x}\right].$$

It suffices to show that $\phi^*(x) \in \mathscr{F}$, for then $s_N \to 0$ as $N \to \infty$, whence $\sum_{n=0}^{\infty} a_n = 0$ as required. Now ϕ^* clearly satisfies (a), so it is enough to show that the conditions of the lemma are fulfilled. Since continuous functions are dense in the integrable functions, we can find continuous functions $g_1(x)$ and $g_2(x)$ such that

$$(10) \qquad g_1(x) < \frac{\phi^*(x) - x}{x(1 - x)} < g_2(x) \qquad \int_0^1 [g_2(x) - g_1(x)]dx < \varepsilon.$$

The functions g_1 and g_2 may then be approximated uniformly by polynomials q_1 and q_2 in such a way that (10) still holds with the g_i's replaced by the q_i's. Putting $p_i(x) = x + x(1 - x)q_i(x)$, $q(x) = q_2(x) - q_1(x)$, we obtain polynomials satisfying the hypothesis of the lemma. This completes the proof.

The subject of Tauberian theorems extends far beyond questions concerning the convergence or divergence of a power series on its circle of convergence. One of the central results in the harmonic analysis of the real line is Wiener's Tauberian theorem, which states that if $f \in L^1(\mathbb{R})$ and the Fourier transform of f never vanishes, then linear combinations of translates of f are dense in $L^1(\mathbb{R})$. The relation between the theorems of Wiener and Littlewood is far from obvious, and it has become customary to deduce the latter from the former by way of explicating the Tauberian character of Wiener's theorem. This deduction is standard and may be found, for instance, in [**60**, pp. 104–106]. The proof involves the function $K(x) = e^{-x}\exp(-e^{-x})$ and uses the fact that the gamma function $\Gamma(z)$ has no zeroes on the line $\operatorname{Re} z = 1$.

Unfortunately, the *deduction* of Littlewood's theorem from Wiener's is longer and significantly more complicated in both conception and detail than Wielandt's proof of the (more general!) Hardy-Littlewood theorem: it is a little like proving that the medians of a triangle are concurrent by invoking the fact that a nested sequence of compact sets has nonvoid intersection. Of course, Wiener's powerful methods have applications in many situations where the simple approximation theory argument we have given does not apply.

One such instance concerns the so-called high indices theorem.

HIGH INDICES THEOREM. *Let $f(z) = \sum_{k=0}^{\infty} a_k z^{n_k}$ be analytic in $|z| < 1$ and suppose that $n_{k+1}/n_k \geqq q > 1$ for all k. If $\lim_{r \to 1-} f(r) = L$ exists, then $\sum_{k=0}^{\infty} a_k = L$.*

This theorem lies considerably deeper than Littlewood's theorem or its extension proved above; it was first proved, by Hardy and Littlewood, in 1925 [**23**], having been conjectured by Littlewood as early as 1910. The novelty of the result lies in the fact that the Tauberian condition (the lacunarity of the sequence of coefficients) involves no bound on the size of the coefficients. It is most instructive to try to apply the ideas used in proving the Tauberian theorems of Fejér and Hardy-Littlewood to the high indices theorem: they all fail miserably. In fact, I am aware of no really simple proof of this result. A particularly attractive argument, marked by considerable ingenuity in the use of such tools as the Phragmén-Lindelöf principle and Blaschke products, has been given by Halász [**21**], following some ideas of the German mathematician Dieter Gaier.

In concluding this section we should like to mention an amusing sidelight. Wielandt's proof of the Hardy-Littlewood theorem shares, with Mordell's proof of the Hadamard gap theorem, the property of being a gem of complex analysis mined by a mathematician whose central interests lay altogether outside analysis. The late Professor Mordell was, of course, one of the world's leading number theorists; Professor Wielandt is a group theorist of international repute. Is there a moral to be drawn here?

11. Category. The usual theorems on convergence of sequences of analytic functions, such as Vitali's convergence theorem [**54**, p. 168], require the uniform boundedness of the sequence in question on compact subsets of the domain. There is, however, a sometimes useful result, due to Osgood, which avoids altogether hypotheses other than simple pointwise convergence.

OSGOOD'S THEOREM [**40**]. *Let D be a domain and let $\{f_n\}$ be a sequence of functions analytic in D. Suppose $f_n(z) \to f(z)$ for each $z \in D$. Then f is analytic in an open set $D_1 \subset D$ which is dense in D, and convergence is uniform on compact subsets of D_1.*

This result has been rediscovered countless times and has on innumerable other occasions brought the experts to grief. Indeed, the question as to whether f must

be analytic *anywhere*, appears (happily, with a correct solution) in the problem section of a recent symposium [**29**, p. 543]. The present formulation suggests — correctly — the use of the Baire category theorem.

Proof of Osgood's Theorem. Let $F_m = \{z\colon \left|f_n(z)\right| \leqq m,\ n = 1,2,3,\cdots\}$. The F_m are clearly relatively closed in D and $\cup F_m = D$. By the Baire category theorem, some F_m must have interior. For this m, the sequence $\{f_n\}$ is uniformly bounded on F_m^0 hence by Vitali's theorem converges uniformly on compact subsets of F_m^0 to an analytic function. Thus f is analytic on F_m^0. Since the argument can be applied to any subdomain R of D — in particular, to an arbitrary disc — it follows that f must be analytic on a dense open subset D_1 of D. That convergence is uniform on compacta contained in D_1 is a standard argument, which we suppress.

A comment is perhaps in order on our use of the Baire category theorem, which states that a complete metric space is not the countable union of closed nowhere dense sets. Obviously, D is *not* complete in the Euclidean metric. However, it is easy to see that D can be given a new metric which induces the same (Euclidean) topology, under which D is complete. Alternatively, one may replace D by a slightly smaller *compact* set K and relativize the argument to K. We should also mention that category arguments appear elsewhere in complex analysis as well. A notable example is the proof of Hartogs' theorem, mentioned earlier in Section 2.

A nice complement to Osgood's theorem is provided by an example of a sequence of entire functions $f_n(z)$ with the property that

$$\lim_{n\to\infty} f_n(z) = \begin{cases} 0 & z \neq 0 \\ 1 & z = 0. \end{cases} \tag{11}$$

There are (at least) two essentially distinct ways of constructing such a sequence. One method is to construct an entire function $F(z)$ such that $F(0) = 1$ and $F(z) \to 0$ as $\left|z\right| \to \infty$ on each ray through the origin. Such functions were first ex-

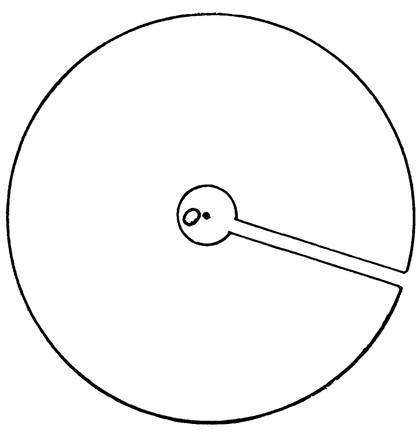


FIG. 3

hibited by Mittag-Leffler; see [**34**] for a detailed discussion and some surprising extensions. The construction of F is, with appropriate hints, a nice and doable exercise and occurs as such in Rudin's text [**49**, pp. 326–327]. Once F has been obtained, one observes that the sequence $f_n(z) = F(nz)$ satisfies (11).

Alternatively, one can apply Runge's theorem to "notched annuli" to construct polynomials satisfying (11). To be explicit, consider the set indicated in figure 3. A moment's reflection will reveal that one can choose a sequence K_n of such sets, with the property that for each $z \in \mathbb{C}\setminus\{0\}$ there exists an integer N such that $z \in K_n$ for all $n \geqq N$. (The inner circle contracts, the outer circle expands, and the notch gets thinner and rotates, tending toward but never reaching its limiting line.) The function

$$g_n(z) = \begin{cases} 0 & z \in K_n \\ 1 & z = 0 \end{cases}$$

clearly extends to a function analytic in a neighborhood of the (disconnected) set $K_n \cup \{0\}$. Since this set does not separate the plane, g may be approximated uniformly (to within $1/n$, say) on $K_n \cup \{0\}$ by a polynomial p_n. These polynomials clearly satisfy (11).

12. Miscellany. The interactions between real and complex analysis are by no means limited to the areas mentioned above. To keep the discussion within manageable limits, we have restricted ourselves to (a subset of) those applications, examples, and aspects of the theory that have not found sustained treatment in the "popular" literature of texts and survey articles. Subjects which are treated elsewhere at adequate length but which deserve mention here by virtue of their interdisciplinary nature include the following:

(a) *The evaluation of real integrals and sums by residue techniques.* This is surely one of the most striking applications of complex function theory to real analysis. Fortunately, any good text on complex analysis will contain a fairly detailed discussion.

(b) *Complex methods in harmonic analysis.* This is a substantial area, which includes topics as diverse as interpolation theorems (see, for instance, [**28**, pp. 93–98]) and theorems of Paley-Wiener type [**43**]. Two of the most attractive recent texts in harmonic analysis [**13**], [**28**] devote whole chapters to this aspect of the theory. Further developments are discussed in the survey article of Weiss [**57**].

(c) *Functional analysis.* Complex variable methods appear here perhaps most notably in the construction of functional calculi for operators on Hilbert space or Banach space. The applications to commutative Banach algebras are particularly substantial; indeed, parts of this last-named subject are virtually coextensive with certain aspects of several complex variable theory. For further references, see [**17**] and [**58**]. In the opposite direction, techniques of functional analysis can be used

to establish many results in function theory; this is the programme of [**48**]. Finally, an honest partnership between complex variables and functional analysis occurs in the study of certain Banach spaces of analytic functions, especially H^p spaces [**12**], [**26**].

(d) *Function theoretic methods in differential equations.* Complex methods occur rather naturally in the study of ordinary differential equations [**8**]. Their appearance in the study of partial differential equations is perhaps more surprising. Yet there are substantial applications, and more than one book [**2**], [**19**] has been devoted to this area. Further applications of function theory to problems in partial differential equations will be found in [**18**]. In a rather different direction, the theory of linear partial differential equations with constant coefficients is intimately connected with the study of certain spaces of entire functions of several complex variables; see [**14**] for an exhaustive treatment.

13. Monodromy. No excursion onto the bypaths of complex analysis would be complete without some mention of the monodromy theorem.

MONODROMY THEOREM. *Let D be a simply connected domain and let $f(z)$ be analytic in a neighborhood of $z_0 \in D$. Then if $f(z)$ can be continued analytically from z_0 along every path lying in D, the continuation gives rise to a single-valued unction analytic on all of D.*

A more general version states that analytic continuation along paths is a homotopy invariant; see, for instance, [**53**]. Like the reflection principle, the monodromy theorem is an essential ingredient in the short proof of Picard's little theorem; in its extended form, it is the central result in the subject of analytic continuation. Yet no theorem of basic complex analysis is more abused or less understood. Indeed, it has been misapplied more than once even by mathematicians of the first rank (and specialists in complex analysis, at that!). One may speculate that a source of at least some of the confusion surrounding this result is the essentially topological, rather than function-theoretic, nature of the theorem.

The sort of error into which one may lapse is best indicated by an explicit example. Let D be a simply connected domain and f a function analytic in D which satisfies $f'(z) \neq 0$ on D. Suppose $R = f(D)$ is also simply connected. Question: Must f be univalent (one-one)? An affirmative answer may be found in [**56**, p. 243] and in other references as well. The argument is as follows. At each point $w_0 \in R$ one may define a local inverse $f_{w_0}^{-1}(w)$ of f, analytic in a neighborhood of w_0. Since R is simply connected, the totality of these functions defines a single-valued analytic function f^{-1} on R, which is a global inverse for f. Thus f must be univalent. Note further that the simple-connectivity of D is quite extraneous to the demonstration.

Unfortunately, the argument given above is altogether incorrect, since the essential hypothesis of the monodromy theorem, that analytic continuation be possible along every path in R, has not been verified. Can the proof be salvaged?

The answer is no. In fact, consider the function $f(z) = \int_0^z e^{\zeta^2} d\zeta$. This f is analytic (entire) on the simply connected domain $\mathbb{C}$ and $f'(z) = e^{z^2}$ is nowhere zero. Clearly, f is not univalent. So it suffices to prove that $f(\mathbb{C})$ is simply connected. We claim $f(\mathbb{C}) = \mathbb{C}$. Indeed, suppose $f(z) \neq w$. Since e^{z^2} is an even function, $f(z)$ is odd, so that if f fails to take on the value w it also misses the value $-w$. If $w \neq 0$, this contradicts Picard's (small) theorem. Since $f(0) = 0$, f takes on every value in the complex plane.

For $D = \mathbb{C}$, any function which satisfies $f'(z) \neq 0$ must be transcendental and hence must (by Picard's theorem) take on most values infinitely often. One can, however, construct a non-univalent, locally univalent function mapping the disc $\Delta = \{z: |z| < 1\}$ onto itself, which takes on no value more than three times. The extremely elegant example given above is due to D. S. Greenstein and appears as a solution to MONTHLY Problem 4740. It is an appropriate note on which to end this survey.

Preparation of this paper was supported in part by NSF GP 28970.

References

1. László Alpár, Sur certaines transformées des séries de puissance absolument convergentes sur la frontière de leur cercle de convergence, Magyar Tud. Akad. Mat. Kutató Int. Közl., 7 (1962) 287–316.

2. Stefan Bergman, Integral Operators in the Theory of Linear Partial Differential Equations, Springer-Verlag, 1969.

3. Arne Beurling, Ensembles exceptionnels, Acta Math., 72 (1940) 1–13.

4. Harold Bohr, Über einen Satz von J. Pál, Acta Sci. Math. (Szeged), 7 (1935) 129–135.

5. Andrew Browder, Introduction to Function Algebras, Benjamin, New York, 1969.

6. C. Carathéodory, Über die gegenseitige Beziehung der Ränder bei der konformer Abbildung des Inneren einer Jordanschen Kurve auf einen Kreis, Math. Ann., 73 (1913) 305–320.

7. ———, Über die Begrenzung einfach zusammenhängender Gebiete, Math. Ann., 73 (1913) 323–370.

8. Earl A. Coddington and Norman Levinson, Theory of Ordinary Differential Equations, Mc-Graw-Hill, New York, 1955.

9. E. F. Collingwood and A. J. Lohwater, The Theory of Cluster Sets, Cambridge University Press, New York, 1966.

10. E. H. Connell, On properties of analytic functions, Duke Math. J., 28 (1961) 73–81.

11. Arnaud Denjoy, Sur la continuité des fonctions analytiques singulières, Bull. Soc. Math. France, 60 (1932) 27–105.

12. Peter L. Duren, Theory of H^p Spaces, Academic Press, New York, 1970.

13. H. Dym and H. P. McKean, Fourier Series and Integrals, Academic Press, New York, 1972.

14. Leon Ehrenpreis, Fourier Analysis in Several Complex Variables, Wiley-Interscience, New York, 1970.

15. L. Fejér, Über gewisse Potenzreihen an der Konvergenzgrenze, Sitzungsber. Bayer. Akad. Wiss. Math. -phys. Kl., no. 3 (1910) 1–17.

16. ———, La convergence sur son cercle de convergence d'une série de puissance effectuant une représentation conforme du cercle sur le plan simple, C. R. Acad. Paris, 156 (1913), 46–49.

17. Theodore W. Gamelin, Uniform Algebras, Prentice-Hall, Englewood Cliffs, N.J., 1969.

18. P. R. Garabedian, Partial Differential Equations, Wiley, New York, 1964.

19. Robert P. Gilbert, Function Theoretic Methods in Partial Differential Equations, Academic Press, New York, 1969.

20. Casper Goffman and Daniel Waterman, Some aspects of Fourier series, this MONTHLY, 77 (1970) 119–133.

21. Gábor Halász, Remarks on a paper of D. Gaier on gap theorems, Acta Sci. Math. (Szeged), 28 (1967) 311–322.

22. G. H. Hardy and J. E. Littlewood, Tauberian theorems concerning power series and Dirichlet's series, whose coefficients are positive, Proc. London Math. Soc., 13 (1914) 174–191.

23. ———, and ———, A further note on the converse of Abel's theorem, Proc. London Math. Soc., 25 (1926) 219–236.

24. Maurice Heins, Selected Topics in the Classical Theory of Functions of a Complex Variable, Holt, Rinehart, and Winston, New York, 1962.

25. Einar Hille, Analytic Function Theory vols. 1 and 2, Ginn, New York, 1959 and 1962.

26. Kenneth Hoffman, Banach Spaces of Analytic Functions, Prentice-Hall, Englewood Cliffs, N. J., 1962.

27. J. Karamata, Über die Hardy-Littlewoodschen Umkehrungen des Abelschen Stetigkeitssatzes, Math. Zeitschr., 32 (1930) 319–320.

28. Yitzhak Katznelson, An Introduction to Harmonic Analysis, Wiley, New York, 1968.

29. Jacob Korevaar, ed., Entire Functions and Related Parts of Analysis, Proc. Symp. Pure Math. vol. 16, Amer. Math. Soc., 1968.

30. M. Lavrentieff, Sur quelques problèmes concernant les fonctions univalentes sur la frontière, Mat. Sb. N. S. 1, (43) (1936) 816–846. (Russian)

31. J. E. Littlewood, The converse of Abel's theorem on power series, Proc. London Math. Soc., 9 (1911) 434–448.

32. J. E. McMillan and George Piranian, The compression of the boundary, to appear.

33. Kikuji Matsumoto, On some boundary problems in the theory of conformal mappings of Jordan domains, Nagoya Math. J., 24 (1964) 129–141.

34. G. Mittag-Leffler, Sur la représentation analytique d'une branche uniforme d'une fonction monogène VI, Acta Math., 42 (1920) 285–308.

35. R. L. Moore and J. R. Kline, On the most general plane closed set through which it is possible to pass a simple continuous arc, Ann. Math., (2) 20 (1919) 218–223.

36. L. J. Mordell, On power series with circle of convergence as a line of essential singularities, J. London Math. Soc., 2 (1927) 146–148.

37. G. Morera, Un teorema fondamentale nella teoria delle funzioni di una variabile complessa, Rend. del R. Instituto Lombardo di Scienze e Lettere, (2) 19 (1886) 304–307.

38. W. F. Osgood, Example of a single-valued function with a natural boundary, whose inverse is also single-valued, Bull. Amer. Math. Soc., 4 (1898) 417–424; Supplementary note, *ibid.* 5 (1898) 17–18.

39. ———, Some points in the elements of the theory of functions, Bull. Amer. Math. Soc., 2 (1896) 296–302.

40. ———, Note on the functions defined by infinite series whose terms are analytic functions of a complex variable, with corresponding theorems for definite integrals, Ann. Math., (2) 3 (1901) 25–34.

41. ———, A Jordan curve of positive area, Trans. Amer. Math. Soc., 4 (1903) 107–112.

42. Jules Pál, Sur des transformations de fonctions qui font converger leurs séries de Fourier, C. R. Acad. Sci. Paris, 158 (1914) 101–103.

43. Raymond E. A. C. Paley and Norbert Wiener, Fourier transforms in the complex domain, Colloq. Publ. no. 19, Amer. Math. Soc., 1934.

44. P. Porcelli and E. H. Connell, A proof of the power series expansion without Cauchy's formula, Bull. Amer. Math. Soc., 67 (1961) 177–181.

45. Alfred Pringsheim, Ueber den Goursat'schen Beweis des Cauchy'schen Integralsatzes, Trans. Amer. Math. Soc., 2 (1901) 413–421.

46. A. H. Read, Higher derivatives of analytic functions from the standpoint of topological analysis, J. London Math. Soc., 36 (1961) 345–352.

47. F. Riesz and M. Riesz, Über die Randwerte einer analytischen Funktion, C. R. du Congrès des Math. Scandinaves, (1916) 27–44.

48. L. A. Rubel and B. A. Taylor, Functional analysis proofs of some theorems in function theory, this MONTHLY, 76 (1969) 483–489.

49. Walter Rudin, Real and Complex Analysis, McGraw-Hill, New York, 1966.

50. R. Salem, On a theorem of Bohr and Pál, Bull. Amer. Math. Soc., 50 (1944) 579–580.

51. Walter J. Schneider, A short proof of a conformal mapping theorem of Matsumoto, Notices Amer. Math. Soc., 16 (1969) 549.

52. Harold S. Shapiro, Smoothing and Approximation of Functions, Van Nostrand, Princeton, N.J., 1969.

53. George Springer, Introduction to Riemann Surfaces, Addison-Wesley, Reading, Mass., 1957.

54. E. C. Titchmarsh, The Theory of Functions, Oxford University Press, New York, 1960.

55. M. Tsuji, Potential Theory in Modern Function Theory, Maruzen, 1959.

56. Hans F. Weinberger, A First Course in Partial Differential Equations with Complex Variables and Transform Methods, Blaisdell, Waltham, Mass., 1965, p. 243.

57. Guido Weiss, Complex methods in harmonic analysis, this MONTHLY, 77 (1970) 465–474.

58. John Wermer, Banach Algebras and Several Complex Variables, Markham, 1971.

59. Helmut Wielandt, Zur Umkehrung des Abelschen Stetigkeitssatzes, Math. Zeitschr., 56 (1952) 206–207.

60. Norbert Wiener, The Fourier Integral and Certain of its Applications, Dover, New York, 1958.

61. Lawrence Zalcman, Null sets for a class of analytic functions, this MONTHLY, 75 (1968) 462–470.

62. ———, Analyticity and the Pompeiu problem, Arch. Rat. Mech. Anal., 47 (1972) 237–254.

63. ———, Mean values and differential equations, Israel J. Math., 14 (1973) 339–352.

64. Lennart Carleson, On convergence and growth of partial sums of Fourier series, Acta Math., 116 (1966) 135–157.

65. W. F. Osgood, On the transformation of the boundary in the case of conformal mapping, Bull. Amer. Math. Soc., 9 (1903) 233–235.

66. William F. Osgood and Edson H. Taylor, Conformal transformations on the boundaries of their regions of definition, Trans. Amer. Math. Soc., 14 (1913) 277–298.

67. J. W. Alexander, II, Functions which map the interior of the unit circle upon simple regions, Ann. Math., (2) 17 (1915) 12–22.

68. Dieter Gaier, Schlichte Potenzreihen, die auf $|z| = 1$ gleichmässig, aber nicht absolut konvergieren, Math. Zeitschr., 57 (1952) 349–350.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MARYLAND, COLLEGE PARK, MD 20742, U.S.A.

VARIATIONS ON A THEOREM BY ARCHIMEDES

MICHAEL GOLOMB

Let P_0 be a point on a parabola C and let P_-, P_+ be endpoints of a chord of C that is parallel to the tangent at P_0. If $|T|$ is the area of the triangle with vertices P_-, P_0, P_+, while $|S|$ is the area of the segment bounded by the arc and the chord with the common endpoints P_-, P_+, then

$$T = \tfrac{3}{4}S.$$

This equation was first formulated and proved by Archimedes (see [**1**]) by his method of exhaustion. In a recent note [**2**] it was shown that $T < \frac{3}{4}S$ if C is an ellipse and $T > \frac{3}{4}S$ if C is a hyperbola, and furthermore, that $\frac{3}{4}$ is the limit of the ratio of T to S as P_-, P_+ approach P_0 on an arbitrary plane curve C of continuous nonzero curvature. The purpose of this note is to establish n-dimensional ($n \geqq 2$) analogues of these results. This is not quite an elementary exercise since the quadratures involved cannot be carried out explicitly.

1. The limit case. The manifolds to be considered are hypersurfaces of the $(n+1)$-dimensional Euclidean space E^{n+1}. Throughout it will be assumed that $n \geqq 2$. Let a Cartesian coordinate system of E^{n+1} be chosen in which a point x has the coordinates $x_1, \cdots, x_{n+1}$. The nonempty set $M \subset E^{n+1}$ will be called a $C^{(2)}$**-surface** if there exist an open set $U \subset E^{n+1}$ and a real-valued function $F \in C^{(2)}(U)$ with partial derivatives $F_k (k = 1, \cdots, n+1)$ such that

$$M = \{x \in U \colon F(x) = 0\} \tag{1.1}$$

and

$$\sum_{k=1}^{n+1} |F_k(x)| > 0, \quad x \in U. \tag{1.2}$$

A $C^{(2)}$surface M, so defined, is, in particular, an n-manifold of class $C^{(2)}$ (see, for example, [**3**]). Of special importance to us is a property of M that generalizes that of positive (Gaussian) curvature of surfaces in E^3. Let $t^i(x^0)$ $(i = 1, \cdots, n)$ be an orthogonal set of vectors of E^{n+1} with components $t^i_k(x^0)$ $(k = 1, \cdots, n+1)$ that span the tangent plane of M at x^0, and let $F_{kl}(x^0)$ $(k, l = 1, \cdots, n+1)$ denote the second order partial derivatives of F at x^0. Then x^0 is said to be an **elliptic point** of M if the $n \times n$ matrix with elements

$$\sum_{k,l=1}^{n+1} F_{kl}(x^0) t^i_k(x^0) t^j_l(x^0), \qquad i,j, = 1, \cdots, n \tag{1.3}$$

is positive definite. It is easily seen that this definition is independent of the choice

from among the systems $\{t^i(x^0)\}$ that span the tangent plane. If F is of the form

$$F(x) = x_{n+1} - f(x_1, \cdots, x_n) \tag{1.4}$$

then the above condition is equivalent to the requirement that the Hessian matrix with elements

$$a_{ij} = \frac{1}{2} \frac{\partial^2 f}{\partial x_i \partial x_j}(x_1^0, \cdots, x_n^0) \tag{1.5}$$

is positive definite. If moreover x^0 is the origin O and the tangent plane at O is $x_{n+1} = 0$, then a neighborhood of $O \in M$ is given by

$$x_{n+1} = \sum_{i,j=1}^{n} a_{ij} x_i x_j + o\left(\sum_{i=1}^{n} x_i^2\right), \tag{1.6}$$

where, as usual, $o(\Sigma x_i^2)$ denotes a remainder term $r(x_1, \cdots, x_n)$ for which $(\Sigma x_i^2)^{-1} r(x_1, \cdots, x_n)$ tends to 0 as $x_1, \cdots, x_n$ tend to 0.

Thus, in the neighborhood of an elliptic point x^0, M is closely approximated by an elliptic paraboloid with its axis along the normal at x^0. By **normal** at x^0 we shall mean the half-line which is the axis of this paraboloid. From a plane parallel to the tangent plane at x^0 and at a distance δ sufficiently small from x^0 the surface cuts out a compact set, which we call a **chord section** and denote as $K(x^0; \delta)$. By $T(x^0; \delta)$ we denote the compact ($n + 1$-dimensional) cone whose base is $K(x^0; \delta)$ and whose vertex is x^0, and by $S(x^0; \delta)$ the compact ($n + 1$-dimensional) **segment** bounded by $K(x^0; \delta)$ and M.

In the following $|S|$ denotes the Euclidean volume of the set $S \subset E^{n+1}$. We now prove

THEOREM 1. *If x^0 is an elliptic point of the $C^{(2)}$-surface M in E^{n+1}, and $T(x^0; \delta)$, $S(x^0; \delta)$ denote the cone and segment defined above then*

$$\lim_{\delta \to 0} \frac{|T(x^0; \delta)|}{|S(x^0; \delta)|} = \frac{1}{2} \frac{n+2}{n+1}. \tag{1.7}$$

Proof. We position the Cartesian $x_1, \cdots, x_{n+1}$ coordinate system so that (1.6) represents M in some neighborhood of $x^0 = 0$. We write z for x_{n+1} and make an orthogonal coordinate transformation $x_i \to y_i$ ($i = 1, \cdots, n$) that reduces the quadratic form in (1.6) to diagonal form. Then M is represented by

$$z = \sum_{i=1}^{n} b_i^2 y_i^2 + o\left(\sum_{i=1}^{n} y_i^2\right) \tag{1.8}$$

near 0, where $b_i > 0$ ($i = 1, \cdots, n$). The chord section $K(x^0; \delta)$ is given by

$$K(x^0; \delta) = \left\{z = \delta, \sum_{i=1}^{n} b_i^2 y_i^2 \leqq \delta + o(\delta)\right\} \tag{1.9}$$

and its projection onto the plane $z = 0$ by

$$K_0(x^0;\delta) = \left\{\sum_{i=1}^{n} b_i^2 y_i^2 \leqq \delta + o(\delta)\right\}. \tag{1.10}$$

Therefore,

$$\left|S(x^0;\delta)\right| = \int_{K_0(x^0;\delta)} \left[\delta - \sum_{i=1}^{n} b_i^2 y_i^2 + o\left(\sum_{i=1}^{n} y_i^2\right)\right] dy, \tag{1.11}$$

where dy denotes the n-dimensional Euclidean volume element. We make the substitution $b_i y_i = z_i$, put $r^2 = \sum z_i^2$, and write ω for the surface area of the unit $(n-1)$-sphere. Then (1.11) becomes

$$\begin{aligned} \left|S(x_0;\delta)\right| &= \frac{\omega}{\Pi b_i} \int_{r \leqq \delta^{\frac{1}{2}} + o(\delta^{\frac{1}{2}})} [\delta - r^2 + o(r^2)] r^{n-1} dr \\ &= \frac{\omega}{\Pi b_i} \left(\frac{1}{n} - \frac{1}{n+2}\right) \delta^{1+n/2} + o(\delta^{1+n/2}). \end{aligned} \tag{1.12}$$

The equation of the lateral surface of the cone $T(x^0;\delta)$ is

$$z = \left(\delta \sum_{i=1}^{n} b_i^2 y_i^2\right)^{\frac{1}{2}} + o\left(\delta^{\frac{1}{2}} \sum_{i=1}^{n} |y_i|\right). \tag{1.13}$$

Therefore, proceeding as above one obtains

$$\begin{aligned} \left|T(x^0;\delta)\right| &= \int_{K_0(x^0;\delta)} \left[\delta - \left(\delta \sum_{i=1}^{n} b_i^2 y_i^2\right)^{\frac{1}{2}} + o\left(\delta^{\frac{1}{2}} \sum_{i=1}^{n} |y_i|\right)\right] dy \\ &= \frac{\omega}{\Pi b_i} \int_{r \leqq \delta^{\frac{1}{2}} + o(\delta^{\frac{1}{2}})} [\delta - \delta^{\frac{1}{2}} r + o(\delta^{\frac{1}{2}} r)] r^{n-1} dr \\ &= \frac{\omega}{\Pi b_i} \left(\frac{1}{n} - \frac{1}{n+1}\right) \delta^{1+n/2} + o(\delta^{1+n/2}). \end{aligned} \tag{1.14}$$

Clearly (1.7) follows from (1.12) and (1.14).

That the hypotheses of Theorem 1 are sharp is seen from the following example. Let M in the y_i, z coordinates be given by

$$z = \left(\sum_{i=1}^{n} y_i^2\right)^{p/2} \tag{1.15}$$

where $p > 1$, and suppose $x^0 = 0$. Then M is not a $C^{(2)}$-surface near x^0 if $p < 2$, and x^0 is not an elliptic point if $p > 2$. Only for $p = 2$ are the hypotheses of Theorem 1 satisfied. One finds

$$\lim_{\delta \to 0} \frac{\left|T(x^0;\delta)\right|}{\left|S(x^0;\delta)\right|} = \frac{1}{p} \cdot \frac{n+p}{n+1}, \tag{1.16}$$

which agrees with (1.7) only for $p = 2$.

2. Elliptic points on quadrics. In this section we consider the question of elliptic points on quadric surfaces. The surface M as defined in (1.1) is a quadric if F is a quadratic polynomial in the variables $x_1, \cdots, x_{n+1}$. By the proper choice of Cartesian coordinates $x_1, \cdots, x_{n+1}$ any quadric is represented by

$$F(x) \equiv \sum_{i=1}^{p} a_i x_i^2 - \sum_{i=p+1}^{q} a_i x_i^2 + \sum_{i=q+1}^{n+1} b_i x_i - c = 0, \tag{2.1}$$

where $a_i > 0$, b_i, c are real constants. The rank of the matrix $(F_{ij}(x^0))$ must be $\geqq n+1$ if x^0 is an elliptic point of M. Since in the present case (F_{ij}) is constant it follows that M can have no elliptic point unless $q \geqq n$. If $q = n$ then (2.1) takes the form

$$\sum_{i=1}^{p} a_i x_i^2 - \sum_{i=p+1}^{n} a_i x_i^2 + b x_{n+1} - c = 0. \tag{2.2}$$

If $b = 0$ in (2.2) (**cylinder**) then an infinite line $x_i = c_i$ $(i = 1, \cdots, n)$ lies in every chord section K $(x^0;\ \delta)$, hence no point x_0 is elliptic. If $b \neq 0$ then (2.2) represents a paraboloid and its vertex $x_1 = \cdots = x_n = 0$, $x_{n+1} = c/b$ is an elliptic point if and only if $p = n$, and in this case all points of M are elliptic.

Next assume $q = n + 1$ in (2.1), hence

$$\sum_{i=1}^{p} a_i x_i^2 - \sum_{i=p+1}^{n+1} a_i x_i^2 = c. \tag{2.3}$$

If $c = 0$ (**cone**) then necessarily $1 \leqq p \leqq n$ and we may assume $p \geqq 2$ (if $p = 1$ interchange x_i and x_{n+1-i}). The point $x^0 = 0$ is excluded since condition (1.2) is violated there. Suppose $x^0 \neq 0$, and consider the intersection of $K(x^0;\ \delta)$ with the linear manifold $x_3 = x_4 = \cdots = x_n = 0$. The remaining coordinates are restricted by

$$a_1 x_1^2 + a_2 x_2^2 - a_{n+1} x_{n+1}^2 = 0,\ c_1 x_1 + c_2 x_2 + c_{n+1} x_{n+1} + d = 0. \tag{2.4}$$

The set of real solutions of (2.4) is either $x_1 = x_2 = x_{n+1} = 0$ or is unbounded. Hence, x^0 cannot be an elliptic point.

We are left to discuss (2.3) with $c \neq 0$. We may assume $c > 0$ (if $c < 0$ interchange x_i and x_{n+1-i}). If then $p = n + 1$ the quadric is an **ellipsoid** and each of its points is elliptic. We distinguish two remaining cases.

a. $p = 1$. Then $x_1^2 \geqq c/a_1$ for all points, hence the surface consists of two continua (sheets), separated by and symmetric with respect to the plane $x_1 = 0$. If $x^0 \in M$, then

$$a_1 x_1^0 x_1 - \delta = \sum_{i=2}^{n+1} a_i x_i^0 x_i \tag{2.5}$$

represents the tangent plane at x^0 if $\delta = c$, and a plane parallel to it for $\delta \neq c$. The Cauchy inequality gives for points in the intersection of (2.3) and (2.5)

$$(a_1x_1^0x_1-\delta)^2 \leqq \sum_{i=1}^{n+1} a_i(x_i^0)^2 \sum_{i=1}^{n+1} a_ix_i^2 = (a_1(x_1^0)^2-c)(a_1x_1^2-c) \tag{2.6}$$

hence

$$ca_1\left(x_1-\frac{\delta}{c}x_1^0\right)^2 \leqq (c^2-\delta^2)\left(1-\frac{a_1}{c}x_1^0\right)^2. \tag{2.7}$$

It follows from (2.7) that the coordinate x_1 and then from (2.6) that also $x_2, \cdots, x_{n+1}$ are bounded for the points of the intersection of (2.3) and (2.5), which therefore must be an ($n-1$-dimensional) ellipsoid. Thus, x^0 is an elliptic point.

b. $2 \leqq p \leqq n$. Then the surface is a continuum. Choose $x^0 \in M$ and consider the intersection of $K(x^0;\delta)$ with the linear manifold $x_3 = x_4 = \cdots = x_n = 0$ again. The remaining coordinates are restricted by

$$\begin{aligned} &a_1x_1^2 + a_2x_2^2 - a_{n+1}x_{n+1}^2 - c = 0, \\ &a_1x_1^0x_1 + a_2x_2^0x_2 - a_{n+1}x_{n+1}^0x_{n+1} - \delta = 0. \end{aligned} \tag{2.8}$$

This is an unbounded set for $\delta > c$. It follows that x^0 is not an elliptic point.

Our discussion has shown that there are exactly 3 (types of) quadrics which have any elliptic points: the elliptic paraboloid, the ellipsoid and the 2-sheeted hyperboloid. Moreover, all the points on these surfaces are elliptic.

3. Segment inequalities for quadrics. After the discussion of Section 2 we are now prepared to prove

THEOREM 2. *If M is an elliptic paraboloid in E^{n+1} then*

$$\frac{|T(x^0;\delta)|}{|S(x^0;\delta)|} = \frac{1}{2}\left(\frac{n+2}{n+1}\right). \tag{3.1}$$

If M is an ellipsoid then

$$\frac{|T(x^0;\delta)|}{|S(x^0;\delta)|} < \frac{1}{2}\left(\frac{n+2}{n+1}\right), \quad \sup_{\delta\to 0} \frac{|T(x^0;\delta)|}{|S(x^0;\delta)|} = \frac{1}{2}\left(\frac{n+2}{n+1}\right). \tag{3.2}$$

If M is one sheet of a 2-sheeted hyperboloid then

$$\frac{|T(x^0;\delta)|}{|S(x^0;\delta)|} > \frac{1}{2}\left(\frac{n+2}{n+1}\right), \quad \inf_{\delta\to 0} \frac{|T(x^0;\delta)|}{|S(x^0;\delta)|} = \frac{1}{2}\left(\frac{n+2}{n+1}\right). \tag{3.3}$$

These relations hold for all $x^0 \in M$ and all $\delta \neq 0$ for which the chord section $K(x^0;\delta)$ is not empty.

Proof. For the paraboloid we choose the representation

$$x_{n+1} = \sum_{i=1}^{n} (a_ix_i^2 + b_ix_i) \tag{3.4}$$

in Cartesian coordinates $x_1, \cdots, x_{n+1}$ with $x^0 = 0$. The chord section $K(x^0; \delta)$ is then given by

$$K(x^0; \delta) = \left\{ \sum_{i=1}^{n} b_i x_i + k\delta = x_{n+1}, \ \sum_{i=1}^{n} (a_i x_i^2 + b_i x_i) \leqq x_{n+1} \right\}, \tag{3.6}$$

where k is some positive constant. Therefore,

$$\begin{aligned} |S(x^0; \delta)| &= \int_{K_0(x^0;\delta)} \left[\sum_{i=1}^{n} b_i x_i + k\delta - \sum_{i=1}^{n} (a_i x_i^2 + b_i x_i) \right] dx \\ &= \int_{\Sigma a_i x_i^2 \leqq k\delta} \left[k\delta - \sum_{i=1}^{n} a_i x_i^2 \right] dx = \frac{\omega}{\Pi a_i} \left(\frac{1}{n} - \frac{1}{n+2} \right) (k\delta)^{1+n/2} \end{aligned} \tag{3.5}$$

where we have made use of (1.12) without the remainder term. The equation of the lateral surface of the cone $T(x^0; \delta)$ is

$$x_{n+1} = \sum_{i=1}^{n} b_i x_i + \left(k\delta \sum_{i=1}^{n} a_i x_i^2 \right)^{\frac{1}{2}} \tag{3.6}$$

and proceeding as above, one obtains

$$\begin{aligned} |T(x^0; \delta)| &= \int_{K_0(x^0;\delta)} \left[\sum_{i=1}^{n} b_i x_i + k\delta - \sum_{i=1}^{n} b_i x_i - \left(k\delta \sum_{i=1}^{n} a_i x_i^2 \right)^{\frac{1}{2}} \right] dx \\ &= \int_{\Sigma a_i x_i^2 \leqq k\delta} \left[k\delta - \left(k\delta \sum_{i=1}^{n} a_i x_i^2 \right)^{\frac{1}{2}} \right] dx \\ &= \frac{\omega}{\Pi a_i} \left(\frac{1}{n} - \frac{1}{n+1} \right) (k\delta)^{1+n/2}. \end{aligned} \tag{3.7}$$

(3.5) together with (3.7) gives (3.1).

For the proof of (3.2) and (3.3) an indirect argument must be used since the involved integrals cannot be explicitly evaluated. We observe that the second parts of (3.2) and (3.3) are immediate consequences of Theorem 1, once the first parts are proved. Let the Cartesian coordinates $x_1, \cdots, x_{n+1}$ be so chosen that $x^0 \in M$ is the origin, M is tangent to the plane $x_{n+1} = 0$ at 0 and the normal at 0 coincides with the positive x_{n+1} axis. Thus, M is represented by

$$\sum_{i,j=1}^{n+1} a_{ij} x_i x_j - c x_{n+1} = 0 \tag{3.8}$$

with $c > 0$. Next, we choose new Cartesian coordinates $y_1, \cdots, y_n, z = x_{n+1}$ such that (3.8) becomes

$$\sum_{i=1}^{n} b_i y_i^2 + b z^2 + 2z \sum_{i=1}^{n} c_i y_i - cz = 0 \tag{3.9}$$

or, equivalently,

$$\sum_{i=1}^{n} b_i\left(y_i + \frac{c_i}{b_i}z\right)^2 + \left(b - \sum_{i=1}^{n}\frac{c_i^2}{b_i}\right)z^2 - cz = 0. \tag{3.10}$$

The coefficients b_i in (3.9) must be positive since x^0 is an elliptic point of M. If M is an ellipsoid (hyperboloid) then the form $\Sigma a_{ij}x_ix_j$ is positive definite (indefinite), hence the coefficient of z^2 in (3.10) is positive (negative). Thus,

$$\begin{aligned} b' = b - \sum_{i=1}^{n}\frac{c_i^2}{b_i} &> 0 \text{ for ellipsoid} \\ &= 0 \text{ for paraboloid} \\ &< 0 \text{ for hyperboloid.} \end{aligned} \tag{3.11}$$

The chord section $K(x^0;\delta)$ is given by

$$K(x^0;\delta) = \left\{z = \delta,\ \sum_{i=1}^{n} b_iy_i^2 + b\delta^2 + 2\delta\sum_{i=1}^{n} c_iy_i - c\delta = 0\right\}. \tag{3.12}$$

Let M^0 be a paraboloid which is also tangent to the plane $z = 0$ at 0 and is represented by

$$\sum_{i=1}^{n} b_iy_i^2 + b^0z^2 + 2z\sum_{i=1}^{n} c_iy_i - c^0z = 0, \tag{3.13}$$

where b^0, c^0 are some new constants. By (3.11) we must have

$$b^0 - \sum_{i=1}^{n}\frac{c_i^2}{b_i} = 0. \tag{3.14}$$

M^0 intersects the plane $z = \delta$ in the same quadric as M does if $b^0\delta^2 - c^0\delta = b\delta^2 - c\delta$, hence by (3.11)

$$c = c^0 + (b - b^0)\delta = c^0 + b'\delta. \tag{3.15}$$

By (3.9) and (3.13) we have

$$\begin{aligned} z &= \frac{1}{c}\sum_{i=1}^{n} b_iy_i^2 + o\left(\sum_{i=1}^{n} y_i^2\right) \text{ near } x^0 \in M \\ &= \frac{1}{c^0}\sum_{i=1}^{n} b_iy_i^2 + o\left(\sum_{i=1}^{n} y_i^2\right) \text{ near } x^0 \in M^0. \end{aligned} \tag{3.16}$$

Using (3.11) and (3.15), we conclude that the ellipsoid (hyperboloid) is below (above) the paraboloid near x^0.

The last statement holds not only near x^0, but for all points x with $0 < z < \delta$. To see this, assume on the contrary that the paraboloid and ellipsoid intersect in some point x^1 with coordinates $z^1, y_1^1, \cdots, y_1^n$, $0 < z^1 < \delta$. Let V be the 2-plane

$$V = \{y_n^1y_i - y_i^1y_n = 0,\ i = 1,\cdots, n-1\}. \tag{3.17}$$

V is orthogonal to the plane $z = 0$, passes through x^0 and x^1, and intersects the boundary of $K(x^0; \delta)$ in two distinct points x^2, x^3, which are also distinct from x^0, x^1. The two conics C^0 and C in which V intersects the paraboloid M^0 and ellipsoid M are tangent to each other at x^0 and intersect in x^1, x^2, x^3. Therefore, C^0 and C must coincide, while it was shown that C^0 is above C near x^0. This contradiction proves that the paraboloid is above the ellipsoid (below the hyperboloid) between the planes $z = 0$ and $z = \delta$. The inequalities (3.2) and (3.3) are immediate consequences and the theorem is proved.

I conjecture a converse of equation (3.1). If M is a $C^{(2)}$-surface in E^{n+1} with elliptic points only for which the ratio $T(x^0; \delta)/S(x^0; \delta)$ is constant then M is an elliptic paraboloid. I have not been able to prove this.

References

1. Carl B. Boyer, A History of Mathematics, Wiley, New York, London, Sydney, 1968, pp. 142–144.
2. Michael Golomb and Hiroshi Haruki, An inequality for elliptic and hyperbolic segments, Math. Mag., 46 (1973) 152–155.
3. W. H. Fleming, Functions of Several Variables, Addison-Wesley, Reading, Mass., 1965.

Division of Mathematical Sciences, Purdue University, Lafayette, IN 47907.

CORRECTION TO "DIFFERENTIATION UNDER THE INTEGRAL SIGN"

Harley Flanders

Professor Wilfred Kaplan informs me that in my article (this Monthly, 80 (1973) 615–627) I should have quoted his *Advanced Calculus* (1952), particularly Problem 6, p. 223 (2nd ed.: p. 290) for a linear flow, and Problems 6–8, pp. 299–300 (2nd ed.: pp. 367–368) for a space flow. I regret this oversight.

Department of Mathematics, Tel Aviv University, Ramat Aviv, Israel.

QUERIES

Edited by A. C. Zitronenbaum

This Department welcomes queries from readers about mathematics at the collegiate level, such as sources for exposition of a particular topic from a special point of view, references to vaguely remembered articles, descriptions of special kinds of courses or teaching methods, methods constructing illustrative examples for exercises of particular kinds (questions on research topics should, in general, be addressed to be the "Queries Department" of the Notices of the American Mathematical Society). Replies will be forwarded to the questioner and may also be edited into a

composite answer for publication in this Department. Consequently all items submitted for consideration for possible publication should include the name and complete mailing address of the person who is to receive the reply. Queries and answers should be sent to A. C. Zitronenbaum, Department of Mathematics, Cornell University, Ithaca, NY 14850.

3. G. V. Berlichingen. Can anybody furnish me with information as to whether the theory of osculating circles and planes has any application in celestial mechanics and the computation of planetary orbits?

4. G. Pedrick. I would like suggestions for source material to be used in an interdisciplinary seminar for juniors and seniors dealing with what mathematics can contribute to ecology and preserving the environment.

MATHEMATICAL NOTES

EDITED BY DAVID ROSELLE

Material for this Department should be sent to David Roselle, Department of Mathematics, Louisiana State University, Baton Rouge, LA 70803.

A CHARACTERIZATION OF CIRCLES AND OTHER CLOSED CURVES

HANS HERDA

The response to the author's research problem (conjecture **I** in [**11**]) on characterization of circles has been so intense that a short survey article is appropriate. More than five years elapsed between discovery of the problem and its publication, but good problems have staying power and (in this case) admit solution, extension and generalization. There are at least a dozen solutions, and almost all use a separate attack.

First, let us restate the problem. In the sequel, curve always means closed, rectifiable curve. Suppose C is a simple plane curve having positive perimeter p. Let x be any point on C. With x associate the unique point x' on C whose distance (measured on C) from x is $p/2$. Denote the line segment joining x and x' by s_x and call s_x the **pseudo-diameter of** C **at** x. Also denote the length of the pseudo-diameter at x, by s_x.

The function $f: C \to R^+$ defined by setting $f(x) = s_x$ is continuous with respect to distance on C. Because C is compact, this implies that $s = \min_{x \in C} s_x$ exists. Call s the **pseudo-diameter of** C. Since C is simple, s is positive. Now consider the ratio p/s. If C is a circle having positive, finite radius, then $p/s = \pi$. The conjecture is:

THEOREM A. *For any simple plane curve C, $p/s \geqq \pi$, and $p/s = \pi$ implies C is a circle.*

Noting that C need not be simple, Ault [1] proves the considerably more general

THEOREM B. *If C is a curve in any real linear inner-product space with perimeter p and pseudo-diameter s, then $p \geqq \pi s$, and $p = \pi s$ implies C is a circle.*

Ault's proof uses trigonometry and should be accessible to undergraduate students with modest backgrounds.

Batten [2] establishes Theorem A using a convexity argument and in the process also shows that if C is convex and has constant pseudo-diameter, then C is continuously differentiable.

Chakerian [4] lets f be a continuous involution without fixed points on a plane curve C and defines $s(x) = |f(x) - x|$ and $s = \min_{x \in C} s(x)$. He then proves:

THEOREM C. *If C is a simple plane curve with perimeter p, then $p \geqq \pi s$, and $p = \pi s$ iff C is a convex curve of constant width and f is the involution such that each chord joining x to $f(x)$ is a diametral chord of C (a chord of C whose endpoints lie on two parallel lines such that C meets each of them and lies in the closed strip which they bound).*

Theorem C implies Theorem A. Chakerian's proof involves a study of the boundary of the smallest convex set containing C and depends on several results expounded in Bonnesen-Fenchel's classic on convex bodies [3].

Davies [5] shows that Theorem A can be established for a simple plane curve C via the Fourier series method used by Hurwitz to solve the isoperimetric problem (see, e.g., [14] or [17] for Hurwitz's method).

Fink [7] proves the stronger

THEOREM D. *The average pseudo-diameter of a simple curve C in the plane (or in E^n) is less than p/π unless C is a circle.*

This proof also uses a Fourier series method. Let $x(s)$ and $y(s)$ be the coordinates of a simple plane curve parametrized by arc length, and let $f = d(x(s))/ds$ and $g = d(y(s))/ds$. Fink interprets (f, g) as weighting the unit circle like a wire with a non-homogeneous mass, with the centroid of the wire at the origin. If the wire is cut into two equal connected masses in every way, then some centroid of half the wire is within $2/\pi$ of the origin

Goodey [8] proves Theorem A using a convexity argument. He shows in a few lines that if C is convex and $s = p/\pi$ then C has both constant width and a unique tangent at each of its points.

Johnson [12] proves Theorem A quickly using integration and vector methods. He also proposes this generalized conjecture: let $X(s)$ be a parametrization of the simple plane curve C by arc length, for $0 \leqq s \leqq p$. Consider $|X(s) - X(s + p/n)|$ for any fixed integer $n \geqq 2$. Let d_n be the minimum of these values on C. Then $p/d_n \geqq \pi/\sin(\pi/n)$.

Lipskie [13] also proves Theorem A with an argument based on integration and vectors, but does not require that C be simple and allows C to lie in E^n.

Short [16] obtains the same result as Lipskie but his argument requires that C be piecewise differentiable.

Wente [19] gives a quick proof of Theorem A which does not require C to be simple; he also uses an integration argument.

Witsenhausen [20] establishes

THEOREM E. *The pseudo-diameter s and the perimeter p of a simple curve in a real normed linear space satisfy $p \geqq gs$, where g is the half-girth of the unit ball. If the space is finite-dimensional the bound is sharp, i.e., there exist curves for which $p = gs$.*

This result depends on two lemmata due to Schäffer [15], whose paper breaks new ground in the inner geometry of normed linear and Banach spaces. Witsenhausen also shows that the author's conjecture **II** in [11] is false.

There are many open problems on geometric inequalities and convex sets, and the field enjoys a lively journal literature. Here is a list of text and reference books: aside from Bonnesen-Fenchel's survey treatise of 1934 [3], the monographs by Eggleston [6] and Hadwiger [9] also deal with E^n; the books by Yaglom-Boltyanskii [21] and Hadwiger-Debrunner-Klee [10] are restricted to the plane; and Valentine's work [18] deals with finite-dimensional normed linear spaces. A taste of recent advances can be obtained by browsing through the 1963 AMS Convexity Symposium volume [22]; Grünbaum's survey article "Measures of symmetry for convex sets" (pp. 233–270) even has a certain affinity to the problem discussed in this paper. Perhaps you, gentle reader, will be tempted to formulate and solve a new problem.

References

1. R. Ault, Metric characterization of circles, this MONTHLY, p. 149.

2. G. W. Batten, Jr., A characterization of circles, written communication.

3. T. Bonnesen and W. Fenchel, Theorie der konvexen Körper, Springer Verlag, Berlin 1934. Chelsea Publ. Co., N. Y. 1948 and 1971.

4. G. D. Chakerian, A characterization of curves of constant width, this MONTHLY, page 153.

5. R. O. Davies, written communication.

6. H. G. Eggleston, Convexity, Cambridge Univ. Press, 1958 and 1969.

7. A. M. Fink, A circle maximizes the minimum pseudo-diameter, written communication.

8. P. R. Goodey, A characterization of circles, Bull. London Math. Soc., 4(1972) 199–201.

9. H. Hadwiger, Altes und Neues über konvexe Körper, Birkhäuser Verlag, Basel und Stuttgart 1955.

10. H. Hadwiger, H. Debrunner, and V. Klee, Combinatorial Geometry in the Plane, Holt, Rinehart & Winston, New York, 1964.

11. H. Herda, A conjectured characterization of circles, this MONTHLY, 78(1971) 888–889.

12. H. H. Johnson, A problem of Herda concerning circles, written communication.

13. L. C. Lipskie, Pseudo-diameters and the length of a closed curve, written communication.

14. W. W. Rogosinski, Fourier Series, Chelsea, New York, 1959.

15. J. J. Schäffer, Inner diameter, perimeter and girth of spheres, Math. Annalen, 173 (1967) 59–82.

16. D. R. Short, A characterization of circles, written communication.
17. I. N. Sneddon, Fourier Series, Routledge and Kegan Paul, London, 1961.
18. F. A. Valentine, Convex Sets, McGraw Hill, New York, 1964.
19. H. C. Wente, A proof of Herda's characterization of circles, written communication.
20. H. S. Witsenhausen, On closed curves in Minkowski spaces, Proc. Amer. Math. Soc., 35(1972) 240–241.
21. I. M. Yaglom and V. A. Boltyanskii, Convex Figures, Holt, Rinehart & Winston, New York, 1961.
22. Convexity, Proceedings of Symposia in Pure Mathematics, Vol. 7, edited by V. L. Klee. Amer. Math. Soc., Providence, R. I., 1963. Corrected reprint 1972.

45 Fremont Street, Lexington, Massachusetts 02173.

METRIC CHARACTERIZATION OF CIRCLES

Robin Ault

1. Introduction. Hans Herda [1] has conjectured that a circle is the only plane curve in which the ratio of circumference to minimal "diameter" is as small as π. We verify this conjecture, here extended to any real Hilbert (even any real linear inner-product) space. Let C be any rectifiable closed curve in such a space, with circumference k. For each point x of C, define the following:

$A(x)$, the point of C whose distance from x (measured along C) is $\frac{1}{2}k$;

$D(x)$, the **diameter** of C through x, as the vector from x to $A(x)$;

$M(x)$, the midpoint of $D(x)$; and

$d(x)$, the length of $D(x)$.

Also, define the **minimum diameter** d of C as $\inf\{d(x) \mid x \in C\}$.

If C is a circle, then $k = \pi d$; noting this, Herda proposed:

Conjecture 1. *For any C, $k \geqq \pi d$.*

Conjecture 2. *If $k = \pi d$, then C is a circle (of diameter d).*

The conjectures are trivial if $d = 0$, so we assume $d \neq 0$. Without loss of generality we assume C has been "normalized" to make $d = 2$. The conjectures then become: $k \geqq 2\pi$; and $k = 2\pi$ implies C is a circle.

2. Proof of first conjecture. Select two points x and y "fairly near each other" on C (specifically, so the vectors $D(x)$ and $D(y)$ form an *acute* angle between their positive directions). Let T denote the vector from $M(x)$ to $M(y)$. Make the following definitions (see figure 1):

$R = \frac{1}{2}d(x) = \operatorname{dist}(x, M(x))$;

$r = \dfrac{d(y) - d(x)}{2}$, so that $R + r = \frac{1}{2}d(y)$;

$2a =$ size of the acute angle between $D(x)$ and $D(y)$ (translated so as to intersect);

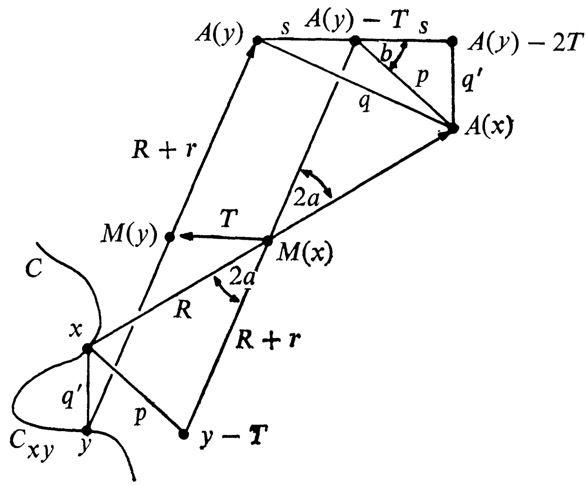


FIG. 1

b = size of the smaller angle (so $0 \leqq b \leqq \pi/2$) between the lines joining $A(x)$ to $A(y) - T$, and $A(y)$ to $A(y) - 2T$;

p = $\operatorname{dist}(x, y - T)$;

q = $\operatorname{dist}(A(x), A(y))$;

q' = $\operatorname{dist}(x, y)$; and

s = length of T.

Applying the law of cosines to the triangle with vertices $x, y - T$, and $M(x)$, we see $p = \sqrt{R^2 + (R + r)^2 - 2R(R + r)\cos(2a)}$, which reduces by elementary trigonometry to

$$p = \sqrt{4(R^2 + Rr)\sin^2 a + r^2}\,. \tag{1}$$

Since R and $R + r$ are both half-diameters, each is at least 1, hence their product $R^2 + Rr$ is at least 1, and (1) yields

$$p \geqq \sqrt{4\sin^2 a + r^2}\,. \tag{2}$$

Again by law of cosines we see that q and q' equal

$$\sqrt{p^2 + s^2 \pm 2ps\cos b}\,,$$

the two signs giving the two correct values; combining this with (2) gives

$$\begin{aligned} \max(q, q') &\geqq \sqrt{4\sin^2 a + r^2 + s^2 + 2ps\cos b} \\ &\geqq \sqrt{4\sin^2 a + r^2 + s^2}\,. \end{aligned}$$

Let C_{xy} denote the arc of C from x to y. Its arc-length $|C_{xy}|$ is at least q'; the arc-

length of C from $A(x)$ to $A(y)$ is at least q; and the two arc-lengths are equal, so each is at least $\max(q, q')$. Thus

$$|C_{xy}| \geqq \sqrt{4\sin^2 a + r^2 + s^2}. \tag{3}$$

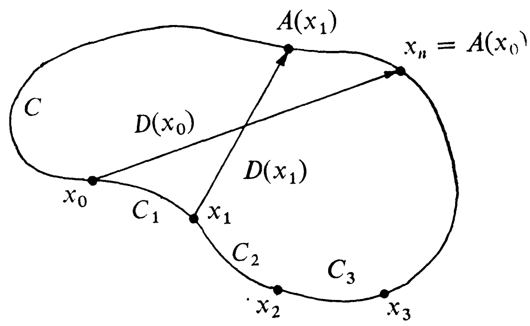


FIG. 2

Orient C and choose on it points $x_0, x_1, \cdots, x_n = A(x_0)$ in that order around C, relatively near each other in succession (see figure 2). For $i = 1, 2, \cdots, n$, define:

C_i = the arc of C from x_{i-1} to x_i;
r_i = $\frac{1}{2}(d(x_i) - d(x_{i-1}))$;
s_i = $\operatorname{dist}(M(x_{i-1}), M(x_i))$; and
$2a_i$ = size of the acute angle between $D(x_{i-1})$ and $D(x_i)$.

The preceding discussion applies to each arc C_i, so (3) gives us

$$|C_i| \geqq \sqrt{4\sin^2 a_i + r_i^2 + s_i^2},$$

hence

$$k/2 = \sum_{i=1}^{n} |C_i| \geqq \sum_{i=1}^{n} \sqrt{4\sin^2 a_i + r_i^2 + s_i^2}.$$

By the generalized triangle inequality this gives

$$k/2 \geqq \sqrt{4\left(\sum_{i=1}^{n} \sin a_i\right)^2 + \left(\sum_{i=1}^{n} |r_i|\right)^2 + \left(\sum_{i=1}^{n} s_i\right)^2}. \tag{4}$$

The angles $2a_i$ measure the changing direction of the vector $D(x_i)$, which reverses direction as i goes from 0 to n; so $\sum_{i=1}^{n} a_i \geqq \pi/2$. Let a_* denote the maximum value of the a_i; note $a_* < \pi/4$. Since

$$\sin a_i \geqq a_i - \frac{a_i^3}{3!} \geqq a_i - a_i^2 \quad (\text{for } 0 \leqq a_i \leqq \pi/4),$$

we deduce:

$$\sum_{i=1}^{n} \sin a_i \geqq \sum_{i=1}^{n} (a_i - a_i^2) \geqq (1 - a_*) \sum_{i=1}^{n} a_i \geqq \frac{\pi}{2}(1 - a_*).$$

Together with (4) this proves

$$k/2 \geqq \sqrt{2\pi(1 - a_*)^2 \sum_{i=1}^{n} a_i + \left(\sum_{i=1}^{n} |r_i|\right)^2 + \left(\sum_{i=1}^{n} s_i\right)^2} \geqq \pi(1 - a_*). \tag{5}$$

By choosing n and the points $x_0, x_1, \cdots, x_n$ appropriately, we can make a_* as close to 0 as desired; hence $k/2 \geqq \pi$, establishing the first conjecture.

3. Proof of second conjecture. Assume $k = 2\pi$; for any selection of the points $x_0, x_1, \cdots, x_n$, we show:

$$\sum_{i=1}^{n} a_i = \pi/2, \tag{6}$$

$$\sum_{i=1}^{n} |r_i| = 0, \tag{7}$$

$$\sum_{i=1}^{n} s_i = 0. \tag{8}$$

Suppose $x_0', x_1', \cdots, x_m' = A(x_0')$ is another selection of points in that order around C, with $x_0 = x_0'$, $x_n = x_m'$, and each x_i equal to some x_j' (so that the x_j' could be called a "refinement" of the x_i). Define a_j', r_j', s_j' (for $j = 1, \cdots, m$), and a_*' in the obvious way. We note:

$$\left\{\begin{aligned} \sum_{j=1}^{m} a_j' &\geqq \sum_{i=1}^{n} a_i, \\ \sum_{j=1}^{m} |r_j'| &\geqq \sum_{i=1}^{n} |r_i|, \\ \sum_{j=1}^{m} s_j' &\geqq \sum_{i=1}^{n} s_i, \end{aligned}\right. \tag{9}$$

because the total variation of diameter-direction, diameter-length, and midpoint-location can not be decreased by division into smaller steps. We know

$$\sum_{i=1}^{n} a_i \geqq \pi/2, \quad \sum_{i=1}^{n} |r_i| \geqq 0, \text{ and } \sum_{i=1}^{n} s_i \geqq 0;$$

if any of these inequalities is strict, we choose a refinement $x_0', x_1', \cdots, x_m'$ of $x_0, x_1, \cdots, x_n$, having a_*' so small that

$$2\pi(1 - a_*')^2 \sum_{i=1}^{n} a_i + \left(\sum_{i=1}^{n} |r_i|\right)^2 + \left(\sum_{i=1}^{n} s_i\right)^2 > \pi^2. \tag{10}$$

Formulas (9) and (10) combine to give

$$2\pi(1-a'_*)^2 \sum_{j=1}^{m} a'_j + \left(\sum_{j=1}^{m} |r'_j|\right)^2 + \left(\sum_{j=1}^{m} s'_j\right)^2 > \pi^2. \tag{11}$$

This contradicts the assumption $k = 2\pi$, since the quantity on the left of (11) is no larger that $k^2/4 = \pi^2$ (square formula 5). Hence equalities (6), (7), and (8) hold.

These equalities are true of *any* selection $x_0, x_1, \cdots, x_n$ (as described). We conclude from (8) that the diameters of C have the same midpoint M; from (7) that they have the same length (which must be 2); hence the points of C are all on the sphere of radius 1 about M. The only path on this sphere, from x_0 to its antipode $A(x_0)$, having length π, is half of a great circle; so formula (6) shows C must consist of two great semi-circles, which in fact form a single great circle because the diameters all pass through M. This verifies the second conjecture.

Reference

1. Hans Herda, A conjectured characterization of circles, this MONTHLY, 78 (1971) 888–889.

DEPARTMENT OF MATHEMATICS, BOSTON STATE COLLEGE, BOSTON, MASSACHUSETTS 02115.

A CHARACTERIZATION OF CURVES OF CONSTANT WIDTH

G. D. CHAKERIAN

Let C be a simple closed rectifiable plane curve, and let f be a continuous involution of C without fixed points. That is, f is a continuous mapping assigning to each point $x \in C$ another point $f(x) \in C$, with $f(f(x)) = x$ and $f(x) \neq x$ for all x. Let $s(x) = |f(x) - x|$ = the length of the chord joining x to $f(x)$, and $s = \min s(x)$, $x \in C$. By the continuity of $s(x)$ and the compactness of C, the minimum value is achieved. The main result of this note is the following theorem.

THEOREM. *If C, f, and s are as above, and $p(C)$ is the perimeter of C, then*

$$p(C) \geqq \pi s. \tag{1}$$

Moreover, equality holds in (1) *if and only if C is a convex curve of constant width and f is the involution such that each chord joining x to $f(x)$ is a diametral chord* (*defined below*) *of C.*

One of the conjectures in [3] was that if f is the particular involution such that x and $f(x)$ bisect the perimeter of C, then (1) holds, with equality possible only for the circle. Our theorem confirms this conjecture since, as proved in [2, Theorem 2.5], the only curve of constant width each of whose diametral chords bisect the perimeter is the circle.

The proof of the theorem depends on some simple results about convex curves

which we now discuss. By a *supporting line* of a plane convex curve K we mean a line meeting K but such that K is contained in one of the closed half-planes determined by that line. If G is any line making angle θ with the horizontal, K will have two supporting lines parallel to G, and the distance between these supporting lines, denoted by $w(\theta)$, is the *width* of K in the direction θ. The function $w(\theta)$ is continuous and, by a theorem of Cauchy (see [**1**, p. 65]), the perimeter of K is given by the formula

$$p(K) = \tfrac{1}{2} \int_0^{2\pi} w(\theta)d\theta. \tag{2}$$

In other words, the perimeter of a plane convex curve is equal to π times its mean width. If $\Delta(K)$ denotes the minimum width of K, $\Delta(K) = \min w(\theta)$, $0 \leqq \theta \leqq 2\pi$, it follows immediately from (2) that

$$p(K) \geqq \pi\Delta(K), \tag{3}$$

and equality can hold in (3) if and only if $w(\theta)$ is constant, that is, K is a curve of constant width. By a *diametral chord* of K we mean a chord of K whose endpoints lie on two parallel supporting lines. It is easily shown (see [**1**, p. 51]) that $\Delta(K)$ is also the minimum length of all diametral chords of K. For the proof of the main theorem it will be important to observe that of all chords of K parallel to a fixed direction, a diametral chord in that direction has maximum length. The following two lemmas will complete our preparation for the proof of the theorem.

LEMMA 1. *If K is the boundary of the smallest convex set containing C, then $p(K) \leqq p(C)$, with equality if and only if $C = K$.*

Proof. The plausibility of the result is seen when one thinks of K as an elastic band looped tightly around C. Since a rigorous proof is difficult to find in the literature, the following argument, based on ideas from integral geometry, might be of interest. If to each line G in the plane we assign coordinates (p, θ), where p is the distance from G to the origin and θ the angle with the horizontal, then a motion invariant measure on the set of lines in the plane is obtained by defining the density $dG = dp\,d\theta$. A formula of Crofton asserts (see [**4**], or [**1**, p. 69])

$$p(C) = \tfrac{1}{2} \int n(G \cap C)dG, \tag{4}$$

where $n(G \cap C)$ is the number of intersections of G with C, and the integration is over all lines. Now observe that any line G meeting K in exactly two points must also meet C in at least two points. Otherwise C would be contained in a closed half-plane H determined by G and hence in a proper closed convex subset of the set bounded by K, contradicting the definition of K. Thus, except possibly for supporting lines of K

(which constitute a set of lines of measure zero), $n(G \cap K) \leqq n(G \cap C)$. Applying (4) we obtain $p(K) \leqq p(C)$. If $C \neq K$, it is not difficult to show that there exists a set of lines G of positive measure for which $n(G \cap K) < n(G \cap C)$, so $p(K) < p(C)$.

LEMMA 2. *Let C be a simple closed plane curve with an involution f as described above. Then given any line G, there is some $x \in C$ such that the chord from x to $f(x)$ is parallel to G.*

Proof. Let $v(x) = f(x) - x$. The lemma follows immediately from observing that the vectors $v(x)$ vary continuously, never vanish, and $v(f(x)) = -v(x)$.

The proof of the main theorem now follows quickly. Let K be the boundary of the smallest convex set containing C, and let $\Delta(K)$ be the minimum width of K. By Lemma 2, there exists $x_0 \in C$ such that the chord joining x_0 to $f(x_0)$ is parallel to a diametral chord of K of length $\Delta(K)$. Thus $s(x_0) = |f(x_0) - x_0|$ is less than or equal to the length of a chord of K parallel to a diametral chord of length $\Delta(K)$, and

$$\Delta(K) \geqq s(x_0) \geqq \min s(x) = s. \tag{5}$$

By Lemma 1, we have $p(C) \geqq p(K)$. Applying (3) and (5) we have,

$$p(C) \geqq p(K) \geqq \pi\Delta(K) \geqq \pi s, \tag{6}$$

establishing (1). If equality holds in (1), then it holds throughout (6). Then $p(C) = p(K)$, so $C = K$. Further, equality then holds in (3), so C is a curve of constant width Δ. Then *every* diametral chord of C has length Δ and $s(x) \leqq \Delta$ for all x (for Δ is then the maximum chord length in every direction). But equality throughout (6) also implies $\Delta = s = \min s(x)$, so $s(x) \geqq \Delta$ for all x. Hence $s(x) \equiv \Delta$, so the chord joining x to $f(x)$ is a diametral chord for all x, completing the proof.

The author would like to acknowledge his indebtedness to G. T. Sallee for stimulating discussions about the main theorem.

References

1. T. Bonnesen and W. Fenchel, Theorie der konvexen Körper, Springer, Berlin, 1934; Chelsea, New York, 1948 and 1971.

2. P. C. Hammer and T. J. Smith, Conditions equivalent to central symmetry of convex curves, Proc. Cambridge Phil. Soc., 60 (1964) 779-785.

3. H. Herda, A conjectured characterization of circles, this MONTHLY, 78 (1971) 888–889.

4. H. Steinhaus, Length, shape and area, Colloq. Math., 3 (1954) 1–13.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA, DAVIS, CALIFORNIA 95616.

RESEARCH PROBLEMS

EDITED BY RICHARD GUY

In this Department the Monthly presents easily stated research problems dealing with notions ordinarily encountered in undergraduate mathematics. Each problem should be accompanied by relevant references (if any are known to the author) and by a brief description of known partial results. Manuscripts should be sent to Richard Guy, Department of Mathematics, Statistics, and Computing Science, The University of Calgary, Calgary 44, Alberta, Canada.

CONJUGATELY PURE SUBGROUP PROBLEMS

BOLA O. BALOGUN

A subgroup H of a group G is said to be **conjugately pure** in G if two elements of H are conjugate in G only if they are already conjugate in H. For example, free factors, direct factors and retracts are conjugately pure. Also every involution in a group generates a conjugately pure subgroup. Of course there are examples of subgroups which are not conjugately pure. For example, let $G = S_3$, the symmetric group on three symbols $\{1,2,3\}$, and $H = \langle(123)\rangle$ the cyclic subgroup generated by (123), then H is not conjugately pure in G. Another non-example is the following: Let p be a prime number and $G = GL(3,p)$, the 3-dimensional general linear group over the field of p elements. Let H be the subgroup of upper triangular matrices with every main diagonal entry 1. Then H is a Sylow p-subgroup of G, and is generated by the two elements

$$a = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad b = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

Since these matrices have the same Jordan canonical form (a 2-dimensional and a 1-dimensional block for the eigenvalue 1), they are similar, that is, they are conjugate in G. However, a and b are not conjugate in H by inspection.

The following lemma gives a summary of the elementary properties of conjugate purity. The proofs are straightforward.

LEMMA 1. (i) *If A is conjugately pure in B, and B is conjugately pure in C, then A is conjugately pure in C.*

(ii) *If H is a normal subgroup of a group G, and A is a conjugately pure subgroup of G containing H, then the factor group A/H is conjugately pure in the factor group G/H.*

(iii) *An ascending union of conjugately pure subgroups of a group G is conjugately pure in G.*

(iv) *If A is an isolated subgroup of a group G (that is, $(x^{-1}Ax)\cap A \neq 1$, $x \in G \Rightarrow x \in A$, see Motzkin-O'Neill-Straus* [**1**]), *then A is conjugately pure in G.*

(v) *If A is a conjugately pure subgroup of G and H is an arbitrary subgroup of G, then $A \cap H$ is conjugately pure in H.*

(vi) *Let X, Y, N be normal subgroups of a group G such that the factor group X/Y is conjugately pure in the factor group G/Y. Then the factor group $N \cap X/N \cap Y$ is conjugately pure in the factor group $G/N \cap Y$.*

A group G is called **hereditarily pure** if every subgroup of G is conjugately pure. Trivially every abelian group is hereditarily pure. Is the converse true? Yes, for finite groups (hence for locally finite groups and residually finite groups), as we shall see below. In general, we do not know.

Let $\mathfrak{F}$ (resp. $\mathfrak{A}, \mathfrak{H}$) denote the class of all finite (resp. abelian, hereditarily pure) groups.

Our problem takes the following form:

Which classes $\mathfrak{X}$ satisfy the inclusion $\mathfrak{X} \cap \mathfrak{H} \subset \mathfrak{A}$?

That $\mathfrak{X} = \mathfrak{F}$ is a solution is the content of the following theorem.

THEOREM. *Every hereditarily pure finite group is abelian.*

For the proof of the theorem we need the following lemma, which is a special case of Frobenius' Theorem, and for which we give an elementary proof to make the paper self-contained.

LEMMA 2. (Cf. Scott [2, Exercise 12.5.15, p. 347]). *Let H be an abelian subgroup of a finite group G such that $H \cap H^x = (1)$ if $x \in G \backslash H$. Then H has a normal complement in G.*

Proof. We construct a homomorphism $T: G \to H$ such that $T(H) = H$ and $\mathrm{Ker}(T) \cap H = 1$. Let R be a complete set of left coset representatives of H in G, so $G = \bigcup_{r \in R} rH$, $r_1 H = r_2 H \Rightarrow r_1 = r_2$. For any $g \in G$, let $\bar{g}$ denote the unique element of $gH \cap R$. If $g \in G$ and $x \in G$, then $\overline{xg} = (x\bar{g})^-$; and $\bar{g}^{-1} g \in H$. We define the function

$$T = T_R : G \to H$$

by

$$T(x) = \prod_{r \in R} \overline{xr}^{\,-1}\, xr,$$

where $x \in G$. Since H is abelian, the order of the factors in the product is immaterial. The rest of the proof is in three steps.

(1) $T = T_R$ *is a homomorphism.* For if $x \in G$, then $x: rH \to xrH$ permutes the cosets of H, so as r ranges over R so does $\overline{xr}$. Since H is abelian, we can write

$$\begin{aligned} T(xy) &= \prod_{r \in R} (\overline{xyr})^{-1}\, xyr \\ &= \prod_{r \in R} [(x\overline{yr})^-]^{-1} x\overline{yr}\; \overline{yr}^{-1} yr \end{aligned}$$

$$= \prod_{r \in R} [(x\overline{yr})^{-}]^{-1} x\overline{yr} \prod_{r \in R} \overline{yr}^{-1} yr$$

$$= T(x)T(y).$$

The homomorphism T, known as the **transfer** of G into H (see Scott [**2**, p. 61]), can be shown to be independent of R, but this fact is not needed here.

(2) *Let $y \in G$. There is a subset $R' = \{r_1, \cdots, r_s\}$ of R and positive integers n_i such that*

$$T(y) = \prod_{i=1}^{s} r_i^{-1} y^{n_i} r_i, \quad \sum_{i=1}^{s} n_i = [G:H],$$

and n_i is minimal such that $r_i^{-1} y^{n_i} r_i \in H$.

Since as r runs through R so does $\overline{yr}, r \to \overline{yr}$ defines a permutation θ in the symmetric group Sym(R) on R. Let $(r_{i_1}, r_{i_2}, \cdots, r_{i_m})$ be a cycle of θ. Then

$$yr_{i_1} = r_{i_2} h_{i_1}, \cdots, yr_{i_{m-1}} = r_{i_m} h_{i_{m-1}}, yr_{i_m} = r_{i_1} h_{i_m},$$

so $h_{i_m} \cdots h_{i_1} = r_{i_1}^{-1} y^m r_{i_1}$.

Moreover, if $k < m$, then

$$r_{i_1}^{-1} y^k r_{i_1} = r_{i_1}^{-1} r_{i_{k+1}} h_{i_k} \cdots h_{i_1} \notin H,$$

so that m is minimal.

(3) *If $1 \neq y \in H$, then $T(y) = y$.* There is exactly one $r_i \in R'$, say r_{i_0}, such that $r_{i_0} \in H$, and the corresponding $n_{i_0} = 1$ by minimality of n_i. But for $r_i \notin H$

$$y^{n_i} \in r_i \quad Hr_i^{-1} \cap H = (1)$$

so $r_i^{-1} y^{n_i} r_i = 1$. Hence $T(y) = y \neq 1$, so $(\operatorname{Ker} T) \cap H = 1$, $G = (\operatorname{Ker} T)H_i$ and the lemma is proved.

Proof of Theorem. Let G be a minimal counterexample. Then all proper subgroups are abelian by Lemma 1(i). Let U be a maximal proper subgroup and $x \notin U$. Then U is abelian and $G = \langle U, x \rangle$. Let $D = U \cap U^x$. From the above, D is normal in G, so, unless $D = (1)$, G/D is abelian, so $G' \subset D$, $U^x \subset UD = U$, $U = U^x$. But then, since U is abelian and pure, x centralizes U, and so $G = \langle U, x \rangle$ is abelian contrary to assumption. Hence $U \cap U^x = (1)$, for all $x \in G \backslash U$. Now by Lemma 2, U has a normal complement, say N. Since N is also pure, U centralizes N, hence G is abelian contradicting the assumption.

Acknowledgment. The author wishes to express his thanks to Professor Ernst G. Straus and the referee for their advice during the writing of this paper.

References

1. T. S. Motzkin, B. O'Neill and E. G. Straus, Isolated subgroups, Michigan Math. J., 20 (1973) 235–248.

2. W. R. Scott, Group theory, Prentice-Hall, New Jersey, 1964.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA, LOS ANGELES, CALIFORNIA, 90024.

CLASSROOM NOTES

EDITED BY DAVID ROSELLE

Material for this Department should be sent to David Roselle, Department of Mathematics, Louisiana State University, Baton Rouge, LA 70803.

THE INTERSECTION OF SUBSPACES

W. S. ERICKSEN

Let V be a vector space of n-tuples over a field F. If W_1 and W_2 are subspaces of V then their intersection is readily determined by the use of a single Hermite normal form. To show this let

$$W_1 = \langle \alpha_1, \cdots, \alpha_j \rangle, \quad W_2 = \langle \beta_1, \cdots, \beta_k \rangle,$$

where α_i and β_i represent column vectors and the symbols $\langle\ \rangle$ denote the span of the included vectors. Form the matrix

$$M = [\alpha_1 \cdots \alpha_j \mid \beta_1 \cdots \beta_k]$$

and row reduce it to its Hermite normal form

$$\bar{M} = [\bar{\alpha}_1 \cdots \bar{\alpha}_j \mid \bar{\beta}_1 \cdots \bar{\beta}_k].$$

This form is characterized by the following conditions, (a) through (d), where "leading entry" is defined to be the first non-zero element of a row.

(a) Every leading entry (of a row with a non-zero element) is 1.

(b) Each column containing a leading entry has all other elements zero.

(c) Each row with all zero elements is below every row with a non-zero element.

(d) If the columns characterized by (b) form the set

$$\bar{S} = \{\bar{\alpha}_{j_1}, \cdots, \bar{\alpha}_{j_r} \mid \bar{\beta}_{k_1}, \cdots, \bar{\beta}_{k_s}\},$$

then $j_1 < j_2 < \cdots < j_r$ and $k_1 < k_2 < \cdots < k_s$, with the element 1 in column i located in its ith row. By these properties the set $\bar{S}$ is a basis for the column space of $\bar{M}$ and each other column of $\bar{M}$ is a linear combination of those vectors in $\bar{S}$ which precede it in $\bar{M}$. Thus if there is an n such that

$$\bar{\beta}_n \in \bar{M} \text{ and } \bar{\beta}_n \notin \bar{S} \tag{1}$$

there are sets of scalars $\{x_1, \cdots, x_r\}$ and $\{y_1, \cdots, y_s\}$ such that

$$\bar{\beta}_n = \sum_{i=1}^{r} x_i \bar{\alpha}_{j_i} + \sum_{i=1}^{s} y_i \bar{\beta}_{k_i}.$$

By properties (a) through (d) the coefficients x_i, y_i are the components of $\bar{\beta}_n$ in the matrix $\bar{M}$. Since linear relations among the columnsof $\bar{M}$ are preserved in passing

by row operations from $\bar{M}$ back to M, it follows that

$$\beta_n - \sum_{i=1}^{s} y_i\beta_{k_i} = \sum_{i=1}^{r} x_i\alpha_{j_i}$$

and consequently

$$\beta_n - \sum_{i=1}^{s} y_i\beta_{k_i} \in W_1 \cap W_2 .$$

Obviously each vector $\bar{\beta}_n$ satisfying the conditions (1) has an associated vector in $W_1 \cap W_2$ and the totality of these associated vectors is a spanning set for $W_1 \cap W_2$.

References

1. Evar D. Nering, Linear Algebra and Matrix Theory, Wiley, New York, 1970, p. 72.
2. Garrett Birkhoff and Saunders MacLane, A Survey of Modern Algebra, Macmillan, New York, 1960, p. 173.

AIR FORCE INSTITUTE OF TECHNOLOGY, WRIGHT-PATTERSON AIR FORCE BASE, OHIO 45433.

NILPOTENT TRANSFORMATIONS AND THE DECOMPOSITION OF A VECTOR SPACE

A. J. INSEL

The object of this paper is to outline what the author believes to be a new and more 'intuitive' proof of the theorem concerning the decomposition of a finite dimensional vector space into a direct sum of cyclic subspaces, with respect to a given nilpotent linear transformation. Typical proofs are to be found in [**1**, p. 161] and [**2**, p. 111]. A typical formulation of the theorem is:

THEOREM. *Let V be a finite dimensional vector space, and let T be a linear transformation on V, nilpotent of index p. Then there exist elements $x_1, x_2, \cdots, x_m$ in V and integers $p = p_1 \geqq p_2 \geqq \cdots \geqq p_m \geqq 1$, such that*

$$T^{p_1}(x_1) = T^{p_2}(x_2) = \cdots = T^{p_m}(x_m) = 0,$$

and

$$\begin{aligned} x_1 &= T^0(x_1),\ T(x_1),\ T^2(x_1), \cdots, T^{p_1-1}(x_1), \\ x_2 &= T^0(x_2),\ T(x_2),\ T^2(x_2), \cdots, T^{p_2-1}(x_2), \\ &\cdots\cdots\cdots\cdots\cdots \\ x_m &= T^0(x_m),\ T(x_m),\ T^2(x_m), \cdots, T^{p_m-1}(x_m) \end{aligned}$$

is a basis for V (allowing the convention that $T^0 = I$, the identity).

We propose to give an outline of our proof to the above theorem, leaving a few

of the details to be filled in by the reader. These unproved details will be stated in the form of lemmas. When presenting this proof to a linear algebra class the author asked his class to prove these lemmas as exercises. For the rest of this note we let R_T denote the range and N_T the null space of T.

LEMMA 1. *Let T be a nilpotent transformation on a vector space V. Suppose $x_1, x_2, \ldots, x_m$ are elements of V and $p_1, p_2, \cdots, p_m$ are nonnegative integers such that $\{T^{p_1}(x_1), T^{p_2}(x_2), \cdots, T^{p_m}(x_m)\}$ is linearly independent and lies in N_T. Then*

$$\{x_1, T(x_1), T^2(x_1), \cdots, T^{p_1}(x_1), x_2, T(x_2), \cdots, x_m, T(x_m), \cdots, T^{p_m}(x_m)\}$$

is also linearly independent.

Proof. Let $x_1, x_2, \cdots, x_m$ and $p_1, p_2, \cdots, p_m$ be as in the statement of the lemma. Let $p = \max\{p_1, p_2, \cdots, p_m\}$. We prove the lemma by mathematical induction on p. If $p = 0$ there is nothing to prove. Suppose the lemma holds whenever $p < q$, where q is some positive integer. Now suppose $q = \max\{p_1, p_2, \cdots, p_m\}$.

Suppose for some linear combination:

$$\sum_{i=1}^{m} \sum_{j=0}^{p_i} a_{ij} T^j(x_i) = 0. \tag{1}$$

Applying T to both sides of equation (1) we obtain:

$$0 = T(0) = \sum_{i=1}^{m} \sum_{j=0}^{p_i} a_{ij} T^{j+1}(x_i), \tag{2}$$

and hence

$$0 = \sum_{i=1}^{m} \sum_{j=0}^{p_i-1} a_{ij} T^j(y_i), \tag{3}$$

where $y_i = T(x_i)$. Applying the induction hypothesis to (3) we conclude that $a_{ij} = 0$ for $1 \leqq i \leqq m$ and $0 \leqq j < p_i$. Hence equation (1) reduces to

$$\sum_{i=1}^{m} a_{ip_i} T^{p_i}(x_i) = 0. \tag{4}$$

By hypothesis (4) implies $a_{ip_i} = 0$ for $1 \leqq i \leqq m$. We conclude that $a_{ij} = 0$ for $1 \leqq i \leqq m$ and $0 \leqq j \leqq p_i$. Hence the lemma follows.

We now proceed with our proof of the theorem, inserting additional lemmas when called for. The proof involves mathematical induction on the index of nilpotency of T. Trivially, the theorem follows when this index has value 1. Suppose the theorem is true when the index of T is $p - 1$ for some integer $p \geqq 2$. Now suppose T has index p. Then R_T is T-invariant, and the restriction $T \mid R_T$ of T to the range of T is nilpotent of index $p - 1$. Hence by the induction hypothesis there exist elements $y_1, y_2, \cdots, y_r$ in R_T and integers $q_1, q_2, \cdots, q_r$ such that $p - 1 \geqq q_1 \geqq q_2 \geqq \cdots \geqq q_r \geqq 1$ and such that the appropriate statement of the theorem (using

y's instead of x's, and q's instead of p's) is satisfied. The set we get is a basis for R_T. Let us denote this basis by A. We now state our second lemma without proof.

LEMMA 2. *$T^{q_1-1}(y_1), T^{q_2-1}(y_2), \cdots, T^{q_r-1}(y_r)$ is a basis for $R_T \cap N_T$.*

Since this set is linearly independent and lies in N_T it can be extended to a basis for all of N_T by the addition of say, a finite set $\{z_1, z_2, \cdots, z_s\}$. By Lemma 1, the union, $A \cup \{z_1, z_2, \cdots, z_s\}$ is linearly independent. In fact, it is easy to show that:

LEMMA 3. *The set $A \cup \{z_1, z_2, \cdots, z_s\}$ is a basis for $R_T + N_T$.*

We are now ready to put the pieces together. Suppose that n is the dimension of V. Then the following formulae are known from earlier in our linear algebra course:

(1) $\dim(R_T + N_T) + \dim(R_T \cap N_T) = \dim(R_T) + \dim(N_T)$, and

(2) $\dim(R_T) + \dim(N_T) = \dim V = n$.

Combining (1) and (2) leads to

(3) $\dim(R_T + N_T) + \dim(R_T \cap N_T) = n$.

Based on our above notation, $\dim(R_T \cap N_T) = r$ and hence

(4) $\dim(R_T + N_T) = n - r$.

Lemma 3 and (4) tell us that we need only adjoin r elements to $A \cup \{z_1, z_2, \cdots, z_s\}$ in the right way to make a basis for all of V. The problem is, how to choose these elements. The answer (which a few of the better students can guess) is: Since $y_1, y_2, \cdots, y_r$ all lie in R_T we can and do choose elements $x_1, x_2, \cdots, x_r$ in V such that $T(x_i) = y_i$ for $i = 1, 2, \cdots, r$. Then by Lemma 1, $\{x_1, x_2, \cdots, x_r\} \cup A \cup \{z_1, z_2, \cdots, z_s\}$ is linearly independent. This set contains r more elements than a basis for $R_T + N_T$ and hence must be a basis for all of V. Letting $m = r + s$, this basis can easily be put in the form given in the statement of the theorem, allowing

$$x_{r+1} = z_1, \cdots, x_m = z_s, p_1 = q_1 + 1, p_2 = q_2 + 1, \cdots, p_r = q_r + 1, p_{r+1} = \cdots = p_m = 1.$$

Naturally, s may be zero. This proves the theorem.

One more note: By a simple modification of the above proof, the decomposition theorem as stated in the sharper form in [**1**] can also be proved.

References

1. D. T. Finkbeiner II, Introduction to Matrices and Linear Transformations, 2nd ed., Freeman, San Francisco, 1966.

2. P. R. Halmos, Finite-Dimensional Vector Spaces, 2nd ed., Van Nostrand, Princeton, 1958.

MATHEMATICS DEPARTMENT, ILLINOIS STATE UNIVERSITY, NORMAL, IL 61761.

PROBLEMS AND SOLUTIONS

EDITED BY EMORY P. STARKE

All problems (both elementary and advanced) proposed for inclusion in this Department should be sent to E. P. Starke, 1000 Kensington Ave., Plainfield, NJ 07060. Proposers of problems are urged to enclose any solutions or information that will assist the editors. Ordinarily, problems in well-known textbooks and results in generally accessible sources are not appropriate for this Department. No solutions (except those accompanying proposals) should be sent to Professor Starke.

ELEMENTARY PROBLEMS

Solutions of Elementary Problems should be sent to Problems Group, Mathematics Department, University of Maine, Orono, ME 04473. To facilitate their consideration, solutions of Elementary Problems in this issue should be typed (with double spacing) and should be mailed before May 31, 1974.

An asterisk (*) means neither the proposer nor the editors supplied a solution.

E2456. *Proposed by H. H. Johnson, University of Washington*

Let C be a simple closed rectifiable curve in the plane, parameterized with respect to arc length s by the vector function $\mathbf{P}(s)$ $(0 \leqq s \leqq L)$. Assume that $\mathbf{P}$ has three continuous derivatives, and that the curvature $k(s)$ never vanishes. If $R(s) = 1/k(s)$ denotes the radius of curvature and if the area enclosed by C is A, show that

$$A \leqq \tfrac{1}{2}\int_0^L R(s)ds + \frac{L}{8}\int_0^L |R'(s)|\,ds.$$

For which curves does equality hold?

E2457. *Proposed by Don Redmond, University of Illinois*

Let $\tau(n)$ denote the number of divisors of the natural number n and let $\theta(n)$ denote the number of decompositions of n into two relatively prime factors. Suppose that f is a multiplicative function and that $g(n) = \sum_{d|n} f(d)$. Show that

$$\sum_{d^2|n} f(d)\tau\left(\frac{n}{d^2}\right) = \sum_{d^2|n} g(d)\,\theta\left(\frac{n}{d^2}\right).$$

E2458. *Proposed by E. O. Buchman, California State University at Fullerton*

An *n-place truth function* is a function of n variables whose arguments and

values come from the set $\{T, F\}$. Show that the number of n-place truth functions which are capable of generating all n-place truth functions is $2^t(2^t - 1)$, where $t = 2^{n-1} - 1$. (Remark: For $n = 2$, the two functions are "not or" and "not and," otherwise known as *Sheffer's stroke.*)

E2459*. *Proposed by A. A. Mullin, U.S. Army Research Office, Arlington, Virginia.*

One is given an unlimited number of perfect one ohm resistors with which to construct a resistance of π ohms to within an accuracy of 10^{-6} ohms. Only series-parallel circuits are allowed. What is the minimum number of resistors necessary?

E2460. *Proposed by D. Meyers and C. F. Pinzka, University of Cincinnati*

A *Pythagorean triple* is a solution of $x^2 + y^2 = z^2$ in natural numbers with $x < y$. Show that given any $n \geqq 0$, there exists a natural number k such that k appears in precisely n distinct triples.

E2461. *Proposed by Andreas Zachariou, University of Athens, Greece*

Let n and x be natural numbers such that x is divisible by only those primes which are larger than n. Show that

$$(x-1)(x^2-1)\cdots(x^{n-1}-1) \equiv 0 \pmod{n!}.$$

SOLUTIONS OF ELEMENTARY PROBLEMS

A Difficult Binomial Coefficient Summation

E2384* [1972, 1034]. *Proposed by H. W. Gould, West Virginia University*

Let n be a nonnegative integer. For $p = 1, 2, \cdots$ define

$$S_p(n) = \sum_{k=0}^{[n/2]} \left[\binom{n}{k} - \binom{n}{k-1}\right]^p,$$

where we make the usual conventions regarding binomial coefficients. Show that $S_3(n)$ is always divisible by $S_1(n)$.

Comment by the Editors. To date, no solution to this problem has been received. This did not totally surprise the editors, for a problem involving binomial coefficients submitted without solution by H. W. Gould is likely to be difficult indeed. The following table of numerical results may be of use to prospective solvers.

n	$S_1(n)$	$S_3(n)$	$S_3(n)/S_1(n)$
1	1	1	1
2	2	2	1
3	3	9	3
4	6	36	6

n	$S_1(n)$	$S_3(n)$	$S_3(n)/S_1(n)$
5	10	190	19
6	20	980	49
7	35	5705	163
8	70	33040	472
9	126	204876	1626
10	252	1268568	5034
11	462	8209278	17769
12	924	53105976	57474
13	1716	354331692	206487
14	3432	2364239592	688881
15	6435	16140234825	2508195
16	12870	110206067400	8563020

A Limit which is the Geometric Mean

E 2389 [1972, 1135]. *Proposed by Zbigniew Fiedorowicz, Illinois Institute of Technology*

Suppose that f is a strictly positive continuous function on the interval $[0,1]$. Show that the following (two-sided) limit exists and find its value:

$$\lim_{\alpha\to 0}\left\{\int_0^1 [f(x)]^\alpha dx\right\}^{1/\alpha}.$$

Can this result be generalized to a wider class of functions?

I. *Solution by Manny Yothers, Lower Stillwater College.* Let

$$F(\alpha) = \int_0^1 [f(x)]^\alpha dx.$$

We shall show that

$$\lim_{\alpha\to 0}\frac{\log F(\alpha)}{\alpha} = \int_0^1 \log f(x)dx,$$

so by the continuity of the exponential function, it will follow that the required limit is

$$\lim_{\alpha\to 0}[F(\alpha)]^{1/\alpha} = \exp\left\{\int_0^1 \log f(x)dx\right\}.$$

Since f is continuous and positive on $[0,1]$, we see that it is both bounded and bounded away from zero; that is, there exist constants m, M such that $0 < m \leqq f(x) \leqq M$ for all $x \in [0,1]$. From this, we have that

$$g(x,\alpha) = [f(x)]^\alpha \text{ and } D_2 g(x,\alpha) = [f(x)]^\alpha \log f(x)$$

are continuous functions on the rectangle $[0,1] \times [-a, a]$, where $a > 0$ is arbitrary. Because of this, F is differentiable at $\alpha = 0$ and differentiation under the integral sign is valid. (See Theorem 10.8, p. 350 of T. M. Apostol, *Calculus*, Vol. II (2nd Ed.), Xerox, 1969.) Let $G(\alpha) = \log F(\alpha)$; then $F(0) = 1$, $G(0) = 0$ and

$$\lim_{\alpha \to 0} \frac{G(\alpha)}{\alpha} = \lim_{\alpha \to 0} \frac{G(\alpha) - G(0)}{\alpha} = G'(0) = \frac{F'(0)}{F(0)} = \int_0^1 \log f(x)\, dx.$$

II. *Solution by Gerald Leibowitz, University of Connecticut.* A well-known result of F. Riesz asserts that if μ is a probability measure and f is a μ-measurable function which is nonnegative almost everywhere, then

$$\text{(1)} \qquad \lim_{p \to 0^+} \left\{ \int f^p d\mu \right\}^{1/p} = \exp \left\{ \int \log f\, d\mu \right\},$$

the geometric mean of f, provided that the indicated integral exists for at least one positive value of p. (See, e.g., Andrew Browder, *Introduction to Function Algebras*, W. A. Benjamin, 1969, pp. 124–126.) If both f and $1/f$ have some positive power which is μ-integrable, then an application of (1) to each yields

$$\lim_{\alpha \to 0} \left\{ \int f^\alpha d\mu \right\}^{1/\alpha} = \exp \left\{ \int \log f\, d\mu \right\}.$$

Also solved by Erol Barbut & James Calvert, Ellen Hertz, L. Kuipers, O. P. Lossers (Netherlands), Jürg Rätz (Switzerland), Kenneth Rosen, William Sánchez, T. T. Trent, W. R. Ugolik, Paul Zwier, and the proposer.

Editor's comment. The result appears to be fairly widely known. Barbut & Calvert, Rosen, and Sánchez all refer to G. Pólya and G. Szegö, *Problems and Theorems in Analysis*, Vol. I, Springer-Verlag, 1972; the problem appears on p. 69 and its solution on p. 250. Rätz calls attention to Hardy, Littlewood and Pólya, *Inequalities*, Cambridge, 1934, pp. 134, ff. This latter reference has an especially complete discussion of the various degenerate cases that can arise (for example, if $f(x) = 0$ identically). Note that in Solution I, the continuity of f was necessary to justify the differentiation under the integral sign, so that this proof does not immediately generalize to more general circumstances.

$2^p + 3^p = s^k$ and $2^m + 3^n = t^2$

E 2396 [1973, 76]. *Proposed by Erwin Just, Bronx Community College*

(A) Prove that $2^p + 3^p$ is not a perfect power if p is prime.

(B) Find all natural numbers m and n such that $2^m + 3^n$ is a perfect square.

Solution by R. S. Stacy, Manzano High School, Albuquerque, New Mexico. (A) If $p = 2$, then $2^2 + 3^2 = 13$ is not a perfect power. Suppose that p is odd; then

$$2^p + 3^p = (2+3) \sum_{k=0}^{p-1} (-1)^k 2^{p-1-k} 3^k,$$

and since $3 \equiv -2 \pmod 5$, the sum is congruent to $\sum_{k=0}^{p-1} 2^{p-1} = 2^{p-1}p \pmod 5$. If $p \neq 5$, then $2^p + 3^p = 5k$, where $k \not\equiv 0 \pmod 5$, so that $2^p + 3^p$ is not a perfect power. However $2^5 + 3^5 = 275$ is obviously not a perfect power.

(B) Suppose that $2^m + 3^n = t^2$. We note that $t^2 \equiv 0$ or $t^2 \equiv 1 \pmod 3$ for all t, so that m cannot be odd, since $2^m \equiv 2 \pmod 3$ for all odd m. In particular, $m \geqq 2$. But t^2 is odd and, being an odd square, must satisfy $t^2 \equiv 1 \pmod 4$. Since $2^m \equiv 0 \pmod 4$, it follows that $3^n \equiv 1 \pmod 4$ so that necessarily n is even, say $n = 2q$. Then $2^m = (t + 3^q)(t - 3^q)$. Since the sum of the two factors is $2t$, and t is odd, the highest power of 2 which can divide both factors is 2 itself. Therefore $t - 3^q = 2$ and $t + 3^q = 2^{m-1}$. Subtracting the first of these equations from the second and dividing by 2 shows that $3^q + 1 = 2^{m-2}$. However, m is even, and obviously $m \neq 2$, so that $3^q + 1 \equiv 0 \pmod 4$ from which it follows that q is odd. If q were greater than 1, we would have

$$3^q + 1 = (3 + 1) \sum_{k=0}^{q-1} (-1)^k 3^{q-1-k},$$

which is 4 times an odd number greater than 1. This is impossible, so that $q = 1$, and the unique solution is $m = 4$, $n = 2$ with $5^2 = 2^4 + 3^2$.

Also solved by I. K. Abruob, Anders Bager (Denmark), Merrill Barnebey, L. C. Bourburgh, Robert Breusch, N. J. Fine, J. D. Gillam, Robert Gilmer, S. H. Greene, M. G. Greening (Australia), J. L. Hunsucker & Jack Nebb, Ralph Jones, V. Linis, O. P. Lossers (Netherlands), D. C. B. Marsh, L. E. Mattics, H. F. Mattson, Jr., C. C. Oursler, G. D. Pritchett, Bernardo Recaman S. (Colombia), St. Olaf Problem Group, Ben Sapolsky, F. W. Saunders, Ken Schilling, M. A. Smith & D. R. Stone, Joel Spencer, D. P. Sumner, Phil Tracy, E. W. Trost (Switzerland), and the proposer. Partial solutions by E. J. Howard and by Carolyn MacDonald.

Affine Transformations with No Invariant Points or Lines

E 2399 [1973, 202]. *Proposed by D. E. Daykin, University of Reading, England*

Let T be an affine transformation of the plane; that is, $Tx = Ax + b$, where A is a 2×2 real matrix and b is a fixed vector. Characterize those affine transformations which have no fixed points and for which $TL \subseteq L$ for no line L.

I. *Solution by D. M. Bloom, Brooklyn College.* For any basis $\{u, v\}$ of the plane and any vector b not belonging to the line $\mathbb{R}u$, the mapping

$$(*) \qquad T\colon cu + dv \mapsto (c + d)u + dv + b$$

has the desired properties. (To prove this, show that for any vector w, the vectors $Tw - w$ and $T^2w - w$ are independent.) To show that there are no other such mappings, let $Tx = Ax + b$ be such a mapping. Since the equation $x = Ax + b$ has no solution, the matrix $A - I$ is singular; since the line $\mathbb{R}b$ is not T-invariant it follows that $A \neq I$. Hence $A - I$ has rank 1 and its Jordan form is either

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 0 & 0 \\ 0 & a \end{pmatrix}, \text{ where } a \neq 0.$$

Let $\{\boldsymbol{u}, \boldsymbol{v}\}$ be the corresponding Jordan basis. In the first case, T has the required form (*), and since the line $\mathbb{R}\boldsymbol{u}$ is not T-invariant, we have $\boldsymbol{b} \notin \mathbb{R}\boldsymbol{u}$. The second case is impossible, for if we write $\boldsymbol{b} = r\boldsymbol{u} + s\boldsymbol{v}$, then the line $\mathbb{R}\boldsymbol{u} - (s/a)\boldsymbol{v}$ is T-invariant.

We remark that, geometrically, the mapping (*) is a shear followed by a translation not in the direction of the shear.

II. *Solution by T. S. Bolis, State University College, Oneonta, New York.* The transformation T has the required property if and only if $(A - I)^2 = O$ and $A\boldsymbol{b} \neq \boldsymbol{b}$, where I denotes the identity matrix.

To prove that these conditions are sufficient, let T satisfy them and consider an arbitrary line $L = \mathbb{R}\boldsymbol{a} + \boldsymbol{c}$, where $\boldsymbol{a}$ and $\boldsymbol{c}$ are arbitrary vectors ($\boldsymbol{a} \neq \boldsymbol{0}$). Suppose to the contrary that $TL \subseteq L$. Then $TL = \mathbb{R}(A\boldsymbol{a}) + A\boldsymbol{c} + \boldsymbol{b} \subseteq L$, implying that $A\boldsymbol{a} = r\boldsymbol{a}$ and $(A - I)\boldsymbol{c} = s\boldsymbol{a} - \boldsymbol{b}$ for some $r, s \in \mathbb{R}$. Since $(A - I)^2 = O$, we get $r = 1$ (since $\boldsymbol{a} \neq \boldsymbol{0}$) and thus $\boldsymbol{0} = (A - I)^2\boldsymbol{c} = -(A - I)\boldsymbol{b}$, implying that $A\boldsymbol{b} = \boldsymbol{b}$, a contradiction. This same contradiction is reached if $\boldsymbol{a} = \boldsymbol{0}$, so that the argument shows also that T has no fixed points.

Conversely, suppose that $T\boldsymbol{x} = A\boldsymbol{x} + \boldsymbol{b}$ leaves invariant no line or point. Certainly $\boldsymbol{b} \neq \boldsymbol{0}$ and $A\boldsymbol{b} \neq \boldsymbol{b}$ since otherwise the line $\mathbb{R}\boldsymbol{b}$ would be invariant under T. Since T has no fixed points, we have $(A - I)\boldsymbol{x} = -\boldsymbol{b}$ satisfied for no vector $\boldsymbol{x}$ so that $A - I$ is singular, i.e., 1 is an eigenvalue of A. If the eigenvalue 1 is of multiplicity 2, then $(A - I)^2 = O$ by the Cayley-Hamilton Theorem and we are through, so suppose to the contrary that A has another eigenvalue $t \neq 1$. Let $\boldsymbol{a}$ and $\boldsymbol{c}$ be eigenvectors of A corresponding to the eigenvalues 1 and t respectively. Then $\boldsymbol{a}$ and $\boldsymbol{c}$ span the plane so that $\boldsymbol{b} = r\boldsymbol{a} + s(1 - t)\boldsymbol{c}$ for some $r, s \in \mathbb{R}$. Consider the line $L = \mathbb{R}\boldsymbol{a} + s\boldsymbol{c}$. Then $TL = \mathbb{R}\boldsymbol{a} + st\boldsymbol{c} + \boldsymbol{b} = \mathbb{R}\boldsymbol{a} + r\boldsymbol{a} + s\boldsymbol{c} = \mathbb{R}\boldsymbol{a} + s\boldsymbol{c} = L$, so that L is invariant under T. This contradiction shows that no such eigenvalue can exist.

We note that the conditions $(A - I)^2 = O$ and $A\boldsymbol{b} \neq \boldsymbol{b}$ hold if and only if A is of the form

$$A = \begin{pmatrix} 1 + r & s \\ t & 1 - r \end{pmatrix},$$

where $r^2 + st = 0$ and s, t are not both 0 and where $\boldsymbol{b}$ is not on the line $tx - ry = rx + sy = 0$. (Since s, t are not both 0, at least one of these expressions is non-trivial.)

Also solved by D. Ž. Djoković, M. J. Hoffman, Byrl Loch, Robert Lumbert, Jürg Rätz (Switzerland), Earl Taft, Mark Vigder, and the proposer.

Editor's comment. Taft obtains generalizations applicable to finite-dimensional spaces over arbitrary fields. The proposer comments that the problem arose in attempting to generalize the

Banach Fixed Point Theorem in the plane to a wider class of mappings; we suggest that readers wanting further details contact the proposer directly.

A Curious Triangle Ratio

E 2400 [1973, 202]. *Proposed by A. W. Walker, Toronto, Canada*

If O, H, I, r, R are the circumcenter, orthocenter, incenter, inradius and circumradius of any scalene triangle T, and P (defined as a limit point if T is right-angled) is the orthocenter of the pedal triangle of H, then the line OI divides the line segment PH internally as $r:R$.

Solution by M. G. Greening, The University of New South Wales, Australia. Set $x = \cos A$, $y = \cos B$, $z = \cos C$; then, using the identity $\cos(B-C) = \cos A + 2\cos B \cos C$, the trilinear coordinates of O, I, H, P can be taken as:

$$O: (Rx, Ry, Rz), \qquad I: (r, r, r), \qquad H: (2Ryz, 2Rzx, 2Rxy),$$

$$P: [R(1-2x^2)(x+2yz), R(1-2y^2)(y+2zx), R(1-2z^2)(z+2xy)].$$

The α-coordinate of the point T dividing PH internally in the ratio $r: R$ is

$$R^2(R+r)^{-1}[(1-2x^2)(x+2yz)+2(x+y+z-1)yz]$$

since $r = R(x+y+z-1)$. Cyclic interchanges of x, y, z give the β- and γ-coordinates of T. As the equation of OI is $\Sigma\,(z-y)\alpha = 0$, T lies on OI if

$$\Sigma\,[(1-2x^2)(x+2yz)(z-y)+2(x+y+z-1)(yz)(z-y)] = 0,$$

the summation being over the cyclic interchange of x, y, z. But the left hand side reduces successively to

$$-2\,\Sigma\, x^3(z-y) - 4xyz\,\Sigma\, x(z-y) + \Sigma\, x(z-y) + 2(x+y+z)\,\Sigma\, yz(z-y)$$

$$= 2[-\Sigma\, x^3(z-y) + (x+y+z)\,\Sigma\, yz(z-y)] = 0,$$

and the result is established.

Also solved by Huseyin Demir (Turkey), and the proposer.

ADVANCED PROBLEMS

All solutions of Advanced Problems should be sent to J. Barlaz, Rutgers — The State University, New Brunswick, N. J. 08903. Solutions of Advanced Problems in this issue should be typed (with double spacing) on separate, signed sheets and should be mailed before May 31, 1974.

An asterisk (*) means neither the proposer nor the editors supplied a solution.

5952*. *Proposed by J. Gilles, University of Charleroi, Belgium*

Prove the following results:

$$\sum_{n=1}^{\infty} \frac{1}{n^3 \sin n\pi\sqrt{2}} = \frac{13\pi^3}{360\sqrt{2}}, \tag{3}$$

$$\Gamma\left(\frac{8}{9}\right) = \frac{9-\sqrt{14}+\sqrt{75-32\sqrt{3}}}{33}. \tag{4}$$

5953*. *Proposed by Paul Erdös, Northern Illinois University, and G. B. Purdy, University of Illinois*

Let G be a planar graph of n vertices $x_1, x_2, \cdots, x_n$. Let $v(x_i)$ be the valency (or degree) of x_i and let

$$v(x_1) \geqq v(x_2) \geqq \cdots \geqq v(x_n).$$

Determine (or estimate) $\sum_{i=1}^{k} v(x_i)$.

5954. *Proposed by C. W. Anderson, University of California at Berkeley*

Is the collection of Lebesgue non-measurable subsets of $[0,1]$ a first category (null) subset of 2^C?

5955*. *Proposed by F. D. Hammer, Berkeley, California*

Is there a differentiable function which takes rationals into rationals but whose derivative takes rationals into irrationals?

5956*. *Proposed by G. J. Michaelides, University of South Florida*

Let Z^+ denote the set of positive integers, with the topology whose basic open sets are sequences $\{an+b\}_{n=0}^{\infty}$ and $(a,b)=1$. It is known that Z^+ with the above topology is Hausdorff. Is there a homeomorphism from Z^+ to Z^+ other than the identity?

5957. *Proposed by Dietrich Marsal, Hannover, Germany*

If the sequence of partial sums

$$f_n(x) = \phi_1(x)\int_a^x \phi_1(t)dt + \cdots + \phi_n(x)\int_a^x \phi_n(t)dt \quad (n = 1, 2, \cdots)$$

of the orthonormal system $\{\phi\} \in L^2[a,b]$ is uniformly bounded and convergent almost everywhere, then $\{\phi\}$ is complete if and only if $\lim_{n\to\infty} f_n(x) = \frac{1}{2}$ on $a \leqq x \leqq b$ almost everywhere.

SOLUTIONS OF ADVANCED PROBLEMS

$\Gamma^{(n)}(1)$

5878 [1972, 1041]. *Proposed by Václav Konečný, Jarvis Christian College, Hawkins, Texas*

Show that

$$\int_0^\infty e^{-x}\ln^2 x\,dx = \gamma^2 + \frac{\pi^2}{6} \text{ and } K - 5/2 < \int_0^\infty e^{-x}\ln^3 x\,dx < K - 9/4,$$

where $K = -\gamma(\gamma^2 + \pi^2/2)$ and γ is Euler's constant.

Solution by Allen Stenger, Student, Pennsylvania State University. We know $\Gamma(z) = \int_0^\infty e^{-x}x^{z-1}dx$, so the integrals we are interested in are

$$\int_0^\infty e^{-x}\log^2 x\,dx = \Gamma''(1) \text{ and } \int_0^\infty e^{-x}\log^3 x\,dx = \Gamma'''(1).$$

From

$$\log\Gamma(z) = -\gamma z - \log z + \sum_{n=1}^\infty \left\{\frac{z}{n} - \log\left(1 + \frac{z}{n}\right)\right\}$$

we get

$$\frac{\Gamma'(z)}{\Gamma(z)} = -\gamma - \frac{1}{z} + \sum_{n=1}^\infty \left(\frac{1}{n} - \frac{1}{n+z}\right),\ \Gamma'(1) = -\gamma,$$

$$\frac{\Gamma(z)\Gamma''(z) - (\Gamma'(z))^2}{(\Gamma(z))^2} = \frac{1}{z^2} + \sum_{n=1}^\infty \frac{1}{(n+z)^2},$$

whence $\Gamma''(1) = \gamma^2 + \pi^2/6$. Similarly, we can differentiate again and solve for $\Gamma'''(1)$, to get $\Gamma'''(1) = -\gamma(\gamma^2 + \pi^2/2) - 2\zeta(3)$. The final inequalities result from

$$\frac{9}{8} = 1 + \frac{1}{2^3} < \zeta(3) < 1 + \frac{1}{2^3} + \int_2^\infty \frac{dx}{x^3} = \frac{5}{4}.$$

The process could be continued and $\int_0^\infty e^{-x}\log^n x\,dx$ $(n = 4, 5, \cdots)$ evaluated in terms of γ, $\zeta(2)$, $\zeta(3), \cdots, \zeta(n)$.

Also solved by Dorothee Aeppli, Günter Bach (Germany), M. G. Beumer (Netherlands), Dieter Bode (Germany), Peter Enis, Ralph Garfield, M. L. Glasser, M. G. Greening (Australia), Sidney Heller, R. A. Hurd, A. A. Jagers (Netherlands), Charlotte Krauthamer (Austria), O. P. Lossers (Netherlands), Alexander Lupas (Roumania), Ram Murty & Kumar Murty, K. R. Penrose, G. S. Rogers, Kenneth Rosen, Hermann Schmidt (Germany), David Shelupsky, F. W. Steutel (Netherlands), N. M. Temme (Netherlands), Phil Tracy, P. H. Young, and the proposer.

Note. Beumer, Glasser, the Murtys, and Shelupsky all note that the solution appears in a generalized form in M. E. Levenson, *A recursion formula for* $\int_0^\infty e^{-t}(\log t)^{n+1}\,dt$, this MONTHLY, 65 (1965) 695.

Condition for a Quadratic

5880 [1972, 1043]. *Proposed by Anon, Erewhon-upon-Yarkon*

Let $f(x)$ be a continuous function on $a < x < b$ such that $f'(x)$ exists at each point. Suppose for each x in this interval there exists a $\delta = \delta_x > 0$ such that

$$\frac{f(x+h) - f(x-h)}{2h} = g(x) \tag{1}$$

for all h satisfying $0 < h < \delta$. Prove that $f(x)$ is a quadratic polynomial. (This generalizes a problem in T. M. Flett, *Mathematical Analysis*, where f''' is assumed to exist.)

Solution by Douglas B. Price, Newport News, Virginia. By adding and subtracting $f(x)$ to the left hand side of (1), and taking the limit as h approaches zero, we see that for every x in (a,b), $g(x) = f'(x)$. Pick x in (a,b) and h such that $0 < h < \delta_x$, and let x' be such that

$$x < x' < \min\{x + \delta_x - h, x + \delta_{x-h}\}. \tag{2}$$

Then we have

$$\frac{f(x'+h) - f(x'-h)}{2h} = \frac{f(x + (x'-x) + h) - f(x - (x'-x) - h)}{2h}$$

$$+ \frac{(x'-x)}{h} \frac{f((x-h) - (x'-x)) - f((x-h) + (x'-x))}{2(x'-x)},$$

$$g(x') = g(x) \cdot \frac{(x'-x+h)}{h} - g(x-h) \cdot \frac{(x'-x)}{h}.$$

Therefore, for x' in the interval defined by (2), $g(x') = f'(x')$ is a linear function of x', hence f must be a quadratic polynomial there.

Let x be in (a,b) and suppose that we have shown that for x' in some interval extending to the right of x, $f(x') = Ax'^2 + Bx' + C$. We would like to show that this same polynomial representation applies for all x' between x and b. If this were not the case, then there would be a point y in (x,b) such that for $x < x' < y$, $f(x') = Ax'^2 + Bx' + C$, but for x' in some interval to the right of y, $f(x') = A'x'^2 + B'x' + C'$, $((A,B,C) \neq (A',B',C'))$. Then for h sufficiently small, we have

$$g(y) = \frac{f(y+h) - f(y-h)}{2h} = \frac{A'(y+h)^2 + B'(y+h) + C' - A(y-h)^2 - B(y-h) - C}{2h},$$

$$g(y) = \frac{(A'-A)h}{2} + \frac{((A'-A)y^2 + (B'-B)y + (C'-C))}{2}\frac{1}{h} \tag{3}$$

$$+ \frac{2(A'+A)y + B' + B}{2}.$$

Since equation (3) holds for h in some interval, and $g(y)$ is constant, the coefficients of the first two terms in the right hand side of (3) must be zero. That is, $A' = A$ and $(B'-B)y + C' - C = 0$. But by looking at left and right derivatives of f at y, we see that $2Ay + B = 2A'y + B'$, so $B = B'$ and $C = C'$. Thus for any x in

(a,b), f is a quadratic polynomial on (x,b). Clearly this can happen only if f is a quadratic polynomial on all of (a,b).

Also solved by A. S. Adikesavan (India), R. G. Bilyeu, D. Borwein, R. O. Davies (England), M. G. Greening (Australia), Rev. William Habakkuk, R. D. Leitch (England), O. P. Lossers (Netherlands), Joel Levy, Brian Peterson, and John Swetits.

The Riesz Decomposition Property for Nonnegative Differentiable Functions

5882 [1962, 1044]. *Proposed by E. S. Langford, University of Maine*

Does the set of differentiable functions on the real line have the Riesz Decomposition Property? I.e., if f_1, f_2, and g are positive differentiable functions such that $f_1 + f_2 \geqq g \geqq 0$, can g be written as $g = g_1 + g_2$, where g_1 and g_2 are differentiable functions which satisfy $f_1 \geqq g_1 \geqq 0$ and $f_2 \geqq g_2 \geqq 0$?

Solution by Neal Felsinger, Yale University. We will assume f_1 and f_2 are nonnegative differentiable functions. Let $f = f_1 + f_2$ and define

$$g_i(x) = \begin{cases} g(x)f_i(x)/f(x) & \text{if } f(x) \neq 0, \\ 0 & \text{if } f(x) = 0 \end{cases}$$

for $i = 1,2$. Clearly $f_i \geqq g_i \geqq 0$ and g_i is differentiable whenever $f(x) \neq 0$. If $f(x_0) = 0$, we have $f_i'(x_0) = 0$ and if $f(x) \neq 0$, $0 \leqq g(x)/f(x) \leqq 1$. Thus

$$\lim_{x \to x_0} (g_i(x) - g_i(x_0))/(x - x_0) = \lim_{x \to x_0} (g(x)/f(x))(f_i(x)/(x - x_0)) = 0.$$

Therefore g_i is differentiable everywhere.

Also solved by K. F. Anderson, R. O. Davies (England), D. P. Giesy, A. A. Jagers (Netherlands), J. W. Shaw, Jr., R. M. Warten, A. C. Zaanen, and the proposer.

Editorial Note. Zaanen proves the property using the equivalent formulation that if $u_1 + u_2 = v_1 + v_2$, then there exist $w_{ij}(i,j=1,2)$ such that $w_{1j} + w_{2j} = v_j$, $w_{i1} + w_{i2} = u_i$, u, v, w, being nonnegative differentiable functions. He sets $w_{ij} = u_i v_j / w$ if $w \neq 0$, and $w_{ij} = 0$ if $w = 0$. He also notes that a reference for the proof is Exercises 15.12 and 15.14 in Luxemburg and Zaanen, *Riesz Spaces*, vol. 1, wherein the problem is considered for the partially ordered vector space of all functions $f(x) = p(x)/q(x)$, $a \leqq x \leqq b$, p and q are real polynomials, $q(x) > 0$. For this space the problem and solution may be found in F. Riesz, *Sur quelques notions fondamentales dans la théorie générale des opérations linéaires*, Annals of Math. 41 (1940), p. 174, ff.

The proposer notes that the question and answer were probably known by Riesz by 1928 and offers a reference to a review by P. F. Conrad (Math. Reviews, V.31, 4843) of the paper by L. Fuchs, *Riesz groups*, Ann. Scuola Norm. Sup. Pisa 3 (19) (1965), pp. 1–34. It is also noted in Fuchs, *Riesz vector spaces and Riesz algebras*, Queen's Papers in Pure and Applied Mathematics, Kingston, Ontario, 1966, that the polynomials, the k-times differentiable functions, and the rational functions all enjoy the Riesz Decomposition Property.

$\zeta(n)$, $\psi^{(n)}$, and an Infinite Series

5885 [1972, 1140]. *Proposed by F. Haring and G. T. Nelson, North Dakota State University*

(a) Show

$$1+\frac{1}{2^2}-\frac{1}{4^2}-\frac{1}{5^2}+\frac{1}{7^2}+\frac{1}{8^2}-\frac{1}{10^2}-\frac{1}{11^2}+\cdots$$
$$=-\frac{2}{27}\pi^2-2\int_0^1\frac{\ln x}{1+x^3}\,dx.$$

(b) Find the sum of the series.

Solution by Günter Bach, Braunschweig, Germany. More generally, we show for any integer $n \geqq 2$

$$\text{(a)}\qquad R_n = 1+\frac{1}{2^n}-\frac{1}{4^n}-\frac{1}{5^n}+\frac{1}{7^n}+\frac{1}{8^n}--++\cdots$$
$$=-\left(1-\frac{1}{2^{n-1}}-\frac{1}{3^n}+\frac{2}{6^n}\right)\zeta(n)+\frac{2(-1)^{n-1}}{(n-1)!}\int_0^1\frac{(\ln x)^{n-1}dx}{1+x^3}$$
$$=\left(1+\frac{1}{2^{n-1}}\right)\frac{(-1)^n}{(n-1)!}\left\{\frac{2}{3^n}\psi^{(n-1)}\left(\frac{1}{3}\right)-\left(1-\frac{1}{3^n}\right)\psi^{(n-1)}(1)\right\},$$

where $\zeta(n) = \sum_{k=0}^{\infty}(1+k)^{-n}$ is Riemann's Zeta-function, and

$$\text{(1)}\qquad \psi(z)=\frac{d\ln\Gamma(z)}{dz}=-\gamma+\sum_{k=0}^{\infty}\left(\frac{1}{k+1}-\frac{1}{z+k}\right),$$

the logarithmic derivative of the Gamma function, γ is Euler's constant.

(b) $R_2 = \frac{1}{3}\{\psi'(\frac{1}{3})-4\psi'(1)\} = 1.171953619345$ with 11 significant digits.

Proof. (a) Put

$$S_n(a,b)=\sum_{k=0}^{\infty}\frac{1}{(a+bk)^n},\quad J_n(\alpha)=\int_0^1\frac{(\ln x)^{nn-1}dx}{1+x^\alpha}$$

(a,α,n positive integers, $n \geqq 2$). It is easily seen that, for any integers t, $m \geqq 1$,

$$\text{(2)}\qquad S_n(ta,tb)=\frac{S_n(a,b)}{t^n},\quad \left(1-\frac{1}{m^n}\right)\zeta(n)=\sum_{\mu=1}^{m-1}S_n(\mu,m).$$

Now

$$\text{(3)}\qquad \begin{aligned} J_n(\alpha) &= (-1)^{n-1}(n-1)!\sum_{k=0}^{\infty}\frac{(-1)^k}{(1+\alpha k)^n}\\ &= (-1)^{n-1}(n-1)!\,\{S_n(1,2\alpha)-S_n(1+\alpha,2\alpha)\}\end{aligned}$$

by substituting $x = e^{-z}$, expanding the denominator of the integrand in an infinite series, and integrating termwise. Further

$R_n = S_n(1,6) + S_n(2,6) - S_n(4,6) - S_n(5,6)$, and using (2) and (3) we have

$$R_n + \left(1 - \frac{1}{6^n}\right)\zeta(n) - 2\frac{(-1)^{n-1}}{(n-1)!}J_n(3) = 2S_n(2,6) + 2S_n(4,6) + S_n(3,6).$$

But the right hand side equals

$$\frac{2}{2^n}\left(1 - \frac{1}{3^n}\right)\zeta(n) + \frac{1}{3^n}\left(1 - \frac{1}{2^n}\right)\zeta(n),$$

and therefore the first representation in (a) holds; from $\zeta(2) = \pi^2/6$ we have $R_2 = -2\pi^2/27 - 2J_2(3)$.

With $A_n = 1 - 1/2^n + 1/4^n - 1/5^n + 1/7^n - + \cdots$, we get at once

$$R_n = \left(1 + \frac{1}{2^{n-1}}\right)A_n. \tag{4}$$

Since $A_n = S_n(1,3) - S_n(2,3) = 2S_n(1,3) - (1 - 1/3^n)\zeta(n)$, it follows that

$$R_n = \left(1 + \frac{1}{2^{n-1}}\right)\left\{2S_n(1,3) - \left(1 - \frac{1}{3^n}\right)S_n(1,1)\right\}. \tag{5}$$

But from (1) we get $\psi^{(n-1)}(z) = (-1)^n(n-1)!\,\sum_{k=0}^{\infty} 1/(z+k)^n$ and consequently

$$S_n(a,b) = \frac{(-1)^n}{(n-1)!\,b^n}\psi^{(n-1)}\left(\frac{a}{b}\right). \tag{6}$$

Therefore the second representation in (a) holds.

(b) Put $n = 2$ in the second representation in (a) and we get

$$R_2 = \tfrac{1}{3}\psi'\left(\frac{1}{3}\right) - \frac{2\pi^2}{9}$$

which is evaluated using a tabulation of $\psi'(z)$ found, e.g., in Abramowitz and Stegun, *Handbook of Mathematical Functions.*

Also solved by Dieter Bode (Germany), T. S. Bolis, L. Carlitz, H. E. Fettis, Ralph Garfield, M. L. Glasser, M. G. Greening (Australia), Václav Konečný, G. T. Nelson, T. J. Osler, F. G. Schmitt, Jr., Daniel Shanks, Robert Spira, A. Zujus, and the proposer.

Note. Using a table in a paper in Vol. 17 (1963) of Mathematics of Computation, pp. 136–154, Shanks evaluates the series as

$$S = 1.1719536193\ 4472944530\ 0781144436.$$

REVIEWS

EDITED BY J. ARTHUR SEEBACH, JR. AND LYNN A. STEEN

with the assistance of the mathematics departments of St. Olaf and Carleton Colleges

COLLABORATING EDITOR FOR FILMS: SEYMOUR SCHUSTER, CARLETON COLLEGE

We invite readers to submit reviews of significant recent college-level mathematics books. We especially encourage reviews based on classroom use, or comparative reviews of several related books. Reviews should ordinarily not exceed two pages (per book) typed double spaced. Manuscripts of reviews as well as books submitted for review should be sent to: Book Review Editor, American Mathematical Monthly, St. Olaf College, Northfield, MN 55057.

Essential Mathematics: A Modern Approach. By Mervin L. Keedy, Marvin L. Bittinger. Addison-Wesley, Reading, Massachusetts, 1972. xvi + 641 pp. $7.95 (P). (Telegraphic Review, August-September 1972.)

Some mathematicians seem ashamed to teach remedial mathematics. It is a shame not to learn this material in the grades. But once the student is in college, the only sensible course is to try to teach him what he must know.

Essential Mathematics is an excellent remedial text on arithmetic, algebra, and trigonometry. I used it for two semesters at a four-year college.

The format is remarkable.There are tear-out exercises with special answer columns. Thus you need not hunt answers on a messy paper. You need not search the text to find what the question was, because it is right there on the exercise sheet. The teacher's manual has large, bold print, corresponding in layout to the exercises. The manual even includes graphs. Thus you can go through a stack of homework in record time.

The exercises are outstanding in content as well as form. Problems are realistic, as in the following poignant example: "A man earned $9600 one year. He received a 6% raise in salary, but the cost of living rose 5.4%. How much additional earning power did he actually get?" Each section has an ample number of problems, none recondite.

Exposition is clear and concise. Concepts proceed from the simple to the complex. Distributivity, associativity, and commutativity receive their just due, but the field axioms never appear in full to frighten weak students. No single section is burdened with too many ideas. For instance, there is a separate section for each method of solving quadratic equations: factoring, completing the square, and the quadratic formula.

The text has a few disadvantages. There is too much material for a two-semester course meeting three times a week. Since no sections are labelled optional, it is hard to decide what to exclude. I skipped determinants, but later sections had problems unexpectedly involving determinants. The slide rule is so interesting and useful that it belongs near the beginning of the course. But the excellent chapter on the slide rule appears at the end, dependent on previous material.

The binding is too flimsy for such a large book and will probably not last a year. The teacher might recommend reinforcement with tape. Warn the students not to use the tear-out exercises until they are sure to stay in the course.

The authors suggest a novel teaching method described by Carol Kipps ["Who's Committed? Who's Involved?" *The Two-Year College Mathematics Journal* 1 (1970), 32–35]. She lectures only five minutes; then the class works in groups. But I found that if I did not explain an idea thoroughly, then virtually nobody learned it. I had to lecture 20–30 minutes, including time for questions.

Regularly I assigned an exercise, collected it at the next session, and returned it at the following meeting. An assistant did much of the grading. With remedial students it is particularly important to supervise homework promptly and routinely. Thus the convenience of the exercises in this book is a tremendous advantage. It is a notable text.

ELIZABETH BERMAN, Rockhurst College

Biomathematics, V. 2: *Introduction to Mathematics for Life Scientists.* By Edward Batschelet. Springer-Verlag, New York, 1971. xiv + 495 pp. $15.60. (Telegraphic Review, October 1972.)

The mathematical boom of the 1960's saw much emphasis on mathematical rigor in freshman courses, and ε's and δ's tended to thrive at the expense of the more historical and applied aspects of the calculus. However, in the last few years there has been what may be regarded as a healthy tendency to de-emphasize rigor and to bring the elementary mathematics programs into line with what the average freshman at a given institution can digest and use. The present book attempts to present mathematics as an instrument of the life sciences, and in particular, of biology and medicine. It is an impressive attempt, but for various reasons the instructors using the book at this University during the past year found it difficult to adapt to.

Perhaps the main difficulty lies in the lack of mathematical content. Although most of the standard topics of freshman calculus are represented, they are for the most part treated superficially, the reason being supposedly that the readers are potential life scientists rather than mathematicians, and therefore can have little interest in seeing the mathematical development. Thus frequently a topic is presented at the beginning of a section with a minimum of mathematical justification, and the remainder of the section is devoted to examples couched in somewhat abstruse terminology and equipped with references and recommendations for further reading from a list of over 200 scientific books and articles. The same situation prevails in the exercises, where the emphasis seems placed more on familiarizing the reader with technical language from various fields than on amplifying and illustrating the mathematical concepts involved. The result is that the instructor invariably finds it necessary to provide examples and assignments from other sources.

Both differential and integral calculus are treated in two chapters around the middle of the book. Virtually no methods of integration (e.g., parts or substitution) are provided. The omission is justified with the statement: "For many life scientists it is hardly worth spending so much time on the technique of integration. If he has to evaluate integrals such as $\int \sqrt{1+x}\,dx$ or $\int x \sin x\,dx$, he may ask a mathematician for help." Likewise a three page section on polynomial approximation is all that is provided in the way of showing how one actually computes values of functions, and no reference to the remainder term is made. One tends to wonder if a scientist whose knowledge of mathematics is this superficial can ever be qualified to apply mathematics to his field in any meaningful way.

Other topics covered, besides the usual ones leading up to the calculus, are ordinary differential equations (where separation of the variables is the only method presented), functions of several variables, probability, and matrices and vectors. All these provide some examples for the personal enlightenment of the instructor but leave him with little that is teachable or assignable in a mathematics classroom. A final chapter on the complex numbers is included so as to treat the differential equation associated with an oscillatory system. Unfortunately the sum rules for sine and cosine, which have been used earlier in the book, are made to appear dependent on the unproven formulas of the chapter, leaving the reader who has not seen a direct treatment with the impression that these rules are of a difficulty beyond the scope of the book.

Perhaps it is a mistake in teaching mathematics at this level to treat any group of students as if they were already budding biologists, economists, chemists, or mathematicians. For the most part they tend rather to be young people one step away from secondary school with no clear idea of what they are going to be, and who, in order to be attracted to mathematics either as a tool or as an end in itself, require a simple and clear exposition of the basic ideas in a language which is as familiar and free from intimidating terminology as possible. One does not form a scientist simply by talking to a person as if he already were one.

Undoubtedly, a biologist would have a different attitude toward the book than that presented here, and perhaps a fairer appraisal would be given by him. However, one must not forget that the book is called "Mathematics for Life Scientists," and as such is intended as a source for learning mathematics. If, on the other hand, the title were "Life Sciences for the Mathematician," a different evaluation could be made. In that case, free from concern for one's obligation as a mathematics teacher to explain mathematics, one could speak of the enjoyment with which one read about such things as the mysterious appearance of the Fibonacci numbers in botany, the maximization problem which the bees seem to have solved through selection in constructing their cells, and the system of differential equations pertaining to the digestive system of a cow.

B. MITCHELL, Rutgers University

TELEGRAPHIC REVIEWS

Telegraphic reviews are designed to give prompt notice of new books with sufficient information to assist our readers in deciding whether to order an examination copy or to suggest library purchase. Possible uses are indicated as follows:

T = textbook P = professional reading
S = supplementary reading L = undergraduate library purchase
13 to 18 = freshman to second year graduate level usage
1 to 4 = appropriate time in semesters to cover text

Asterisks (*) or question marks (?) denote special positive or negative emphasis, respectively. Publishers are denoted by standard abbreviations; complete addresses may be found in *Books in Print*.

GENERAL, L, *Index to Translations Selected by the American Mathematical Society 1966-1973*. AMS, 1973, iii + 93 pp, $6.60. Covers *AMS Translations*, Series 2, Vol. 51-100, and *Selected Translations in Probability and Statistics*, Vols. 6-13. Includes both author and subject indices. LAS

GENERAL, T(13-14), L, *Mathematics, A Liberal Arts Approach*. Malcolm Graham. HarBrace J, 1973, x + 278 pp, $8.95. Math for *thinking* poets. Designed for non-math majors, this book covers more areas of mathematics than usual, including topology, elementary abstract algebra, and computers in addition to such more common topics as probability, logic, sets, and number systems. About the only omission is geometry and elementary linear algebra. Exercises are excellent, but no bibliography or references for further reading. If the instructor is willing to provide further references for curious students, this will be an excellent text. Also reasonably priced. PJM

GENERAL, P, *English-Greek Mathematical Dictionary*. C.P. Tzelekis. Athens, 1973, ix + 235 pp, (P). A nicely done dictionary with a rather specialized purpose: to translate modern English mathematics at the college and beginning graduate level into modern Greek. JAS

GENERAL, P, L, *Scientific Truth and Statistical Method*. Marcello Boldrini. Transl: Ruth Kendall. Hafner, 1972, xiv + 264 pp, $18.50. "Quid est veritas? ...This book is entirely concerned with Pilate's question and ends by accepting...the answer of Jesus." A serious reflection on the philosophy of science encompassing the relations of science with language, the nature of axioms, deduction and induction, probability, statistics, and the methodological structure of the natural sciences. PJC

BASIC, T(13: 1), *Arithmetic: Beginning Math for College Students*. Edward C. Ortell, Sanderson M. Smith. Holbrook Pr, 1973, xi + 227 pp, $9.95. A text written for the student who needs training in elementary arithmetic. LLK

PRECALCULUS, T(13: 1), *Precalculus Mathematics: Algebra, Trigonometry, and Analytic Geometry*. Margaret L. Lial, Charles D. Miller. Scott F, 1973, 419 pp, $10.50. A good selection of topics, with sufficient exercises and a few problems. The authors' claim is that rigor and sophistication increase but there is little in the text which enables the student to progress in his ability to interpret sophisticated problems. LLK

PRECALCULUS, T(13: 1), *Analytic Geometry.* Eugene D. Nichols, Robert Kalin. HR&W, 1972, x + 406 pp, $8.50 (P). A traditional approach to analytic geometry in the first 6 chapters, followed by vectors in 2 and 3 dimensions, solid analytic geometry and finally n-dimensional concepts. LLK

PRECALCULUS, T(13: 1), *Modern Algebra and Trigonometry, Third Edition.* Eldridge P. Vance. A-W, 1973, 436 pp, $10.75. Third edition modifications include more illustrations and problems. Chapters 1 and 2 could have been deleted. Surely some background can be assumed. LLK

PRECALCULUS, T(13: 1), *Elementary Functions: A Precalculus Primer.* Israel H. Rose. Scott F, 1973, 324 pp, $9.75. Derivatives and continuity are included on an intuitive level, but there is not enough work with the algebra necessary to handle the derivative when introduced in the next course. LLK

EDUCATION, P**, S*, *A Response to Managerial Education.* Ed: Gerald R. Rising, Burt A. Kaufman. A Special Issue of *Educational Technology*, Nov., 1973. Nine urgent exhortations concerning the ominous implications for education in general and for mathematics education in particular of the current emphasis on competency-based education using explicit behavioral and managerial objectives. "...where the principle is adopted that the only matters worth pursuing...are those which we can measure, then our objectives will be hopelessly restricted and biased..." Authors include Gerald Rising, Peter Hilton, Gail Young, Peter Braunfeld, Burt Kaufman and Vincent Haag.LAS

EDUCATION, S**, L*, *Topics for Mathematics Clubs.* Ed: LeRoy C. Dalton, Henry D. Snyder. NCTM, 1973, vi + 106 pp, $2.80 (P). 10 brief chapters giving background, examples, conjectures, theorems, projects and references for such topics as Fibonacci sequences, groups, topology, Boolean algebra, non-Euclidean geometry. An extremely handy resource. LAS

EDUCATION, P, L*, *Developments in Mathematical Education: Proceedings of the Second International Congress on Mathematical Education.* Ed: A.G. Howson. Cambridge U Pr, 1973, ix + 318 pp, $14.50. Reports and papers from the August, 1972 I.C.M.I. conference at Exeter, England, including contributions by Pólya, Piaget, Freudenthal, and Thom. LAS

EDUCATION, P, L, *Mathematics As An Educational Task.* Hans Freudenthal. Reidel, 1973, xii + 680 pp, $19.50 (P). A monumental personal philosophy of mathematics education which attempts to distinguish the fine line between the "letter that kills and the spirit that gives life." Essentially a rearrangement of previous papers--a few reprinted *verbatim*--into 19 somewhat repetitious chapters. In an appendix, the author develops his criticism of Piaget's educational philosophy and, more emphatically, of those who have misused Piaget's name. LAS

EDUCATION, L, *Conference on Graduate Training of Mathematics Teachers.* Canadian Math. Congress, 1972, v + 110 pp, (P). Eight brief essays plus panel discussion from a 1969 conference at Sir George Williams U. LAS

EDUCATION, P, *Beiträge zum Mathematikunterricht 1972*. Ed: Helmut Freund, Peter Sorger. Hermann Schroedel, 1973. *Teil 1*, 192 pp, DM 39,80; *Teil 2*, 224 pp, DM 44,80 (P). Proceedings of the April 1972 conference of educators in Kiel. 43 papers on the teaching of mathematics and the mathematics taught. Considerable mathematical content as well as pedagogical remarks that would be of interest through the college level in the U.S. JAS

HISTORY, *Hermann Minkowski Briefe an David Hilbert*. L. Rüdenberg, H. Zassenhaus. Springer-Verlag, 1973, 165 pp, $11.90 (P). One side--the other has not been found--of a correspondence lasting from 1885 to 1908. Some mathematics, some mathematical gossip. Price seems high. J.D.-B.

HISTORY, S, P, L***, *A Source Book in Classical Analysis*. Ed: Garrett Birkhoff. Harvard U Pr, 1973, xii + 470 pp, $25. 81 brief selections from the nineteenth century masters Cauchy, Jordan, Gauss, Abel, Fourier, Weierstrass, ..., Poincaré, Klein, Lie, each translated into English and arranged by topic into 13 chapters with introductory interpretive essays. A magnificent contribution to this distinguished series of *Source Books*. LAS

HISTORY, L**, *Lecons Sur L'Intégration*. Henri Lebesgue. Chelsea, 1973, xii + 340 pp, $9.50. An unabridged reprint on special alkaline paper of the 1928 second edition with minor corrections. (The first edition was published in 1903.) LAS

HISTORY, S, P*, L, *Probability and Statistical Inference in Ancient and Medieval Jewish Literature*. Nachum L. Rabinovitch. U of Toronto Pr, 1973, xiii + 205 pp, $12.50. Mathematical folklore falsely asserts that probability and statistical reasoning arose in gambling context in recent centuries. In fact, this book exhibits a history reaching to biblical times manifested in juridical problems of the Talmud. Talmudic sources and rabbinical commentaries considered all variety of modern conceptions of probability; formulated the arithmetic of probabilities, combinations and permutations; discussed related logical and philosophical ramification; and arrived at elementary decision-theoretic guidelines. PJC

HISTORY, P, *Mathematik Bei Den Juden*. Moritz Steinschneider. Georg Olms, 1964, 221 pp. A highly technical (historically speaking) history and bibliography of Jewish mathematical and scientific literature up to the sixteenth century. JAS

HISTORY, S*, L**, *A Computer Perspective*. Charles and Ray Eames. Harvard U Pr, 1973, 175 pp, $15. A fascinating pictorial record of Eames' unique exhibition of computer source documents, machines and memorabilia. Items range from Babbage to von Neumann, and include lesser-known contributions from such areas as business, literature, and agriculture. LAS

HISTORY, S(16-18), P, L**, *Sets and Integration: An Outline of the Development*. D. van Dalen, A.F. Monna. Wolters-Noordhoff, 1972, viii + 162 pp, $16.75. Two independent detailed historical essays, both replete with quotations from and references to source documents. Van Dalen's essay pursues set theory from pre-Cantor to post-Cohen; Monna's essay outlines integration from Eudoxus to Haar. LAS

HISTORY, L**, *Selected Works of Giuseppe Peano*. Ed: Hubert C. Kennedy. U of Toronto Pr, 1973, xi + 249 pp, $12.50. First English translation of 21 papers and excerpts selected for their interest to modern readers and for their representation of Peano's major achievements. Includes a brief biographical sketch and a complete list of Peano's publications. LAS

COMBINATORICS, T?(17-18: 1), P, L. *Constructive Combinatorics*. Earl Glen Whitehead, Jr. Courant Inst., 1973, v + 130 pp, $3.50 (P). Five chapters of lecture notes: finite planes and square designs (Latin squares, Room squares), difference sets, Hadamard matrices, block designs (with Steiner triple systems as a special case of SB/BD), and sum-free sets. Short and sketchy for a text, but good use of recent literature plus exposition of some new results. PJC

NUMBER THEORY, P. *Analytic Number Theory*. Ed: Harold G. Diamond. Proc. of Symp. in Pure Math., V. XXIV. AMS, 1973, vii + 340 pp, $23. 30 invited lectures from March 1972 symposium at St. Louis University. LAS

NUMBER THEORY, T(14-16). *Explorations in Number Theory*. Jeanne Agnew. Brooks/Cole, 1972, xi + 308 pp, $10.95. An interesting text, written with care and great enthusiasm; theorems and proofs are well-motivated and illustrated profusely with examples; thousands of interesting problems at all levels of difficulty are included; definitely worth considering as a text for an undergraduate number theory course. Some topics: fundamental theorem of arithmetic; quadratic reciprocity; distribution of primes; diophantine equations; Waring's problem; number theoretic functions; p-adic numbers. SG

NUMBER THEORY, P. *The Distribution of Prime Numbers: Large Sieves and Zero-Density Theorems*. M.N. Huxley. Oxford U Pr, 1972, x + 128 pp, $21. A sort of supplement to Davenport's *Multiplicative Number Theory* (TR October 1970). Topics: uniform distribution; large sieve inequalities; zeta and L-functions; prime number theorem; sieves; functional equations; Bombieri's theorem; Vinogradov's three-primes theorem; gaps between prime numbers. An expensive addition to anyone's library. SG

NUMBER THEORY, P, L. *Number Theory Tables*. Brother Alfred Brousseau. Fibonacci Assoc., 1973, xi + 230 pp, $12.50. A collection of forty tables. Some examples: Gaussian primes; cubic and quartic residues; amicable numbers; Bernoulli numbers; Fermat and Mersenne numbers. SG

LINEAR ALGEBRA, T(15-16: 2), S, L. *Einführung in die Algebra*. H. Lüneburg. Springer-Verlag, 1973, vii + 289 pp, $8.60 (P). An introduction to linear algebra, which begins with the elements of group, ring and field theory and ends with the theory of finitely generated modules over principal ideal rings. Problems, most of them nontrivial. JD-B

LINEAR ALGEBRA, T(14-16: 1), S, L. *Mathematik für Ökonomen II: Lineare Algebra*. M.J. Beckmann, H.P. Künzi. Springer-Verlag, 1973, xii + 160 pp, $4.10 (P). The second of three volumes on mathematics for economists. Deals largely with linear algebra, but also has chapters on linear difference equations, input-output theory and linear optimization. Assumes no advanced mathematics, but is rather abstract and sophisticated. No problems. JD-B

LINEAR ALGEBRA, T?(14-15: 1), S*, L. *Principles of Linear Algebra.* J. Larrieu. Transl: Marina Smith-Kok. Gordon, 1972, v + 273 pp, $13.50. With only one comprehensive set of 40 exercises for the whole book and very few examples in the usual sense the average passive student would "learn" nothing. But the original French flavor comes through: no-nonsense, clear exposition of the important ideas and a focus on some solid mathematics. The key tools are diagonalization of matrices and decomposition of a space into appropriate subspaces. The applications are to quadratic forms and to probability and statistics. JAS

LINEAR ALGEBRA, T(14-15: 1), S(14-16), P, L*. *Linear Algebra Through Its Applications.* T.J. Fletcher. Van-N-Rein, 1972, x + 274 pp, $9.95 (P). A sophisticated introduction to the theory of linear algebra through selected applications in geometry, chemistry, mechanics, statistics and other areas. A radical and valuable contrast to the isomorphic introductory texts on elementary linear algebra. The numerous applications, however, contribute equally to the interest and to the difficulty of this book. Excellent references at the end of each chapter. LAS

LINEAR ALGEBRA, S(14). *Matrices and Vector Spaces.* F. Brickell. Ed: L. Marder. Prob. Solvers, No. 8. Allen & Unwin, 1972, 88 pp, $4.65. Small book of solved problems and exercises with answers to supplement standard texts of lecture courses. LLK

LINEAR ALGEBRA, S(14). *Matrix Algebra for Statistical Applications.* Walter L. Sullins. Interstate Printers, 1973, viii + 108 pp, $2.50 (P). Short nonrigorous introduction to matrices with a chapter on special topics--correlation matrices, variance-covariance matrices and multiple regression. LLK

LINEAR ALGEBRA T(14: 1). *Linear Algebra.* Crist Dixon. Van-N-Rein, 1971, ix + 278 pp, $9.75. Theoretical approach to linear algebra. Introductory chapter contains induction, complex numbers and solutions to linear equations without reference to matrices. The book continues with vector spaces, the theory of linear transformations, and some geometric concepts in chapter 9 under inner product spaces. LLK

ALGEBRA, T(15-16: 1), L. *First Course in Group Theory.* P.B. Bhattacharya, S.K. Jain. Wiley Eastern, 1972, 97 pp, $3.95 (P). TR published October 1973 misquoted price as $10. Correct price is $3.95. Distributed in U.S. by Halsted Press.

ALGEBRA, P. *Mathieu Groups.* Philip J. Greenberg. Courant Inst, 1973, iv + 189 pp, $4.75 (P). A systematic exposition on the historical background, construction, subgroups, automorphisms, characterizations, and presentations of the Mathieu groups M_{11}, M_{12}, M_{22}, M_{23}, M_{24}. Appendices describe all the non-Sylow subgroups of each. DFA

ALGEBRA, T(16-17), P. *A First Course of Homological Algebra.* D.G. Northcott. Cambridge U Pr, 1973, xi + 206 pp, $12.50. Concrete category theory (category = category of modules over a ring). Standard homological algebra, with a section on local homological algebra. Plenty of exercises and solutions to exercises. Possible for senior level course/seminar or special topics in algebra if abstract algebra course does modules instead of abelian groups. PJM

ALGEBRA, T(15-17), S*, L*, *Classical Galois Theory With Examples, Second Edition.* Lisl Gaal. Chelsea, 1971, viii + 248 pp, $7.50. Reprint of a unique book, first published in 1971 (TR August 1971); contains extensive worked out examples of Galois groups, together with their application to solutions of solvable equations. LAS

ALGEBRA, S(18), P, *Jordan Algebras and Algebraic Groups.* T.A. Springer. Ergebnisse der Math., B. 75. Springer-Verlag, 1973, vii + 169 pp, $17.80. A treatment of parts of the theory of Jordan algebras, making extensive use of linear algebraic groups. An essential notion is that of a J-structure, which except for characteristic 2 is essentially the same thing as a Jordan algebra. JD-B

CALCULUS, T(13: 1), *Calculus with the Computer: A Laboratory Manual.* L. Carl Leinbach. P-H, 1974, xii + 205 pp, $4.95 (P). A supplement for a calculus course. Projects fall into three categories: definitions of limit, differential and integral; applications such as population growth and ecological simulation; and elementary numberical techniques (Newton's method and Simpson's rule). LLK

CALCULUS, T(14-16: 1, 2), L, *Advanced Calculus, Second Edition.* Wilfred Kaplan. A-W, 1973, xv + 709 pp, $15.75. Linear algebra concepts are introduced and used in the treatment of differential calculus and ordinary differential equations; except for this, the addition of a few new sections, and the deletion of some of the complex variable theory, this edition is virtually a reprinting of the 1952 one. Suitable for many audiences. DFA

CALCULUS, S(13-14), L, *Mathematics for Engineers and Scientists: A Students' Course Book, V. I.* A.C. Bajpai, *et al.* Wiley, 1973, iv + 785 pp, $9.50 (P). Programmed instruction on functions, analytic geometry, differentiation and applications, infinite series, integration and applications, determinants, matrices, vectors, complex numbers, elementary scalar ordinary differential equations. Problems are of the "drill" variety only. Some "answer frames" are detailed, others have no instructional value. DFA

CALCULUS, T(13: 2), *A First Course in Calculus, Third Edition.* Serge Lang. A-W, 1973, xii + 498 pp, $10.95. The first four chapters from the author's *Second Course* have been tacked on and there are some (but not many) new examples and exercises; otherwise, this is a replay of the second edition. Of 22 errors this reviewer found in teaching from the first half of that earlier work, 14 remain. Rather than this, you should require used copies of the second edition. DFA

REAL ANALYSIS, T(16-17), P, L, *Intégration.* Roger Descombes. Hermann, 1972, 208 pp, 58F. A real analysis book for seniors with topology or first year graduate students. Approach essentially that of Daniell; covers Lebesgue and Radon measures, L^p spaces and Fourier transforms and series. The French is straightforward; the Bibliography refers to about 40 percent French articles, the rest English. PJM

COMPLEX ANALYSIS, P, *Lectures on Riemann Surfaces, Jacobi Varieties.* R.C. Gunning. Princeton U Pr, 1972, vi + 189 pp, $4.50 (P). A sequel to the same author's *Lectures on Riemann Surfaces.* A marking of a Riemann surface is a collection of data related to a triangulation (or polygonalization) of the covering space of the Riemann surface. The Jacobi variety arises as the quotient manifold of a subset of C^{2g} associated with a marking. Advanced graduate seminar material. PJM

DIFFERENTIAL EQUATIONS, S(18), P, L, *Lectures on Linear Partial Differential Equations.* Louis Nirenberg. CBMS, No. 17. AMS, 1973, v + 58 pp, $4 (P). For the nonexpert. Reduction of first order operators to canonical form, pseudo-differential operators and uniqueness in the Cauchy problem, wave front sets and propagation of singularities. From expository lectures at Texas Tech in May 1972. DFA

DIFFERENTIAL EQUATIONS, T(17: 1), S, P, *Topological Dynamics and Ordinary Differential Equations.* George R. Sell. Van-N-Rein, 1971, ix + 199 pp, $6.95 (P). Various notions of topological dynamics (almost periodicity, ω-limit sets, etc.) are studied. The main emphasis is on the exploitation of these concepts in the study of ordinary differential equations. The flow normally considered is then the set of solution trajectories of the differential equations. JJ

DIFFERENTIAL EQUATIONS, T(16-17: 2), L, *Random Differential Equations in Science and Engineering.* T.T. Soong. Math. in Sci. and Eng., V. 103. Acad Pr, 1973, xiii + 327 pp, $19.50. A study of methods of solution of differential equations in which the parameters are stochastically specified. The random nature of these parameters is introduced in stages: in the initial conditions, in the driving functions, and finally in the coefficients of the differential equations. Although the author insists on an understanding of stochastic processes as a prerequisite, he nevertheless provides a rather complete review of this area. JJ

DIFFERENTIAL EQUATIONS, T(16-17: 2), L, *Partial Differential Equations of Mathematical Physics.* Tyn Myint-U. Am Elsev, 1973, xiv + 365 pp, $15.95. A standard introduction to partial differential equations. The wave, heat, and Laplace equations are presented in a very readable style. Manipulative technique is emphasized rather than theory. JJ

DIFFERENTIAL EQUATIONS, P*, *Theory of Bifurcations of Dynamic Systems on a Plane.* A.A. Andronov, *et al.* Wiley, 1973, xiv + 482 pp, $42.50. This book is a companion volume to *Qualitative Theory of Second-Order Dynamic Systems* by Andronov, *et al.* The first half is concerned with structural stability, a condition in which small changes in the system parameters leave the qualitative pattern of the solution trajectories topologically invariant. The second half deals with bifurcations, a phenomenon whereby small changes in the system parameters result in the movement of the solution structure from one topological pattern to another. JJ

DIFFERENTIAL EQUATIONS, S*(15-17), L, *Invariant Imbedding and Its Applications to Ordinary Differential Equations: An Introduction.* Melvin R. Scott. A-W, 1973, xvii + 215 pp, $11.50 (P). The author does not use the term "imbedding" in the usual topological sense, i.e., the realization (locally, at least) of one topological space as a subset of another. Rather he uses the term to imply the replacement of a given problem with a parametrized group of problems. He then endeavors, within the context of differential equations, to demonstrate the superiority of this technique over more classical approaches. A very interesting "second viewpoint" on the subject. JJ

DIFFERENTIAL EQUATIONS, S, *Computational Methods in Ordinary Differential Equations.* J.D. Lambert. Wiley, 1973, xv + 278 pp, $15.50. Suitable for reference for numerical methods of solution for differential equations. Not a complete text for numerical analysis. LLK

DIFFERENTIAL EQUATIONS, T?(17-18: 2), P. *The Theory of Partial Differential Equations.* Sigeru Mizohata. Cambridge U Pr, 1973, xii + 490 pp, $37.50. Elliptic and hyperbolic partial differential equations are studied with the emphasis placed on abstract theory as opposed to practical technique. Translated from the 1965 original Japanese edition. The book is exorbitantly priced. JJ

DIFFERENTIAL EQUATIONS, P*. *Analysis of Discretization Methods for Ordinary Differential Equations.* Hans J. Stetter. Springer Tracts in Nat. Philo., V. 23. Springer-Verlag, 1973, xvi + 388 pp, $44.40. A very abstract study of the premise that the solution to a differential equation may be approximated by restricting the variables to discrete sets. The problem is viewed as reducing the search for an element of an infinite dimensional function space to related searches in a sequence of finite-dimensional Banach spaces. For this reason (and the outrageous price), the book is not recommended as a source of practical solution methods. JJ

DIFFERENTIAL EQUATIONS, T(16-17: 1), S, L. *Notes on Nonlinear Systems.* J.K. Aggarwal. Van-N-Rein, 1972, 214 pp, $3.95 (P); $6.95. Having illustrated the distinction between linear and non-linear systems, the author introduces, via phase-plane analysis, the concepts of stability and limit cycles. Analytical study of these points is followed by a discussion of digital computer methods of approximating solutions. The book affords the student an early opportunity to gain an appreciation of the importance of the qualitative structure of solution trajectories. JJ

DIFFERENTIAL EQUATIONS, S*(17-18), P*. *Stable and Random Motions in Dynamical Systems: With Special Emphasis on Celestial Mechanics.* Jürgen Moser. Princeton U Pr, 1973, viii + 198 pp, $7.50 (P). The qualitative behavior of the solutions to the N-body problem of celestial mechanics is discussed. Stability and statistical behavior of the solution trajectories are studied, with well-conceived illustrations providing helpful intuitive guidance. JJ

DIFFERENTIAL EQUATIONS, T(17-18: 1), *Ecuaciones en Derivadas Parciales y su Resolución Numérica.* Alberto Dou, Alfredo Mendizábal. ETS de Ingenieros de Caminos (Madrid), 1973, xiii + 450 pp. Linear partial differential equations of the second order with constant coefficients. Rigorous mathematical consideration of physically-based problems, i. e., methods of mathematical physics: wave equation, elliptic equations, heat equation, plus theoretical considerations of numerical methods (no computer programs). PJC

DIFFERENTIAL EQUATIONS, T(17: 2), L. *Techniques in Partial Differential Equations.* Clive R. Chester. McGraw, 1971, xvi + 440 pp, $14.95. An introduction to partial differential equations, for advanced engineering students and working applied scientists. Final chapters study overdetermined systems, variational and transform methods, integral equations. Intuitive approach, with emphasis on motivation. Lots of exercises, over 300 references for further study. DFA

DIFFERENTIAL EQUATIONS, P. *Proceedings of a Seminar on Applications of Differential Equations to Mechanics and Physics.* Ed: Vadim Kokov. Math. Ser. No. 10. Texas Tech U, 1973, 173 pp, $4.25 (P). Several expository and research papers. LAS

DIFFERENTIAL EQUATIONS, P, *Non-Homogeneous Boundary Value Problems and Applications, V. III.* J.L. Lions, E. Magenes. Transl; P. Kenneth. Grund. math. Wissenschaften, B. 183, Springer-Verlag, 1973, xii + 308 pp, $28.90. Studies linear, non-homogeneous problems involving elliptic, abstract evolution, parabolic, hyperbolic, Petrowski and Schroedinger operators in spaces of distributions or of analytic functionals or of Gevrey-type ultra-distributions. Main developments use regularity and trace theorems. Presentation and discussion of many open problems. DFA

DIFFERENTIAL EQUATIONS, P, *Initial Value Methods for Boundary Value Problems: Theory and Application of Invariant Imbedding.* Gunter H. Meyer. Math. in Sci. and Eng., V. 100. Acad Pr, 1973, xiii + 220 pp, $14.50. For the working scientist. Linear and non-linear two- and multi-point problems with fixed and free boundary and interface conditions, linear infinite-dimensional problems. Numerical solution of many nontrivial technical problems. Balance among abstract theory, geometric interpretation, applications. DFA

DIFFERENTIAL EQUATIONS, S(14), *Ordinary Differential Equations.* John Heading. Ed: L. Marder. Prob. Solvers, No. 1. Allen & Unwin, 1971, iii + 92 pp, $4.25. A very small book consisting of worked problems illustrating methods of solution for differential equations, followed by exercises with answers. Not suitable as a text. LLK

DIFFERENTIAL EQUATIONS, T(17-18: 1), S, P, *Gewöhnliche Differentialgleichungen: Die Grundlagen der Theorie im Reellen und Komplexen.* F. W. Schäfke, D. Schmidt. Springer-Verlag, 1973, 163 pp, $5.50 (P). A brief, modern and difficult sketch of the theory of ordinary differential equations, both real and complex. The reader is expected to know something, for example, about Banach spaces and analytic continuation. Some problems, none of them routine. JD-B

DIFFERENTIAL EQUATIONS, T(14-15: 1), *Differential Equations and Related Topics for Science and Engineering.* Robert W. Hunt. Brooks/Cole, 1973, x + 286 pp, $9.95. A first course in applied mathematics presented in a direct nonrigorous style. All the physical applications as well as some accompanying concepts are developed as exercises. LLK

NUMERICAL ANALYSIS, T*(14: 1), S*, L, *A First Course in Numerical Analysis.* M.A. Wolfe. Van-N-Rein, 1972, ix + 156 pp, $3.50 (P). Numerical techniques for the solution of algebraic equations in one variable, sets of linear algebraic equations, and first order differential equations are presented. The methods are developed in an algorithmic fashion, which facilitates their implementation on a digital computer. JJ

NUMERICAL ANALYSIS, T(17: 1), P, L, *Imbedding Methods in Applied Mathematics.* John Casti, Robert Kalaba. A-W, 1973, xiv + 306 pp, $8.50 (P); $16. Uses invariant imbedding to get initial value representations for broad classes of boundary value problems. Considers finite difference equations, two-point boundary value problems, Fredholm integral equations with displacement-type kernels, variational problems. Mathematically formal. Many examples. Prerequisite: first course in ordinary differential equations. Final chapter discusses some applications in the physical sciences. DFA

NUMERICAL ANALYSIS, P, *An Analysis of the Finite Element Method.* Gilbert Strang, George J. Fix. P-H, 1973, xiv + 306 pp, $16. Definitely aimed at the engineer as well as the mathematician. Presentation of the general theory, examination of questions about the approximations, admissibility of trial functions and stability, study of the method's uses in eigenvalue and initial-value problems and ones with singularities. Useful for both mathematical practitioner and theoretician. DFA

NUMERICAL ANALYSIS, T(17-18: 1), P, *Approximationstheorie.* Lothar Collatz, Werner Krabs. Teubner, 1973, 208 pp, (P). A textbook, though there are few problems, on the general theory of Chebyshev approximation. Many applications. Assumes considerable mathematical knowledge and maturity. JD-B

NUMERICAL ANALYSIS, T(16-17: 1), S, *Einführung in die Numerische Mathematik II.* J. Stoer, R. Bulirsch. Springer-Verlag, 1973, ix + 286 pp, $5.50 (P). The second volume of an introduction to numerical analysis. (The first, by Stoer alone, received a TR in April 1973.) Treats characteristic values, initial and boundary value problems for ordinary differential equations, and systems of linear (algebraic) equations. Bibliography and problems at the end of each chapter. JD-B

NUMERICAL ANALYSIS, T(15-16: 1, 2), S, L, *Numerical Methods for Scientists and Engineers, Second Edition.* R.W. Hamming. McGraw, 1973, ix + 721 pp, $14.95. Thoroughly revised version of the well known 1962 original edition. An interesting, informative survey of classical (polynomial) and modern (Fourier) methods, amply seasoned with informal insight which is, according to the motto of the book, the real purpose of computing. LAS

FUNCTIONAL ANALYSIS, S(18), P, L, *Normed Linear Spaces, Third Edition.* Mahlon M. Day. Ergebnisse der Math., B. 21. Springer-Verlag, 1973, viii + 211 pp, $17.30. A "compressed introduction", originally written in 1957. Major changes in this edition are additions to several sections, new sections on weak compactness and metric geometries, and an index of citations. LAS

FUNCTIONAL ANALYSIS, P, *Extension Theory of Formally Normal and Symmetric Subspaces.* Earl A. Coddington. Mem. of AMS, No. 134. AMS, 1973, iv + 80 pp, $2.90 (P). Formally normal and normal extensions of formally normal subspaces,and symmetric and self-adjoint extensions of symmetric subspaces of the direct sum of a Hilbert space with itself. Applications to subspaces of an orthogonal sum of two Hilbert spaces and to the spectral theory of non-densely defined ordinary differential operators and of subspaces generated by pairs of symmetric operators. DFA

FUNCTIONAL ANALYSIS, T(16-17: 2), L*, *Elements of Functional Analysis.* A.L. Brown, A. Page. Van-N-Rein, 1970, xi + 394 pp, £6.50. Covers the standard material of a first course in functional or real analysis. Additional material on Fréchet differential calculus (on Banach spaces) and on spectral theory is included. A defect, of which the authors are aware, is the absence of any study of the L_p spaces. JJ

FUNCTIONAL ANALYSIS, P, *Theory of Nonlinear Operators*. Ed: M. Kučera. Acad Pr, 1973, 207 pp, $14. Eleven lectures delivered at a summer school on spectral analysis of nonlinear operators held in September 1971 at Babylon, Czechoslovakia. Eight are in English, two in German, one in French; some are concerned with solvability of certain nonlinear operator equations. DFA

FUNCTIONAL ANALYSIS, P, *Analyse Fonctionnelle, Tome III: Espaces Fonctionnels Usuels*. H.G. Garnir, *et al*. Birkhauser Verlag, 1973, 375 pp, $32. Application of the results of Volume I and II to particular function spaces: sequences, measurable, continuous and differentiable functions. Many good exercises. French is of medium difficulty; most technical terms are cognates. PJM

FUNCTIONAL ANALYSIS, P, *Approximate Solution of Operator Equations*. M.A. Krasnosel'skii, *et al*. Transl: D. Louvish. Wolters-Noordhoff, 1972, xii + 484 pp, Dfl. 95,00. Iterative processes and their application to linear and smooth nonlinear operator equations; projection methods (including their use in the eigenvalue problem); approximate methods in the theory of branching of small solutions. High exercise density makes it very usable in an advanced graduate course or seminar. Well over 300 references. DFA

FUNCTIONAL ANALYSIS, T(17-18), P, *Monads and Their Eilenberg-Moore Algebras in Functional Analysis*. Zbigniew Semadeni. Queen's U, 1973, iii + 98 pp, $3.50 (P). Studies Banach spaces as Eilenberg-Moore algebras of the free Banach space monad. Good for an applied category theory course, or for a general overview of some results from functional analysis. Reasonably priced and well written and includes some exercises. PJM

FUNCTIONAL ANALYSIS, S(18), P, *Function Algebras*. Ion Suciu. Transl: Mihaela Mihailescu. Academiei Bucuresti, 1973, 274 pp, (P). An exposition of fundamental results in the theory of function algebras oriented towards the representation of these algebras by spaces of linear operators. Translated and revised from the original 1968 Romanian edition. LAS

OPTIMIZATION, T(15-17), S, L, *Flows in Transportation Networks*. Renfrey B. Potts, Robert M. Oliver. Acad Pr, 1972, xi + 192 pp, $13.15. A systematic elementary survey of the more important transportation models focused on fundamental conservation and extremal principles. Annotated references and problems conclude each chapter. Three appendices provide theoretical details omitted in the main text, and a fourth contains answers to the problems. LAS

OPTIMIZATION, P, *Some Aspects of the Optimal Control of Distributed Parameter Systems*. J.L. Lions. CBMS Reg. Conf. in Math., No. 6. SIAM, 1972, vi + 92 pp, $6.20 (P). Variational inequalities, optimal control problems for linear and nonlinear distributed parameter systems, existence theorems for geometrical optimization problems, applications of boundary layers in singular perturbations. Remarks on numerical methods. From an August 1971 conference in College Park, Maryland. DFA

OPTIMIZATION, S(14-15), L, *Theory of Games and Strategies*. Richard I. Levin, Robert B. Desjardins. International Textbook, 1970, xi + 132 pp, $2.50 (P). A brief yet non-trivial introduction to 2- and

n-person games, unfortunately devoid of realistic examples or any exercises. It does, however, contain many worked out artificial games which illustrate simple algebraic and geometric means of solutions. Good bibliography. LAS

OPTIMIZATION, T(17: 2), P, *Optimization Methods for Large-Scale Systems...with Applications.* Ed: David A. Wismer. McGraw, 1971, xii + 335 pp, $19.50. A survey of optimization techniques with special emphasis given to methods of reducing the complexity or dimensionality of large systems. Familiarity with system description via "state space" matrix notation appears to be a prerequisite. JJ

ANALYSIS, P, *Lecture Notes in Mathematics-329: Multipliers for (C, α)-Bounded Fourier Expansions in Banach Spaces and Approximation Theory.* Walter Trebels. Springer-Verlag, 1973, vi + 103 pp, $7.20 (P). Approximation theory via abstract Fourier series in Banach spaces. Assumes the uniform boundedness of the Cesàro means of order α (for some $\alpha \geq 0$, perhaps fractional) for the expansion. Applications to one-dimensional and multiple trigonometric, Laguerre, Hermite, and Jacobi series and to surface spherical harmonics. DFA

ANALYSIS, P, *Thin Sets in Harmonic Analysis.* Ed: L.-Å. Lindahl, F. Poulsen. Dekker, 1971, ix + 185 pp, $9.50 (P). The proceedings of a special seminar at the Mittag-Leffler Institute during the year 1969-70. JAS

ANALYSIS, P, L, *The Theory of Bernoulli Shifts.* Paul Shields. U of Chicago Pr, 1973, x + 118 pp, $6.50; $2.50 (P). Ornstein's result that two Bernoulli shifts are isomorphic if they share the same entropy is derived in detail. Background required for reading the presentation is minimal. JJ

ANALYSIS, P*, *Lecture Notes in Economics and Mathematical Systems-67: Lectures on Mathematical Theory of Extremum Problems.* I.V. Girsanov. Springer-Verlag, 1972, 136 pp, $5.10 (P). The optimum operating point of a process is often given by the extremum of some function. The author describes general schemes for locating such an extremum. These conditions are illustrated with a series of problems, including selections from optimal control theory and linear programming. A rather complete review of the prerequisite topics in functional anlaysis precedes the main work. JJ

ANALYSIS, P, *Lecture Notes in Mathematics-295: Ensembles Analytiques, Capacites, Measures de Hausdorff.* Claude Dellacherie. Springer-Verlag, 1972, xii + 123 pp, $5.10 (P). Mostly expository. In the author's words a "*brochure publicitaire*" for a certain viewpoint for the theory of analytic sets--namely in the context of compact metrizable spaces. JAS

ANALYSIS, P, *American Mathematical Society Translations, Series 2, V. 101.* B.M. Levitan, *et al.* AMS, 1973, iii + 250 pp, $19.60. Six papers on analysis.

GEOMETRY, P, *Geometry of Submanifolds.* Bang-yen Chen. Dekker, 1973, vii + 298 pp, $17.50. An exposition of recent results on submanifolds. The author discusses minimal submanifolds; umbilical submanifolds; conformally flat submanifolds; submanifolds with parallel mean curvature vector, geometric inequalities. Familiarity with differential geometry is presumed. SG

GEOMETRY, P, *Aufbau der Geometrie aus dem Spiegelungsbegriff.* Friedrich Bachmann. Grund. math. Wissenschaften, B. 96, Springer-Verlag, 1973, xvi + 374 pp, $35.10. A treatise on the development of plane metric (absolute) geometry from group-theoretic axioms. Contains the text of the first edition (1958), plus an account (with a complete bibliography) of results obtained since then. JD-B

GEOMETRY, P, *Lecture Notes in Mathematics-286: On Automorphisms of Siegel Domains.* Shingo Murakami. Springer-Verlag, 1972, 95 pp, $5.10 (P). Geometric results about certain domains in complex n-space obtained by an analysis of the automorphism groups of these domains. An expository work summarizing and unifying a number of recent results in the field. JAS

TOPOLOGY, P*, *CW-Complexes, Homology Theory.* Renzo A. Piccinini. Queen's U, 1973, iii + 129 pp, $4 (P). The third chapter containing results on extensions of homology theories is supported by two very helpful chapters on CW-complexes in a categorical context and generalized homology theories. Careful exposition clarifies a lot of folklore. JAS

TOPOLOGY, T(18: 1, 2), P, *Cohomology and Differential Forms.* Izu Vaisman. Transl: Samuel I. Goldberg. Dekker, 1973, vii + 284 pp, $19.75. Starting with some very general definitions of categories and functors (e.g., a pseudo-category is a "category" with partial composition), and using these to obtain nice formulations of differentials (local categories) and atlases, the author presents De Rham cohomology and other results of differential topology and geometry in a very elegant, up-to-date setting. Skip over the section on categories and functors--the details are mainly technical and on a first reading confusing. An excellent book, usable for a second year graduate course with a little additional geometric intuition added according to the taste of the instructor. PJM

TOPOLOGY, T*(17: 2, 3), P, *Connections, Curvature, and Cohomology, V. II: Lie Groups, Principal Bundles, and Characteristic Classes.* Werner Greub, *et al.* Acad Pr, 1973, xxi + 541 pp, $35. Using the machinery of Volume 1 (TR January 1973) the authors study fibre bundles with structure group or Lie group. Characteristic (cohomology) classes of such bundles are defined and finally used to relate the geometry to the topology via the Gauss-Bonnet-Chern Theorem. An excellent book which could be used independently of Volume 1 (a summary and table of contents of Volume 1 are included). Very good up-to-date bibliography. Problems are difficult but plentiful. PJM

STATISTICS, P, L, *Russian-English/English-Russian Glossary of Statistical Terms.* Samuel Kotz. Oliver & Boyd, 1971, vii + 87 pp, $9.95. Based on the Kendall and Bruckland *Dictionary of Statistical Terms*, this glossary includes about 2500 terms. Prepared under the auspices of the International Statistical Institute. LAS

STATISTICS, T(13: 1, 2), *Introduction to Statistics.* Herbert Friedman. Random House, 1972, xi + 334 pp, $9.50. Presupposes only high school algebra. Starts off with statistical inference and treats probability very lightly. Considers many non-parametric tests and more in analysis of variance and design of experiments than is usual at this level. No Bayesian methods. FLW

STATISTICS, T(13: 1), *Fundamental Statistical Concepts*. Frederic E. Fischer. Canfield Pr, 1973, ix + 371 pp, $9.95. Presupposes only high school algebra. Descriptive statistics (including partial correlation), probability (but no discussion of subjective probabilities), and statistical inference (including several non-parametric tests). FLW

STATISTICS, T(13: 1, 2), *Elements of Statistics; An Introduction to Probability and Statistical Inference*. Donald R. Byrkit. Van-N-Rein, 1972, xi + 324 pp, $9.50; *Commentary and Solutions Manual for Elements of Statistics*, iv + 145 pp, (P). Presupposes only high school algebra. Standard topics. Exposition often lacks precision and clarity. FLW

STATISTICS, T(13-14: 1, 2), S, L, *Statistical Analysis for Business and Economics, Second Edition*. Leonard J. Kazmier. McGraw, 1973, xv + 623 pp, $8.95 (P). Programmed text. Presupposes only high school algebra. Descriptive statistics, probability, statistical inference, decision theory, time series, index numbers, and computers (with Fortran programs for basic statistics). FLW

STATISTICS, T(13-14: 1, 2), *Statistical Methods, Third Edition*. Allen L. Edwards. HR&W, 1973, viii + 312 pp, $9. Presupposes only high school algebra. Covers a great many topics very briefly. Most of the examples are from psychology. No Bayesian methods. FLW

STATISTICS, P, *Selected Translations in Mathematical Statistics and Probability, V. 12*. AMS, 1973, v + 312 pp, $21.50.

STATISTICS, T(18), P, L, *The Advanced Theory of Statistics, V. 2, Third Edition*. Maurice G. Kendall, Alan Stuart. Hafner, 1973, x + 723 pp, $29.95. Revision of the 1967 edition of the second volume of the authors' three volume treatise. Topics include the theory of estimation and testing hypotheses, statistical relationship, distribution-free methods and sequential analysis. RSK

STATISTICS, S, *Practical Exercises in Probability and Statistics*. N.A. Rahman. Hafner, 1972, xiv + 338 pp, $18.95. Companion volume to the author's *A Course in Theoretical Statistics* (Griffin, 1968); parallels the author's *Exercises in Probability and Statistics* (TR February 1968) which contained theoretical exercises. Book consists solely of 200 exercises, together with answers and extensive hints for their solutions. Price is high. RSK

STATISTICS, T(13: 2), *Introductory Statistics for the Behavioral Sciences, Second Edition*. Robert K. Young, Donald J. Veldman. HR&W, 1972, xii + 559 pp, $9.95. A descriptive, non-mathematical text, distinguished by its programmed exercises. Each of the 18 sections is followed by numerous (60-125) frames to drill the students. Differs from the first edition (1965) by the addition of problem sets after the programs. May be considered for a self-paced course. TAV

STATISTICS, T(13-14: 1), S, *Biostatistics: An Introductory Text, Eighth Printing*. Avram Goldstein. Macmillan, 1971, ix + 272 pp, $10.95. Standard topics illustrated with biological or medical examples. Presupposes only high school algebra. Few derivations but good discussions of assumptions. No Bayesian methods. FLW

STATISTICS, T(14: 1), S, *Probability and Experimental Errors in Science: An Elementary Survey.* Lyman G. Parratt. Dover, 1971, xv + 255 pp, $3 (P). Unaltered Dover edition of the 1961 Wiley text. Written for scientists to increase their understanding of science, it treats those concepts of probability and statistics that are most applicable to scientific measurements, emphasizing the normal and Poisson distributions. Some background in calculus is desirable. RSK

STATISTICS, P, *Discriminant Analysis and Applications.* Ed: T. Cacoullos. Acad Pr, 1973, xviii + 434 pp, $16.50. Proceedings of a NATO institute held in Athens in June, 1972. Includes an extensive classification and review paper by S. Das Gupta and a supposedly exhaustive bibliography of books and papers through 1972. LAS

STATISTICS, P, *Multivariate Statistical Inference.* Ed: D.G. Kabe, R.P. Gupta. North-Holland, 1973, x + 258 pp, $15. Invited lectures together with some contributed papers from 1972 research seminar at Dalhousie U. LAS

STATISTICS, T*(16-17: 1), P, L, *Nonparametric Statistical Methods.* Myles Hollander, Douglas A. Wolfe. Wiley, 1973, xviii + 503 pp, $18.95. An applied book--centered around problems encountered in actual experiments in a variety of fields. Presupposes at least an introductory course in non-mathematical statistics. Includes not only hypothesis testing, but methods for point estimators and confidence intervals, plus multiple comparison procedures. Topics: dichotomous data, one- and two-sample location problems, one- and two-way layouts, two-way dispersion, independence, regression. PJC

COMPUTER SCIENCE, S**, *My Computer Likes Me...when i speak in BASIC.* Dymax, P.O. Box 310, Menlo Park, Calif., 1972, $1.49 (P). A flamboyant, graphic introduction to BASIC skillfully employing a free-form typography to highlight important details, and illustrated with a continually evolving population growth problem. Covers only basic BASIC in an unusually interesting format. (The same publishers produce a similar zany newsletter for the *People's Computer Company*, $4/year.) LAS

COMPUTER SCIENCE, T*(14-16: 1), S, L, *Computer Approaches to Mathematical Problems.* Jurg Nievergelt, *et al.* P-H, 1974, xiii + 257 pp, $8.95. Six independent essays on arithmetic expressions, combinatorial computing, game playing, random processes, computing with numbers, and machine limitations. Intended as a middle level introduction to important computational techniques and to the role of the computer in helping to solve a wide variety of problems. Exercises and references conclude each essay. LAS

COMPUTER SCIENCE, T*(15-18), P, L, *Discrete Mathematical Structures and Their Applications.* Harold S. Stone. SRA, 1973, vii + 345 pp, $11.95. An exceptional book. Intended for students of computer science and computer engineering, it is also quite suitable for a second undergraduate algebra course, a graduate level course, and independent reading courses. Ideas are well motivated and clearly explained; problems are wide ranging and very interesting. Topics: groups, rings; finite fields; Polya enumeration theory; codes; finite state machines; Boolean algebras; applications of group theory and Boolean algebra to computer design. SG

COMPUTER SCIENCE, T(13: 2), *Problems Solving with Computers.* Paul Calter. McGraw, 1973, xii + 187 pp, $5.95 (P). Procedures for and techniques of problem solving. A good collection of projects and exercises for a course in computer programming. LLK

COMPUTER SCIENCE, T(17-18: 1, 2), P, *Introduction to Mathematical Techniques in Pattern Recognition.* Harry C. Andrews. Wiley, 1972, xiii + 242 pp, $11.50. Summarizes and unifies research on theoretical (not heuristic or structural) aspects of pattern recognition. Very good chapter bibliographies. Topics: features selection, distribution-free classification, statistical classification, nonsupervised learning, and sequential learning. PJC

COMPUTER SCIENCE, S(13-18), P, L, *Modern Business Data Processing.* Daniel D. Benice. P-H, 1973, x + 613 pp, $10.95. A well-rounded summary of both batch and time-sharing in the business world. Part 2 gives a brief outline of APL, BASIC, COBOL, FORTRAN, PL/1, RPG and IBM ASSEMBLER. There are many good exercises and selected solutions in the back. Very clearly written and recommended for general reading, but not for a course. RB

COMPUTER SCIENCE, T(15: 1, 2), S(13-18), P, L, *Introduction to Computers and Computer Programming.* Lawrence J. Prince, David F. Nyman. P-H, 1972, xiii + 209 pp, $9.95. This is an excellent introduction to the internal makeup of computers, to base 2, 8, 10 and 16 number systems and to a general assembly language. Teaches the types of things programmers should know about other than high level languages. Not enough problems and exercises and no solutions. RB

COMPUTER SCIENCE, T(13-15), *Elements of Computer Science.* James W. Estes, B. Robert Ellis. Canfield Pr, 1973, viii + 376 pp, $10.95. A broad elementary introduction to algorithms, flowcharts and computer technology designed to parallel any standard course in a particular programming language. Problems and references conclude each chapter. LAS

COMPUTER SCIENCE, P, *Complexity of Sequential and Parallel Numerical Algorithms.* Ed: J.F. Traub. Acad Pr, 1973, ix + 300 pp, $11.50. Proceedings of a symposium held at Carnegie-Mellon U., May, 1973. LAS

COMPUTER SCIENCE, S*, L, *Fortran IV Pocket Handbook.* Daniel E. Alexander, Andrew C. Messer. McGraw, 1972, 91 pp, $1.95 (P). Clear and brief outline of FORTRAN, well suited to quick referencing. Not recommended as a textbook since it is too small to contain long examples and problem solutions. RB

COMPUTER SCIENCE, S, P**, L, *Minicomputers for Engineers and Scientists.* Granino A. Korn. McGraw, 1973, xiv + 303 pp, $17.75. This excellent book contains many of the state-of-the-art facts which are so hard to pick up piece-by-piece. Book is oriented toward hardware and lower level languages like assembler. Contains good sections on interfacing between the minicomputer and the outside world. RB

COMPUTER SCIENCE, S(14), L, *Computer Dictionary.* Donald D. Spencer. Abacus, 1973, 57 pp, $3.95 (P). A small interesting reference to computer science terms. Not recommended for classroom use, but may be handy in a computer center. RB

COMPUTER SCIENCE, T(15-17), P, L. *Symbolic Logic and Mechanical Theorem Proving.* Chin-Liang Chang, Richard Char-Tung Lee. Acad Pr, 1973, xiii + 331 pp, $17.50. Symbolic logic, not for the mathematician or the philosopher but for the computer scientist. The interest is not on axiomatic development, but rather on techniques of mechanical theorem proving and applications to artificial intelligence. Many examples, complete bibliography. Exudes the spirit of a rapidly developing field of research. LCL

COMPUTER SCIENCE, T, S, P. *Flowcharting Concepts and Data Processing Techniques, A Self-Instructional Guide.* Mark N. Wayne. Canfield Pr, 1973, vii + 248 pp, $4.50 (P). Well suited to teaching data processing technique and flowcharting without specifying any particular language. Has good sections on I/O buffers and contains a nice glossary of commonly used terms. No solution section, but plenty of flowcharting examples. Recommended for short data processing course. RB

COMPUTER SCIENCE, S(15), P*, L. *Introduction to Computer Programming for the Social Sciences.* Peter B. Harkins, *et al.* Allyn, 1973, x + 258 pp, $5.95 (P). Has short introductory section on FORTRAN, but is not meant to be a FORTRAN text. The rest of the book contains 23 complete programs, documented and explained, from the social sciences. RB

COMPUTER SCIENCE, T(13-14), S. *PL/I for Business Applications.* Mary Ellen Anderson. P-H, 1973, xviii + 397 pp, $13.95. A clearly written and easily understood text on the elementary features of PL-1. Advanced topics have been omitted. There are nice self review sections throughout the book and exercises without solutions in each chapter. Might best be used as a self-study guide or as a supplementary text. The appendix contains a handy keyword outline. RB

COMPUTER SCIENCE, T(15-16), P, L. *PL/1 for Business Applications.* Leonard E. Edwards. Reston, 1973, xi + 480 pp, $10.95; $8.95 (P). Excellent text for a business computer course. Each chapter has a number of lengthy problems with solutions in the back. Good on record-oriented I/O; includes many helpful flowcharts. However, the computer print-out examples are very hazy and hard to read. Contains many advanced topics such as indexed and regional files and list processing. RB

COMPUTER SCIENCE, S*(13-15), L. *A Collection of Programming Problems and Techniques.* H.A. Maurer, M.R. Williams. P-H, 1972, x + 256 pp, $6.95 (P). A rich collection of several hundred interesting problems, sparsely interspersed with verbal algorithms and followed by an appendix with hints, sample outputs, or (often) estimates on the length of solutions. Unusually complete index. LAS

COMPUTER SCIENCE, T(14), L. *Introduction to Computer Programming for Biological Scientists.* Howard Orr, *et al.* Allyn, 1973, xiii + 396 pp, $6.95 (P). Useful to a wider audience than the title indicates. Many of the 51 illustrative FORTRAN programs are essentially statistical or numerical. The description of the language is rather brief and specialized, but includes most constructs. A balance between writing programs and using library routines. Perpetuates some nonstandard terminology. RWN

COMPUTER SCIENCE, S. *Computer Appreciation.* T.F. Fry. Philo. Library, 1971, viii + 237 pp, $15. Intended for business students. History, hardware, software and data processing applications and management. RWN

COMPUTER SCIENCE, T(13-14), S, L, *Simplified Fortran Programming: With Companion Problems*. Lisa Rosenblatt, Judah Rosenblatt. A-W, 1973, vi + 229 pp, $3.95 (P). Too little space is given to FORTRAN, and time sharing is stressed instead of batch processing. However, it makes a nice companion book with its BASIC counterpart by the same authors. There are answers given to selected problems. Recommended for practice in problem working. RB

COMPUTER SCIENCE, T(13: 1), S(13), *Communicating with the Computer: Introductory Experiences FORTRAN IV*. Zeney P. Jacobs, *et al.* Holbrook Pr, 1973, vi + 346 pp, $6.95 (P). Has many good examples and makes excellent use of many flowcharts. All programs have been run on an IBM 1130. There is no answer section. Recommended for an elementary course in which the book leads the student along step by step. RB

COMPUTER SCIENCE, S, L, *Computer Science: Projects and Study Problems*. Alexandra I. Forsythe, *et al.* Wiley, 1973, xiv + 292 pp, $6.50 (P). 13 language-independent programming projects and numerous related and unrelated exercises. Intended for use with an introductory course.RWN

COMPUTER SCIENCE, T*(16: 1), S, P, L, *The Art of Computer Programming, V. 3: Sorting and Searching*. Donald E. Knuth. A-W, 1973, xi + 722 pp, $19.50. The most nearly complete reference to date on sorting and searching. About 2/3 on sorting techniques and 1/3 on searching plus hundreds of problems ranging from 00 (easy) to 50 (research). The variety and, at times, depth of mathematics used in the analyses make this delightful reading as well as a useful guide for the practitioner. RWN

COMPUTER SCIENCE, T(15-17: 1, 2), L, *Data-Structures and Programming*. Malcolm C. Harrison. Scott F, 1973, 322 pp, $10.95. Major concepts for programming: string processing, evaluation of expressions, sorting, searching, hash coding, recursion and macros. Uses an extension of Fortran with machine language capabilities for most examples and also considers the capabilities and storage allocation for PL/1, Lisp, Snobol, APL, SETL, Algol 68 and Balm. Introduces and uses many types of data structures, especially lists. RWN

COMPUTER SCIENCE, P, *Lecture Notes in Economics and Mathematical Systems-75: 1. Fachtagung über Programmiersprachen*. Ed: M. Beckmann, *et al.* Springer-Verlag, 1972, vii + 280 pp, $7.70 (P). Fifteen papers from the March 1971 symposium in Munich. JAS

COMPUTER SCIENCE, T(1), *Fundamental Programming Concepts*. Jonathan L. Gross, Walter S. Brainerd. Har-Row, 1972, x + 304 pp, $8.95. BASIC programming. Interesting applications in diverse areas. Introduces the student to such concepts as strings, stacks, graphs, simulation and artificial intelligence. In general, needs to be supplemented by a manual on the local version. RWN

COMPUTER SCIENCE, S(15-17), L, *Computer Science*. Ed; Alfonso F. Cardenas, *et al.* Wiley, 1972, xii + 522 pp, $19.95. 14 authoritative survey papers on hardware technology, design, arithmetic, systems, on-line computing, graphic display, networks, analog computation, languages, translators and applications. Could be considered for a senior seminar. RWN

COMPUTER SCIENCE, T?, S?, *Background Math for a Computer World.* Ruth Ashley. Wiley, 1973, xi + 286 pp, $3.95 (P). Mechanical, sterile, self-teaching guide to algorithmic thinking. Context: basic arithmetic, probability and statistics, matrices. LCL

COMPUTER SCIENCE, S(15-17), L*, *Games Playing with Computers.* A.G. Bell. Allen & Unwin, 1972, 204 pp, $12.65. An unusual introduction to programming, using games (*alias* "dynamic techniques of search and evaluation in a multi-dimensional problem space") such as Nim, Poker, Chess, and Go. The programming challenges are nontrivial, so it is hard to imagine a beginning student learning to program from the book; but it is equally hard to imagine a budding computer scientist putting the book down. LAS

COMPUTER SCIENCE, T(15), S, P, L, *Introduction to PL/1 Programming and PL/C.* Marilyn Bohl, Arline Walter. SRA, 1973, vii + 280 pp, $5.95 (P). Easy to read. Explains many details relevant to other programming languages. Many short examples, but not enough long and complex ones. There are answers to selected problems. The index on the inside front cover is very helpful and outlines the basic commands. RB

SYSTEMS THEORY, P, *Lecture Notes in Economics and Mathematical Systems-82: Resolution Space Operators and Systems.* R. Saeks. Springer-Verlag, 1973, x + 267 pp, $8.20 (P). Systems problems, such as optimization, stability, controllability, are studied within the mathematical framework of a resolution space. In the simpler applications, the resolution space amounts to a Hilbert space of functions together with an operator which truncates the function in a selective manner.JJ

SYSTEMS THEORY, S(16-17), *Linear Systems.* W.A. Coppel. Notes on Pure Math., No. 6. Australian Natl. U, 1972, vii + 85 pp, $3.75 (P). An introductory survey of linear systems, using the standard "state space" approach. Problems of controllability and observability are covered; systems are characterized in terms of their impulse response; the question of optimal control is considered. A great deal of space is devoted to matrix manipulation technique. JJ

SYSTEMS THEORY, T(16-17: 1), L, *Introduction to Systems Theory.* Stephen W. Director, Ronald A. Rohrer. McGraw, 1972, xii + 441 pp, $16.50. An introductory "state space" approach to first and second order systems. The text is highly self-contained, and proceeds from the primitive definitions through stability concepts and numerical techniques, such as the fast Fourier transform. JJ

APPLICATIONS, P, *Decomposition of Large-Scale Problems.* Ed: David M. Himmelblau. North-Holland, 1973, ix + 571 pp, $34.75. Proceedings of the July, 1972 conference at Cambridge, England. A well edited broad spectrum of papers. JAS

APPLICATIONS, P, *Transactions of the Eighteenth Conference of Army Mathematicians.* US Army, 1973, xv + 812 pp, (P). Papers from the May 24-26, 1972 Conference at Feltman Research Laboratory, Picatinney Arsenal, Dover, New Jersey. Contains an agenda listing all papers given in addition to those published here. JAS

APPLICATIONS (ARCHAEOLOGY), P, L*, *Mathematics in the Archaeological and Historical Sciences.* Ed: F.R. Hodson, *et al.* Edinburgh U Pr, 1971, ix + 565 pp, $36. Over 50 papers from a 1970 Anglo-Romanian

conference on such topics as taxonomy, multidimensional scaling, seriation, and evolutionary trees. Includes several brief survey papers, some theoretical models, and many specific applications. An exciting sample of uncommon applications. LAS

APPLICATIONS (BIOLOGY), T(16-17), P, L, *Numerical Taxonomy: The Principles and Practice of Numerical Classification.* Peter H.A. Sneath, Robert R. Sokal. Freeman, 1973, xv + 573 pp, $19.50. Taxonomy--a mixture of statistics, computer science and special techniques--is the theoretical study of the ordering of objects (especially biological organisms) into groups on the basis of their relationships. This text, a complete rewriting of the authors' 1963 *Principles of Numerical Taxonomy*, reflects the dominant influence of computerized data analysis of the past decade. Largely a compendium of references and techniques, it has a massive bibliography together with appendices which provide lists of comparable studies. LAS

APPLICATIONS (BUSINESS), S(14-16), L, *Operations Research: An Introduction to Modern Applications.* Ed: William C. House. Auerbach, 1972, xi + 365 pp, $19.95. An anthology of reprints from managerial journals providing many examples of mathematical programming, mostly at an elementary mathematical level. LAS

APPLICATIONS (CONTROL THEORY), P, *Control and Dynamic Systems: Advances in Theory and Applications, V. 9.* Ed: C.T. Leondes. Acad Pr, 1973, xvi + 514 pp, $16. Five survey articles continuing the series of volumes formerly titled "Advances in Control Systems." LAS

APPLICATIONS (CONTROL THEORY), P, *Control and Dynamic Systems, V. 10.* Ed: C.T. Leondes. Acad Pr, 1973, xvii + 527 pp, $19.50. Seven expository essays on optimization techniques, aircraft control, and traffic modelling. LAS

APPLICATIONS (CONTROL THEORY), P, *Recent Developments in Control Theory.* Ed: Nam P. Bhatia. SIAM, 1972, 177 pp, $8.85. 13 selected papers from NSF Regional Conference on control theory at U. of Maryland, August, 1971. LAS

APPLICATIONS (ECONOMICS), P, *Mathematical Methods in Investment and Finance.* Ed: Giorgio P. Szegö, Karl Shell. North-Holland, 1972, x + 665 pp, $24.25. 29 papers from an international symposium held in Venice in September, 1971. Most focus on models of portfolio selection. LAS

APPLICATIONS (ENGINEERING), P, *The Numerical Control of Machine Tools: Basic Principles, Systems Analysis and Industrial Applications.* Dr.-Ing Wilhelm Simon. Crane, Russak, 1970, xiv + 544 pp, $49.50.

Reviewers Whose Initials Appear Above

David F. Appleyard, Carleton; Ralph Bjork, St. Olaf; Paul J. Campbell, St. Olaf; John Dyer-Bennet, Carleton; Steven Galovich, Carleton, James Johnson, St. Olaf; Lorraine L. Keller, St. Olaf; Richard S. Kleber, St. Olaf; Loren C. Larson, St. Olaf; Pierre J. Malraison, Carleton; R.W. Nau, Carleton; J. Arthur Seebach, Jr., St. Olaf; Lynn A. Steen, St. Olaf; T.A. Vessey, St. Olaf; Frank L. Wolf, Carleton.

NEWS AND NOTICES

EDITED BY RAOUL HAILPERN, SUNY at Buffalo

Readers are invited to contribute to the general interest of this Department by sending news items to Mathematical Association of America, 1225 Connecticut Avenue, NW, Washington, D. C. 20036. Items must be submitted at least two months before publication can take place.

PERSONAL ITEMS

Brock University: Assistant Professors S. C. Chang and V. B. Headley have been promoted to Associate Professors.

Mississippi State University: Associate Professor A. C. Grimes has been promoted to Professor; Assistant Professors J. R. Graef and P. W. Spikes have been promoted to Associate Professors.

Dr. Jack Alanen has been appointed Senior Lecturer and Chairman of Department in the Computing Center at the University of Nairobi, Kenya.

Dr. L. K. Arnold of Daniel H. Wagner, Associates, has been promoted to Senior Associate.

Assistant Professor P. C. Eklof, Stanford University, has been appointed Associate Professor at the University of California, Irvine.

Dr. George Epstein has been appointed Professor at Indiana University, Bloomington.

Associate Professor Janos Galambos, Temple University, has been promoted to Professor.

Assistant Professor K. J. Heuvers, Michigan Technological University, has been promoted to Associate Professor.

Assistant Professor M. H. Moore, University of Florida, has joined Vector Research, Inc., Ann Arbor, as a Senior Systems Analyst.

Associate Professor R. E. Pippert, Purdue University at Fort Wayne, has been promoted to Professor.

Associate Professor Augustus Frank Bausch, Kalamazoo College, died on June 11, 1973, at the age of 53. He was a member of the Association for seven years.

Dr. Helen Calkins, Cornell University, died on June 17, 1970, at the age of 77. She was a member of the Association for forty-seven years.

Professor Brewster H. Gere, Hamilton College, died on July 16, 1973, at the age of 62. He was a member of the Association for thirty-six years.

Professor Pasquale Porcelli, Louisiana State University, died on December 7, 1972, at the age of 46. He was a member of the Association for twenty-six years.

Professor Myron F. Rosskopf, Columbia University, Teachers College, died on January 31, 1973, at the age of 65. He was a member of the Association for forty-four years.

UNIVERSITY OF WISCONSIN-MADISON ASSUMES EDITING AND MANAGEMENT OF DELTA

The Mathematics Department of the University of Wisconsin-Madison announces that in cooperation with the Mathematics Department of the University of Wisconsin-Extension it has assumed the editing and management of the undergraduate mathematics journal DELTA, as of January 1, 1974. The editorial board will be constituted as follows: Chief

Editor, Steven Bauman; Associate Editors, R. A. Brualdi, D. W. Crowe, J. E. Hall, R. S. Luthar. The journal will be published twice a year with an annual subscription rate of $3.00. Subscriptions can be obtained by writing to the business address: DELTA, Extension Mathematics Department, 432 North Lake Street, Madison, Wisconsin 53706.

The editorial policy of the journal is: "Delta is an undergraduate journal of mathematics which is devoted to serving both students and teachers at two and four year colleges and universities. We welcome articles of wide interest which contain original research, new proofs of old theorems, expositions of recent developments in various mathematical fields, fresh findings in the history of mathematics, and stimulating ideas about mathematics education. A high standard of exposition will be maintained." Editorial correspondence should be addressed to: Chief Editor, DELTA, University of Wisconsin, 213 Van Vleck Hall, Madison, Wisconsin 53706.

LOGISTICS RESEARCH CONFERENCE

The Office of Naval Research and the George Washington University, with the cooperation of the Air Force Office of Scientific Research and the Army Research Office, announce a Logistics Research Conference to be held at the George Washington University, Washington, D. C., on 8–10 May 1974. The main objectives of the Conference are to survey major developments and difficulties in government, industrial and military logistics research and applications since World War II, and to assess outstanding current problems and promising new research techniques.

Areas of research activity have been categorized as follows: (1) applications of mathematical programming; (2) applied case studies; (3) design of systems; (4) inventory systems; (5) data collection, representation, and analysis; (6) measurement of performance; (7) probabilistic methods; (8) production and procurement; (9) reliability, maintainability, and availability; (10) simulation; (11) statistical methods; and (12) transportation and scheduling.

Contributed papers are welcome. Abstracts and inquiries may be addressed to Ms. Henrietta Jones, Department of Operations Research, The George Washington University, Washington, DC 20006. (Phone: 202/676-7504.) Further information may also be obtained from Professors A. V. Fiacco (202/676-7511), W. H. Marlow (202/676-7503), or Henry Solomon (202/676-7521) at the University, or from Mr. Marvin Denicoff (202/692-4304) at the Office of Naval Research.

THE FIRST NATIONAL CONVENTION FOR TYC FACULTY

The Editors of the Mathematics Associations of Two-Year Colleges Journal (MATYCJ) have announced The First National Convention for two-year college mathematics educators to be held on April 25 and 26, 1974, at the Essex House in New York City.

During the seven-year history of MATYCJ, this publication has grown to a 52-page journal published three times a year and read by over 2,500 educators and mathematicians in the United States, Puerto Rico, Canada, and Australia. To help meet the need to increase communications among two-year college mathematics teachers *The MATYC Journal* proudly announces the First National Convention.

Topics of Discussion will be:

Establishment of a permanent study group for TYC remedial mathematics
National Science Foundation–Funds for TYC's
Possible relationships between the MAA and the TYC educator

Curriculum problems relative to college size
The why and how of publishing for TYC's
Teacher Exchange and Placement
Classroom techniques for TYC's
Future National Conventions

Guest speakers include Herb Gross, Allyn Washington, R. P. Boas, A. B. Willcox, William Rice, Louis Roethel, E. F. Beckenbach, Louis Auslander, Mary Dolciani, and Allan Tucker.

For a detailed brochure and schedule of events, please write: Director of Special Projects, The MATYC Journal, Department of Mathematics and Computer Science, Nassau Community College, Garden City, NY 11530.

THE INTERNATIONAL CONGRESS OF MATHEMATICIANS VANCOUVER, CANADA — AUGUST 21–29, 1974

First Announcement

The Organizing Committee is pleased to announce that the next International Congress of Mathematicians will be held in Vancouver during August 21–29, 1974.

SCIENTIFIC PROGRAM. The work of the Congress will be divided into twenty sections. There will be approximately 16 invited one-hour expository lectures and approximately 150 invited 45-minute specialist talks. Members of the Congress will be given an opportunity to present 15-minute oral communications of contributed papers and to organize small informal mathematical seminars on their own initiative, either in advance or at the time of the Congress. Instructions concerning abstracts of contributed papers and organization of seminars will be included in the second announcement.

All formal lectures will be given at the University of British Columbia, but some of the informal seminars are expected to take place at Simon Fraser University and the University of Victoria.

LANGUAGES: English, French, German, and Russian are the designated languages of the Congress.

LOCAL ARRANGEMENTS: Accommodations for approximately 3000 persons will be available in residences at the University of British Columbia. Based on single occupancy, the daily rate for these is expected to be approximately $13 (Can.) including meals. In addition, rooms in hotels primarily located in downtown Vancouver will be available. The rate for single rooms without meals in hotels is expected to be in the range of $12–$35 (Can.).

A special bus service connecting the University of British Columbia with downtown and Simon Fraser University will be arranged for the members of the Congress. This is in addition to the regular public bus service.

Vancouver is considered one of the most scenic cities in the world with magnificent view of mountains and sea. A series of local tours, day excursions, and more extended trips intended to show some of the spectacular scenery is being arranged.

Details and reservation forms will accompany the second announcement.

TRAVEL: Group charter flights will be available to the Congress from a number of centres. They are being coordinated by World Tours Ltd. (affiliated with American Express), 425 Howe Street, Vancouver 1, Canada, who may be contacted for further information.

Details on travel and pre- and post-convention tours will also accompany the second announcement.

FURTHER COMMUNICATIONS: If you wish to receive any further announcements concerning the Congress, please write to: International Congress of Mathematicians, The University of British Columbia, Vancouver 8, Canada. (Cable address: Mathematix.)

MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

MAY MEETING OF THE MICHIGAN SECTION

The annual meeting of the Michigan Section of the MAA was held at Alma College, Alma, Michigan, on May 4 and 5, 1973. There were approximately 150 people in attendance. This was the third time that the Michigan Section has used the two-day format, and the results were again encouraging.

Professor Don Lewis, of the University of Michigan, Chairman of the section, presided at the business meeting. In addition to various reports the Section approved a trial high-school visiting-lectureship program, a Summer Campout in the Upper Peninsula, and an adjustment in the Michigan Mathematics Prize Competition. Officers elected for the coming year were: George Feeman, Oakland University, Chairman; A. V. Martin, Grand Valley State College, Vice-Chairman; and Yousef Alavi, Western Michigan University, Secretary-Treasurer.

The program included invited speakers, contributed papers, student papers, a panel discussion, luncheons, and a dinner banquet.

The following lectures were presented:

Principal Address (Part I): *The Language of Categories in Undergraduate and High School Mathematics*, by P. J. Hilton, Battelle Research Center.

Principal Address (Part II): *The Language of Categories in Undergraduate and High School Mathematics*, by P. J. Hilton, Battelle Research Center.

Hour Address: *Hermite-Birkhoff Interpolation or What $n+1$ conditions determine a polynomial of degree n?*, by R. A. DeVore, Oakland University.

Hour Address: *Infinitesimals and Their Uses*, by L. D. Krugler, The University of Michigan-Flint.

Contributed Papers:

Optimal Rational Starting Approximations for Iterative Schemes, by John Gibson, Alma College.

The Traveling Salesman Problem, by Sam Savage, General Motors Research Laboratories.

Probabilistic Metric Spaces, by Michael Weiss, Wayne State University.

On Panconnected Graphs, by James Williamson, Western Michigan University.

Local Connectivity in Graphs, by Donald Vander Jagt, Grand Valley State College.

Sticks and Strings, by William Hammerle, Oakland University.

Orthogonal Polynomials Associated with Iteration, by Daniel Nussbaum, Saginaw Valley College.

A Nonlinear Property Tax Model, by Michael Skaff, University of Detroit.

Epimorphisms and Monomorphisms in the Homotopy Category, by David Handel, Wayne State University.

Student Papers:

Reconstructing Graphs, by Mary Irvin and Elaine Houtman, Western Michigan University.

Shortest Confidence Interval for Standard Deviation of Normal Distributions, by Roger Crisman, Hope College.

Computer Assisted Instruction Developments in a Course in Discrete Structures, by Ronald Ball, Ted Dyson, John Dayton, Michigan State University.

Symposium on the use of the computer in Mathematics Courses:

Statistics, by Herbert Dershem, Hope College.
Number Theory, by Donald Malm, Oakland University.
Education, by Rose Novey, Saginaw Valley College.
Linear Algebra, by John Van Iwaarden, Hope College and Florida State University.
General Mathematics, by M. A. Rahimi, Michigan State University.
Prospects for the Future, by Elliot Tanis, Hope College.

The Banquet Address was given by Professor George Hay, University of Michigan.

YOUSEF ALAVI, *Secretary-Treasurer*

JUNE MEETING OF THE PACIFIC NORTHWEST SECTION

The annual meeting of the Pacific Northwest Section of the MAA was held at Western Washington State College, Bellingham, on June 15, 1973. The meeting was held jointly with the Northwest sections of AMS and SIAM. There were 95 MAA registrants.

Chairman Sheldon Rio presided over the business meeting at which the following officers were elected: Chairman, John Reay, Western Washington State College; First vice-chairman, Jim Jordan, Washington State University; Second vice-chairman, Norman Barton, Vancouver City College; Chairman elect, Ted White, Everett Community College.

The session on June 15 included the following invited papers:

1. *The most significant advances in convexity in the last decade*, by David Larman, University College, London.
2. *The challenge spectrum in the various mathematical sciences*, by John Kenelly, Clemson University.
3. *Venn diagrams and independent sets*, by Branko Grünbaum, University of Washington.

The session on June 16 was directed toward community colleges and included invited papers:

1. *Evaluation of mathematics education innovation*, by Greg Thomas, Monmouth, Oregon.
2. *Evaluating mathematics education innovative proposals*, by Jennis Bapst, Mt. Hood Community College.
3. *Exchange of experience relative to evaluating mathematics education*, by Howard Zink, Lane Community College.
4. *Programming liberal arts students*, by Daniel Lambert, Boise State College.
5. *Computer computation of integrals*, by Arne Broman, Western Wash. State College.

JAMES CALVERT, *Secretary-Treasurer*

CALENDAR OF FUTURE MEETINGS

Summer Meeting 1974: There will be no joint summer meeting in 1974, in order that mathematicians may attend the International Congress of Mathematicians to be held in Vancouver, British Columbia, August 21–29, 1974.

Fifty-eighth Annual Meeting, Washington, D. C., January 25–27, 1975.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Editorial Director.

ALLEGHENY MOUNTAIN, Allegheny College, Meadville, Pennsylvania, May 3–4, 1974.

FLORIDA, University of Florida, Gainesville, March 8–9, 1974.

ILLINOIS, Knox College, Galesburg, May 10-11, 1974.

INDIANA, Rose-Hulman Institute of Technology, Terre Haute, April 27, 1974.

IOWA, Upper Iowa College, Fayette, April 19, 1974.

KANSAS, Ottawa University, Ottawa, Spring 1974.

KENTUCKY

LOUISIANA-MISSISSIPPI

MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA

METROPOLITAN NEW YORK, College of Mount St. Vincent, Riverdale, April 28, 1974.

MICHIGAN, Central Michigan University, Mount Pleasant, May 3–4, 1974.

MISSOURI, University of Missouri at Rolla, Rolla, March 29–30, 1974.

NEBRASKA, University of South Dakota, Vermillion, April 19–20, 1974.

NEW JERSEY

NORTH CENTRAL, South Dakota State University, Brookings, April 27, 1974.

NORTHEASTERN

NORTHERN CALIFORNIA, Chabot College, Hayward, February 1975.

OHIO, Muskingum College, New Concord, May 3–4, 1974.

OKLAHOMA-ARKANSAS, University of Arkansas, Little Rock, April 5–6, 1974.

PACIFIC NORTHWEST, University of British Columbia, Vancouver, August 21–24, 1974 (business meeting only — no general meeting).

PHILADELPHIA

ROCKY MOUNTAIN, Colorado School of Mines, Golden, April 26–27, 1974.

SEAWAY, Union College, Schenectady, April 27, 1974.

SOUTHEASTERN, University of Tennessee at Knoxville, March 29–30, 1974.

SOUTHERN CALIFORNIA, Harvey Mudd College, Claremont, March 2, 1974.

SOUTHWESTERN, New Mexico State University, Las Cruces, April 5–6, 1974.

TEXAS, University of Texas, Austin, April 5–6, 1974.

WISCONSIN, Marquette University, Milwaukes, May 3–4, 1974.

FUTURE MEETINGS OF OTHER ORGANIZATIONS

AMERICAN ASSOCIATION FOR THE ADVANCEMENT OF SCIENCE, San Francisco, February 24–March 1, 1974.

AMERICAN MATHEMATICAL SOCIETY, Washington, D. C., January 23–26, 1975.

AMERICAN SOCIETY FOR ENGINEERING EDUCATION, Rensselaer Polytechnic Institute, Troy, New York, June 17–20, 1974.

ASSOCIATION FOR COMPUTING MACHINERY, San Diego, California, November 12–16, 1974.

ASSOCIATION FOR SYMBOLIC LOGIC, Biltmore Hotel, New York City, April 12–13, 1974.

FIBONACCI ASSOCIATION

INSTITUTE OF MATHEMATICAL STATISTICS

MU ALPHA THETA, University of Arkansas, Fayetteville, August 4–7, 1974.

NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS, Atlantic City, New Jersey, April 17–20, 1974.

OPERATIONS RESEARCH SOCIETY OF AMERICA, Boston, April 22–24, 1974.

PI MU EPSILON, Western Michigan University, Kalamazoo, August 19–20, 1975.

SCHOOL SCIENCE AND MATHEMATICS ASSOCIATION, Sheraton-Gibson Hotel, Cincinnati, Ohio, November 7–9, 1974.

SOCIETY FOR INDUSTRIAL AND APPLIED MATHEMATICS, Montana State University, Bozeman, June 24–26, 1974.

Most mathematicians and many students can recall at least one authentic mathematical gem, an insight, a proof, a solution so crystal clear and elementary that it can be appreciated by a person with limited mathematical training, yet so elegant, so typical of mathematical reasoning at its best, as to fascinate even the research mathematician. But the store of such gems, while rich, is not large. How many can you remember? The overworked curve of constant width? A tour of the bridges of Königsberg? Euler's prime insight?

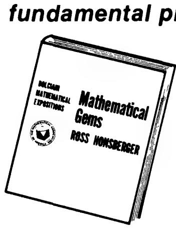
Can a whole book qualify as a gem? The requirements are strict. The appeal must be extremely wide, spanning the entire spectrum of the membership of the Mathematical Association of America, to use a convenient yardstick. Can an *entire book* sustain such breadth and elegance? The products of such inspired expositors as Polya, Halmos, Rademacher and Toeplitz, Courant and Robbins come quickly to mind. But after that the list grows small indeed.

Will others be found? We think so and to give substance to that conviction, the MAA is proud to announce the creation of a new series of books,

The DOLCIANI MATHEMATICAL EXPOSITIONS,

dedicated to the proposition that mathematical ideas of such wide appeal exist and masterful expositions of them can be found by a diligent gemologist. The new series is made possible by a generous gift to the Association from Professor Mary P. Dolciani of the Mathematics Department at Hunter College, herself a talented expositor.

The first of the Dolciani Mathematical Expositions is a convincing existence proof of the fundamental proposition. Its title is most fitting for the inaugural volume, and the identity of the author will surprise no one who knows him. We are proud to announce the publication of:



DOLCIANI MATHEMATICAL EXPOSITIONS, NO. I

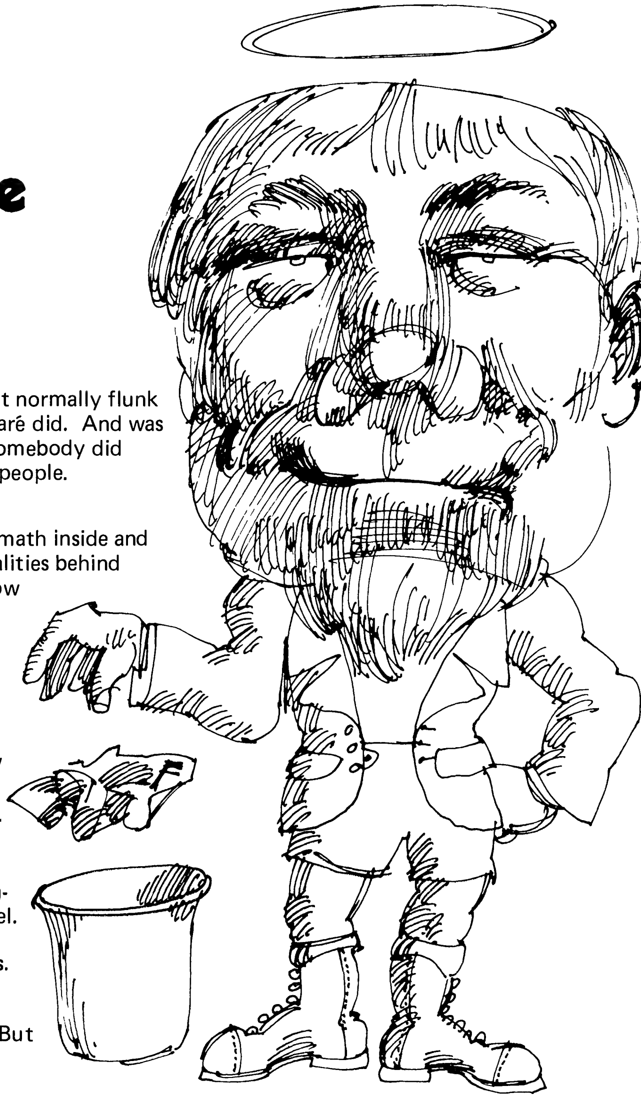
MATHEMATICAL GEMS

by Ross Honsberger

Honsberger has collected a treasure trove of true gems from elementary combinatorics, number theory, and geometry and has written a sprightly essay about each. Each essay contains ***historical commentary, interesting anecdotes, mathematical background,*** and a ***careful discussion*** built around a famous problem, theorem, or the work of an eminent mathematician. It also contains a collection of fascinating and illuminating ***exercises*** related to the topic of the essay. **Mathematical Gems** can be read and appreciated by anyone with an interest in mathematics supported by a good high school training. However, it will delight and challenge students, teachers and research mathematicians alike. If you appreciate mathematical elegance and demonstrations of outstanding ingenuity, you will not be able to put it down once you have opened the covers.

An idea of Gems' many facets is revealed by the titles of Honsberger's essays. See the facing page for a table of contents, complete information on how to order MATHEMATICAL GEMS and comments

Poincaré's brown nose saved him a red face.



A mathematical genius doesn't normally flunk final exams. But Henri Poincaré did. And was passed anyhow. Obviously, somebody did some fast talking to the right people.

What's the point?

At Addison-Wesley we know math inside and out. Its ideas and the personalities behind those ideas. And we know how to get the message across. We publish college texts that blend theory and application, so students come to understand mathematics. Thoroughly. And that makes math a whole new numbers game.

To illustrate, we offer **Mathematical Analysis, Second Edition,** by Tom Apostol. It's a text for advanced calculus students at the junior/senior level. In its first edition, the book was in use for almost 20 years. Now, it's been completely rewritten and updated via new chapters and new problems. But the original flexibility still remains.

Elementary Geometry from an Advanced Standpoint, Second Edition, by Edwin Moise, is another exceptional text. The concentration on thorough, logical elucidation of geometry makes it a good choice for math majors or prospective high school teachers. It follows recommendations of the CUPM Teacher Training Panel too.

Algebra and Trigonometry: A Diagnostic Approach is a workbook supplement written by Robert Sackett of Erie Community College. With it, students can diagnose their own weaknesses and more easily make the transition from the fundamentals of arithmetic to the concepts of algebra and trig.

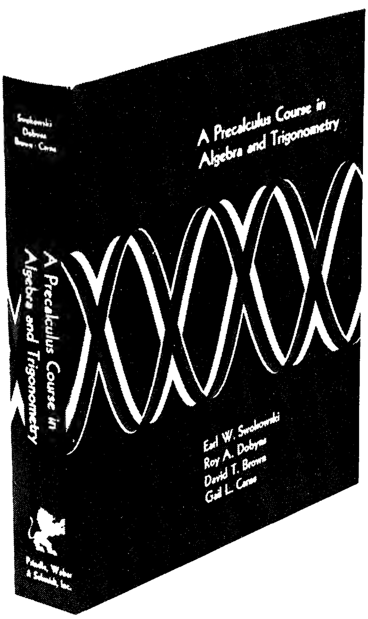
In **First Year Calculus,** Ethan Bolker and Joseph Kitchen have developed a cogent, readable discussion of single variable calculus. Their treatment is intuitive. And they take an informal approach that sometimes refers to math beyond the scope of the course. This keeps the course lively and more interesting. Still, there's solid preparation for a later encounter with formalism. Lots of problems too, both routine and difficult.

So if you want math books that get through to students, you want Addison-Wesley math books.

College Division
ADDISON-WESLEY PUBLISHING COMPANY, INC.
Reading, Massachusetts 01867

WHERE MATH WARMS UP.

A Precalculus Course in Algebra and Trigonometry



Earl W. Swokowski, *Marquette University*
Roy A. Dobyns, *Clayton Junior College*
David T. Brown, *University of Pittsburgh*
Gail L. Carns, *University of Pittsburgh*

A Precalculus Course in Algebra and Trigonometry is a unified course divided into 17 study units. Each unit contains (a) a clear and concise exposition of the theory, generously laced with illustrative diagrams and worked-out examples; (b) a linear program consisting of multiple-choice questions relating directly to the exposition, designed to reinforce the key topics in the study unit; (c) a student study guide that highlights important topics; and (d) sample self-tests accompanied by completely worked-out solutions to the problems, designed to determine whether the student has mastered the material in the study unit.

Whether used in a self-paced course or in a standard lecture presentation, *A Precalculus Course in Algebra and Trigonometry* is a complete and unique course of study.

CONTENTS

Prindle, Weber & Schmidt, Incorporated
20 Newbury Street
Boston, Massachusetts 02116

For each study unit there are 10 forms of test available to the mathematics department (170 tests in all).

Each form of test is similar in scope and difficulty to the Sample Test contained in the study unit.

Ideal for self-pacing. Perfectly suitable for a precalculus lecture course.

Exposition

is an algebraic expression in x. If we find the solution set of
en the solutions of the original equation can be obtained by
to the expression u. The method is illustrated in the following

nple 4. Solve $x^{2/3} + x^{1/3} - 6 = 0$.
ition: If we let $u = x^{1/3}$, then the equ

$$u^2 + u - 6 = 0$$

$$(u + 3)(u - 2) =$$

his we obtain $u = x^{1/3} = -3$ or u
ns give us $x = -27$ or $x = 8$ (Why?
eck: Letting $x = -27$ in the given e
$7)^{1/3} - 6 = 9 - 3 - 6 = 0$. Thus
is a solution. Hence the solution se

ample 5. Solve $x^4 - 3x^2 + 1 = 0$.
lution: Letting $x^2 = u$, we have
atic formula, we obtain

Programmed Supplement

Q73 Which one of the following is the solution set of the ineq $> 7x - 10$?

$\{x\|x < 7\}$	Frame A
$\{x\|x < 6\}$	Frame B
$\{x\|x > 7\}$	Frame C

A YOUR ANSWER: $\{x|x < 7\}$.

You are correct.

$$5x + 4 > 7x - 10$$
$$-2x > -14 \quad \text{adding } -7x - 4 \text{ to both expressions}$$
$$x < 7 \quad \text{multiplying by } -\tfrac{1}{2} \text{ and using Theoren}$$

B YOUR ANSWER: $\{x|x < 6\}$.

Your calculations are off. Return to frame Q73 and try aga

C YOUR ANSWER: $\{x|x > 7\}$.

No. You applied Theorem (2.9) incorrectly. If you multiply by a real number, then the inequality is reversed.

$$5x + 4 > 7x - 10$$
$$-2x > -14 \quad \text{adding } -7x - 4 \text{ to both expressions}$$
$$x < 7 \quad \text{multiplying by } -\tfrac{1}{2} \text{ and using Theorem (2.9}$$

After thinking about thi... Q73 and select the correct

Things to Study

OMEWORK PROBLEMS

ction 4: 1–5, 9, 11, 13, 15, 17, 19
ction 5: 1, 3, 5, 7, 9, 10, 13, 15, 17, 19
ction 6: 1, 3, 5, 7, 9, 11, 12, 19, 20
ction 7: 9, 11, 13, 15, 17 (written)

INGS TO STUDY

fore taking the sample test, review the f

Remember, if you square an equation o may introduce new solutions. Thus yo not extraneous.

nple 1.

$\sqrt{2x}$
$2x +$

Sample Test

Find the solution(s) of each of the following equations or in

1. $\sqrt{16 - 15x} = 2x + 3$

2. $\sqrt{\dfrac{x}{3} + \dfrac{2}{3}} = \sqrt{\dfrac{1}{2} - \dfrac{x}{4}}$

THE AMERICAN MATHEMATICAL MONTHLY

(FOUNDED IN 1894 BY BENJAMIN F. FINKEL)

THE OFFICIAL JOURNAL OF

THE MATHEMATICAL ASSOCIATION OF AMERICA

VOLUME 81 NUMBER 3

CODEN: AMMYAE

CONTENTS

(*Continued on inside cover*)

MARCH 1974

NOTICE TO AUTHORS

Research papers per se are unsuitable; see statement of policy (Vol. 81, p. 1).

Please follow the format in current issues of the MONTHLY. Manuscripts must be legibly typewritten or reproduced from typewritten copy, double spaced with wide margins. Three copies should be submitted to the appropriate editor and one kept by the author as protection against loss. The author's full address *must* appear at the end of the manuscript.

Backlog: Main Articles 17 months, Math. Notes 29 months, Research Problems 19 months, Classroom Notes 24 months, Math. Education 19 months.

EDITORIAL CORRESPONDENCE AND MAIN ARTICLES: to ALEX ROSENBERG, Department of Mathematics, Cornell University, Ithaca, N.Y. 14850; NOTES, etc.: to the corresponding Associate Editor; ADVERTISING CORRESPONDENCE: to RAOUL HAILPERN, Mathematical Association of America, SUNY at Buffalo, Buffalo, N. Y. 14214; CHANGE OF ADDRESS and SUBSCRIPTIONS: to A. B. WILLCOX, Mathematical Association of America, 1225 Connecticut Ave., N.W., Washington, D.C. 20036.

Annual dues for members of the Association (including a subscription to the American Mathematical Monthly) are $14.00 for each of the first two years of membership and $18.00 thereafter. Student Membership is available with annual dues of $9.00. For nonmembers the subscription price is $18.00.

PUBLISHED BY THE ASSOCIATION at Washington, D. C., and Menasha, Wisconsin, during the months of January, February, March, April, May, June–July, August–September, October, November, December.

Second-class postage paid at Washington, D. C., and additional mailing offices.

PRINTED IN THE UNITED STATES OF AMERICA

SINGULARITIES AND PLANE MAPS

JAMES CALLAHAN

1. Introduction. Take a piece of fabric or sheet rubber, crumple it, and press it on a flat surface. What different shapes can the sheet assume around a given point? In general, three possibilities occur:

(*a*) the sheet lies flat and smooth;

(*b*) the point appears on a fold line of the sheet;

(*c*) a pleat is being formed at the point.

Figure 1 gives examples of each kind of point.

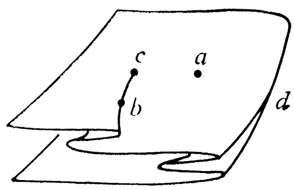


FIG. 1

A bit of experimenting will show that there are other possibilities; point *d* in Figure 1 is a new type, for instance. Further experimenting, however, leads to the conclusion that arbitrarily small alterations in the position of the fabric can eliminate any points not of the first three types. The new shape at *d* disappears, for example, as soon as the bottom fold is pulled to the right by any amount whatsoever (Figure 2).

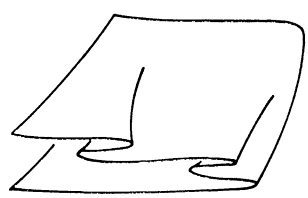


FIG. 2

In fact, by comparing these figures we can see that the point *d* arises when a pleat is run into a fold. On the other hand, points of types *a*, *b*, and *c* cannot be made to disappear by small perturbations; the position of a pleat or a fold can be affected, but not its presence.

Points of type *a* are the simplest and most common. We shall call them *regular* points. All other points are *singular*, to indicate both their scarcity and special

nature. Our observations about the pressed cloth can now be summarized: almost all points are regular, and the only singular points which are *stable* (i.e., unremovable by small perturbations) are folds and pleats. If we think of the application of the sheet to a flat surface as a function mapping one plane to another — Brouwer's fixed point theorem for a disc is sometimes illustrated this way — the preceding observations amount to an informal paraphrase of a theorem proved by H. Whitney in 1955 [**27**]. Whitney and R. Thom, a major contributor to the subject whose work began about this same time, were interested in treating the following questions, among others (see also [**1**, **17**, **28**]):

According to the inverse function theorem, a differentiable map $f: \mathbb{R}^2 \to \mathbb{R}^2$ is a homeomorphism in a neighborhood of a point at which the Jacobian of the map is non-zero; what structure does f have around a point where the Jacobian *is* zero?

Is there a finite classification of the different structural patterns (e.g., homeomorphism, fold, pleat, etc.) which a function can exhibit locally?

The term homeomorphism has a standard meaning, but what is a fold or a pleat; specifically, is there a function for each structural type which has some simple, or normal, form? (Later we shall see that the identity map is the normal form of a homeomorphism.)

How is a fold, for example, to be detected in a function which is not in normal form?

Finally, a normal form is supposed to be stable in the sense that small perturbations do not alter its structure; what does it mean to perturb a function?

Suitably modified, these questions can be applied to the larger collections of differentiable functions $f: U \to \mathbb{R}^p$, where U is an open subset of $\mathbb{R}^n$; or $f: N \to P$, where N and P are manifolds. However, we shall be concerned mainly with what happens in low dimensions, because those results can be visualized and are typical of more general ones.

2. Equivalence of Maps. The shape, or structure, or geometric character of a function around a point is the central idea of the preceding section. Analogies with applications of sheets to surfaces are suggestive, but they remain too imprecise to be useful. We shall have to trade intuition for precision, a common practice in mathematics; a *shape* will be defined to be a class of geometrically equivalent functions, just as a cardinal number is a class of equivalent sets, the equivalence determined by the notion of a one-to-one correspondence. With this understood, the problem is now to define an equivalence relation on functions which captures essential geometric features as well as a one-to-one correspondence captures the idea of number. The following example can suggest how to proceed.

Consider the two maps from $\mathbb{R}^2$ to $\mathbb{R}^2$ given in terms of coordinates as

$$f: \begin{array}{l} u = x \\ v = y^2 \end{array} \qquad g: \begin{array}{l} \phi = \xi - \eta - (\xi + \eta)^2 \\ \psi = 2(\xi - \eta) + (\xi + \eta)^2. \end{array}$$

As Figure 3 shows, the map f folds the xy-plane along the x-axis and applies this doubled-up sheet to the half-plane $v \geqq 0$, the fold line itself appearing on the u-axis. Lines parallel to the x-axis end up parallel to the u-axis, but in general the distance between a pair of lines is different from the distance between its image pair. Lines parallel to the y-axis are bent double but their spacing is preserved.

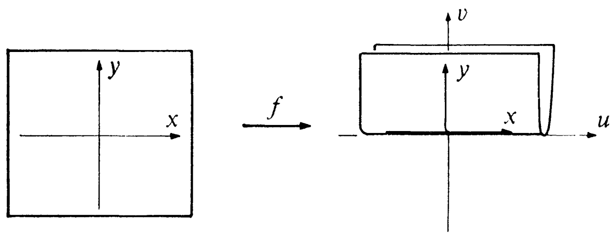


FIG. 3

The map g is only slightly harder to describe. Since

$$\psi - 2\phi = 3(\xi + \eta)^2 \geqq 0,$$

we conclude first that $\psi = 2\phi$ is the image of the line $\xi + \eta = 0$, and second that the rest of the image of the $\xi\eta$-plane must lie to the northwest of $\psi = 2\phi$. This means that g folds the $\xi\eta$-plane along the line $\xi + \eta = 0$ (the dotted line in figure 4) and applies this doubled-up plane to the half-plane $\psi \geqq 2\phi$. The image of the $\xi - \eta$ coordinate grid is curvilinear; a line parallel to either axis is mapped to a parabola tangent to $\psi = 2\phi$.

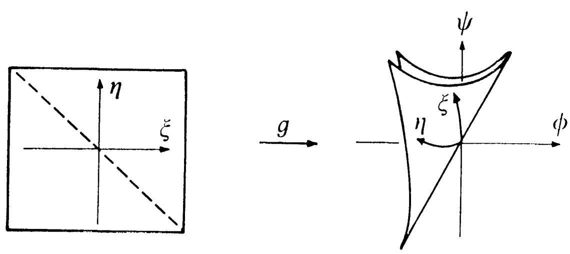


FIG. 4

So both these maps are folds. They have the same geometric character, and differ only in the form of their coordinate expressions. If this really is the only difference, it should be possible, by changing coordinates, to make g look like f even formally. We can do this using the particular coordinate changes

$$h\colon \begin{array}{l} x = \xi - \eta \\ y = \xi + \eta \end{array} \qquad k\colon \begin{array}{l} u = \frac{1}{3}(\phi + \psi) \\ v = \frac{1}{3}(\psi - 2\phi). \end{array}$$

Starting from the original map

$$g: \begin{aligned} \phi &= \xi - \eta - (\xi + \eta)^2 \\ \psi &= 2(\xi - \eta) + (\xi + \eta)^2, \end{aligned}$$

we get

$$k \circ g \circ h^{-1} = f: \begin{aligned} u &= \tfrac{1}{3}[\xi - \eta - (\xi + \eta)^2 + 2(\xi - \eta) + (\xi + \eta)^2] = x \\ v &= \tfrac{1}{3}[2(\xi - \eta) + (\xi + \eta)^2 - 2(\xi - \eta) + 2(\xi + \eta)^2] = y^2. \end{aligned}$$

In other words, the diagram

$$\begin{tikzcd} \mathbb{R}^2 \; ; \; x, y \arrow[r, "f"] & \mathbb{R}^2 \; ; \; u, v \\ \mathbb{R}^2 \; ; \; \xi, \eta \arrow[u, "h"] \arrow[r, "g"'] & \mathbb{R}^2 \; ; \; \phi, \psi \arrow[u, "k"'] \end{tikzcd}$$

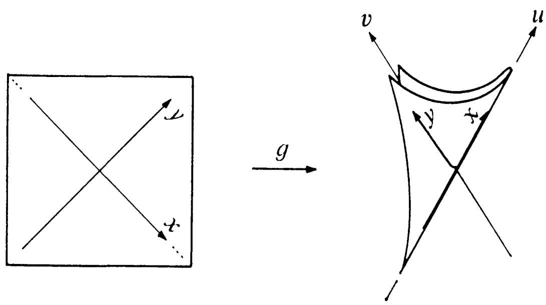


FIG. 5

is commutative. Figure 5 shows the source and target of g with their new coordinate systems. A line parallel to the x-axis has its image parallel to the u-axis, while a line parallel to the y-axis ends up bent double but still parallel to the v-axis. Except for the "babushka" look of the image (which happens because the source is still cropped parallel to the now-vanished ξ- and η-axes) and the fact that the $u - v$ frame is not rectilinear (again, because it is superimposed on the vanished rectilinear $\phi - \psi$ frame), this map is visually indistinguishable from f.

We now have a preliminary definition: two maps f and g are *equivalent* if there are coordinate changes h and k making the commutative diagram

$$\begin{tikzcd} \mathbb{R}^n \arrow[r, "f"] & \mathbb{R}^p \\ \mathbb{R}^n \arrow[u, "h"] \arrow[r, "g"'] & \mathbb{R}^p \arrow[u, "k"'] \end{tikzcd} .$$

This definition seems fine, but what exactly is a coordinate change? Let us first notice that the spaces on which these maps are defined do not have a preordained inner product structure, so that rectilinearity of a coordinate system is an accidental feature of our representation of it. Rectilinearity is rather a relationship between two different coordinate systems, and exists when the coordinate change is given by an orthogonal linear map. Although the coordinate changes in our example were linear, there is no reason to make this assumption generally.

In fact, the theory needs a rich collection of coordinate changes to make sure that two maps with the same geometric structure will be equivalent according to the definition just given. Another example will illustrate this point. Consider the real-valued functions

$$f: u = |x| \qquad\qquad g: \phi = \xi^2.$$

These maps have the same shape. Each folds its source double at the origin. Their difference is metric: g squares the distance of a point from the origin while f leaves it unaltered. Either pair of coordinate changes

$$h: x = \begin{matrix} \xi^2, & \xi \geqq 0 \\ -\xi^2, & \xi < 0 \end{matrix} \qquad\qquad k: u = \phi$$

or

$$h: x = \xi \qquad\qquad k: u = \begin{matrix} \sqrt{\phi}, & \phi \geqq 0 \\ \phi, & \phi < 0 \end{matrix}$$

yields a commutative diagram:

$$\begin{tikzcd} \mathbb{R} \arrow[r, "f"] & \mathbb{R} \\ \mathbb{R} \arrow[u, "h"] \arrow[r, "g"'] & \mathbb{R} \arrow[u, "k"'] \end{tikzcd}.$$

These functions are portrayed in figure 6, not by graphs, but by analogy with the

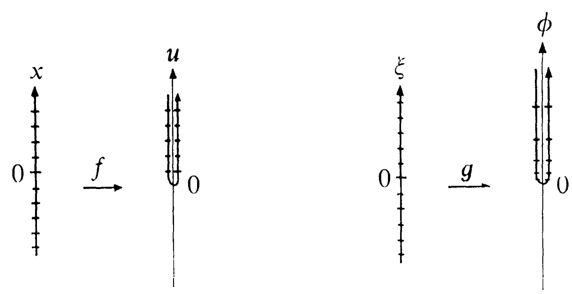


FIG. 6

earlier figures. The purpose of the cross-marking is to illustrate metric differences. For instance, notice how g "bunches up" points near the origin; this is another, and for us more useful, way of saying that the usual graph of g is almost horizontal near the origin.

It is not possible for f and g to be equivalent using linear coordinate changes. The reason is this: on the positive half-axis, f itself is linear, and would continue to be under any linear coordinate changes. Since g is non-linear, no commutative diagram can exist. But these maps do have the same shape and so ought to be equivalent. Henceforth, we allow arbitrary homeomorphisms to serve as coordinate changes; this defines *topological equivalence.*

Maps are said to be *smoothly* or *analytically equivalent* if the homeomorphisms linking them are smooth or real analytic, respectively. These relations yield progressively finer discriminations; that is, equivalence classes are smaller and more numerous. For example, the functions $f(x) = |x|$ and $g(x) = x^2$ are topologically but not smoothly (or analytically) equivalent. To see this, notice that any function smoothly equivalent to g will, like g itself, be differentiable. This rules out f. In effect, smooth equivalence, unlike topological, is sensitive to the "bunching up" which g creates near the origin. We shall deal with both kinds of equivalence, often without distinguishing between them, although topological intuition is simpler because "a fold is a fold is a fold."

The following definition summarizes the discussion thus far and uses the most general settings in which the terms make sense.

DEFINITION. *Let $f, g\colon X \to Y$ be continuous maps of topological spaces. Then f and g are* **topologically equivalent** *if there are homeomorphisms h and k which make the following diagram commutative.*

$$\begin{tikzcd} X \arrow[r, "f"] & Y \\ X \arrow[u, "h"] \arrow[r, "g"'] & Y \arrow[u, "k"'] \end{tikzcd}$$

If X and Y are smooth manifolds and f and g are smooth maps, then f and g are **smoothly equivalent** *if there are smooth diffeomorphisms h and k which make the above diagram commutative.*

REMARK. In this paper, **smooth** means infinitely differentiable. There are other variants of this definition for C^r or analytic manifolds and maps, but the two given will suffice here. A **diffeomorphism** is a smooth homeomorphism which has a smooth inverse. A variety of treatments of the basic ideas of differential analysis which appear in this paper can be found in the texts by Buck [**4**], Hu [**6**], Milnor [**12, 13**], Spivak [**15**], and Wallace [**25**].

Until now we have discussed the shape of a function on its whole domain, even though it is useful and perhaps more natural to work locally. For example, it is not difficult to show that $y = x^3 - 3x$ has folds at the points $x = \pm 1$ (i.e., is equivalent to $y = x^2$ in a neighborhood of each point) and in a neighborhood of every other point is a homeomorphism. On the other hand, this function is globally equivalent to neither a homeomorphism nor a (single) fold. In fact, polynomials with different numbers of relative extrema will be mutually inequivalent, so there are infinitely many topological equivalence classes of functions, considered globally. The local situation is, on the contrary, quite simple. Except for cases to be dismissed later as unstable, there are only two equivalence classes of local real-valued functions of a real variable: homeomorphisms and folds. After all, if a string is crumpled and then squeezed into a line, at any point the string is either straight or folded back on itself.

In turning our attention to the local behavior of a function f on a neighborhood of a point x in its domain, it is natural to identify all functions which agree with f near x. What results when this is done is called the **germ of f at x**; it is an equivalence class of functions, each defined on some neighborhood of x, and two such functions are said to be (germ-) equivalent if, when restricted to some common subneighborhood of x, they agree. (See [**6**, p. 18] for details.) We denote the germ of f at x by $(f; x, y)$; the point x is called the **source** of the germ and $y = f(x)$ the **target**. The target is well-defined because all functions which are germ-equivalent to f at x take the same value at x. We write $(f; x, y)\colon X, x \to Y, y$ if $f\colon X \to Y$ or even if f is defined only on a neighborhood of x in X. The composite of two germs is well-defined, and we have

$$(g; y, z) \circ (f; x, y) = (g \circ f; x, z).$$

DEFINITION. *The germs $(f_1; x_1, y_1)$ and $(f_2; x_2, y_2)$ are* **topologically (smoothly) equivalent** *if there are germs $(h; x_1, x_2)$ and $(k; y_1, y_2)$ of homeomorphisms (smooth diffeomorphisms) which make the following diagram commutative.*

$$\begin{tikzcd} X, x_2 \arrow[r, "(f_2;\, x_2, y_2)"] & Y, y_2 \\ X, x_1 \arrow[u, "(h;\, x_1, x_2)"] \arrow[r, "(f_1;\, x_1, y_1)"'] & Y, y_1 \arrow[u, "(k;\, y_1, y_2)"'] \end{tikzcd}$$

For example, the pair of germs $(y = x^2; 0, 0)$ and $(y = x^3 - 3x;\ 1, -2)$ are smoothly and topologically equivalent, as are the pair $(y = x; 0, 0)$ and

$$(y = x^3 - 3x; a, a^3 - 3a),$$

where a is any real number different from ± 1.

It follows from the definitions that the two germs $(f_1; x_1, y_1)$ and $(f_2; x_2, y_2)$ are equivalent if and only if the functions f_1 and f_2 are equivalent, when restricted to suitable neighborhoods of x_1 and x_2 respectively. We shall often say that a map f_1

at x_1 is equivalent to another map f_2 at x_2 when the corresponding germs are equivalent.

Thus the language of map-germs provides a way of discussing local properties of functions, and we use it in the following definition to describe what was vaguely and variously called the shape, geometric form, or structural pattern of a function at a point.

DEFINITION. *The* **local topological** (*respectively*, **smooth**) **type** *of the function* $f: X \to Y$ *at the point* x *is the topological* (*respectively*, *smooth*) *equivalence class of the germ* $(f; x, f(x))$.

3. Singularities. The problem raised in the first section — how to analyze and classify the local topological types of arbitrary continuous functions — is much too difficult, and the theory which has developed over the last twenty years seems to be concerned exclusively with the much smaller category of differentiable manifolds and maps. This is quite natural. For one thing, local geometry is the same at every point on a manifold, so any differences in the local behavior of a function are to be attributed to the function itself. But most of all, it has been the infinitesimal version of a function, as given by its differential, which has helped to clarify local behavior. This section will develop these ideas.

Let $f: N \to P$ be a smooth map, where N and P are smooth manifolds of dimensions n and p, respectively. Every neighborhood of a point x on N has a subneighborhood diffeomorphic to $\mathbb{R}^n$, and the diffeomorphism can be chosen to carry x to 0. Similar statements are true for any point y in P. Consequently every map-germ $(f; x, y): N, x \to P, y$ has a smooth representative (which we also call f) of the form $f: \mathbb{R}^n, 0 \to \mathbb{R}^p, 0$. It is customary to call such a function a **local map** because, although it is defined on all of $\mathbb{R}^n$, we are interested only in its germ-equivalence class at the origin. Hence we shall say that two local maps are topologically or smoothly equivalent even if they are equivalent only when restricted to some smaller neighborhoods of the origin.

Because we shall treat mainly local questions in the remainder of this survey, we can concentrate on maps $f: \mathbb{R}^n \to \mathbb{R}^p$ of Euclidean spaces (usually without the local map restriction $f(0) = 0$).

So suppose that $(x_1, \cdots, x_n)$ and $(y_1, \cdots, y_p)$ are coordinates for $\mathbb{R}^n$ and $\mathbb{R}^p$ respectively, and that the coordinate functions for $f: \mathbb{R}^n \to \mathbb{R}^p$ are

$$y_i = f_i(x_1, \cdots, x_n) \qquad i = 1, \cdots, p.$$

Then the differential of f at a point $b = (b_1, \cdots, b_n)$ is the linear map $\mathrm{df}_b: \mathbb{R}^n \to \mathbb{R}^p$ given by the $p \times n$ matrix

$$\left(\frac{\partial f_i}{\partial x_j}(b)\right) \qquad \begin{array}{l} i = 1, \cdots, p \\ j = 1, \cdots, n. \end{array}$$

The rank of df_b is the rank of this matrix; the maximum value it can assume is the

smaller of the two numbers n and p. We shall also call the rank of df_b the **rank of** f **at** b.

DEFINITION. *If f has maximal rank at b then b is a* **regular point** *of f; otherwise it is a* **singular point**, *or a* **singularity**, *of f. If the rank of f at b is less than p, b is a* **critical point** *of f.*

The terms *singular* and *critical* are often used interchangeably, perhaps because they coincide for $n \geqq p$. Sard's fundamental theorem [**13**], for example, more strictly applies to critical points. On the other hand, questions about equivalence of maps lead to the study of singularities, as the first theorem shows.

THEOREM 1. *If b is a regular point of $f : \mathbb{R}^n \to \mathbb{R}^p$, then the germ of f at b is smoothly (and hence topologically) equivalent to df_b, which in turn is equivalent to one of the linear maps (normal forms)*

$$\begin{aligned} y_1 &= x_1 \\ &\vdots \\ y_p &= x_p \end{aligned} \quad \text{if } n \geqq p; \qquad \begin{aligned} y_1 &= x_1 \\ &\vdots \\ y_n &= x_n \\ y_{n+1} &= 0 \\ &\vdots \\ y_p &= 0 \end{aligned} \quad \text{if } n < p.$$

The proof of the first part uses the inverse function theorem and is a generalization of it. The second part, giving normal forms for linear maps, is a standard result from linear algebra [**2**, page 234]. The theorem implies that all other smooth or topological types must appear at singular points, and the goal of the local theory thus becomes the classification of map singularities.

Unlike what is true for it at a regular point, a local map is generally not equivalent to its differential or any other linear map at a singular point. For example, the folding map: $y = x^2$, and its derivative at the origin: $y = 0$, are inequivalent. Nevertheless, the rank of a map at a singular point is a useful invariant, as the next theorem shows. Its proof follows directly from the definitions.

DEFINITION. *A map f: $\mathbb{R}^n \to \mathbb{R}^p$ has a* **singularity of type** S_k *at the point b if the rank of f at b is* $\min(n, p) - k$. *The number k is called the* **deficiency** *of the singularity, and a regular point has deficiency* 0.

THEOREM 2. *Two smoothly equivalent local maps have the same singularity type S_k at* 0.

Both the topological version and the converse of this theorem are false, the topological version because $y = x$ and $y = x^3$ are topologically equivalent while 0

is a regular point for the first map and an S_1 singularity for the second. The converse is false because $y = x^2$ and $y = x^3$ are topologically and smoothly inequivalent even though 0 is an S_1 singularity for both.

Thus the classification of local types by the singularity classes S_k remains incomplete. There are at least two ways to improve the situation: we can use second and higher order derivatives to make finer distinctions between singularities, and we can eliminate some local types as unstable. The main lemma of Morse theory, which follows, is a good example of the first procedure. We shall take up the question of stability in the next section.

Let us now see how Morse theory contributes to the classification of singularities of real-valued maps. If $b = (b_1, \cdots, b_n)$ is a singular point (or critical point, as it is more usually called) of $f: \mathbb{R}^n \to \mathbb{R}$, $y = f(x_1, \cdots, x_n)$, then the **Hessian of** f at b is defined to be the quadratic form

$$\mathrm{Hf}_b(z) = \sum_{i,j=1}^{n} \frac{\partial^2 f}{\partial x_i \, \partial x_j}(b)\, z_i z_j, \quad z = (z_1, \cdots, z_n).$$

If the matrix of coefficients of this form is invertible, the critical point is said to be **non-degenerate,** and in any case the number of negative eigenvalues of the matrix is called the **index** of the critical point. We can now state the relevant part of Morse's lemma. Notice the similarities to Theorem 1.

THEOREM 3 [**12**, page 6]. *If the point b is a non-degenerate critical point of index r for the map $f: \mathbb{R}^n \to \mathbb{R}$, then the germ of f at b is smoothly equivalent to its Hessian* Hf_b, *which in turn is equivalent to the normal form*

$$y = -x_1^2 - \cdots - x_r^2 + x_{r+1}^2 + \cdots + x_n^2.$$

The target diffeomorphism $w = -y$ makes the normal forms of index r and $n - r$ equivalent, so only about half these forms represent distinct local types. The precise number is $\frac{1}{2}(n+1)$ or $\frac{1}{2}(n+2)$, whichever is an integer. This is just the number of quadratic n-forms having different non-negative signature $n - 2r$ [**2**, page 272]. Incidentally, Theorem 3 proves that the map-germs $(x^2; 0, 0)$ and $(x^3 - 3x; \pm 1, \mp 2)$ of section 2 are smoothly equivalent.

4. Stability. The simplest local map not yet classified is $y = x^3$, which has a degenerate singularity at the origin. This map is unstable in the sense that the singularity can be made to disappear by altering the function slightly to one of the form $y = x^3 + \varepsilon x$. (See figure 7; the two singularities which appear when $\varepsilon < 0$ are non-degenerate.)

The kind of instability x^3 exhibits at the origin can not occur at a regular point or a non-degenerate critical point. For suppose b is a regular point of a function $f: \mathbb{R} \to \mathbb{R}$; then $f'(b) \neq 0$. If g is a function near f and g' is near f' (using the terminology of the introduction, we could call g a perturbation of f), then $g'(b) \neq 0$ as well, so b is a regular point of g. There is a similar argument for a function $f: \mathbb{R}^n \to \mathbb{R}^p$;

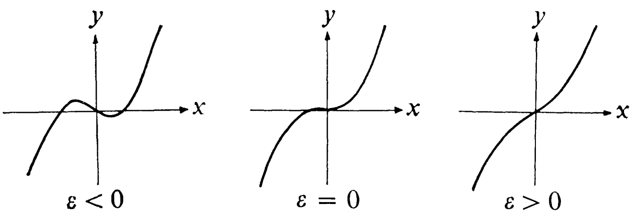


FIG. 7

this time a certain square submatrix of df_b has non-zero determinant, and this property is shared by any function close to f (if its differential is likewise close to df).

It is only slightly more difficult to show that a non-degenerate critical point is stable. Again for simplicity let us take $y = x^2$ as the model, and suppose that g approximates $y = x^2$ out to the second derivative. Thus g'' is everywhere positive because it is near $y \equiv 2$, and so g' is monotone increasing. Since g' is near $y = 2x$ and is increasing, it has a unique root b which is therefore the only critical point of g. This critical point is non-degenerate because $g''(b)$ is positive. Hence the map-germs $(y = x^2; 0, 0)$ and $(g;\, b, g(b))$ are smoothly equivalent (see figure 8) and $y = x^2$ is stable.

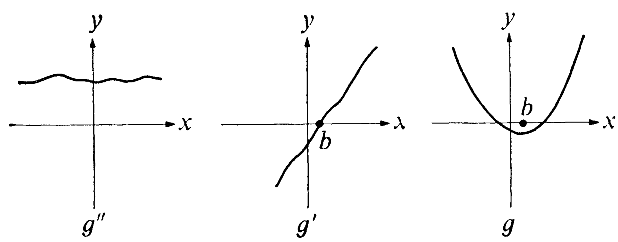


FIG. 8

We should point out here that g must approximate $y = x^2$ out to the second derivative in order to guarantee that it has the same local type as $y = x^2$. For we can choose a smooth function g which is zero on a small neighborhood of the origin but which approximates $y = x^2$ and its first derivative (see figure 9, and notice that g'' does *not* approximate $y \equiv 2$). Since any function equivalent to g will also be constant on some open set, g does not have the topological type of $y = x^2$.

We are saying that a function is stable if its local type is the same as all of its neighbors', and functions are neighbors only if they, and all their derivatives up to some predetermined order, are close. This idea of stability accords with common usage: an organism in a certain environment is stable if the organism's gross behavior is unaffected by mild changes in that environment. (Of course the feedback mechanisms which produce stability for the organism have no mathematical analog. However, the connection between mathematics and biology is not so tenuous as it may appear; the theory of map singularities and stability is the basis of Thom's work on morphogenesis [**18, 19, 21, 23**]).

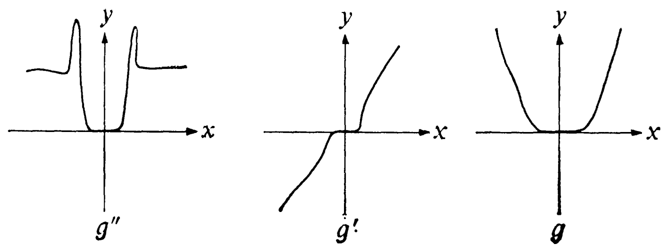


FIG. 9

Let us get back to the question of neighboring functions. The various function space topologies which are used to study stability are somewhat complicated (see [7; **9**, part II]), but we can illustrate their general character with a special case. Let $C^\infty(N, P)$ denote the collection of smooth maps $f\colon N \to P$; in particular, $C^\infty(\mathbb{R}^n, \mathbb{R}^p)$ is a vector space. For any $m = 0, 1, \cdots, \infty$, the **uniform C^m-topology** on $C^\infty(\mathbb{R}^n, \mathbb{R}^p)$ is determined by a neighborhood base at the origin. The basic sets are

$$B(\varepsilon) = \left\{ f \text{ in } C^\infty : \max \left| \frac{\partial^q f_i}{\partial^{q_1} x_1 \cdots \partial^{q_n} x_n} \right| < \varepsilon \right\}$$

for every $\varepsilon > 0$. Here f_i are the coordinate functions of f, and $q = q_1 + \cdots + q_n$; the maximum is taken over all x in $\mathbb{R}^n$, all i between 1 and p, and all partial derivatives of order $q \leqq m$.

The ordinary **C^m-topology** on $C^\infty(\mathbb{R}^n, \mathbb{R}^p)$ is the topology of convergence on compact sets, using the uniform C^m-topology. With reasonable care, these topologies can be carried over to $C^\infty(N, P)$. Essentially, every function f in $C^\infty(N, P)$ has a neighborhood which looks like $C^\infty(\mathbb{R}^n, \mathbb{R}^p)$. From now on, we shall assume that each of the function spaces $C^\infty(N, P)$ and $C^\infty(\mathbb{R}^n, \mathbb{R}^p)$ is endowed with its C^∞-topology.

DEFINITION. *A map $f\colon N \to P$ is* **stable** (*respectively,* **topologically stable**) *if every map g sufficiently close to f in $C^\infty(N, P)$ is smoothly* (*respectively, topologically*) *equivalent to f.*

In other words, a map is stable if it is an interior point of its equivalence class in $C^\infty(N, P)$. The notion of stability is therefore a general kind of link between topology and algebra.

DEFINITION. *Suppose b is a point of $\mathbb{R}^n$. A map $f\colon \mathbb{R}^n \to \mathbb{R}^p$ is* (**topologically**) **stable at** *b if, for every map g sufficiently close to f, there is a point b_g for which the map-germs $(f; b, f(b))$ and $(g; b_g, g(b_g))$ are smoothly* (*topologically*) *equivalent.*

The local definition is awkward, but the example $y = x^2$ has already shown why the source b_g of the map-germ of g must in general vary with g.

Notice that, either locally or globally, a stable map is topologically stable.

Although the global theory is quite extensive, we shall concentrate on local stability. It is somewhat simpler to state local results, and much of the global theory

amounts to patching together local facts. For example, a real-valued function is (globally) stable if it is stable at every point of its domain and if the images of its critical points are all different. To see why this last condition is needed, consider $f(x) = x^4 - 2x^2$. The critical points $x = \pm 1$ have the same image $f(\pm 1) = -1$, so any function smoothly equivalent to f will have a pair of critical points with the same image. (Remember, coordinate changes do not alter geometric facts.) However, there are functions arbitrarily near f—for example, $g(x) = x^4 - 2x^2 + \varepsilon x$—whose critical points all have different images. Such functions are globally inequivalent to f, which is consequently unstable. The situation is similar in higher dimensions. For instance, the map which is implied by Figure 1 is unstable because a pleat and fold have the same image at d. The perturbation in figure 2 is stable because all pleats and folds have been properly separated.

Another reason for working locally is to get back to the study of singularities. Toward the end of the last section it was argued that there are too many singularity types for a decent classification to exist. Eliminating unstable singularities (i.e., singularities of maps which turn out to be unstable) should make the classification problem easier, but two questions arise. First, is the elimination not too drastic? More precisely, are the stable types dense (by definition, they form an open set), so that an unstable singularity can always be "slipped into" the classification by altering it slightly into a stable type? The second question is obvious: Is the elimination worthwhile? Is there a complete classification of stable singularities?

The classical results provide affirmative answers for the specific dimensions they treat. The results are summarized in the following theorem.

THEOREM 4. *The table below is an exhaustive list of stable types of local maps* $f: \mathbb{R}^n, 0 \to \mathbb{R}^p, 0$ *for three different collections of dimension pairs* n, p. *In each of these three cases, the stable types are dense.*

Case		Type	Normal form
$2n \leqq p$	Whitney [26]	regular point	$y_i = x_i \quad i = 1, \cdots, n$ $y_i = 0 \quad i = n+1, \cdots, p$
$p = 1$	Morse [12]	regular point	$y = x_1$
		non-degenerate critical point of index j	$y = -x_1^2 - \cdots - x_j^2 + x_{j+1}^2 + \cdots + x_n^2$
$n = p = 2$	Whitney [27]	regular point	$y_1 = x_1$ $y_2 = x_2$
		fold point	$y_1 = x_1$ $y_2 = x_2^2$
		cusp point	$y_1 = x_1$ $y_2 = x_2^3 - x_1 x_2$

The third case is the theorem of Whitney on plane maps which was informally described in the introduction. Let us study the normal form of the cusp point:

$$f: \begin{array}{l} y_1 = x_1 \\ y_2 = x_2^3 - x_1 x_2. \end{array}$$

The differential is

$$df = \begin{pmatrix} 1 & 0 \\ -x_2 & 3x_2^2 - x_1 \end{pmatrix},$$

so every point on the parabola $3x_2^2 = x_1$ has singularity type S_1, and every other point is regular.

The map f carries any line $x_1 = a$ into another line $y_1 = a$ homeomorphically if a is negative, because there are no singular points in the half-plane $x_1 < 0$. If a is positive, there are two singular points on the line $x_1 = a$, and these are both folds. Figure 10, which is not quite a graph of f but can be interpreted like one, shows that the x-plane is carried onto the y-plane by pleating it at the origin. The pair of folds forming the edges of the pleat lie on the two halves of the parabola. Their image is a curve with a cusp, which explains Whitney's name for the type.

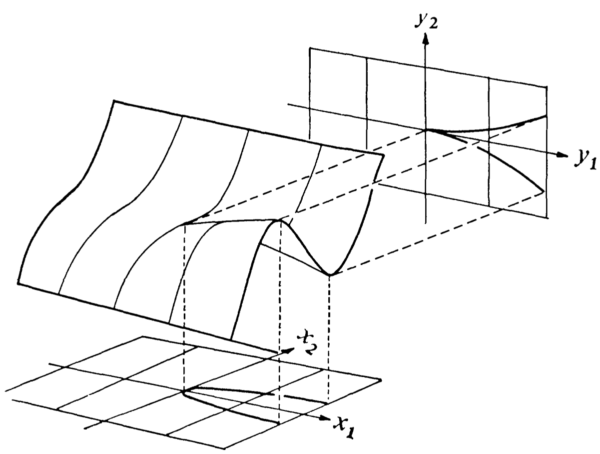


FIG. 10

We can use this example to show that the division into the singularity classes S_k can be refined to give more information. Notice first that all points on the singular set $3x_2^2 = x_1$ are ordinary fold points, except for the origin, where the local type of the map is entirely different; yet these points are all lumped together in the class S_1. Suppose we restrict f to the parabola—which we can call S_1—using x_2 as variable:

$$f \mid S_1: \begin{array}{l} y_1 = 3x_2^2 \\ y_2 = -2x_2^3. \end{array}$$

The differential of this map is

$$d(f|S_1) = \begin{pmatrix} 6x_2 \\ -6x_2^2 \end{pmatrix},$$

so every point on the parabola is a regular point of $f|S_1$, except the origin, which is a singularity of type S_1. The usual notation for the regular points on the parabola is $S_{1,0}$, for the origin $S_{1,1}$. More generally, $S_{k,h}$ is the set of singular points of deficiency h of the map $f|S_k$, where S_k is the set of singular points of deficiency k of the map f. This process can be iterated in an obvious way, but there is one difficulty: in order to talk about the singular sets of $f|S_k$, it is necessary that S_k itself be a manifold. Boardman [3] has resolved these problems and made an extensive study of iterated singular sets.

For the Whitney cusp, S_1 is a manifold so the iterated singular sets $S_{1,0}$ and $S_{1,1}$ do make sense and are even manifolds. Moreover, the iteration provides exactly the right degree of discrimination between singularities: all regular points are in the sets S_0, all folds are in $S_{1,0}$, and the cusp point is by itself in $S_{1,1}$.

In [28], Whitney extends the classification of stable map-germs to all dimensions $n, p \leqq 5$, plus $p = 2n - 1$ and $p = 2n - 2$. In all these cases, the stable types are dense.

The first indication of trouble in the classification program was given by Thom in [7]; see also [1]. He showed that the stable types are not dense in any of the cases $n = p = k^2$, where k is an integer greater than 3. Mather, in a paper appropriately titled "The Nice Dimensions" [9, VI], has gone on to determine all cases for which the stable germs are dense. Figure 11, copied from his paper, displays the nice dimensions as points in the n, p plane. He even gives a complete classification of the stable types in the nice region: each type is characterized by a certain algebra, which can in turn be used to reconstruct a normal form for the type.

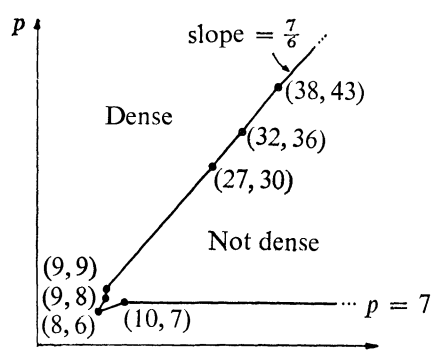


FIG. 11

Thom [17] and Mather [8] have shown that there is a complete topological classification for all dimensions.

We can pause here to remark that the work described so far explicates most of the problems outlined in the introduction.

5. Instability and the bifurcation set. The classification of stable map-germs is only half the story. Recall that a map f in $C^\infty(N, P)$ is unstable if every neighborhood of f contains maps inequivalent to f. We shall take up two questions which have received considerable attention: exactly which equivalence classes are found near a given unstable map; and how are those equivalence classes situated in the function space?

To develop these questions, we consider some examples. The simplest unstable map is $y = x^3$, and it even possesses all the essential features of more complicated ones.

THEOREM 5. *Let g be any function sufficiently near $y = x^3$. Then g has an inflection at a point x_0 near 0 and the smooth local type of g at x_0 is determined by the sign of $g'(x_0)$. Specifically, the map-germ $(g; x_0, g(x_0))$ is smoothly equivalent to the local map $y = x^3 + g'(x_0)x$.*

Thom's proof [**23**; page 51] uses the C^∞-version of the Weierstrass preparation theorem which was first proved by B. Malgrange. (There is an interesting legend surrounding this theorem. In about 1960, Thom saw that it would be a valuable tool in the study of singularities, but the existing analytic proofs could not be carried over to the C^∞-case. Malgrange, when the theorem was first described to him by Thom, believed it to be false, but some time later he had a complete proof [**10, 11**]. Since then a number of proofs have appeared. Arnold [**1**] and C. T. C. Wall [**24**] discuss the importance of Malgrange's preparation theorem for the theory of singularities.)

Here is a brief plausibility argument for Theorem 5. By Taylor's theorem, any function near $y = x^3$ looks like

$$u = a_0 + a_1 t + a_2 t^2 + (1 + a_3) t^3 + R(t),$$

where R and all the a's are small. The inflection point is found by solving $u'' = 0$, and it can be checked that this equation has a unique solution x_0 near 0. Expanding about this inflection point and setting $x = t - x_0$ gives

$$u = a_0^* + a_1^* x + (1 + a_3^*) x^3 + R^*(x).$$

The coordinate change $y = (u - a_0^*)/(1 + a_3^*)$ gives a further reduction to

$$y = a_1^{**} x + x^3 + R^{**}(x).$$

Finally, R^{**} is like x^4 so it does not influence the local type and can be ignored. Since a_1^{**} is a positive multiple of $g'(x_0)$, we have shown that g at x_0 is equivalent to $x^3 + g'(x_0)x$.

The theorem describes how equivalence classes are situated around $y = x^3$ in

$C^\infty(\mathbb{R}, \mathbb{R})$. To begin, the equivalence class of $y = x^3$ is determined by the condition $g'(x_0) = 0$ ("horizontal inflection"), and this is a single constraint on the function g.

Let us digress a moment to recall that in $\mathbb{R}^m$ a single constraint $F(x_1, \cdots, x_m) = 0$ has for its locus a submanifold of dimension $m - 1$, alternatively, of codimension 1—at least if the constraint F is reasonably nice. More generally, k independent constraints $F_1(x) = \cdots = F_k(x) = 0$ specify a submanifold of codimension k.

All these facts carry over to an infinite-dimensional space, so Theorem 5 can be interpreted as saying that the smooth equivalence class of $y = x^3$ is a submanifold of $C^\infty(\mathbb{R}, \mathbb{R})$ of codimension 1. On one side of this submanifold is the open set (= stable equivalence class) of maps of type $x^3 + x$; on the other is the open set of maps of type $x^3 - x$.

Figure 12 is a three-dimensional slice of $C^\infty(\mathbb{R}, \mathbb{R})$ which shows the three equivalence classes found in a neighborhood of $y = x^3$. That is, the figure shows the intersection of a neighborhood of x^3 with a three-dimensional hyperplane W, in such a way that W is transverse to each of the equivalence classes C near x^3. This means essentially that W is never tangent to C, and has the important consequence that the codimensions (but not the dimensions!) of C and $W \cap C$ are the same.

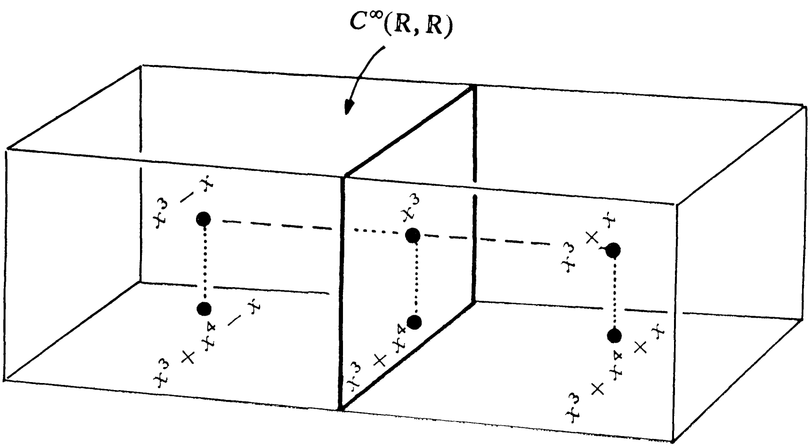


FIG. 12

The next simplest unstable function is $y = x^4$. The following result is an analog of Theorem 5 and can be proven in the same way.

THEOREM 6. *Let g be any function sufficiently close to $y = x^4$. There is a point x_0 near 0 for which $g'''(x_0) = 0$, and the germ of g at x_0 is smoothly equivalent to a polynomial local map $y = x^4 + ux^2 + vx$.*

To see what a neighborhood of x^4 looks like in $C^\infty(\mathbb{R}, \mathbb{R})$, we use the theorem to assign to each function g the pair of coefficients (u, v) of its equivalent polynomial. This defines a projection

$$\pi \colon C^\infty(\mathbb{R}, \mathbb{R}) \to \mathbb{R}^2$$
$$g \mapsto (u, v)$$

in which maps having the same image are equivalent.

The next step is to determine the local types of the polynomials $x^4 + ux^2 + vx$ corresponding to the points (u, v). Since a function is unstable when its derivative has a double root (this is the non-degeneracy condition of Morse), we want to consider the cubic equation

$$4x^3 + 2ux + v = 0.$$

According to the classical solution of Cardan, this equation has a double root when (u, v) lies on the cusp

$$27v^2 + 8u^3 = 0.$$

The cubic then has three roots—i.e., the original polynomial $x^4 + ux^2 + vx$ has three non-degenerate critical points—when (u, v) lies inside the cusp, but only one (real) root outside. It follows that for any point on the cusp different from the origin, the polynomial $x^4 + ux^2 + vx$ is unstable because it has an x^3-type of singularity.

In addition to these, there are other unstable maps near x^4, which have only non-degenerate critical points. In fact, any polynomial of the form $x^4 - a^2x^2$ is unstable, for reasons already outlined in the last section for the particular function $x^4 - 2x^2$. A polynomial $x^4 + ux^2 + vx$ is in this unstable class if it satisfies the single condition $v = 0$ (and also $u < 0$, but this describes an open set and thus does not alter codimension), so the class itself has codimension 1 in $C^\infty(\mathbb{R}, \mathbb{R})$. Although classes like this one are not properly part of the study of map singularities because instability occurs for global rather than local reasons, they must be considered in any complete description of the partitioning of a function space into equivalence classes.

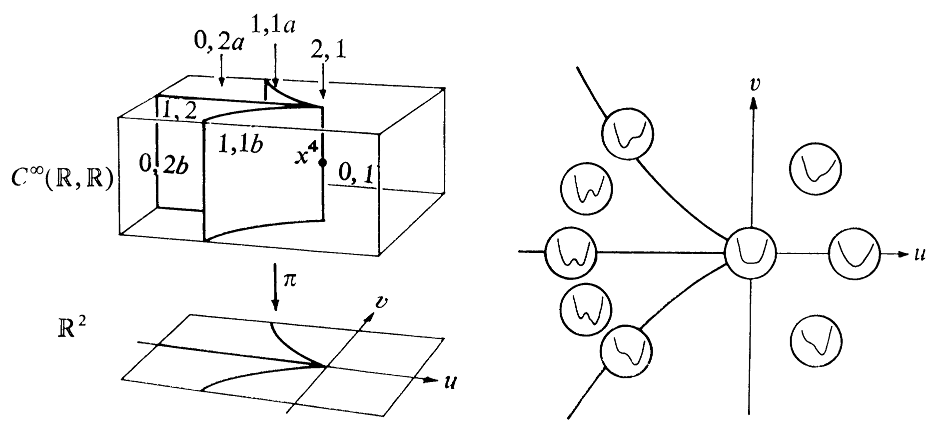


FIG. 13

Figure 13 shows how a neighborhood of $y = x^4$ in $C^\infty(\mathbb{R}, \mathbb{R})$ is partitioned. The equivalence classes are

0, 1: stable maps with one minimum;
0, 2: stable maps with two unequal minima;
1, 1: unstable maps with an x^3-type singularity;
1, 2: unstable maps with two equal minima;
2, 1: unstable maps with an x^4-type singularity.

The first index is the codimension of the class and the second is the number of minima possessed by any function in the class. The classes 0, 2 and 1, 1 each have two components, designated a and b in the diagram. A sketch of a function in each class is also included. Crossing the cusp line creates or destroys a minimum, while crossing the negative u-axis causes the location of the absolute minimum to jump.

The most complicated unstable singularity we can picture easily (and this is questionable!) is $y = x^5$.

Following the pattern of the two preceding examples, any function near x^5 will have a germ equivalent to a polynomial of the form

$$y = x^5 + ux^3 + vx^2 + wx.$$

Hence, the equivalence class of x^5 is determined by the three conditions $u = v = w = 0$, so it has codimension 3 in $C^\infty(\mathbb{R}, \mathbb{R})$. In the squashed-down three-dimensional version of this function space which we have been using, the equivalence class of x^5 must therefore appear as a point; however, every class near x^5 is determined by fewer than three conditions so it appears as either a curve, a surface, or an open set.

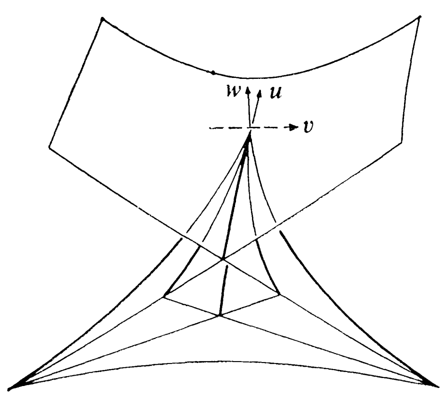


FIG. 14

Figure 14, which shows all the equivalence classes around x^5, suggests why Thom has named $y = x^5$ the **swallowtail** singularity. (Only the six heavy lines in the figure represent classes of condimension 2.) Since the polynomials

$$x^5 + ux^3 \pm vx^2 + wx$$

are equivalent, any point-set in figure 14 is equivalent to its mirror image across the plane $v = 0$. Thus there are five stable classes (but six disjoint open sets), six unstable classes of codimension 1, four unstable classes of codimension 2, and one unstable class of codimension 3.

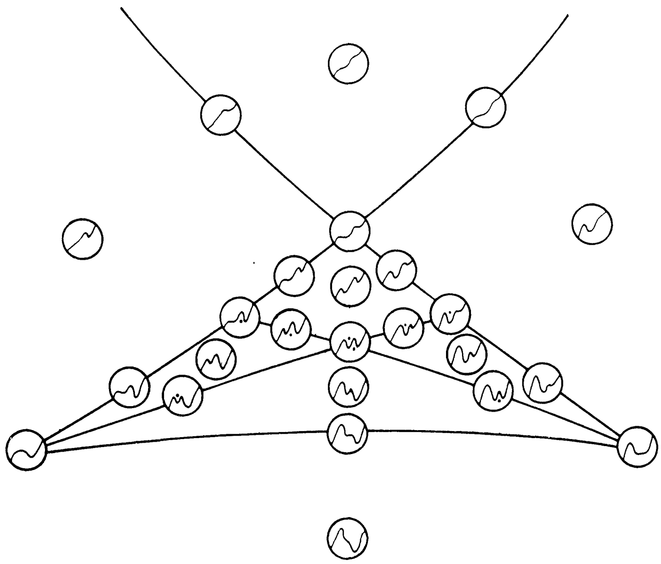


FIG. 15

Figure 15 shows a representative of each component of each class on the swallowtail side of x^5. Functions in the classes of different codimension have the following features:

0: either none, or two, or four non-degenerate critical points;

1: either an x^3-type singularity, or a pair of equal minima, or a pair of equal maxima;

2: either an x^4-type singularity, or two x^3-type singularities, or two equal minima and two equal maxima, or an x^3-type singularity appearing at the same level as a minimum or a maximum;

3: an x^5-type singularity.

Incidentally, the sketches in [**19**] and [**23**] incorrectly show that the set of functions with two equal minima (it begins at the lower-right-hand cusp) terminates at the node of the swallowtail, instead of on its left-hand side.

For the general case $y = x^n$, it can be shown that every nearby function has a germ equivalent to a polynomial

$$y = x^n + a_{n-2}x^{n-2} + \cdots + a_1 x.$$

Hence the codimension of the singularity x^n is $n - 2$ and this is greater than 3 when

$n > 5$. It follows that we cannot get a complete picture of the disposition of the equivalence classes around x^n. Nevertheless, Thom [**23**, chapter 5; **19**] has obtained numerous two- and three-dimensional slices of $C^\infty(\mathbb{R}, \mathbb{R})$ near $y = x^6$ and, because of the shape of one of these slices, has named x^6 the **butterfly** singularity. Using a computer, T. Woodcock at the University of Warwick has done the same thing for $y = x^7$; he calls it the **wigwam** singularity.

We end this section with a summary of terms, most of them due to Thom. Although the preceding examples were all functions of a single variable, everything we have said can be carried over to arbitrary maps f in $C^\infty(\mathbb{R}^n, \mathbb{R}^p)$, or even in $C^\infty(N, P)$. (The set $C^\infty(N, P)$ is not a vector space, but it is an infinite-dimensional manifold, and the equivalence classes it contains are still submanifolds.)

The **bifurcation set** of $C^\infty(N, P)$ is the collection Σ of unstable maps. As the examples suggest, Σ is a disjoint union of manifolds, or **strata**; Σ itself is called a **stratified set**. This stratification is a certain kind of decomposition of the function space, analogous to a triangulation. Whitney, Thom, and others [**29**, **20**, **14**] have investigated the abstract theory of stratified sets, and Thom [**22**] in particular has urged that the study of the bifurcation set be considered one of the basic parts of functional analysis.

A **singularity** is a smooth equivalence class of map-germs; this is the sense in which the term is used in a statement like "$y = x^5$ is the swallowtail singularity." Since the distinction between regular and singular points is not maintained by coordinate changes which are merely homeomorphisms, we shall provide no analogous term for a topological equivalence class.

The **codimension** of a singularity is the codimension of its equivalence class. E.g., the singularity $y = x^n$ has codimension $n - 2$, while $y = 0$ and $y = \exp(-x^{-2})$ both have infinite codimension.

Suppose that a singularity, represented by the local map-germ $f: \mathbb{R}^n, 0 \to \mathbb{R}^p, 0$, has finite codimension k. An **unfolding** of this singularity is a smooth map

$$F: \mathbb{R}^k, 0 \to C^\infty(\mathbb{R}^n, \mathbb{R}^p), f$$

which is transverse to all the strata of Σ near f. (Roughly speaking, f is transverse to a stratum if the image $F(\mathbb{R}^k)$ is not tangent to it. The number of strata near a finite-codimensional singularity is always finite.) The map F is what we called earlier a "slice"; it is a k-dimensional family of maps containing a representative of every smooth equivalence class found near f.

These definitions are somewhat provisional and have not yet appeared in the literature in any final form. Nevertheless, they meet our needs in this survey.

6. Stabilizing plane maps. According to Whitney's theorem, any germ of a map from one plane to another is either a diffeomorphism, a fold, a cusp, or else is unstable, in which case an arbitrarily small change will convert it into one of the three stable types. In this section, we shall consider a number of unstable map-germs f:

$\mathbb{R}^2, 0 \to \mathbb{R}^2, 0$; the object is to see exactly how such a map can be "stabilized" by small perturbations.

The quarto map. The coordinate functions for this map are

$$f: \begin{aligned} u &= x^2 \\ v &= y^2; \end{aligned}$$

it is unstable because the origin has singularity type S_2 (the differential of f at the origin has rank 0). All other points on the x-axis and the y-axis are ordinary S_1-type folds. We can think of f as folding the xy-plane first along the x-axis, then along the y-axis, and applying the resulting quarto sheet to the first quadrant in the uv-plane. This is illustrated in figure 16. The images of the fold lines lie on the coordinate axes of the uv-plane, but for clarity the figure separates these images. Also, it makes no difference which axis is folded first—our choice was arbitrary.

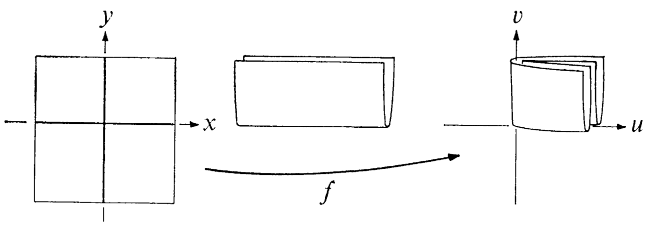


FIG. 16

One unfolding of the quarto map is given by the family

$$F: \begin{aligned} u &= x^2 + 2ay \\ v &= y^2 + 2bx. \end{aligned}$$

A map in this family is stable only if a and b are both different from 0 (figure 17 shows the image of F in the case $a = 0$, $b \neq 0$); in fact, every stable map is equivalent to one in which $a = b$.

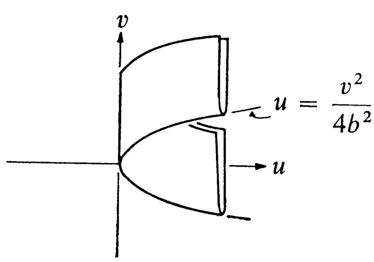


FIG. 17

Let us study the case $a = b$ in more detail. Since

$$dF = \begin{pmatrix} 2x & 2a \\ 2a & 2y \end{pmatrix},$$

the singularity set S_1 is the hyperbola $xy = a^2$. The only singular point of $F \mid S_1$ is found at (a, a), and it is an $S_{1,1}$-type Whitney cusp. The effect of the map F is illustrated by Figure 18 (with $a < 0$): pleat the xy-plane along the branch of the hyperbola which contains the cusp, fold it along the other branch, and then apply the result to the uv-plane. As a approaches 0, the pleat and the fold coalesce to give the original quarto fold.

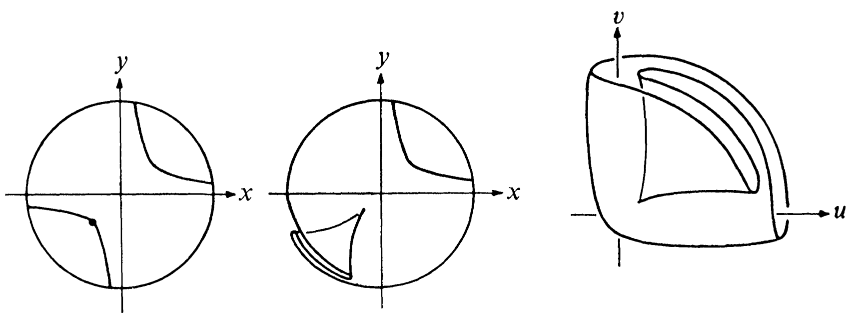


FIG. 18

It was not necessary here to have a complete unfolding to find all stable types — they were adequately described by a single parameter. Likewise, in each of the succeeding examples we shall merely embed the given map in a family which exhibits all nearby stable types.

The complex squaring map. This is the familiar map $w = u + iv = (x + iy)^2 = z^2$ of the complex plane. Its real form is

$$f\colon \begin{aligned} u &= x^2 - y^2 \\ v &= 2xy. \end{aligned}$$

The origin is once again the only singularity; it is of type S_2 and is commonly referred to in complex analysis as a *branch point* (of the double-valued function $z = \sqrt{w}$). It can be shown that every stable map near f is equivalent to one of the form

$$F\colon \begin{aligned} u &= x^2 - y^2 + 2ax \\ v &= 2xy - 2ay. \end{aligned}$$

Notice that f is unstable only as a real map: $w = z^2$ is stable under complex-analytic coordinate changes. There is no contradiction because the perturbation F is not complex-analytic; its complex form is $w = z^2 + 2a\bar{z}$.

To see what form the stable map F takes, it is helpful to have first a picture of the action of the squaring map different from the one usually seen.

Beginning along the positive x-axis, partition the xy-plane into three 120° sectors. Fan out each sector to double its size, and position them over the uv-plane as shown in Figure 19. The distance of any point from the origin should also be squared; this bunches up material around the origin, so to speak, and provides the slack which will allow a perturbation to alter the local type of the map. Now apply the three large sectors to the uv-plane, and line up the seams so that the whole composite is a two-to-one map away from the origin.

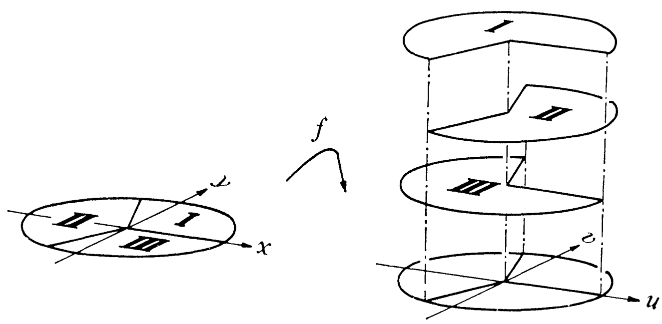


FIG. 19

For the perturbation map F, the singularity set S_1 is the circle $x^2 + y^2 = a^2$; the intersection of this circle with each of the three sector edges will be a Whitney $S_{1,1}$ cusp point, and every other point on the circle will be an ordinary fold. These are the only singular points of F. In particular, the origin is a regular point.

In Figure 20, we can see how F alters the action of f. As each sector is fanned out to double its size, it is folded along the circular arc S_1 and this fold is pushed past the origin and into the region previously unoccupied by the double-sized sector. A ''half-pleat'' is thus formed at each of the two edges. Since the image of a sector edge is unchanged by F, the altered sectors can be patched together exactly as they were for the original map f. The pairs of half-pleats then match up to form three ordinary pleats, and the image of the circle of singular points is a hypocycloid with three cusps. The interior of the circle is carried by an orientation-reversing diffeomorphism to the interior of the hypocycloid. Since the image of the outer part of every sector also covers the hypocycloid, every point inside the hypocycloid has four preimages; every point outside continues to have two.

A flabby pleat. This example is rather complicated, but it is useful because it ties together the previous two and because it relates to Thom's theory of catastrophes. The map is

$$f: \begin{aligned} u &= x^3 + 2xy \\ v &= y^3 + x^2. \end{aligned}$$

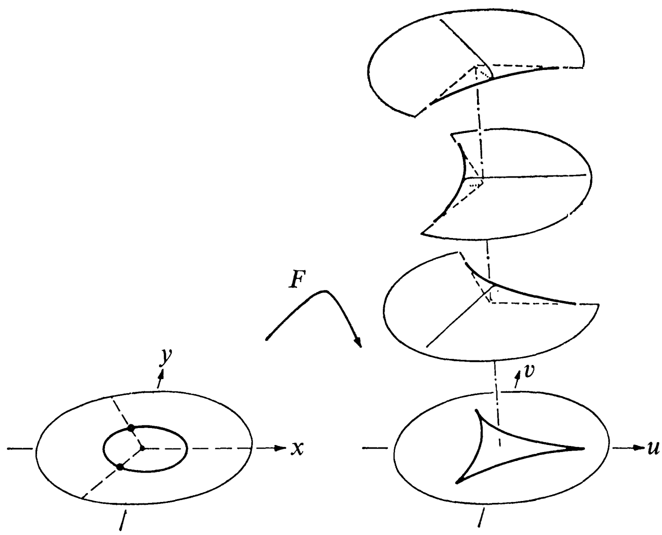


FIG. 20

The singular set, which has three components, is the locus of

$$6y^3 - x^2(4 - 9y^2) = 0.$$

Every singular point except the origin is an ordinary fold. As figure 21 shows, the map carries a small neighborhood of the origin into the target by pleating it.

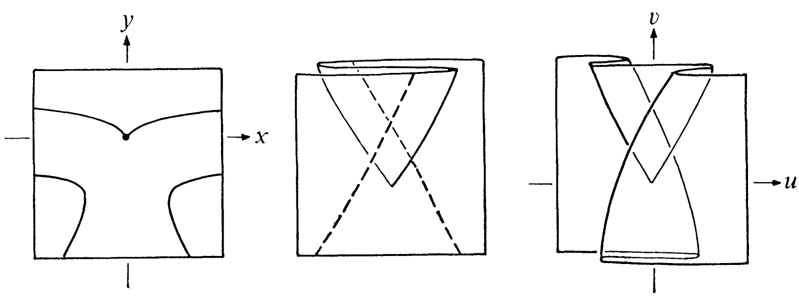


FIG. 21

However, f is not stable, and in particular, it is not equivalent to the Whitney cusp. In effect, it piles up a lot of source material near the origin—the pleat is flabby and can be distorted by a perturbation. This contrasts with the Whitney cusp which is taut and impossible to distort. These distinctions are quantitative and cannot be seen in the illustrations.

Every stable map near f is equivalent to one of the form

$$F: \begin{aligned} u &= x^3 + 2xy \\ v &= y^3 + x^2 + ay + by^2. \end{aligned}$$

The singular set of F is given by

$$6y^3 + 4by^2 + 2ay - x^2(4 - 9y^2 - 6by - 3a) = 0.$$

For small values of a and b, two of the components of this set are ordinary fold lines like the lower two components of the singular set of f. Since they are not involved in what happens near the origin, we shall ignore them and omit them from the illustrations.

The local type of F depends on the sign of b. For $b < 0$, the quadratic terms of F (which determine the type) are just the functions defining the complex-squaring map, while for $b > 0$, those terms are equivalent, under linear coordinate changes, to the functions defining the quarto map. How exactly does the pleat f unfold to each of these maps?

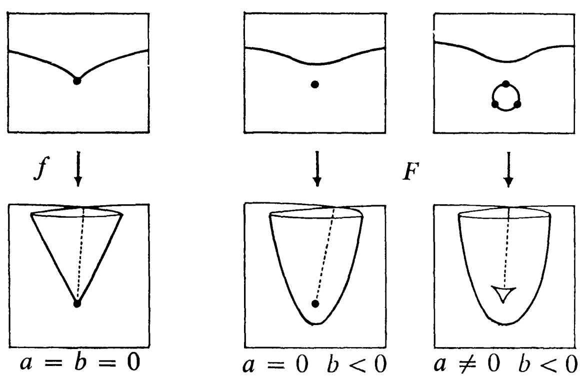


FIG. 22

Let us consider the case $b < 0$, $a = 0$ first. It is helpful to think of the pleat image of f as a flattened out conical pocket, with a seam of self-intersection extending up from the origin; see Figure 22. The front flap of the pleat has merely been pushed to the back; as an image set, it is unchanged. We get the image of F by filling the conical pocket with water and watching it sag under the weight. The seam and the origin stay fixed, so points in the target just below the origin now have three pre-images. If we peel off the top layer of the pocket, the origin is revealed as a branch

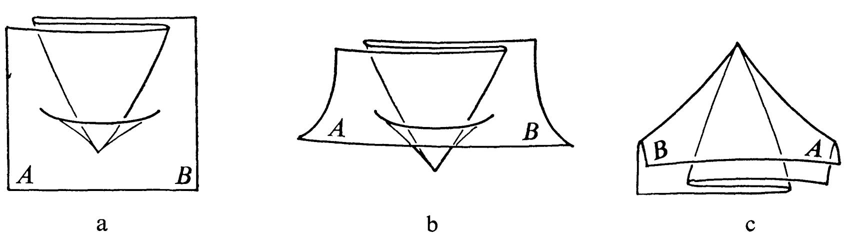


FIG. 23

point of a double covering. The map is not yet stable, but the unfolding is completed as it was for the complex-squaring map by taking $a \neq 0$.

When $b > 0$ and $a = 0$ that part of the source material near the origin and below the pleat gets shoved up past the point of the pleat. This is shown in figure 23a. We can reverse the process to recover the original map f by pulling down the sheet at the points marked A and B. By peeking under the top sheet, as in figures 23b and 23c, we can see that F is indeed equivalent to the (unstable) quarto map.

The unfolding can be completed in two ways when the parameter a is allowed to vary. In either case, the singular set splits into two components, and the component passing through the origin retains the Whitney cusp (cf., the discussion of the quarto map). In figure 24, we have $a > 0$ and the unbounded component has the cusp; the compact component forms a pouch which harbors it. If $a < 0$, it is helpful to return to the picture of the flabby pleat as a flattened cone. Figure 25 then shows the effect of the successive perturbations $b > 0$, $a = 0$ and $b > 0$, $a < 0$.

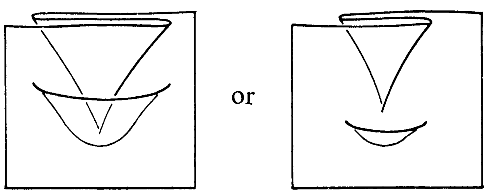


FIG. 24

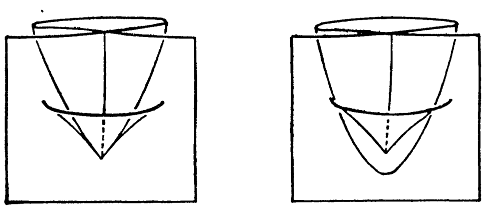


FIG. 25

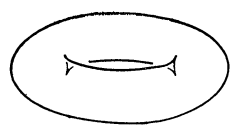


FIG. 26

The image of the singular set of a map is sometimes called its **apparent contour**. It gives a general idea of the outline of the image set in the target. In addition to the examples already discussed, we mention that the usual rendering of a torus on a flat sheet of paper or a blackboard, as in figure 26, is just the apparent contour of a certain map $F: T^2 \to \mathbb{R}^2$ of the torus. See [**14**, chapter 4].

7. An application: the parabolic umbilic. With a view toward biological application, Thom has classified (in [**23**]; see also [**19, 21**]) all unstable map-germs f: $\mathbb{R}^n, 0 \to \mathbb{R}, 0$ whose codimension does not exceed 4. There are exactly seven, and n can always be taken to be 1 or 2. The four (or fewer) parameters U involved in the unfolding F_U of such a singularity are interpreted as the space-time coordinates of a biochemical process. As we saw in section 5, the equivalence classes of maps F_U

induce a stratification of the parameter space U (cf. the swallowtail, figure 14). In Thom's interpretation, the process has a more or less constant character on each stratum, in particular on each open stratum. Thus any change in the process will be sudden and will occur on one of the positive-codimensional strata. These discontinuities Thom calls the **catastrophe points** of the process. This catastrophe theory provides another impetus for the study of bifurcation and unfolding. See also [**30**].

We now use the analysis of the flabby pleat to study the catastrophe set of the most complicated of the seven map-germs, the parabolic umbilic:

$$f(x,y) = \tfrac{1}{4}(x^4 + y^4) + x^2 y.$$

It has codimension 4 and can be unfolded by

$$F_U(x,y) = f(x,y) + wy^2 + ty^3 - ux - vy; \;\; U = (u,v,w,t).$$

As we have seen, such a map is unstable if it has either two equal extrema or else a degenerate critical point. We shall discuss only the second kind of instability.

Let $\bar{U} = (\bar{u}, \bar{v}, \bar{w}, \bar{t})$ be a fixed choice of the parameters: $(\bar{x}, \bar{y})$ is a critical point of $F_{\bar{U}}$ if $dF_{\bar{U}}(\bar{x}, \bar{y}) = 0$. This means

$$\begin{aligned} \bar{u} &= \bar{x}^3 + 2\bar{x}\bar{y} \\ \bar{v} &= \bar{y}^3 + \bar{x}^2 + 2\,\bar{w}\bar{y} + 3\,\bar{t}\bar{y}^2. \end{aligned}$$

Restated, $(\bar{x}, \bar{y})$ is a critical point of $F_{\bar{U}}$ if $(\bar{x}, \bar{y})$ is a preimage of the point $(\bar{u}, \bar{v})$ under the map

$$\begin{aligned} T_{\bar{w},\bar{t}}\colon \; & \mathbb{R}^2 \to \mathbb{R}^2 \\ u &= x^3 + 2xy \\ v &= y^3 + x^2 + 2\bar{w}y + 3\bar{t}y^2. \end{aligned}$$

Notice that $T_{w,t}$ is the stabilization of the flabby pleat which was discussed in the last section.

By definition, the critical point is degenerate when the Hessian matrix $H(F_{\bar{U}})_{(\bar{x},\bar{y})}$ is non-invertible. But $H(F_{\bar{U}})_{(\bar{x},\bar{y})}$ is also the matrix of the differential $dT_{\bar{w},\bar{t}}$, so the degenerate critical points of $F_{\bar{U}}$ are precisely the singular points of $T_{\bar{w},\bar{t}}$. Thus F_U is unstable whenever (u,v) is in the apparent contour of $T_{w,t}$. These contours represent the intersection of the umbilic's catastrophe set (a three-dimensional stratified set in (u, v, w, t)-space) with the (u,v)-plane and its parallel translates. Some of them are shown in figure 27.

The parabolic umbilic is analyzed at length in [**5**]; it appears there in a slightly different form.

More advanced surveys of the whole subject of singularities of maps are given by Arnold [**1**] and Wall [**25**]; they also provide extensive bibliographies.

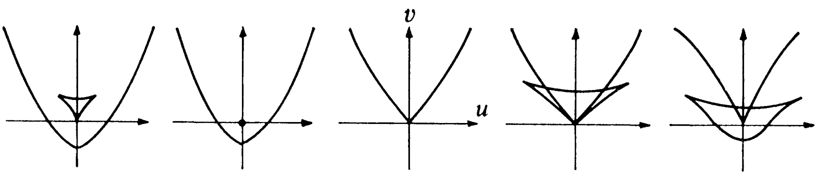


FIG. 27

References

1. V. I. Arnold, Singularities of smooth mappings, Russian Math Surveys, (1969) 1–43; translated from Uspekhi Math. Nauk, 23 (1968) 3–44.

2. G. Birkhoff and S. MacLane, A Survey of Modern Algebra, Macmillan, New York, revised edition 1953.

3. J. M. Boardman, Singularities of differentiable maps, Inst. Hautes Études Sci. Publ. Math., 33 (1967) 383–419.

4. R. C. Buck, Advanced Calculus, McGraw-Hill, New York, second edition 1965.

5. A. N. Godwin, Three dimensional pictures for Thom's parabolic umbilic, I. H. É. S. Publ. Math., 40 (1971) 117–138.

6. S. T. Hu, Differentiable Manifolds, Holt, Rinehart and Winston, New York, 1969.

7. H. Levine, Singularities of Differentiable Mappings, I, Mathematics, Institute, Bonn, 1959; reprinted in [24].

8. J. Mather, Notes on Topological Stability, mimeographed, Harvard, 1970.

9. ———, Stability of C^∞-mappings:

I The division theorem, Ann. of Math, 87 (1968) 89–104;

II Infinitesimal stability implies stability, Ann. of Math., 89 (1969) 254–291.

III Finitely determined map germs, I. H. É. S. Publ. Math., 35 (1968) 127–156;

IV Classification of stable germs by $\mathbb{R}$-algebras, I. H. É. S. Publ. Math., 37 (1969) 223–248;

V Transversality, Advances in Math., 4 (1970) 301–336;

VI The nice dimensions, in [24], 207–253.

10. B. Malgrange, Le théorème de préparation en géométrie différentiable, Séminaire H. Cartan (1962–63), nos. 11, 12, 13, 22.

11. ———, Ideals of Differentiable Functions, Oxford, 1965.

12. J. Milnor, Morse Theory, Princeton, 1963.

13. ———, Topology from the Differentiable Viewpoint, U. of Virginia, Charlottesville, 1965.

14. F. Pham, Introduction à l'Étude Topologique des Singularités de Landau, Gauthier-Villars, Paris, 1967.

15. M. Spivak, Calculus on Manifolds, Benjamin, New York, 1965.

16. R. Thom, Les singularités des applications différentiables, Ann. Inst. Fourier, 6 (1956) 43–87.

17. ———, Local topological properties of differentiable mappings, in Bombay Colloquium on Differentiable Analysis, Oxford,1964.

18. ———, Une théorie dynamique de la morphogénèse, in Towards a Theoretical Biology 1: Prolegomena, Aldine, Chicago, 1968.

19. ———, Topological models in biology, Topology, 8 (1969) 313–335.

20. ———, Ensembles et morphismes stratifiés, Bull. Am. Math. Soc., 75 (1969) 240–284.

21. ———, Topologie et Linguistique, in Essays on Topology and Related Topics, Springer, New York, 1970.

22. R. Thom, The bifurcation subset of a space of maps, Manifolds — Amsterdam 1970, Springer Lecture Notes in Math., 197 (1971) 202–208.

23. ———, Stabilité Structurelle et Morphogénèse, Benjamin/Addison-Wesley, Reading, Mass. 1973.

24. C. T. C. Wall, editor, Proceedings of the Liverpool Singularities Symposium I and II, Springer Lecture Notes in Math. 192, 209 (1971).

25. A. H. Wallace, Differential Topology, First Steps, Benjamin, New York, 1968.

26. H. Whitney, The self-intersections of a smooth n-manifold in $2n$-space, Ann. of Math., 45 (1944) 220–246.

27. ———, On singularities of mappings of Euclidean spaces, I. Mappings of the plane into the plane, Ann. of Math., 62 (1955) 374–410.

28. ———, On singularities of mappings of Euclidean spaces, in Symposium International de Topologia Algebraica, Mexico City, 1958.

29. ———, Local properties of analytic varieties, in Jubilee Volume: Differential and Combinatorial Topology, Princeton, 1965.

30. E. C. Zeeman, The geometry of catastrophe, The Times Literary Supplement, 10 Dec. 1971.

DEPARTMENT OF MATHEMATICS, SMITH COLLEGE, NORTHAMPTON, MA 01060.

THE BAY RESTAURANT — A LINEAR STORAGE PROBLEM

D. R. WOODALL

Summary. Suppose $l \geqq l_2 \geqq l_1 > 0$. Suppose that, at random times, items whose lengths are between l_1 and l_2 are inserted into and withdrawn from a linear store, subject to the restrictions that at no moment do we wish to store items whose total length exceeds l, and that items may not be moved within the store between their insertion and their withdrawal. The minimum length of store necessary for this to be possible is at least $\frac{1}{4}l(\log_2(l_2/l_1) - 4)$ and at most $l(\log_2(l_2/l_1) + 3)$. It is one of the more surprising features of this theory that both of these bounds tend to infinity as $l_1 \to 0$.

The research reported here has been sponsored in part by the Science Research Council of the United Kingdom.

Throughout the paper the symbol $:=$ or $=:$ indicates that the equation in which it occurs acts as the definition of (some part of) the expression on the same side of the equality sign as the colon. The symbol ▌ denotes the end (or absence) of a proof.

1. Introduction. At the recent informal combinatorial conference at Royal Holloway College, a certain amount of time was devoted to a discussion of the Bay Restaurant. This novel establishment (Proprietor J. H. Conway) contains a single table, which runs the whole length of the establishment, and the patrons all sit on the same side of the table, facing the Bay. If a party of people come into the res-

taurant and demand to be seated together as a group, the Proprietor is obliged to seat them in consecutive seats; but then, of course, they must all leave together: it would be most impolite for some of them to walk out and leave other members of the party still eating in the restaurant. However, once they are outside, there is nothing to stop them forming up into other groups and coming back into the restaurant for coffee. The problem is this: if the restaurant has seats for N people at the table, what is the smallest number $f(N)$ of people who, by going in and out of the restaurant in different sized parties a sufficient number of times, can eventually defeat the Proprietor by sending in a party of people whom he cannot seat in consecutive seats (without asking people already in the restaurant to move — an unpardonable breach of etiquette)? We shall first obtain the weak bounds

$$[2\sqrt{N}] \leqq f(N) \leqq [\tfrac{1}{2}N] + 2,$$

where $[x]$ denotes the greatest integer less than or equal to x, and we shall then prove the better bounds

$$\frac{N}{\log_2 N} \leqq f(N) \leqq 8\,\frac{N}{\log_2 N}.$$

The upper bound can be improved somewhat. The precise results that we shall prove are the following. If the General outside, who is organising the parties of people coming into the restaurant, has k people available, then if $N > k(\log_2 k + 3)$ the Proprietor can always frustrate the General by the simple process of always seating a party as far to the left as it will go, whereas if $N < \frac{1}{4}k(\log_2 k - 3)$ the General can always defeat the Proprietor. This last result shows that

$$f(N) < (4 + o(1))\frac{N}{\log_2 N} \quad \text{as} \quad N \to \infty.$$

It seems quite likely that we have still lost a factor of 2 on each side, and that $f(N)$ is asymptotically $2(N/\log_2 N)$.

The methods used to prove the above results show also that, if all the parties sent into the restaurant have at least k_1 and at most k_2 ($\leqq k$) people, then the minimum length of table needed to accommodate them is at least $\frac{1}{4}k(\log_2(k_2/k_1) - 4)$ and at most $k(\log_2(k_2/k_1) + 3)$. Of course, it is not important that the total number of people available should be at most k, only that at most k are ever in the restaurant at any one time. The continuous analogue of this result (obtained by successive approximations of the form $l \simeq k/t$, $l_1 \simeq k_1/t$, $l_2 \simeq k_2/t$, where k, k_1, k_2 and $t \to \infty$) is thus the linear storage result mentioned in the summary. If the ratio l_2/l_1 is small, as in the problem of roadside parking, we can improve somewhat on the bounds given. If we mark off the road into parking bays of length l_2, we need a total length of road of ll_2/l_1. If $l_2/l_1 < 2$, this is certainly a better strategy than parking every car as far to the left as it will go, but I suspect that it is not as good if $l_2/l_1 > 2$. In any case, this strategy is to ensure that we can park every car provided that the

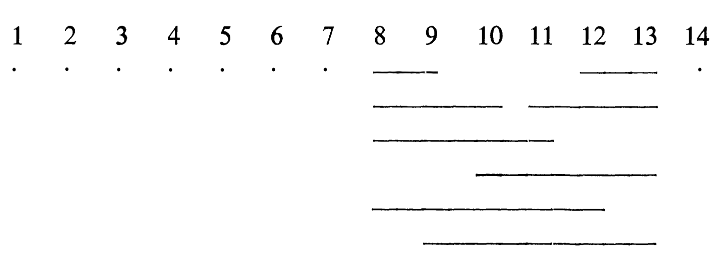
total length of car to be parked at any moment is at most l, and it is not necessarily the strategy that gives the maximum probability of being able to park each car if the total length of car is unlimited.

2. The weak bounds. Conway denies having founded the Bay Restaurant, and blames this on another member of the conference, who also denies it. It is clear, however, that it was Conway who was responsible for taking on two members of staff, the Dumb Waiter and the Proprietor's Mate. (The fact that he apparently neglected to take on a chef as well may account for the fact that the restaurant does not appear to be very well known.) The Dumb Waiter seats people at random, wherever they will fit in, with no strategy at all. The Proprietor's Mate adopts the strategy of seating each party as far to the left as it will go. These gentlemen will be very useful to us in our investigations. (The Proprietor, of course, is infinitely intelligent, and always adopts the best strategy possible.)

For example, we note that we need at least $[2\sqrt{N}]$ people in order to defeat even the Dumb Waiter (and hence, *a fortiori*, to defeat the Proprietor). For consider the party that finally defeats him. If this contains p people, then there is no space of length p in the restaurant, or he would be able to fit in a party of size p. So the number of people sitting in the restaurant is at least $[N/p]$. Thus the number of people we need is at least $\min_{p \leqq N}(p + [N/p]) \geqq [2\sqrt{N}]$.

The upper bound $[\frac{1}{2}N] + 2$ is also easy to prove. For if we send this number of people in singly, then when they are seated there must be at least one person on each side of the centre-line. If we leave behind the person closest to the centre-line on each side of it, withdraw the remaining $[\frac{1}{2}N]$ people, and send in a single party of $[\frac{1}{2}N]$, it is clear that the Proprietor cannot seat them in consecutive seats.

If $N \leqq 7$, the bounds $\min_{p \leqq N}(p + [N/p])$ and $[\frac{1}{2}N] + 2$ coincide. At the conference we spent some time considering other small values of N (using beer bottles to represent people, and taking it in turns to be the Proprietor and the General), and we succeeded in establishing that $[\frac{1}{2}N] + 2$ is the correct answer if $N \leqq 17$. It seems likely that 10 people can defeat a table of 18, but we did not prove this. As an example, the proof that 8 people cannot defeat a table of 14 uses the following diagram:



The Proprietor always seats a party of people in a position where they are partitioned by lines of the diagram, a point representing a line of length 1. For example, a group

of three people could be seated in seats 8 9 10, since there is a line of length 3 under these numbers, or in positions 7 8 9, since there is a line of length 1 under 7 and one of length 2 under 8 9, or in positions 5 6 7, where there are three lines of length 1, but not in positions 6 7 8 or 9 10 11. It is not difficult to see that the Proprietor can always follow these rules, and hence that he can never be defeated. The reader is invited to construct a similar diagram to prove that 9 people cannot defeat a table of 16: I have not been completely successful, and have an extra rule in my strategy as well as those derived directly from the diagram. It would also be of interest to know whether or not 10 people can defeat a table of 18.

3. The better bounds. The starting-point for the investigations in this section was the observation that if k (the number of people available) is equal to 2^r, where r is an integer, then the General can defeat the Proprietor's Mate unless $N \geqq (r+1)2^{r-1}$. He does this by sending in 2^r single people, whom the Proprietor's Mate seats in positions 1 to 2^r. He then pulls out numbers $1, 3, \cdots, 2^r - 1$ and sends in 2^{r-1} people in pairs, whom the P.M. seats in positions $2^r + 1$ to $3 \cdot 2^{r-1}$. He then pulls out numbers $2, 6, \cdots, 2^r - 2$ and the first, third, $\cdots$, $(2^{r-2} - 1)$-th pairs, and sends in 2^{r-1} people in fours, whom the P.M. seats in positions $3 \cdot 2^{r-1} + 1$ to $4 \cdot 2^{r-1}$. The process terminates with him sending in a single party of 2^{r-1}, whom the P.M. will seat, if he can, in positions $r \cdot 2^{r-1} + 1$ to $(r+1)2^{r-1}$. The P.M. is thus defeated if $N < (r+1)2^{r-1} \simeq \frac{1}{2}k \log_2 k$. This does not tell us anything directly about $f(N)$, but it suggests $2(N/\log_2 N)$ as a possible answer, and the two strategies involved in this example are the bases of the two theorems that follow. We first require a lemma. As usual, N will denote the number of seats at the table and k the number of people at our disposal.

LEMMA 1.1 *Label the seats $1, 2, \cdots, N$ from the left. Suppose that the Proprietor's Mate is in charge of the restaurant from the start, and that at some stage we send in a party of m people whom the Proprietor's Mate seats in positions $r - m + 1, r - m + 2, \cdots, r$. Then $m \geqq 2^{(r/k)-3}$.*

Proof by induction on r. The result is clearly true if $r \leqq 3k$: so suppose $r > 3k$, and suppose $m < 2^{(r/k)-3}$. Then, at the time when we send in our party of m people, each gap of consecutive empty seats to the left of $r - m + 1$ has length at most $2^{(r/k)-3}$. Suppose that, at this moment, one party of people sitting in the restaurant ends in position r_1, and the next in position r_2, where $r_1 \leqq r_2 \leqq r - m$. Then the gap between them has length at most $2^{(r/k)-3}$, and the second party mentioned occupies a block of seats of length at least $2^{(r_2/k)-3}$, by the induction hypothesis. It follows that

$$2^{(r_2/k)-3} \geqq (r_2 - r_1)h(r_2) \geqq \int_{r_1}^{r_2} h(t)dt,$$

where

$$h(t) := \frac{2^{(t/k)-3}}{2^{(r/k)-3} + 2^{(t/k)-3}}.$$

Thus the total number of people, k, is at least as large as

$$\begin{aligned} 3kh(3k) + \int_{3k}^{r-m} h(t)dt + m &> 3kh(3k) + \int_{3k}^{r} h(t)dt \\ &= 3kh(3k) + (k/\ln 2)\left[\ln(2^{(r/k)-3} + 2^{(t/k)-3})\right]_{3k}^{r} \\ &= 3kh(3k) + (k/\ln 2)(\ln 2^{(r/k)-2} - \ln(2^{(r/k)-3} + 1)) \\ &= k(1 + 3h(3k) - (1/\ln 2)\ln(1 + 2^{3-(r/k)})). \end{aligned}$$

To obtain the required contradiction, it thus suffices to prove that

$$3h(3k) = 3\,\frac{1}{2^{(r/k)-3}+1} > (1/\ln 2)\ln(1 + 2^{3-(r/k)}).$$

But $r > 3k$ and $\ln 2 > \frac{2}{3}$, so the right-hand side is less than $\frac{3}{2} \cdot 2^{3-(r/k)}$, and it suffices to prove that $2 > 1 + 2^{3-(r/k)}$, which is obvious. This completes the proof of the lemma. ∎

THEOREM 1. *We require at least $N/\log_2 N$ people to defeat the Proprietor's Mate (and, a fortiori, to defeat the Proprietor).*

Proof. By the lemma, if we send in a party of m people, the Proprietor's Mate can always seat them, provided $m < 2^{(N/k)-3}$. But we cannot send in a party of more than k people. Thus the P.M. can seat any party that we send in, provided that $k < 2^{(N/k)-3}$, i.e., $N > k(\log_2 k + 3)$.

If we write $g(k) := k(\log_2 k + 3)$, we note that

$$\frac{k \log_2 g(k)}{g(k)} \geqq 1$$

if $k \geqq 2^5$, and that

$$k < \frac{N}{\log_2 N} \leqq k\,\frac{\log_2 g(k)}{g(k)}\,\frac{N}{\log_2 N}$$

therefore implies $N > g(k)$. Thus we cannot defeat the Proprietor's Mate if $k < N/\log_2 N$ and $k \geqq 2^5$. But if $k < N/\log_2 N$ and $k < 2^5$, then $N < 2^8$ and $k < 2\sqrt{N}$, and, since the Proprietor's Mate never puts a single person near the right-hand end, it is clear by the Dumb Waiter argument that we must have at least $[2\sqrt{N}] + 1 > 2\sqrt{N}$ people to defeat him. This completes the proof of Theorem 1. ∎

At this point we introduce the General's Mate (another General), who adopts the following strategy. If he has at least $[\frac{1}{2}N] + 2$ people at his disposal, he defeats the Proprietor in the obvious way. Otherwise he starts with everyone outside the restaurant, and his ith move is as follows. He divides all the people outside the restaurant into parties of 2^{i-1}, with at most $2^{i-1} - 1$ left over. He sends all the parties of 2^{i-1} into the restaurant for coffee, leaving the remainder outside eating ice creams.

When the parties have all been seated, he labels *all* the parties in the restaurant (including the ones that were there already, if any) $1, 2, 3, \cdots$ in order from the left. He then withdraws either the odd-numbered parties or the even-numbered parties, whichever contain between them at least half the people in the restaurant.

THEOREM 2. *The General's Mate (and hence, a fortiori, the General) will defeat the Proprietor if* $k > 8(N/\log_2 N)$.

Proof. Note that if $k \leqq 2^5$ (say) and $k > 8(N/\log_2 N)$, then certainly $N < 2^5$, and so $k > \frac{8}{5}N > [\frac{1}{2}N] + 2$. Thus the General's Mate can certainly defeat the Proprietor in this case. So we may suppose that $k > 2^5$. Suppose that $2^r \leqq k < 2^{r-1}$, where r is an integer.

We note first, by induction on i, that the number of parties sitting in the restaurant after the ith move is at least $(i+1)k2^{-(i+1)} - 2$. For this is clearly true if $i = 1$; and if $i > 1$ and at least $ik2^{-i} - 2$ parties remain after the $(i-1)$-th move, then we first boost the number to at least $(i+1)k2^{-i} - 3$ by sending in at least $k2^{-i} - 1$ parties of size 2^{i-1} (since we have at least $\frac{1}{2}k$ people outside the restaurant between moves), and we then reduce the number to not less than $(i+1)k2^{-(i+1)} - 2$ by removing alternate parties.

The second point to note is that if, after the ith move, a party of 2^p people and a party of 2^q people are adjacent in the restaurant, i.e., are not separated by any other party, then the distance between their centres P and Q is at least 2^{i-1}. We prove this also by induction on i. Consider the party of 2^s people with centre S (say) that was between the 2^p and the 2^q before alternate parties were removed. If p or $s = i - 1$ (which must be the case if $i = 1$) then the distance PS is clearly greater than 2^{i-2}; otherwise $PS \geqq 2^{i-2}$ by the induction hypothesis. Similarly $SQ \geqq 2^{i-2}$, and so $PQ \geqq 2^{i-1}$ as required.

It follows that the total length of restaurant occupied after the ith move is at least

$$2^{i-1}((i+1)k2^{-(i+1)} - 3) = \tfrac{1}{4}(i+1)k - 3 \cdot 2^{i-1}.$$

In particular, taking $i = r - 1$, the total length of restaurant occupied after the $(r-1)$-th move must be at least $\frac{1}{4}rk - 3 \cdot 2^{r-2}$. Thus we have defeated the Proprietor unless $N \geqq \frac{1}{4}rk - 3 \cdot 2^{r-2} \geqq \frac{1}{4}k(\log_2 k - 3)$. (This last inequality comes from simple differentiation.)

If we write $g(k) := \frac{1}{4}k(\log_2 k - 3)$, we note that

$$8 > \frac{k \log_2 g(k)}{g(k)}$$

if $k > 2^5$, and that

$$k > 8\,\frac{N}{\log_2 N} > k\,\frac{\log_2 g(k)}{g(k)}\;\frac{N}{\log_2 N}$$

therefore implies $N < g(k)$. Thus the General's Mate defeats the Proprietor if

$k > 8(N/\log_2 N)$ and $k > 2^5$; and we saw at the beginning of the proof that this is true if $k \leqq 2^5$. This completes the proof of Theorem 2. ∎

4. The generalization. Suppose now that every party of people must contain at least k_1 and at most k_2 people. The analogue of the conclusion of Lemma 1.1 is that $m \geqq k_1 2^{(r/k)-3}$. The proof is exactly the same as that of Lemma 1.1. The Proprietor's Mate can thus seat any party that we send in, provided that $k_2 < k_1 2^{(N/k)-3}$, i.e., $N > k(\log_2(k_2/k_1) + 3)$.

The analogue of Theorem 2 is also not difficult to obtain. We define a 'pseudo-person' to be a group of k_1 people, and we work with $[k/k_1]$ 'pseudo-people'. Suppose that we terminate the General's Mate's strategy after the

$$([\log_2(k_2/k_1)] - 1)\text{-th}$$

move, in which we sent in parties of 'pseudo-people' of size $2^{[\log_2(k_2/k_1)]-2}$, i.e., parties of people of size $k_1 2^{[\log_2(k_2/k_1)]-2}$. We thus defeat the Proprietor unless

$$\begin{aligned} N &\geqq \tfrac{1}{4}k_1[k/k_1][\log_2(k_2/k_1)] - 3k_1 2^{[\log_2(k_2/k_1)]-2} \\ &\geqq \tfrac{1}{4}k_1[k/k_1](\log_2[k_2/k_1] - 3), \end{aligned}$$

(observing that $k \geqq k_2$ and differentiating as before but now with respect to k_2/k_1),

$$\geqq \tfrac{1}{4}k(\log_2(k_2/k_1) - 4)$$

whenever this is positive.

Added in proof. I am indebted to R. L. Graham, of Bell Laboratories, for the following information about the history of this problem. It was first mentioned, as a problem on the storage of information in a computer, by M. D. McIlroy, also of Bell Laboratories, in a short note called *Some problems in dynamic storage allocation*, which was circulated in February 1968. Shortly afterwards, Graham obtained some bounds on the length of store necessary, which were never published. Later that year, J. M. Robson obtained some better bounds, which he has subsequently published under the title *An estimate of the store size necessary for dynamic storage allocation* in J. Assoc. Comp. Machinery, 18 (1971) 416–423. His upper bound is similar to mine, but his lower bound is slightly better: in view of his Theorem 3, the "1/4" in my lower bound can be increased to "4/13", and the "–4" can also be improved slightly. His methods are basically similar to mine, but are slightly more sophisticated. The "Bay Restaurant" formulation of the problem seems to be due to Conway.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF NOTTINGHAM, NOTTINGHAM, ENGLAND.

ELEMENTARY CONSEQUENCES OF THE NONCONTRACTIBILITY OF THE CIRCLE

ROBERT F. BROWN

Most topology texts prove the noncontractibility of the circle as an easy corollary of the computation of its fundamental group or its homology groups. Thus, this property of the circle has become part of a relatively advanced subject: algebraic topology.

The noncontractibility of the circle can be proved quite easily using only material from a traditional undergraduate point-set topology course. Furthermore, a large number of interesting and significant consequences of this fact can be obtained without additional mathematical prerequisites. Therefore, a fascinating area of elementary topology has been inaccessible to a substantial number of people who do, in fact, have sufficient background in topology to understand and appreciate it. The purpose of this paper is to try to correct this oversight.

1. Noncontractibility of the circle. Maps $f, g\colon X \to Y$ are *homotopic* if there is a map (*homotopy*) $h: X \times I \to Y$, where $I = [0,1]$, such that $H(x,0) = f(x)$ and $H(x,1) = g(x)$ for all x in X. A space X is *contractible* if the identity map on X is homotopic to a constant map. Such a homotopy is called a *contraction* of X.

Denote the circle of radius one with center at the origin of the plane by S^1 and the real numbers by R.

The following result has been attributed to Eilenberg [**2**, p. 361]. It is also proved as Theorem 1 in [**4**].

THEOREM. *If $f: S^1 \to S^1$ is a map homotopic to the constant map, then there exists a map $\phi: S^1 \to R$ such that $f(x) = e^{i\phi(x)}$ for all $x \in S^1$.*

Proof. Suppose that $g: S^1 \to S^1$ is a map for which there is a map $\phi: S^1 \to R$ with $g(x) = e^{i\phi(x)}$for all $x \in S^1$. We claim that if $h: S^1 \to S^1$ is a map such that $|g(x) - h(x)| < 2$ for all $x \in S^1$, then $f(x) = e^{i\psi(x)}$ for some map $\psi: S^1 \to R$. To prove the claim, note that $|g(x) - h(x)| < 2$ implies $h(x) \neq -g(x)$ so $h(x)/g(x) \neq -1$. Let $\lambda(x)$ be the number of radians in the angle between 1 and $h(x)/g(x)$ if $h(x)/g(x)$ is above the x-axis, and the negative of that number if $h(x)/g(x)$ is below. Then $h(x)/g(x) = e^{i\lambda(x)}$, or

$$h(x) = g(x)e^{i\lambda(x)} = e^{i(\phi(x)+\lambda(x))} = e^{i\psi(x)},$$

which verifies the claim. Now let $H: S^1 \times I \to S^1$ be a homotopy such that $H(x,0) = x_0$ (constant) and $H(x,1) = f(x)$. By uniform continuity, there exists $\delta > 0$ such that $|H(x,t) - H(x,t')| < 2$, for $|t - t'| < \delta$ and all $x \in S^1$. Partition $[0,1]$ by $0 = t_0 < t_1 < \cdots < t_{n-1} < t_n = 1$ so that $|t_{i+1} - t_i| < \delta$. A constant map from S^1 to itself is of the form $e^{i\phi(x)}$ where $\phi: S^1 \to R$ is constant, so $H(x, t_1) = e^{i\phi_1(x)}$ for some $\phi_1: S^1 \to R$. Repeating the argument $n - 1$ times completes the proof.

THEOREM. *The circle is not contractible.* (Note [**2**, p. 364].)

Proof. Assuming the contrary, the previous theorem implies the existence of a map $\phi : S^1 \to R$ such that $x = e^{i\phi(x)}$ for all $x \in S^1$. The map ϕ is one-to-one and so the function $g : S^1 \to \{-1, 1\}$ given by

$$g(x) = \frac{\phi(x) - \phi(-x)}{|\phi(x) - \phi(-x)|}$$

is well-defined and continuous. But $g(-x) = -g(x)$, so g takes the connected space S^1onto a disconnected space; which is impossible.

2. Elementary consequences. Let R^2 denote the plane, let D^2 be the disc of radius one centered at the origin, and consider S^1 as the boundary of D^2.

A. (Knaster, Kuratowski and Mazurkiewicz [7]). *Given a map $f: D^2 \to R^2$ such that $f(S^1) \subseteq D^2$, there exists $x_0 \in D^2$ with $f(x_0) = x_0$.*

Proof. (Benjamin Halpern) Define $r : R^2 - 0 \to S^1$ by $r(x) = x/|x|$. If $f(x) \neq x$ for all $x \in D^2$, then $H : S^1 \times I \to S^1$ defined by

$$H(x,t) = \begin{cases} r[x - 2tf(x)] & \text{if } 0 \leqq t \leqq \frac{1}{2} \\ r[(2-2t)x - f((2-2t)x)] & \text{if } \frac{1}{2} \leqq t \leqq 1 \end{cases}$$

would be a contraction of S^1, contrary to what has been established. (Note that if $t < \frac{1}{2}$ then $2t|f(x)| < |x|$ so H is well-defined.)

B. (Brouwer Fixed Point Theorem). *Given a map $f: D^2 \to D^2$, there exists $x_0 \in D^2$ with $f(x_0) = x_0$.*

C. *Let $I^2 = I \times I$ and define $B_1 = \{1\} \times I$, $B_1' = \{0\} \times I$, $B_2 = I \times \{1\}$, $B_2' = I \times \{0\}$. Suppose that A_1 and A_2 are closed subsets of I^2 such that, for $i = 1, 2$, B_i and B_i' are in different components of $I^2 - A_i$, then $A_1 \cap A_2 \neq \varnothing$.*

There is an elementary proof of this statement on pages 40–41 of [**6**] — as the case $n = 2$ of "Proposition D." We shall not repeat the argument here.

If $A \subseteq X$ and $f: X \to A$ is a map such that $f(a) = a$ for all $a \in A$, then f is a *retraction* of X onto A.

D. *There is no retraction of D^2 onto S^1.*

Proof. Suppose a retraction $f: D^2 \to S^1$ exists. Let $g : S^1 \to S^1$ be a nontrivial rotation, then $gf: D^2 \to S^1 \subseteq D^2$ contradicts the Brouwer theorem.

Let R^3 denote three-dimensional euclidean space. Two circles J and K in R^3 are *unlinked* if there exists a map $f: D^2 \to R^3 - K$, such that the restriction of f to S^1 is a homeomorphism onto J. Otherwise J and K are *linked.*

E. *Let J be the circle of radius one in the plane $y = 0$ with center* $(1, 0, 0)$, *and*

let K be the circle of radius one in the plane $z = 0$ with center $(0,0,0)$. *Then J and K are linked.*

Proof. Let A be the half-plane of points $(x, 0, z)$ such that $x \geqq 0$. Then radial projection σ from $(1,0,0,)$ retracts $A - (1,0,0)$ onto J. The map $g : R^3 \to R^3$ defined by

$$g(x, y, z) = ((x^2 + y^2)^{\frac{1}{2}}, 0, z)$$

retracts $R^3 - K$ onto $A - (1,0,0)$. Suppose J and K are unlinked, and let $f : D^2 \to R^3 - K$ be a map such that $h : S^1 \to J$, the restriction of f, is a homeomorphism. The retraction

$$h^{-1}\sigma g f : D^2 \to S^1$$

contradicts the previous result.

The noncontractibility of the circle and Theorems A–E are all closely related and can be proved one from another without difficulty. (Compare the remark on page 41 of [**6**].)

F. *If $f : D^2 \to R^2$ is a map such that for $x \in S^1$, either $f(x) = (0,0)$, or x does not lie on the ray from the origin through $f(x)$, then $f(x_0) = x_0$ for some $x_0 \in D^2$* (see [**2**, p. 353]).

Proof. Define $g : D^2 \to D^2$ by

$$g(x) = \begin{cases} f(x) & \text{if } f(x) \in D^2 \\ f(x)/|f(x)| & \text{otherwise.} \end{cases}$$

Then $g(x_0) = x_0$ for some $x_0 \in D^2$ by the Brouwer theorem. If $f(x_0) \notin D^2$, then $x_0 = g(x_0) \in S^1$. But $g(x_0)$ does lie on the ray from the origin through $f(x_0)$, so $f(x_0)$ must be in D^2 and therefore $f(x_0) = x_0$.

G. (Fundamental Theorem of Algebra). *Every nonconstant polynomial with complex coefficients has a complex root.*

An elementary proof, based on the Brouwer theorem, can be found in [**4**, Theorem 6].

Given a map $f : X \to Y$, *the mapping cylinder* $M(f)$ of the map is the quotient space of the disjoint union $(X \times I) \cup Y$ under the equivalence relation: $(s, 1) \sim f(x)$ for all $x \in X$. Note that there is an embedding $i : Y \to M(f)$ taking $y \in Y$ to its equivalence class $[y]$.

H. *The circle is not a mapping cylinder*; *that is, there are no spaces X, Y and map $f : X \to Y$ such that $M(f)$ is homeomorphic to S^1* (see [**5**, p. 159]).

Proof. Suppose there is a homeomorphism $h : M(f) \to S^1$ for some map $f : X \to Y$.

Let $x_0 \in X$, then $h(x_0, 0) \notin hi(Y)$. Define $H : M(f) \times I \to M(f)$ by

$$H([y], s) = [y] \qquad \text{for } y \in Y,$$
$$H([x,t], s) = [x, (1-s)t + s] \qquad \text{for } (x,t) \in X \times I.$$

Then $H([x,t], 0) = [x,t]$. Define $H_1 : M(f) \to i(Y)$ by setting $H_1([y]) = [y]$ and $H_1([x,t]) = [x, 1]$. Then H_1 is a retraction. The existence of the homotopy H proves that the identity map on S^1 is homotopic to the composition:

$$S^1 \xrightarrow{h^{-1}} M(f) \xrightarrow{H_1} i(Y) \xrightarrow{h} hi(Y) \subseteq (S^1 - h(x_0, 0)) \subseteq S^1.$$

On the other hand, $S^1 - h(x_0, 0)$ is homeomorphic to R and therefore contractible, so the composition is homotopic to a constant map. Thus we obtain a contraction of S^1, contrary to the first section of this paper.

Let C denote the complex numbers topologized by identifying C with R^2. Let $C^{n+1} - 0$ be the space of all $(n+1)$-tuples $(z_0, z_1, \cdots, z_n)$ of complex numbers such that $\sum_{j=0}^{n} z_j \bar{z}_j \neq 0$. *Complex projective n-space* CP^n is the quotient space of $C^{n+1} - 0$ under the equivalence relation $\sim$, where $(z_0, z_1, \cdots, z_n) \sim (z'_0, z'_1, \cdots, z'_n)$ if and only if

$$(z'_0, z'_1, \cdots, z'_n) = \lambda(z_0, z_1, \cdots, z_n) = (\lambda z_0, \lambda z_1, \cdots, \lambda z_n)$$

for some $\lambda \in C$. Denote the equivalence class containing $(z_0, z_1, \cdots, z_n)$ by $[z_0, z_1, \cdots, z_n]$.

I. *It is not possible to choose representatives of the elements of* CP^n *in a continuous manner; that is, there is no map* $\sigma : CP^n \to C^{n+1} - 0$ *such that* $\sigma[z_0, z_1, \cdots, z_n]$ *is a member of* $[z_0, z_1, \cdots, z_n]$.

Proof. Let $p : C^{n+1} - 0 \to CP^n$ be the quotient map. The identification of C with R^2 makes $S^1 = \{z \in C \mid z\bar{z} = 1\}$. Assume that σ exists and define $f : S^1 \to C^{n+1} - 0$ by $f(z) = z^{-1}\sigma[z, z, \cdots, z]$. For $(z_0, z_1, \cdots, z_n)$ in $C^{n+1} - 0$, we know that $\sigma[z_0, z_1, \cdots, z_n] = \lambda(z_0, z_1, \cdots, z_n)$ for some nonzero complex number λ. Define $g : C^{n+1} - 0 \to S^1$ by $g(z_0, z_1, \cdots, z_n) = \lambda / |\lambda|$. Now, for $z \in S^1$,

$$\begin{aligned}\sigma p(f(z)) &= \sigma p(z^{-1}\sigma[z, z, \cdots, z]) \\ &= \sigma p \sigma[z, z, \cdots, z] \\ &= \sigma[z, z, \cdots, z] = z f(z).\end{aligned}$$

Thus, since $|z| = 1$, we have $gf(z) = z$ for all $z \in S^1$. Let $\Delta = \{(z, z, \cdots, z) \in C^{n+1} - 0\}$ and define $H : \Delta \times I \to C^{n+1} - 0$ by

$$H((z, z, \cdots, z), t) = \begin{cases} ((1-2t)z + 2t, z, \cdots, z) & \text{if } 0 \leqq t \leqq \frac{1}{2} \\ (1, (2-2t)z, \cdots, (2-2t)z) & \text{if } \frac{1}{2} \leqq t \leqq 1. \end{cases}$$

Then H is a homotopy between the inclusion map of Δ in $C^{n+1} - 0$ and a constant map. Observe that $f(S^1) \subseteq \Delta$ and define $G : S^1 \times I \to S^1$ by $G(z,t) = gH(f(z),t)$. Then $G(z,0) = gf(z) = z$ while $G(z,1) = g(1,0,\cdots,0)$, so that G would be a contraction of S^1. But this is impossible.

J. (Jordan Curve Theorem) *If S is a subset of R^2 homeomorphic to the circle, then $R^2 - S$ has precisely two components.*

An elementary proof can be found in [**2**, p. 362].

3. Generalizations. It is possible to continue the program of this paper and prove the noncontractibility of all n-spheres S^n without explicitly using algebraic topology. For example, there are proofs, based on the theory of the "degree modulo two" of a map of spheres, in [**6**, pp. 37–40] and [**8**, pp. 20–25]. Another alternative is to begin either with the combinatorial proof of the Brouwer Fixed Point Theorem in n-dimensions based on Sperner's Lemma ([**1**, pp. 155–169] or [**11**]), with Tucker's proof [**12**] of Brouwer's Theorem, or with Milnor's proof [**8**, p. 24] of the same theorem. Then the Brouwer theorem obviously implies that there is no retraction of D^n onto S^{n-1} (compare Theorem D of Section 2). Suppose that S^n were contractible and let $H : S^n \times I \to S^n$ be a map where $H(x,0) = x_0$ and $H(x,1) = x$, for all $x \in S^n$. Define $g : D^n \to S^n \times I$ by letting $g(x) = (x/|x|, |x|)$ if x is not the origin and sending the origin to $(x_0, 0)$. Then g is not continuous, but Hg is continuous — in fact, Hg is a retraction of D^n onto S^{n-1}. We conclude that S^n is not contractible.

Assuming that S^n is not contractible, the statements and proofs of Theorems A, B, C, D, F and H of Section 2 generalize to arbitrary dimensions with only trivial changes. Theorem E generalizes to linked n-spheres in R^{2n+1}; with obvious modifications of the proof. There is a form of the Fundamental Theorem of Algebra (Theorem G) for polynomials with quaternion coefficients or with Cayley number coefficients, but its proof requires some algebraic topology [**3**, p. 308]. Theorem I is essentially the fact that the Hopf fibering of S^3 over S^2 and, more generally, the fibre bundle S^{2n+1} over CP^n, admits no cross-section. Assuming S^3 noncontractible, the same proof will show that the fibre bundle S^{4n+3} over quaternionic projective n-space, in particular the Hopf fibering of S^7 over S^4, has the same property (see [**10**, pp. 106–108]). Theorem J extends to all dimensions (the Jordan-Brouwer Separation Theorem [**9**, p. 198]), but I know of no elementary proof. However, the geometric proof of a weaker statement — that if $S \subseteq R^2$ is homeomorphic to S^1 then $R^2 - S$ is disconnected — does generalize [**2**, p. 358].

Supported in part by National Science Foundation Grant GP-29639.

The author thanks the editor (H.F.) and the referee for their help in the preparation of this paper.

References

1. P. Aleksandrov, Combinatorial topology Vol. 1, Graylock Press, New York, 1956.
2. J. Dugundji, Topology, Allyn and Bacon, Boston, 1966.

3. S. Eilenberg and N. Steenrod, Foundations of algebraic topology, Princeton, N. J., 1952.
4. M. Fort, Some properties of continuous functions, this MONTHLY, 59 (1952) 372–375.
5. J. Hocking and G. Young, Topology, Addison-Wesley, Reading, Mass., 1961.
6. W. Hurewicz and H. Wallman, Dimension theory, Princeton, N. J., 1941.
7. B. Knaster, K. Kuratowski and S. Mazurkiewicz, Ein Beweis des Fixpunktsatzes für n-dimensionale Simplexe, Fund. Math., 14 (1929) 132–137.
8. J. Milnor, Topology from a differentiable viewpoint, Univ. of Virginia, 1965.
9. E. Spanier, Algebraic topology, McGraw-Hill, New York, 1966.
10. N. Steenrod, The topology of fibre bundles, Princeton, N. J., 1951.
11. C. Tompkins, Sperner's lemma and some extensions, Applied Combinatorial Math. (E. Beckenbach, Ed.), Wiley, New York, 1964, 416–455.
12. A. Tucker, Some topological properties of disk and sphere, Proc. First Canadian Math. Congress, 1945, 285–309.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA, LOS ANGELES, CALIFORNIA 90024.

THE SECOND U.S.A. MATHEMATICAL OLYMPIAD

S. GREITZER

The U.S.A. Mathematical Olympiad is a new venture, whose purpose is to attempt to discover secondary school students with superior mathematical talent — who possess creativity and inventiveness as well as computational skills. Participation is limited to about 100 students selected mainly from the Honor Roll of the Annual High School Mathematics Examination, plus a few recommended students from those states which sponsor their own High School mathematics competitions. The Olympiad consists of five essay-type problems, requiring mathematical power on the part of the participants.

1. Introduction. In 1973, 123 invitations were sent out, and 107 completed acceptances were received. (An acceptance is "complete" when the student agrees to participate *and* the school agrees to administer the test.) The Second U.S.A. Mathematical Olympiad took place on May 1, 1973. It is reproduced below.

SECOND U.S.A. MATHEMATICAL OLYMPIAD — MAY 1, 1973

1. Two points, P and Q, lie in the interior of a regular tetrahedron $ABCD$. Prove that angle $PAQ < 60°$.

2. Let $\{X_n\}$ and $\{Y_n\}$ denote two sequences of integers defined as follows:

$$X_0 = 1,\ X_1 = 1,\quad X_{n+1} = X_n + 2X_{n-1}\quad (n = 1, 2, 3, \cdots),$$

$$Y_0 = 1,\ Y_1 = 7,\quad Y_{n+1} = 2Y_n + 3Y_{n-1}\quad (n = 1, 2, 3, \cdots).$$

Thus, the first few terms of the two sequences are:

$$X: 1, 1, 3, 5, 11, 21, \cdots$$

$$Y: 1, 7, 17, 55, 161, 487, \cdots$$

Prove that, except for "1", there is no term which occurs in both sequences.

3. Three distinct vertices are chosen at random from the vertices of a given regular polygon of $(2n + 1)$ sides. If all such choices are equally likely, what is the probability that the center of the given polygon lies in the interior of the triangle determined by the three chosen random points?

4. Determine all the roots, real or complex, of the system of simultaneous equations

$$\begin{aligned} x + y + z &= 3 \\ x^2 + y^2 + z^2 &= 3 \\ x^5 + y^5 + z^5 &= 3. \end{aligned}$$

5. Show that the cube roots of three distinct prime numbers cannot be three terms (not necessarily consecutive) of an arithmetic progression.

(Solutions to the problems will appear in a forthcoming issue of the MATHEMATICS TEACHER.)

2. **Analysis:** All papers were returned by May 8, graded on May 11, and the top 25 papers regraded. Final results were mailed to the participating schools on May 25. These results are tabulated below:

H.S. EXAM SCORES

Olymp.	80–84.75	85–89.75	90–94.75	95–99.75	100–104.75	105–109.75	110–114.75	115–119.75	120–124.75	135–139.75
80–89										1
70–79					1					
60–69				1		1				
50–59	1		2				1		1	
40–49		1		1	6	3				
30–39		4	2	3		2	1			
20–29	8	9	4	5	1	1		1		
10–19	1	5		6	1	1	1			
0–9	1	8	3	4		2				

The eight finalists selected to receive awards had scores indicated in the rectangle at the upper part of the table.

The correlation between scores on the Annual High School Mathematics Examination and the Second Olympiad is 0.433, much better than the 0.24 for the First Olympiad, but still too low to indicate a meaningful correlation. Again, since all eight finalists scored at the top, we may conclude that students rated superior on the Olympiad are superior on the High School Mathematics Examination, but that the converse is not necessarily true.

In the hope that it will be found useful to teachers, a table of grades attained by participants on each problem is provided:

PROBLEM NUMBER

score	1	2	3	4	5
21–25		6	3		
16–20	3	16	36	3	8
11–15	13	7	3	1	2
6–10	14	6	9	4	10
1–5	38	5	35	84	46
0	39	67	21	15	41

(Note: A score above 20 indicates extra credit for ingenuity.)

As was done last year, the eight top scorers were elected finalists, to receive awards at ceremonies to be held in Washington, D.C. Through the generous support of International Business Machines, Inc., these students were assembled in Washington on June 26. In three days of activities, they toured the city, visited the Smithsonian Institution and the National Bureau of Standards, and received their awards at the National Academy of Sciences. At this reception, each finalist received an HP-35 calculator as a gift from the Hewlett-Packard Company, an engraved sterling silver tray, a set of books and a \$100 bond from I.B.M.

The principal speaker at the reception was Dr. Saunders Mac Lane, past president of the MAA, present president of the AMS and vice-president elect of the National Academy of Sciences. The reception was followed by a dinner in honor of the finalists at the Academy.

In a report of its activities to the Mathematical Association of America, the Olympiad Committee acknowledged with thanks the support and help provided by these organizations which made the award ceremonies so successful, and also acknowledged the efforts of the committee which graded the papers: Michael Aissen, John Bender, Richard Bumby, Philip Guza, L. M. Kelly, Sol Leader, B. Muckenhoupt, and H. Zimmerberg. Finally, our thanks for the cooperation of the members of the High School Contest Committee and the regional directors whose assistance made the Second Olympiad run so smoothly.

The list of finalists is presented below:

1	Sheldon Katz	Brooklyn Technical H.S.	Brooklyn, N.Y.
2	Eric S. Lander	Stuyvesant High School	Manhattan, N.Y.
3	Gerhard Arenstorf	Peabody Demonstration School	Nashville, Tenn.
4	Martin D. Hirsch	Grant High School	Van Nuys, Calif.
5	David J. Anick	Ranney School	New Shrewsbury, N.J.
6	Ernest S. Davis	Classical High School	Providence, R.I.
6	Bruce E. Hajek	Willowbrook H.S.	Villa Park, Ill.
7	Karl C. Rubin	Woodrow Wilson H.S.	Washington, D.C.

As the numbers indicate, there was a tie for sixth place.

The Third U.S.A. Mathematical Olympiad will take place on Tuesday, May 7, 1974. The Olympiad Committee Consists of the following members: Samuel L. Greitzer (chairman), Alfred Kalfus, Murray S. Klamkin, P. A. Paige, C. C. Rousseau, Nura D. Turner.

DEPARTMENT OF MATHEMATICS, RUTGERS — THE STATE UNIVERSITY, NEWARK, N. J. 07102.

QUERIES

EDITED BY A. C. ZITRONENBAUM

This Department welcomes queries from readers about mathematics at the collegiate level, such as sources for exposition of a particular topic from a special point of view, references to vaguely remembered articles, descriptions of special kinds of courses or teaching methods, methods for constructing illustrative examples for exercises of particular kinds (questions on research topics should, in general, be addressed to the "Queries Department" of the American Mathematical Society). Replies will be forwarded to the questioner and may also be edited into a composite answer for publication in this Department. Consequently all items submitted for consideration for possible publication should include the name and complete mailing address of the person who is to receive the reply. Queries and answers should be sent to A. C. Zitronenbaum, Department of Mathematics, Cornell University, Ithaca, N. Y. 14850.

5. R. P. Boas, Jr. In discussing the applications of Stokes' theorem it is important to know that a two-sided surface can be spanned into any given simple closed curve in 3-space (even if it is knotted). Does anybody know of a proof of this fact (due, I believe, to van Kampen) that is accessible at the sophomore calculus level? (For a pictûre in the simplest knotted case, see Steinhaus, Mathematical Snapshots, 1950 ed. p. 233, 1960 ed. p. 295.)

6. P. Mielke. I would like to learn results of any controlled experiments that studied whether there was any measurable difference when introductory *calculus* was taught in the lecture-quiz manner rather than in the traditional small-class manner.

MATHEMATICAL NOTES

EDITED BY DAVID ROSELLE

Material for this Department should be sent to David Roselle, Department of Mathematics, Louisiana State University, Baton Rouge, LA 70803.

ON THE NUMBER OF TIMES AN INTEGER OCCURS AS A BINOMIAL COEFFICIENT

H. L. ABBOTT, P. ERDÖS, AND D. HANSON

Let $N(t)$ denote the number of times the integer $t > 1$ occurs as a binomial coefficient; that is, $N(t)$ is the number of solutions of $t = \binom{n}{r}$ in integers n and r. We have $N(2) = 1$, $N(3) = N(4) = N(5) = 2$, $N(6) = 3$, etc. In a recent note in the research problems section of the MONTHLY, D. Singmaster [1] proved that

$$N(t) = O(\log t). \tag{1}$$

He conjectured that $N(t) = O(1)$ but pointed out that this conjecture, if it is in fact true, is perhaps very deep. In [1] and [5], Singmaster points out that $N(t) = 6$ for the following values of $t \leqq 2^{48}$; $t = 120, 210, 1540, 7140, 11628$ and 24310. It has been shown by Singmaster [5] and D. Lind [6] that $N(t) \geqq 6$ infinitely often. Singmaster has verified that the only value of $t \leqq 2^{48}$ for which $N(t) \geqq 8$ is $t = 3003$, for which $N(t) = 8$.

In this note we obtain some additional information about the behavior of $N(t)$. In Theorem 1 we prove that the average and normal order of $N(t)$ is 2; in fact, we prove somewhat more than this, namely, the number of integers t, $1 < t \leqq x$, for which $N(t) > 2$ is $O(\sqrt{x})$. (See [4] p. 263 and p. 356, for the definitions of average and normal order.) In Theorem 2 we give an upper bound for $N(t)$ in terms of the number of distinct prime factors of t. Our main result is Theorem 3, in which we show that (1) can be improved to $N(t) = O(\log t/\log\log t)$. Finally, in Theorem 4, we consider the related problem of determining the number of representations of an integer as a product of consecutive integers.

THEOREM 1. *The average and normal order of* $N(t) = 2$.

Proof. For integral x, let n be defined by $\binom{2n-2}{n-1} < x \leqq \binom{2n}{n}$ so that $n = O(\log x)$. We have

$$\sum_{1<t\leqq x} N(t) = 2 \sum_{\substack{1<\binom{m}{r}\leqq x\\ 2r\leqq m}} 1 - \sum_{1<\binom{2k}{k}\leqq x} 1$$

$$= 2\left\{ \sum_{1<\binom{m}{1}\leq x} 1 + \sum_{1<\binom{m}{2}\leq x} 1 + \sum_{\substack{1<\binom{m}{r}\leq x \\ 3\leq r\leq m/2}} 1 \right\} - \sum_{1<\binom{2k}{k}\leq x} 1 \tag{2}$$

$$= 2x + 2\sqrt{2}x^{1/2} + O(x^{1/3}n)$$

$$= 2x + 2\sqrt{2}x^{1/2} + O(x^{1/3}\log x).$$

It follows that the average order of $N(t)$ is 2.

Let $f(x)$ be the number of integers t, $1 < t \leqq x$, such that $N(t) = 2$ and $g(x)$ the number such that $N(t) > 2$, so that $f(x) + g(x) = x - 2$. We have

$$\begin{aligned} \sum_{1<t\leqq x} N(t) &\geqq 2f(x) + 3g(x) + 1 \\ &= 2(x - 2 - g(x)) + 3g(x) + 1 \\ &= 2x + 2g(x) - 3. \end{aligned} \tag{3}$$

It follows from (2) and (3) that $g(x) = O(x^{1/2})$ and this implies that the normal order of $N(t)$ is 2.

THEOREM 2. *Let $w(t)$ denote the number of distinct prime factors of the integer $t > 1$. For all t satisfying $w(t) < \log t/\log\log t$ we have*

$$N(t) < \frac{2w(t)\log t}{\log t - w(t)\log\log t}. \tag{4}$$

Proof. The theorem can be verified directly for $t \leqq 20$. In what follows we therefore assume $t \geqq 21$. Let $k = k(t)$ be the largest integer for which $t = \binom{n}{k}$ for some $n \geqq 2k$. Then clearly

$$N(t) \leqq 2k. \tag{5}$$

By an easy induction argument we have, for $k \geqq 4$, $t = \binom{n}{k} \geqq \binom{2k}{k} \geqq e^k$. Since we are assuming $t \geqq 21 > e^3$, the inequality $t \geqq e^k$ holds for all $k \geqq 1$. Equivalently,

$$k \leqq \log t \text{ and } \log k \leqq \log\log t. \tag{6}$$

Let P^α be the highest power of the prime P which divides t. Then, according to the well-known theorem of Legendre,

$$\alpha = \sum_{i=1}^{[\log_P n]} \left\{ \left[\frac{n}{P^i}\right] - \left[\frac{n-k}{P^i}\right] - \left[\frac{k}{P^i}\right] \right\}.$$

Each term in the sum on the right is either 0 or 1. The number of non-zero terms is therefore α and we must have

$$P^{\alpha} \leqq n. \tag{7}$$

From $t = \binom{n}{k}$ and the inequality $\binom{n}{k} \geqq \left(\frac{n}{k}\right)^k$, we obtain

$$n \leqq kt^{1/k} \tag{8}$$

and from (7) and (8) it follows that

$$t = \Pi P^{\alpha} \leqq n^{w(t)} \leqq k^{w(t)} t^{w(t)/k}.$$

If we take logarithms and substitute from the second inequality in (6) we get, after some manipulations,

$$k \leqq \frac{w(t)\log t}{\log t - w(t)\log\log t},$$

and this, together with (5), yields (4). This completes the proof of Theorem 2.

We come now to our main result.

THEOREM 3. $N(t) = O(\log t/\log\log t)$.

Proof. We shall need to make use of the following deep result of A. E. Ingham [**2**] on the distribution of the primes: If $\alpha \geqq 5/8$, there is a prime between x and $x + x^{\alpha}$ for all sufficiently large x.

For a given integer t, let $S = \{n\colon t = \binom{n}{k}$ for some $k \leqq n/2\}$. Write $S = S_1 \cup S_2$ where $S_1 = \{n\colon n \in S, n > (\log t)^{6/5}\}$ and $S_2 = \{n\colon n \in S, n \leqq (\log t)^{6/5}\}$. We first estimate the size of S_1. Let $n \in S_1$ and let $t = \binom{n}{k}$. We have at our disposal the following inequalities:

$$t = \binom{n}{k} \geqq \left(\frac{n}{k}\right)^k \tag{9}$$

$$t \geqq e^k \text{ (see the proof of Theorem 2)} \tag{10}$$

$$n > (\log t)^{6/5}. \tag{11}$$

Thus

$$k \leqq \frac{\log t}{\log n/k} \leqq \frac{\log t}{\log(n/\log t)} \leqq \frac{\log t}{\log(\log t)^{1/5}}$$

$$= O\left(\frac{\log t}{\log\log t}\right),$$

where we have used, successively, (9), (10) and (11). It follows that

$$|S_1| = O(\log t/\log\log t).$$

Next we must estimate the size of S_2. Let N be the largest number in S_2 and let $t = \binom{N}{K}$. We have the inequalities

$$N \leqq (\log t)^{6/5} \text{ and } t \leqq N^K$$

from which we get $N \leqq (K \log N)^{6/5}$. This in turn implies, for N sufficiently large,

$$N \leqq K^{8/5} < K^{8/5} + K,$$

and it is easy to see that this last inequality implies

$$(N - K) + (N - K)^{5/8} \leqq N.$$

We are now in a position to apply the theorem of Ingham. By this theorem, there is a largest prime P satisfying $K \leqq N - K < P \leqq N$. It follows that P divides t and hence that $n \geqq P$ for all $n \in S_2$. Hence all of the numbers in S_2 lie between P and N. The number of numbers in S_2 is thus

$$|S_2| \leqq N - P \leqq P^{5/8} \leqq N^{5/8} \leqq (\log t)^{3/4} = O(\log t/\log\log t),$$

where, in obtaining the second inequality, we again appeal to Ingham's result. This completes the proof of Theorem 3.

We remark that if one makes use of the unproved conjecture of Cramér [3] asserting that there is a prime between x and $x + (\log x)^2$ for all sufficiently large x, then our argument gives $N(t) = O((\log t)^{2/3+\varepsilon})$. The proof is basically the same as before, except that one puts $S_1 = \{n: n \in S,\ \log n > (\log t)^{1/3-\varepsilon}\}$. We omit the rather laborious details of the argument.

We conclude with a brief discussion of a somewhat related problem. Let $G(t)$ denote the number of representations of the positive integer t as a product of consecutive integers; that is, $G(t)$ is the number of solutions of $t = (n+1)(n+2)\cdots(n+l)$ in integers n and l. For any such solution we have $t \geqq l!$ and consequently we get $G(t) = O(\log t/\log\log t)$. For this problem, however, we can get a substantially stronger result.

THEOREM 4. $G(t) = O(\sqrt{\log t})$.

Proof. Let $S = \{l: t = (n+1)(n+2)\cdots(n+l) \text{ for some } n\}$. Let L_0 be the largest number in S and let

$$S_1 = \{l: l \in S, L_0 - C(\log t)^{1/2} < l \leqq L_0\} \text{ and } S_2 = \{l: l \in S, l \leqq L_0 - C(\log t)^{1/2}\}.$$

C is a constant. It is clear that $|S_1| \leqq C(\log t)^{1/2}$. It remains to estimate the size of $|S_2|$. Let 2^α be the highest power of 2 which divides t. Then, for some constant C_1,

$$\alpha \geqq \sum_{j=1}^{\infty} \left[\frac{L_0}{2^j}\right] \geqq L_0 - C_1 \log L_0. \tag{12}$$

Let L be the largest number in S_2 and let $t = (N+1)(N+2)\cdots(N+L)$. Let 2^β be the highest power of 2 which divides one of $(N+1), (N+2), \cdots, (N+L)$, say $N+k$. Then

(13) $$\alpha = \beta + \sum_{j=1}^{\infty}\left[\frac{L-k}{2^j}\right] + \sum_{j=1}^{\infty}\left[\frac{k-1}{2^j}\right].$$

In fact (13) follows from the observation that the first sum on the right is the exponent to which 2 divides the product $(N+k+1)(N+k+2)\cdots(N+L)$, while the second sum is the exponent to which 2 divides the product $(N+1)(N+2)\cdots(N+k-1)$. It follows from (13) that

(14) $$\alpha \leqq \beta + \sum_{j=1}^{\infty}\left[\frac{L}{2^j}\right] \leqq \beta + L.$$

Thus,

(15) $$\begin{aligned}\beta &\geqq \alpha - L\\ &\geqq (L_0 - C_1 \log L_0) - (L_0 - C(\log t)^{1/2})\\ &\geqq C(\log t)^{1/2} - C_1 \log L_0\\ &\geqq C_2(\log t)^{1/2},\end{aligned}$$

where we have used (14), (12), the definition of S_2 and the estimate $L_0 = O(\log t)$. We need two further inequalities; the first of which is obvious. These are

(16) $$(N+1)^L \leqq t$$

and, for t sufficiently large,

(17) $$N + 1 \geqq 2^{\beta-1}.$$

To obtain (17) we simply have to notice that $N + L \geqq N + k \geqq 2^\beta$, so that $N+1 \geqq 2^\beta - (L-1)$ and (17) now follows from (15) and the fact that $L = O(\log t)$.

It now follows from (15), (16) and (17) that $L \leqq C_3 (\log t)^{1/2}$, where C_3 is a positive constant depending on C_2, and hence on C. This completes the proof of Theorem 4.

We remark that by choosing $C = (1+\varepsilon)(\log 2)^{-1/2}$, our argument yields $G(t) < (2+\varepsilon)(\log t/\log 2)^{1/2}$ for every $\varepsilon > 0$, provided $t \geqq t_0(\varepsilon)$.

References

1. D. Singmaster, How often does an integer occur as a binomial coefficient, this MONTHLY, 78 (1971) 385–386.

2. A. E. Ingham, On the difference between consecutive primes, Quart. J. Math. Oxford, 8 (1937) 255–266.

3. H. Cramér, On the order of magnitude of the difference between consecutive prime numbers, Acta. Arith., 2 (1936) 23–46.

4. G. H. Hardy and E. M. Wright, Introduction to the Theory of Numbers, Oxford University Press, New York. 1960.

5. D. Singmaster, Repeated binomial coefficients and Fibonacci Numbers, (to appear in Fibonacci Quarterly).

6. Douglas Lind, The quadratic field $Q(\sqrt{5})$ and a certain diophantine equation, Fibonacci Quarterly, 6 (1968) 86–94.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ALBERTA, EDMONTON, ALBERTA, CANADA.
DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALGARY, CALGARY, ALBERTA, CANADA.
DEPARTMENT OF MATHEMATICS, UNIVERSITY OF SASKATCHEWAN, REGINA, SASKATCHEWAN, CANADA.

SOME PROPERTIES OF AUTOMORPHISMS WITH AN APPLICATION TO MATRICES

R. HIRSHON

In this note we use an elementary property of group automorphisms to show the following:

THEOREM 1. *If A is a square matrix with integer entries and determinant* 1 *or* -1, *then given any positive integer k there exists an integer $n = f(k)$ such that each entry of A^n which is not on the main diagonal is divisible by k.*

THEOREM 2. *Let C be a periodic finitely generated group whose automorphism group is periodic. If C is a subgroup of finite index of the group B, then the automorphism group of B is periodic.*

Before proceeding with the proof of the above, note the following:

(1) Let G be a group and α an automorphism of G. Let E be a subgroup of G such that the groups $E\alpha^w$, $w = 0, 1, 2, \cdots$ are finite in number. Then there exists a positive integer j with $E\alpha^j = E$. For we may choose positive integers i and k with $i < k$ and $E\alpha^i = E\alpha^k$. Since α is an automorphism this implies $E\alpha^{k-i} = E$.

(2) If $X_1, X_2, X_3, \cdots, X_r$ are free generators of a free abelian group F and if $A = [a_{ij}]$ is a square matrix of order r with integer entries and determinant 1 or -1, the mapping

$$X_i \to \sum_{j=1}^{r} a_{ij} X_j, \quad i = 1, 2, \cdots, r,$$

induces an automorphism A_* of F.

For the proof of Theorem 1, suppose that A is of order r. In the notation of the above remark, let F_i be the unique subgroup of index k in F which contains the generators X_j with $j \neq i$. Since there are only finitely many subgroups of a given index in a finitely generated group, ([**1**], volume 2, p. 56) the subgroups $F_i A_*^j, j = 0, 1, 2, \cdots$ are finite in number. Choose a positive integer n_i with $F_i A_*^{n_i} = F_i$.

Then if n is the product of the n_i, $F_i A_*^n = F_i$ for all i. This implies that A^n has the desired property.

For the proof of Theorem 2, let α be an automorphism of B. Choose $j > 0$ with $C\alpha^j = C$. Choose $k > 0$ such that $\gamma = \alpha^{jk}$ is the identity on C. Choose a normal subgroup N of B with $N \subset C$ and $[B\colon N]$ finite. Hence γ induces an automorphism γ_* on B/N and if $\gamma_*^d = 1$ then $b\gamma^d = b \bmod N$ for all b in B. But if $b\gamma^d = bn$, $n \in N$ and if n has order c, this implies $b\gamma^{dc} = b$. Since B is finitely generated we may find a suitable power of γ, γ^i, which fixes each generator of B. Hence $\alpha^{jki} = 1$.

In relation to the automorphism group of torsion free groups we have

THEOREM 3. *Let B be a finitely generated group such that every element of B has at most one jth root for all positive integers j. Then if B has a subgroup of finite index whose automorphism group is periodic, the automorphism group of B is periodic.*

Proof. Let C be a subgroup of finite index in B such that the automorphism group of C is periodic. Let α be an automorphism of B. Choose a positive integer r with $C\alpha^r = C$. Hence α^r induces an automorphism of C and hence some power of α^r, say α^{rt}, induces the identity automorphism on C. If $X \in B$ choose a positive integer s with $X^s \in C$. Hence if $X\alpha^{rt} = X_1$, $X^s = X_1^s$ from which we conclude $X = X_1$.

Groups whose elements have at most one jth root for any j are discussed in [1], volume 2, p. 242.

Reference

1. A. G. Kurosh, The Theory of Groups, Chelsea, New York, 1960.

DEPARTMENT OF MATHEMATICS, POLYTECHNIC INSTITUTE OF BROOKLYN, 333 JAY STREET, BROOKLYN, N. Y. 11201.

GENERALIZED PYTHAGOREAN THEOREM

D. R. CONANT AND W. A. BEYER

A partial survey of the literature reveals surprisingly few generalizations of the Pythagorean theorem. Pappus (A.D. 300) generalized it to allow arbitrary parallelograms on the legs of the right triangle. The law of cosines is certainly a generalization. J. Faulhaber (1622; see [1]) extended the law of cosines to a relation between face areas of a tetrahedron. Martin Gardner [2] mentions a generalization of the Pythagorean theorem in which arbitrary figures are allowed on the sides of a right triangle provided the figures are similar and have corresponding sides on the triangle. In this case the area of the figure on the hypotenuse equals the sum of the areas

of the other two figures. We have not seen the paper by Wirszup [3]. One could also consider generalizations to other polygons. In this note the Pythagorean theorem is generalized to Lebesgue measurable sets in m ($\leqq n$)-dimensional flats in n-dimensional Euclidean space R^n. An extension to n-dimensional complex complete inner product vector space C^n is indicated, the result being different from that for R^{2n}.

Let $\{a_i\}_{1 \leqq i \leqq m}$, where $m \leqq n$, be a set in R^n. Let A be the matrix whose columns are the coordinates of $a_1, \cdots, a_m$ with respect to an orthonormal basis X. The volume V_m of the parallelepiped spanned by $\{a_i\}$ is given by $\sqrt{(\det A^t A)}$, (Birkhoff and MacLane [4]).

DEFINITION 1. A **coordinate subspace with respect to a basis** X **of** R^n is a subspace for which a subset of X is a basis.

DEFINITION 2. An m-**dimensional flat** in R^n is a translate of an m-dimensional subspace of R^n

A fine-mesh sequential covering class of open parallelepipeds with measure V_m in an m-dimensional flat (Munroe [5, p. 60]) defines m-dimensional Lebesgue measure μ_m.

THEOREM. *The square of the μ_m-measure of a measurable set S in an m-dimensional flat in R^n is the sum of the squares of the μ_m-measures of the $\binom{n}{m}$ orthogonal projections of S on all the m-dimensional coordinate subspaces of R^n with respect to an orthonormal basis X.*

Proof. One has

$$\det A^t A = \sum_H \det A_H^t A_H \tag{1}$$

by the Cauchy-Binet Theorem (Muir [6] or Bourbaki [7]) where H is a subset of m elements of the integers $[1, n]$ and A_H is the $m \times m$ minor of A obtained by taking the rows of A with indices from H. The variable H runs over the $\binom{n}{m}$ sets of m elements of $[1, n]$.

Since the orthogonal projection of a parallelepiped in R^n is a parallelepiped, the determinants $\det A_H^t A_H$ in (1) are the squares of the measures of the projections of the parallelepiped spanned by $\{a_i\}$ onto the m-dimensional coordinate subspaces of R^n with respect to X. Thus the theorem is proved when S is a parallelepiped.

Suppose S is a countable (possibly finite) union of disjoint open parallelepipeds $\{P_i\}$ with $\mu_m(P_i) > 0$. Put $c_i = \mu_m(P_i)$ and $b_i^j = \mu_m(T_m^j P_i)$ where $T_m^j P_i$ is the projection of P_i on the jth m-dimensional coordinate subspace. Observe that $b_k^j / b_l^j = c_k / c_l$ for every j. Making use of this, one has the following calculation where in the finite case the infinite sums are replaced by appropriate finite sums. (The symbol $\sum_{k<l}$ is an abbreviation for $\sum_{k=1}^{\infty} \sum_{l=k+1}^{\infty}$.)

$$\sum_{j=1}^{\binom{n}{m}} \left(\sum_{i=1}^{\infty} b_i^j\right)^2 = \sum_{j=1}^{\binom{n}{m}} \sum_{i=1}^{\infty} \left(b_i^j\right)^2 + 2 \sum_{j=1}^{\binom{n}{m}} \sum_{k<l} b_k^j b_l^j$$

$$= \sum_{i=1}^{\infty} \sum_{j=1}^{\binom{n}{m}} (b_i^j)^2 + 2 \sum_{k<l} \sum_{j=1}^{\binom{n}{m}} b_k^j b_l^j$$

$$= \sum_{i=1}^{\infty} c_i^2 + 2 \sum_{k<l} \sum_{j=1}^{\binom{n}{m}} (b_l^j)^2 \frac{c_k}{c_l}$$

$$= \sum_{i=1}^{\infty} c_i^2 + 2 \sum_{k<l} c_k c_l = \left(\sum_{i=1}^{\infty} c_i\right)^2.$$

Thus the theorem holds for S, a countable union of disjoint open parallelepipeds, and therefore for open sets.

Let Q be a fixed m-dimensional flat in R^n. For each μ_m-measurable set S in Q, define $(v(S))^2$ to be the sum of the squares of the μ_m measures of the $\binom{n}{m}$ orthogonal projections of S on all the m-dimensional coordinate subspaces of R^n with respect to X. If Y is an m-dimensional coordinate subspace and P is an orthogonal projection onto Y, then $P: Q \to Y$ is either a homeomorphism, and hence carries μ_m-measurable sets to μ_m-measurable sets, or $P(Q)$ is of μ_m-measure zero; thus v is well defined. It is easily seen that v is a countably subadditive monotone set function. By the argument in the last paragraph, $v(U) = \mu_m(U)$ for all open sets U. Thus for μ_m-measurable E,

$$v(E) \leqq \inf \mu_m(U) = \mu_m(E),$$

the inf being over all open U containing E. Now given $\varepsilon > 0$, choose U with $\mu_m(U \sim E) < \varepsilon$. See Munroe [**5**, p. 65]. Then by subadditivity,

$$\mu_m(E) \leqq \mu_m(U) = v(U) \leqq v(E) + v(U \sim E) \leqq v(E) + \mu_m(U \sim E) \leqq v(E) + \varepsilon.$$

Hence $v(E) = \mu_m(E)$.

If R^n is replaced by the complex space C^n and μ_m is replaced by μ_m^C (in the formula for V_m, A^tA is replaced by $A^t\bar{A}$), the statement and proof of the theorem hold provided $(\ )^t(\ )$ is replaced by $(\ )^t(^-)$. However, the formula for the measure is anomalous since, for instance, if $n = m = 2$ and $a_1 = (1,0)$, $a_2 = (i,0)$, then $V_2 = 0$. The anomaly results because C^1 is usually treated as R^2 in measure theory.

Work supported by the U.S. Atomic Energy Commission.

The authors are grateful to the referee for improvements in their paper and to Martin Gardner for references.

References

1. Mathematisches Wörterbuch, Akademie-Verlag GMBH, Berlin, 1961, Band I, p. 1007.

2. M. Gardner, Simple proofs of the Pythagorean theorem, and sundry other matters, Mathematical Games in *Scientific American*, 211 (Oct. 1964) 118–126.

3. I. Wirszup, A generalization of the Pythagorean theorem, Math. Student Journal, 2 (April 1955) 2.

4. G. Birkhoff and S. MacLane, A Survey of Modern Algebra, 3rd edition, Macmillan, New York, 1965, p. 289.

5. M. E. Munroe, Introduction to Measure and Integration, 2nd edition, Addison-Wesley, Reading, Mass., 1971.

6. T. Muir, A Treatise on the Theory of Determinants, Revised and enlarged by W. H. Metzler, Dover, New York, 1960, pp. 213–215.

7. N. Bourbaki, Eléments de Mathématique, Algèbre, Chapitre 3, Algèbre Multilinéaire, Actualités Scientifiques et Industrielles, No. 1044, Nouvelle édition, Hermann, Paris, 1958, p. 103.

LOS ALAMOS SCIENTIFIC LABORATORY, UNIVERSITY OF CALIFORNIA, LOS ALAMOS, NEW MEXICO 87544.

RESEARCH PROBLEMS

EDITED BY RICHARD GUY

In this Department the Monthly presents easily stated research problems dealing with notions ordinarily encountered in undergraduate mathematics. Each problem should be accompanied by relevant references (if any are known to the author) and by a brief description of known partial results. Manuscripts should be sent to Richard Guy, Department of Mathematics, Statistics, and Computing Science, The University of Calgary, Calgary 44, Alberta, Canada, T2N 1N4.

SOME PROBLEMS FOR CONVEX SETS

HARALD BERGSTRÖM

For an application in probability theory I had reason to consider the following problem: Let A be a bounded closed convex set in R^k (real k-dimensional Euclidean space) and suppose that A has interior points. Given $\phi \geqq 0$, divide the complement A^c of A into $k+1$ disjoint sets B_r, $0 \leqq r \leqq k$, such that

$$B_r + \beta \boldsymbol{q}_r \subset B_r, \qquad 0 \leqq r \leqq k, \tag{1}$$

for $\beta > 0$ and for any direction $\boldsymbol{q}_r$ which forms at most the angle ϕ with some fixed direction $\boldsymbol{q}_r^{(0)}$ for each B_r. It is clear that this is possible when $\phi = 0$; indeed, the partitioning of A^c into *two* sets B_r is then possible. But when $\phi > 0$ several problems arise, some of which are open and non-trivial even for $k = 2$.

Problem 1. Given A, does there exist an angle ϕ (> 0), depending on A and k, such that a subdivision $A^c = \bigcup_{r=0}^{k} B_r$, satysfying (1), is possible?

Problem 2. Does there exist an angle ϕ (> 0), depending only on k, and solving problem 1 (for all A)?

Problem 3. If the answer to problem 2 is affirmative, what is the largest such ϕ? For $k = 2$, the maximum such ϕ is $\pi/6$.

Problem 4. Given $\phi_0 \geqq 0$, what can be said about the class $\Gamma(\phi_0)$ of convex sets satisfying (1) with $\phi = \phi_0$?

One possible way of dividing A^c into subsets is to inscribe a simplex S in A. The rays from the centroid of S to the inner points of a facet determine a cone and so R^k is divided into $k+1$ cones. The boundaries of the cones may be joined to the various cones so that R^k is partitioned into $k+1$ disjoint cones. The intersections of A^c with these cones form a partition of A^c into disjoint sets. A more appropriate partition of A^c, however, seems to be the following.

Let S be a regular simplex in R^k and $\boldsymbol{p}_j$, $0 \leqq j \leqq k$, the vectors from its centroid to its vertices, i.e.,

$$\sum_{j=0}^{k} \boldsymbol{p}_j = \mathbf{0}. \tag{2}$$

Without loss of generality we may assume $|\boldsymbol{p}_j| = 1$, so that

$$\boldsymbol{p}_j \cdot \boldsymbol{p}_r = -\frac{1}{k} \qquad (j \neq r). \tag{3}$$

All directions in R^k can be divided into $k+1$ disjoint sets

$$C_r = \left\{ \boldsymbol{p} \,\middle|\, \boldsymbol{p} = \sum_{j=0}^{k} \alpha_j \boldsymbol{p}_j, \quad \alpha_j > 0 \ \ (j < r), \ \alpha_j \geqq 0 \ \ (j > r), \ \ \alpha_r = 0 \right\}.$$

Corresponding to the classes C_r the set A^c is divided into $k+1$ disjoint sets B_r in the following way: any point $\boldsymbol{x} \in A^c$ has a shortest distance to A and this distance is equal to the distance between $\boldsymbol{x}$ and a uniquely determined point $\boldsymbol{y}_x \in A$. If the direction $\boldsymbol{x} - \boldsymbol{y}_x$ belongs to C_r we say that $\boldsymbol{x} \in B_r$. Take $\boldsymbol{q}_j^{(0)} = -\boldsymbol{p}_j$ in (1). We shall show that a translation $\boldsymbol{x} \to \boldsymbol{x} + \beta \boldsymbol{q}$, $\beta > 0$, cannot move a point $\boldsymbol{x} \in B_r$ into A if $\boldsymbol{q}$ forms an angle with $-\boldsymbol{p}_r$ at most ϕ, where

$$\sin \phi = 1/k. \tag{4}$$

It is sufficient to show that this translation cannot bring $\boldsymbol{x} \in A \cap \bar{B}_r$ into A^0 where $\bar{B}_r$ is the closure of B_r and A^0 is the interior of A. But if $\boldsymbol{x}$ is a point on the boundary of $A \cap \bar{B}_r$ there is a hyperplane of support at $\boldsymbol{x}$ to A and this hyperplane has a direction $\boldsymbol{p}$ belonging to C_r where $\boldsymbol{p} = \Sigma \alpha_j \boldsymbol{p}_j$, $\alpha_j \geqq 0$, $\alpha_r = 0$. Let $\boldsymbol{q}$ form an angle χ with $\boldsymbol{q}_r^{(0)}$ so that angle $(\boldsymbol{q}, \boldsymbol{q}_r^{(0)}) = \chi \leqq \phi = \arcsin 1/k$. Since angle $(\boldsymbol{p}_j, \boldsymbol{q}_r^{(0)}) = \arccos 1/k = \pi/2 - \phi$ $(j \neq r)$ we have angle $(\boldsymbol{q}, \boldsymbol{p}_j) \leqq \chi + \pi/2 - \phi \leqq \pi/2$, i.e., $\boldsymbol{q} \cdot \boldsymbol{p}_j \geqq 0$. Hence $\boldsymbol{q} \cdot \boldsymbol{p} = \Sigma_j \alpha_j \ (\boldsymbol{q} \cdot \boldsymbol{p}_j) \geqq 0$ and the translation $\boldsymbol{x} \to \boldsymbol{x} + \beta \boldsymbol{q}$ cannot bring $\boldsymbol{x} \in A \cap \bar{B}_r$ into A^0.

There remains the problem: whether a point of B_r (or $\bar{B}_r$) could move into B_s

for some $s \neq r$. Unfortunately this is possible in general; can it be avoided for some orientation of S? It is natural to consider the class $\Gamma' = \Gamma(\arcsin 1/k)$ of all convex sets for which the answer is yes.

Problem 5. How large is Γ'? Does it contain all convex sets? (It does for $k = 2$.) Eggleston conjectures that this is not true for any larger k. In my application I have only to consider arbitrary closed convex polyhedra and I believed that I had proved that Γ' contained these, but my proof was incomplete.

Problem 6. Does Γ' contain all closed convex polyhedra?

The referee observes that the problem is equivalent to the following "illumination problem": Is it possible to subdivide the boundary of A into $k+1$ parts $A_0, \cdots, A_k$ and find directions $\boldsymbol{q}_0^{(0)}, \cdots, \boldsymbol{q}_k^{(0)}$ such that all points of A_j be illuminated by all rays that make an angle at most ϕ with $\boldsymbol{q}_j^{(0)}$? Here a point $\boldsymbol{x}$ is said to be illuminated by a ray of direction $\boldsymbol{q}$ if, for all $\alpha > 0$, we have $\boldsymbol{x} - \alpha\boldsymbol{q} \notin \operatorname{int} A$. In this formulation the problems discussed here become related to the illumination problems by Boltyanskii [**1**] and others, which correspond to $\phi = 0$ and the stricter requirement $\boldsymbol{x} - \alpha\boldsymbol{q} \notin A$ for all $\alpha > 0$.

I thank H. G. Eggleston, J. Schaer and the referee for considerable help with the preparation of this paper.

Reference

1. V. G. Boltyanskii, A problem about the illumination of the boundary of a convex body, (in Russian), Izvest. Moldav. Filial. Akad. Nauk, SSSR, 10 (1960) 79–86.

DEPARTMENT OF MATHEMATICS, CHALMERS UNIVERSITY OF TECHNOLOGY, UNIVERSITY OF GÖTEBORG, S-402 20 GÖTEBORG 5. SWEDEN.

A PROBLEM CONCERNING WEAKLY COMPLETELY CONTINUOUS A^*-ALGEBRAS

B. D. MALVIYA

All algebras under consideration are over the field of complex numbers. We follow the notations and terminology of Rickart's book [**6**]. Recall that a bounded linear operator T from a Banach space X into a Banach space Y is called weakly completely continuous (weakly compact) if it maps bounded sets of X into weakly sequentially compact sets of Y. A Banach algebra A is said to be weakly completely continuous if its left and right regular representations consist of weakly completely continuous operators; see [**6**, p. 284]. It is well known that the set of all weakly completely continuous elements of $B(X)$, the algebra of bounded operators on a Banach space X, form a closed two-sided ideal in the uniform operator topology of $B(X)$. (See Dunford and Schwartz [**2**], p. 484, Corollary 6.)

Following Yood [8] (see also Barnes [1]) we say that an algebra A is a modular annihilator algebra if

(1) $R(M) = \{x \in A : Mx = (0)\} \neq 0$ and $R(A) = 0$,

(2) $L(N) = \{x \in A : xN = (0)\} \neq 0$ and $L(A) = 0$,

for every maximal modular left ideal M and every maximal modular right ideal N of A.

The properties of weakly completely continuous B^*- and A^*-algebras have been studied extensively by Yoshinga and Ogasawara [4]. Before posing the actual problem we shall prove the following:

PROPOSITION: *A modular annihilator B^*-algebra is weakly completely continuous.*

Proof. In view of [8, p. 42, Theorem 4.1] A is a dual B^*-algebra. A result of Kaplansky [3, p. 221, Lemma 2.3] gives us that the dual B^*-algebra A is the $B^*(\infty)$-sum (see [3], p. 221 for the definition of $B^*(\infty)$-sum) of C^*-algebras, each of which consists of the algebra of completely continuous operators on a Hilbert space. The conclusion of our proposition then follows from [4, p. 21, Theorem 6].

Recently Tomiuk and Wong [7, Theorem 6.1, p. 56] proved that an annihilator A^*-algebra is weakly completely continuous. Incidently, this theorem was also proved by Olubummo [5, Theorem 2.1, p. 906] using very simple techniques of Banach algebras.

We now state the

PROBLEM: *Is a modular annihilator A^*-algebra weakly completely continuous?*

References

1. B. A. Barnes, Modular annihilator algebras, Canad. J. Math., 18 (1966) 566–578.

2. N. Dunford and J. T. Schwartz, Linear operators, Part I, Interscience Publishers, New York, 1958.

3. I. Kaplansky, The structure theorem of certain operator algebras, Trans. Amer. Math. Soc. 70 (1951) 219–255.

4. T. Ogasawara and K. Yoshinga, Weakly completely continuous Banach *-algebras, J. Sci.. Hiroshima Univ. Ser. A., 18 (1954), 15–36.

5. A. Olubummo, Weakly compact $B^{\#}$-algebras, Proc. Amer. Math. Soc., 14 (1963) 905–908.

6. C. E. Rickart, General theory of Banach algebras, Princeton, N. J., 1960.

7. B. J. Tomiuk and P. K.Wong, Annihilator and complemented Banach *-algebras, J. Austral. Math. Soc., 13 (1972) 47–66.

8. B. Yood, Ideals in topological rings, Canad. J. Math., 16 (1964) 28–45.

DEPARTMENT OF MATHEMATICS, NORTH TEXAS STATE UNIVERSITY, DENTON, TEXAS 76203.

CLASSROOM NOTES

EDITED BY DAVID ROSELLE

Material for this Department should be sent to David Roselle, Department of Mathematics, Louisiana State University, Baton Rouge, LA 70803.

SOME COROLLARIES TO AN INTEGRAL INEQUALITY

C. H. KIMBERLING

Exercise 9–14 in Apostol's *Mathematical Analysis* includes the following theorem. Suppose functions f, g and fg are Riemann-Stieltjes integrable with respect to a nondecreasing bounded integrator α, over a closed (possibly unbounded) interval $[a,b]$. Then

$$\left\{\int_a^b f(x)d\alpha(x)\right\}\left\{\int_a^b g(x)d\alpha(x)\right\} \leqq [\alpha(b)-\alpha(a)]\int_a^b f(x)g(x)\,d\alpha(x)$$

if both f and g are nondecreasing (or both nonincreasing) on $[a,b]$. The reverse inequality holds if f is nondecreasing and g nonincreasing on $[a,b]$.

Recently there appeared in *Mathematics Magazine* an application of this theorem to probability theory. It was shown that "covariance can be interpreted as a measure of the joint variation of two random variables in the same or opposite directions."

The theorem has another application with connections in probability theory. First, a function f is defined to be **completely monotone** from $[0,\infty)$ into $(0,1]$ if it is continuous on $[0,\infty)$ and has alternating derivatives there: $(-1)^n f^{(n)} \geqq 0$ for $n = 0,1,\cdots$. For such f, we have $f(x+y) \geqq f(x)f(y)$ for all nonnegative x and y. A proof is given in [3].

The theorem easily provides a sum inequality: Suppose $n \geqq 1$. If $0 \leqq a_1 \leqq \cdots \leqq a_n$ and $0 \leqq b_1 \leqq \cdots \leqq b_n$ and $\Sigma c_i \leqq 1$, where $c_i > 0$ for $1 \leqq i \leqq n$, then

$$(\Sigma a_i c_i)\,(\Sigma b_i c_i) \leqq \Sigma a_i b_i c_i \quad \text{(all sums from 1 to } n).$$

If the ordering of the b_i's is reversed, then so is the inequality.

An easy consequence of this sum inequality follows. If $p(x)$ is a polynomial having only nonnegative coefficients, then for nonnegative x and y, $p(x)p(y) \leqq p(1)p(xy)$ if x and y are both less than 1 or both greater than 1, and the inequality is reversed otherwise.

This argument extends to certain power series. For example, from

$$\tan x = \sum_{k=1}^{\infty} \frac{2^{2k}(2^{2k}-1)}{(2k)!}\left|B_{2k}\right| x^{2k-1} \qquad (0 \leqq x < \pi/2),$$

where the B_{2k}'s denote Bernoulli numbers, we obtain $\tan x \tan y \leqq \tan 1 \tan xy$

whenever $0 \leqq x \leqq 1$ and $0 \leqq y \leqq 1$, or $1 \leqq x < \pi/2$ and $1 \leqq y < \pi/2$, and the inequality is reversed in the remaining cases. Similar inequalities hold for $\sinh x$ and $\cosh x$. Another example is this: $2\sin^{-1}x \sin^{-1}y \leqq \sin^{-1}xy$ whenever $0 \leqq x \leqq 1$ and $0 \leqq y \leqq 1$.

References

1. Tom M. Apostol, Mathematical Analysis, Addison-Wesley, Reading, Mass., 1958.
2. Javad Behboodian, Covariance of monotone functions, Math. Mag., 45 (1972) 158.
3. C. H. Kimberling, A probabilistic interpretation of complete monotonicity, Aequationes Mathematicae (to appear).

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF EVANSVILLE, EVANSVILLE, INDIANA 47701.

INFINITE RINGS WITH ALL PROPER SUBRINGS FINITE

THOMAS J. LAFFEY

Let p be a prime number. We denote by $C(p^n)$ the cyclic group of order p^n and by $C(p^\infty)$ the union $\bigcup_{n=1}^{\infty} C(p^n)$. Then $C(p^\infty)$, which is called the p-**quasicyclic group,** is an example of an infinite abelian group all of whose proper subgroups are finite. It is easy to show that if G is an infinite abelian group all of whose proper subgroups are finite, then G is isomorphic to $C(p^\infty)$ for some prime p. The question of the existence of a nonabelian infinite group all of whose proper subgroups are finite is called Schmidt's Problem and appears to be still open. We make $C(p^\infty)$ into a ring, which we again denote by $C(p^\infty)$, by writing it additively and defining all products to be zero. The ring $C(p^\infty)$ is an example of an infinite ring all of whose proper subrings are finite.

Again, given primes p, q, we consider the union

$$G_{p,q} = \bigcup_{n=0}^{\infty} \mathrm{GF}(p^{q^n})$$

of the finite fields $\mathrm{GF}(p^{q^n})$. Now $G_{p,q}$ is an infinite field all of whose proper subfields are of the form $\mathrm{GF}(p^{q^n})$ for some $n \geqq 0$. Also all nonzero subrings of $G_{p,q}$ are subfields. Thus $G_{p,q}$ is another example of an infinite ring all of whose proper subrings are finite.

In this paper we prove the following results.

THEOREM 1. *Let R be an infinite ring all of whose proper subrings are finite. Then either*

(i) $R^2 = \{0\}$ *and* $R = C(p^\infty)$ *for some prime* p,

or

(ii) $R = G_{p,q}$ *for some primes* p, q.

THEOREM 2. *If R is a ring with* A.C.C. *and* D.C.C. *on subrings, then R is finite.*

Theorem 2 has been proved in a different way by T. Szele [4] and by several other authors.

Proofs of the results. We first show that Theorem 2 is a consequence of Theorem 1.

Assume that Theorem 2 is false and let R be a counterexample. Let K be the set of infinite subrings of R. Then K is nonempty, since R itself is an element of K. By Zorn's Lemma and the fact that R has D.C.C., K has a minimal element, S say, under inclusion. Now S is an infinite ring all of whose proper subrings are finite. By Theorem 1, S is either $C(p^\infty)$ or $G_{p,q}$. Since $C(p^\infty)$ and $G_{p,q}$ do not have the A.C.C. on subrings, we have a contradiction.

We now prove Theorem 1.

Let R be an infinite ring all of whose proper subrings are finite. If $R^2 = \{0\}$, then $(R, +)$ is an infinite abelian group all of whose proper subgroups, being the proper subrings of R, are finite. Thus $(R, +) = C(p^\infty)$ for some prime p and the theorem follows.

Assume now that $R^2 \neq \{0\}$. Let $x \in R$ with $xR \neq \{0\}$. If xR is finite, then

$$\{r \in R \mid xr = 0\}$$

is infinite and therefore equals R and thus $xR = \{0\}$, contrary to our choice of x. So xR is infinite and thus $xR = R$. Hence $Rx \neq \{0\}$ and thus $Rx = R$. Hence R has an identity. Now if $0 \neq y \in R$, then $yR \neq \{0\}$, so $yR = R = Ry$ and y has an inverse. Thus R is a division ring.

We next show that R is commutative. This is obvious if R is generated by one element. Assume that each element of R generates a finite subring of R. If $x \in R$, then there exists $n(x) > 1$ such that $x^{n(x)} = x$. The commutativity of R now follows immediately from a result of Jacobson (Herstein [3], Theorem (7.D), page 324). Alternatively we may deduce the commutativity of R from Wedderburn's Theorem (that all finite division rings are commutative) and some linear algebra as follows: Assume that R is not commutative and let $x, y \in R$ with $xy \neq yx$. Let F be the subring of R generated by x. Then F is a finite field and we may look on R as a (left) vector space over F. Let V be the F-subspace of R spanned by

$$\{y, yx, yx^2, \cdots, yx^m, \cdots\}.$$

Since $x^{|F|} = x$, V is finite dimensional over F. Next the map $t: V \to V$ defined by $t(v) = vx (v \in V)$ is a linear transformation of V and $t^{|F|} = t$. Thus all the eigenvalues of t lie in F, and the minimal polynomial of t has no repeated roots. If x is the only eigenvalue of t, then $t(v) = xv$ for all $v \in V$ and this contradicts the fact that $t(y) \neq xy$. Hence t has an eigenvalue $a \neq x$ and there exists a nonzero element $v \in V$ with $t(v) = av$. Now $vxv^{-1} = a \in F$ and hence the subring generated by $\{v, x\}$ is a noncommutative finite division ring, thus contradicting Wedderburn's Theorem.

We now know that R is a field. Since the field of rational numbers has proper

infinite subrings, R has characteristic p for some prime p. Thus R is an infinite dimensional extension of GF(p). Also each element $x \in R$ is algebraic over GF(p), since otherwise the subring generated by x would be a proper infinite subring of R.

Let $T = \{n \mid n$ is a positive integer for which there exists an element

$$x \in R \text{ such that } x^{p^n} = x \text{ but } x^{p^m} \neq x \quad \text{for } 1 \leqq m < n\}.$$

Since R is an infinite dimensional algebraic extension of GF(p), T is infinite. Let q be a prime in T and let

$$U = \{n \in T \mid q \text{ does not divide } n\}.$$

If U is infinite, then, since U contains the least common multiple of each pair of elements it contains, U contains an infinite sequence $\{d_n\}$ such that for each n, d_n is a proper divisor of d_{n+1}. Hence R contains

$$R_0 = \bigcup_{n=1}^{\infty} \mathrm{GF}(p^{d_n}).$$

Since R_0 is infinite, $R_0 = R$. Let $x \in R$ be such that $x^p \neq x$ but $x^{p^q} = x$. Now $x \in \mathrm{GF}(p^{d_n})$ for some n and thus q divides d_n, contrary to the definition of U. Hence U is finite. But now, since T is infinite, T contains q^n for all $n \geqq 1$. Thus R contains $G_{p,q}$ and the result follows.

Concluding Remarks. Theorem 1 above can be considered as a special case of the general problem of determining all rings, all of whose proper subrings have a particular property P. References [**1**] and [**2**] contain results on this problem and again the ring $C(p^\infty)$ plays a role in the situations considered therein.

References

1. R. Gilmer, R. Lea, M. O'Malley, Rings whose proper subrings have property P, Acta. Sci. Math. Szeged, 33 (1972) 69–75.

2. R. Gilmer, M. O'Malley, Non-noetherian rings for which each proper subring is noetherian, Math. Scand., (to appear).

3. I. N. Herstein, Topics in Algebra, Blaisdell, New York, 1964.

4. T. Szele, On a finiteness criterion for modules, Publ. Math. Debrecen, 3 (1954) 253–256.

DEPARTMENT OF MATHEMATICS, NORTHERN ILLINOIS UNIVERSITY, DE KALB, ILLINOIS 60115.

mathematics education, but I think we have made a significant start. Perhaps others may want to help.

The research described inthis article was supported by National Science Foundation grant GW-6796.

References

1. G. Pólya, Mathematical discovery, Vol. I, Wiley, New York, 1962.

2. J. M. Scandura, Deterministic theorizing in structural learning: Three levels of empiricism, Journal of Structural Learning, (1971) 3, 21–53. (a)

3. ———, Mathematics: Concrete behavioral foundations, Harper and Row, New York, 1971. (b)

4. ———, An algorithmic base for curriculum construction in mathematics (A theory of mathematical knowledge). In W. E. Lamon, (Ed.) Learning and the nature of mathematics, Science Research Associates, Inc., Chicago, 1972.

5. ———, Structural learning I: The theory and empirical research, Gordon and Breach, New York, 1973.

6. J. M. Scandura, J. H. Durnin, W. Ehrenpreis, and G. Luger, An algorithmic approach to Mathematics: Concrete behavioral foundations, Harper and Row, New York, 1971.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF PENNSYLVANIA, PHILADELPHIA, PENNSYLVANIA 19104.

PROBLEMS AND SOLUTIONS

EDITED BY EMORY P. STARKE

All problems (both elementary and advanced) proposed for inclusion in this Department should be sent to E. P. Starke, 1000 Kensington Ave., Plainfield, NJ 07060. Proposers of problems are urged to enclose any solutions or information that will assist the editors. Ordinarily, problems in well-known textbooks and results in generally accessible sources are not appropriate for this Department. No solutions (except those accompanying proposals) should be sent to Professor Starke.

ELEMENTARY PROBLEMS

Solutions of Elementary Problems should be sent to Problems Group, Mathematics Department,

University of Maine, Orono, ME 04473. To facilitate their consideration, solutions of Elementary Problems in this issue should be typed (with double spacing) and should be mailed before June 30, 1974.

An asterisk (*) means neither the proposer nor the editors supplied a solution.

E 2434 [1973, 943]. **Correction.** Instead of $\lim(pa_n + qb_n)$ read $\lim(pa_n + qb_n)^n$.

E 2462. *Proposed by Huseyin Demir, Middle East Technical University, Ankara, Turkey*

Let P be a point interior to the triangle $A_1A_2A_3$. Denote by R_i the distance from P to the vertex A_i, and denote by r_i the distance from P to the side a_i opposite to A_i. The Erdös-Mordell inequality asserts that

$$R_1 + R_2 + R_3 \geqq 2(r_1 + r_2 + r_3).$$

Prove that the above inequality holds for every point P in the plane of $A_1A_2A_3$ when we make the interpretation $R_i \geqq 0$ always and r_i is positive or negative depending on whether P and A_i are on the same side of a_i or on opposite sides.

E 2463*. *Proposed by R. N. Gupta, Punjab University, India*

Let p be a prime greater than 3. Since the set $S = \{1, 2, \cdots, p-1\}$ forms a group under multiplication mod p, for every $k \in S$ there exists (a unique) $x_k \in S$ such that $kx_k \equiv 1 \pmod{p}$. Thus integers $n_1, n_2, \cdots, n_{p-1}$ are specified such that

$$kx_k = 1 + n_k p$$

for $k = 1, 2, \cdots, p-1$. Show that

$$\sum_{k=1}^{p-1} kn_k \equiv \tfrac{1}{2}(p-1) \pmod{p}.$$

E 2464. *Proposed by Erwin Just, Bronx Community College*

Solve the following Diophantine equations:

(1) $x^2(x^2 + y) = y^{m+1}$,

(2) $x^2(x^2 + y^2) = y^{m+1}$.

E 2465. *Proposed by Claude Anderson, University of California at Berkeley*

If A is a subset of N, the natural numbers, then the *density* of A, denoted $d(A)$, is defined by the following limit (when it exists):

$$d(A) = \lim_{n\to\infty} \frac{A_n}{n},$$

where A_n is the number of elements of A which do not exceed n. Let $\varepsilon > 0$ be arbitrary. Show that there exist A, B, with $d(A) < \varepsilon$, $d(B) < \varepsilon$, yet $d(A + B) = 1$, where $A + B = \{a + b : a \in A,\ b \in B\}$. Is it possible to have $d(A + B) = 1$ and $d(A) = d(B) = 0$?

E 2466. *Proposed by Claude Anderson, University of California at Berkeley*

With the notation of the previous problem, show that it is possible to have

$$d(A) = d(B) = 0,$$

yet $d(AB) = 1$, where $AB = \{ab : a \in A,\ b \in B\}$.

E 2467. *Proposed by Bruce Reznick and Michael Yoder, Jet Propulsion Laboratory*

Suppose that $a \geqq 3$ and let P_n be an nth degree polynomial with real coefficients. Prove that

$$\max \left| a^i - P_n(i) \right| \geqq 1,$$

the maximum being taken over all integers i satisfying $0 \leqq i \leqq n + 1$.

SOLUTIONS OF ELEMENTARY PROBLEMS

Sum of Two Squares in Galois Fields

E 2377 [1972, 906]. *Proposed by Lawrence Somer, University of Illinois*

Find the number of essentially different ways that an element of the finite field $GF(p^n)$ can be represented as the sum of two squares.

Solution by J. G. Mauldon, Amherst College. Let $f_0(p^n)$ [resp. $f_1(p^n); f^*(p^n)$] denote the number of essentially different representations of zero [resp. a nonzero square; a nonsquare] as a sum of two squares in $GF(p^n)$. Since the multiplicative group of $GF(p^n)$ is cyclic, these functions are well defined and (for odd p) the number of nonzero squares is equal to the number of nonsquares.

If $p = 2$ it is easy to see that every element of $GF(2^n)$ is a square, that $f_0(2^n) = 2^n$ and that $f_1(2^n) = 2^{n-1}$, since $x + y = 0$ if and only if $x = y$.

If $p \neq 2$ we prove that $f_1(p^n) = [(p^n + 10)/8]$, $f^*(p^n) = [(p^n + 6)/8]$ and $f_0(p^n) = 1$ or $(p^n + 3)/4$ according as $p^n \equiv 3$ or $1 \pmod 4$.

Write $q = p^n$ and note the identity

$$f_0 + \tfrac{1}{2}(q-1)(f_1 + f^*) = (q + 1)(q + 3)/8,$$

obtained by counting in two different ways the total number of essentially different sums of two squares.

If -1 is a square, there are $\frac{1}{2}(q-1)/2$ essentially different representations of zero as the sum of two nonzero squares, so that $f_0 = 1 + (q-1)/4$ and $f_1 + f^* = (q+3)/4$. If -1 is not a square, then $f_0 = 1$ and $f_1 + f^* = (q+5)/4$. Hence, in either case $f_1 + f^* = [(q+6)/4]$ and since, as we shall see,

$$f_1 - f^* = 0 \text{ or } 1 \tag{1}$$

according as $f_1 + f^*$ is even or odd, these equations suffice to determine f_1, f^* and f_0.

Proof of (1). If every sum of two squares were a square, the set of squares in GF(q) would be a subfield with $\frac{1}{2}(q+1)$ elements, which is impossible. Consequently $f^* \geqq 1$ and every element of GF(q) admits at least one representation as a sum of two squares. Let G be the multiplicative group of GF(q) and let M denote the multiplicative group of nonsingular matrices of the form

$$\begin{bmatrix} a & b \\ -b & a \end{bmatrix}$$

with $a, b \in \mathrm{GF}(q)$. Then the homomorphism of M onto G defined by $m \to \det(m)$ shows that the number N of matrices in M with specified nonzero determinant d is independent of d. Each such matrix yields a representation of d as a sum of two squares and, conversely, each such representation yields eight such matrices if $ab(a^2 - b^2) \neq 0$ and four if $ab(a^2 - b^2) = 0$. The possibility $ab = 0$ contributes one to f_1 and the possibility $a^2 = b^2$ contributes one either to f_1 or to f^* according as 2 is or is not a square. In the latter case $f_1 = f^*$ and in the former case $4 + 4 + 8(f_1 - 2) = N = 8f^*$, so that $f_1 = f^* + 1$. This completes the proof of (1) and hence the stated main result.

Also solved by Michael Doob, M. G. Greening (Australia), Phil Tracy, and the proposer.

Characterizing the Centroid of a Simplex

E 2394 [1973, 75]. *Proposed by S. L. Greitzer, Rutgers University, and M. S. Klamkin, Ford Motor Company*

A line is drawn through the centroid G of a simplex $A_0, A_1, \cdots, A_n$ intersecting the faces (extended if necessary) in the points $B_0, B_1, \cdots, B_n$ respectively. Show that

$$\sum_{i=0}^{n} \frac{1}{GB_i} = 0,$$

where GB_i denotes the directed distance from G to B_i. Show also that the above property characterizes the point G as the centroid; i.e., if the above sum vanishes for all arbitrary lines, then G is the centroid.

This generalizes known results for triangles and tetrahedrons.

Solution by Mildred L. Stancl, Nichols College, Dudley, Massachusetts. Let G be an arbitrary point which does not lie on a face of a simplex $A_0A_1 \cdots A_n$. (The word *face* throughout means *extended face*, i.e., the $(n-1)$-dimensional affine subspace spanned by n of the points $A_0, A_1, \cdots, A_n$.) Let L be an arbitrary line through G and let $B_0, B_1, \cdots, B_n$ be defined as follows: If L intersects the face opposite A_i, let B_i be the point of intersection and call B_i *finite.* If L does not intersect the face opposite A_i, let B_i be a fictitious point and call B_i *infinite.* Define $1/GB_i$ to be zero if B_i is infinite.

Since the fact that $\sum 1/GB_i = 0$ is immediate if all B_i are infinite, assume that L intersects at least one face in the point B, where B is one of $B_0, B_1, \cdots, B_n$. (Unless otherwise noted, all summations run from 0 to n.) Then B is finite and for each B_j which is finite the following statements hold:

(i) The jth barycentric coordinate of B_j is zero.

(ii) $B_j = (1-t_j)G + t_jB$ where t_j is a unique, nonzero, real number.

(iii) $GB_j = t_jGB$ where GB is nonzero.

Now let G be the barycenter (centroid) of the simplex, so that $G = (n+1)^{-1} \sum A_k$ and let $b_0, b_1, \cdots, b_n$ be the barycentric coordinates of B so that $B = \sum b_kA_k$. If B_r is an infinite point, then the rth barycentric coordinate of B is $(n+1)^{-1}$. This remark is verified by noting the existence of points P and Q lying in the face opposite A_r such that B is a point of the line segment with endpoints A_r and P, and G is a point of the line segment with endpoints A_r and Q. The fact that L does not intersect the face opposite A_r means that the line segment with endpoints P and Q lying in that face is parallel to the line segment with endpoints B and G lying in L. Thus if $s = (n+1)^{-1}$, then

$$B = (1-s)P + sA_r, \qquad G = (1-s)Q + sA_r.$$

The equality $b_r = (n+1)^{-1}$ follows since P and Q have rth barycentric coordinate zero. Since $\sum b_k = 1$, the following statement holds:

(iv) If G is the barycenter, if exactly N $(1 \leqq N \leqq n+1)$ of the points $B_0, B_1, \cdots, B_n$ are finite, and if B with barycentric coordinates $b_0, b_1, \cdots, b_n$ is one of the finite points, then $\sum_j' b_j = N/(n+1)$.

(The notation $\sum_j'$ throughout this solution means the summation over those j's for which B_j is finite.)

If B_j is a finite point, statement (ii) implies that

$$B_j = \sum \left[\frac{1}{n+1}(1-t_j) + t_jb_k\right] A_k$$

and (i) implies that

$$t_j = \frac{1}{1-(n+1)b_j}.$$

Since $\Sigma\ 1/GB_i = \Sigma_j' 1/GB_j$, statement (iii) implies that

$$\Sigma \frac{1}{GB_i} = \frac{1}{GB} \ \Sigma_j'[1-(n+1)b_j].$$

It now follows from (iv) that $\Sigma\, 1/GB_i = 0$.

Conversely, if G is any point such that $\Sigma\ 1/GB_i = 0$ for all lines through G, then G does not lie on a face of $A_0A_1\cdots A_n$, for otherwise the denominator of at least one of the summands would vanish. Let A_k be any one of $A_0, A_1, \cdots, A_n$ and consider the line through G and A_k. The point A_k lies in all but one of the faces; hence, n of $B_0, B_1, \cdots, B_n$ are equal to A_k and are finite. The remaining one, B_k, is also finite, for otherwise the sum of the reciprocals of the directed distances would be n/GA_k which is nonzero. Let $g_0, g_1, \cdots, g_n$ be the barycentric coordinates of G and let $a_0, a_1, \cdots, a_n$ be the barycentric coordinates of A_k. Then statements (ii) and (iii) imply

$$t_i = \frac{g_i}{g_i - a_i}.$$

Statement (iii) implies that

$$0 = \Sigma \frac{1}{GB_i} = \frac{1}{GA_k}\left(n + \frac{1}{t_k}\right) = \frac{1}{GA_k}\left(n + 1 - \frac{a_k}{g_k}\right).$$

Since $a_k = 1$, it follows that $g_k = (n+1)^{-1}$ and G is the barycenter.

Also solved by M. G. Greening (Australia), G. Tsintsifas (Greece), and the proposers.

Anti-Closed Subsets of Integers

E 2402 [1973, 203]. *Proposed by David Newman, Hartford, Connecticut*

Let $n > 7$ be an integer and let $N = \{2,3,4,\cdots,2n\}$. Show that N has precisely $n+5$ n-element subsets S with the property that if i and j are distinct elements of S, then $i + j \notin S$.

I. *Solution by D. R. Stone, Georgia Southern College.* Let us call a subset satisfying the given conditions *anti-closed*; we prove that N_n has $n+5$ n-element anti-closed subsets, one $(n+1)$-element anti-closed subset and no k-element anti-closed subsets if $k > n+1$.

In fact, the $n+5$ n-element anti-closed subsets are

$$S_0 = \{n+1, n+2, \cdots, 2n\},$$

$$S_k = (\{n\} \cup S_0) - \{n+k\} \text{ for } k = 1,2,\cdots,n,$$

$$S_{n+1} = (\{n-1\} \cup S_0) - \{2n\},$$

$$S_{n+2} = (\{n-1\} \cup S_0) - \{n+1\},$$
$$S_{n+3} = (\{n-1, n\} \cup S_0) - \{2n-1, 2n\},$$
$$S_{n+4} = (\{n-1, n\} \cup S_0) - \{n+1, 2n-1\},$$

and the $(n+1)$-element anti-closed subset is $U = \{n, n+1, \cdots, 2n\}$.

The proof is by induction on n. The case $n = 8$ can be checked out directly. Now assuming that the S_i and U are the desired subsets of N_n, let T_i $(i = 0, 1, 2, \cdots, n+5)$ and V be the analogous subsets of N_{n+1}. If T is an $(n+1)$-element anti-closed subset of N_{n+1}, then considering the four possible cases with $2n+1$ or $2n+2$ in T or not in T and using the obvious induction steps, one sees that T is one of the T_i. For example, if $2n+1 \in T$ and $2n+2 \notin T$, then $T - \{2n+1\}$ is an n-element anti-closed subset of N_n and thus equals some S_i. Considering the possibilities one has T equals T_{n+1} or T_{n+2}. The case with $2n+1$ and $2n+2$ both in T splits into subcases dependent on the inclusion or exclusion of $n+1$.

Similar arguments show that V is the only $(n+2)$-element anti-closed subset of N_{n+1} and there are no such subsets having more than $n+2$ elements.

(The original problem is true for $n = 3$, but the induction step only works for $n \geqq 4$. N_4, N_5, N_6 and N_7 have too many anti-closed subsets.)

II. *Comment by Anders Bager, Hjørring, Denmark.* For $n = 4, 5, 6, 7$, there are n-element anti-closed subsets not of the type described above. By a systematic (though computer-less) search I have found the following:

2 3 4 8	2 3 4 8 9	2 3 6 7 11 12	3 4 5 6 12 13 14
2 3 6 7	2 3 4 9 10	3 4 5 10 11 12	
2 3 7 8	2 3 6 7 10		
2 4 5 8	2 5 6 9 10		
2 4 7 8	3 4 8 9 10.		

Also solved by M. T. Bird, D. P. Sumner, Phil Tracy, and the proposer. Six partial solutions were received.

Editor's comment. Several solvers used a backwards induction, but inducted all the way back to $n = 3$, not noticing that the induction breaks down in attempting to go from $n = 8$ to $n = 7$. The case of $n = 8$ was dismissed by most solvers with an "It can be shown ..." remark. Only Tracy supplied all the details.

The Smith College Diploma Problem

E 2404 [1973, 316]. *Proposed by Russell Maurer, Harvard Medical School*

At Smith College, the graduation exercises traditionally proceed as follows: Although each diploma is made out to a particular girl, all the diplomas are initially given out at random. All of the girls who do not get their own diplomas then form a

circle, and each passes the diploma she has to the girl on her right. Those who now have their own diplomas drop out, and the remaining girls again pass their diplomas to the right, and so on. This procedure is repeated until each girl has her own diploma. If there are n girls in the graduating class what is the probability that it takes precisely k passes before each girl has her own diploma?

Solution by Don West, State University College of New York at Plattsburgh. Number the girls $1, 2, 3, \cdots, n$ from left to right and number the diplomas to correspond to their owners. Then an arrangement of diplomas corresponds to a permutation σ of $\{1, 2, \cdots, n\}$, where $\sigma(i)$ is the number of the girl holding diploma i. In arrangement σ, the distance from diploma i to its owner is $d(\sigma, i) = i - \sigma(i)$ (reduced modulo n).

The first result that we show is that the number of shifts required to sort an arrangement σ is the average (over i) of the distances $d(\sigma, i)$. To prove this, notice that the number of shifts required to sort an arrangement σ is unchanged if we modify the procedure so that instead of dropping out, a girl who finds her own diploma holds it, and in subsequent shifts she transfers whatever diploma she receives to the girl on her right. By this scheme the number of transfers in any shift is n, and the distance $d(\sigma, i)$ is the number of transfers required to return diploma i to its owner. Then for arrangement σ requiring k shifts, we see that $nk = \sum_{i=1}^{n} d(\sigma, i)$ so that k is the average of the distances.

Let $A(n, k)$ denote the number of arrangements requiring k shifts and among these, let $D(n, k)$ be the number of derangements (arrangements with no fixed points). Clearly, $A(0,0) = D(0,0) = 1$ and $A(n, k) = D(n, k) = 0$ if $k \geqq n > 0$ or $k < 0$.

The next thing that we show is that

$$D(n, k) = D(n, n - k). \tag{1}$$

The proof is based on the fact that if a diploma is not yet at its destination, then its right distance plus its left distance is n. Let ρ be the reversing permutation $\rho(i) = n + 1 - i$; then the function $\sigma \to \rho\sigma\rho$ is one-to-one, and if $\sigma(i) \neq i$, then $d(\rho\sigma\rho, i) = n - d(\sigma, \rho(i))$. If σ is a derangement, then so is $\rho\sigma\rho$ (and conversely) and we take the average of the distances for $\rho\sigma\rho$ to show that if σ requires k shifts, then $\rho\sigma\rho$ requires $n - k$ shifts.

Another result before our main theorem is that

$$A(n, k) = A(n, n - 1 - k) \text{ for } n > 0. \tag{2}$$

The idea here is that if the right distance of a diploma is k and we move the diploma one space to the left, its new left distance is $n - 1 - k$. Let $\lambda(i) = i - 1$ (reduced mod n); then the function $\sigma \to \rho\lambda\sigma\rho$ is one-to-one and $d(\rho\lambda\sigma\rho, i) = n - 1 - d(\sigma, \rho(i))$ so if σ requires k shifts, we average and show that $\rho\lambda\sigma\rho$ requires $n - 1 - k$ shifts.

Clearly

$$A(n, k) = \sum_{i=0}^{n} \binom{n}{i} D(i, k),$$

so we know [G. Berman and K. D. Fryer, *Introduction to Combinatorics*, Academic Press, New York, 1972, p. 116] that

$$D(n,k) = \sum_{p=0}^{n} (-1)^{n-p}\binom{n}{p} A(p,k).$$

The fact that the symmetries in (1) and (2) are different allows us, given $A(n,k)$ for all n, to compute $A(n,n-1-k)$, then $D(n,n-1-k)$, next $D(n,k+1)$ and finally $A(n,k+1)$. These substitutions yield

$$A(n,\, k+1) = \sum_{i=0}^{n} \binom{n}{i} \sum_{p=0}^{i} (-1)^{i-p}\binom{i}{p} A(p, i-k-1);$$

since n cannot be 0 for (2) to apply, we must pull off the terms with $p = 0$:

$$A(n,k+1) = \sum_{i=0}^{n} (-1)^{i}\binom{n}{i} A(0,i-k-1) + \sum_{i=1}^{n} \binom{n}{i} \sum_{p=1}^{i} (-1)^{i-p}\binom{i}{p} A(p,i-k-1).$$

Now using (2) and the fact that $A(0,t) = 1$ if $t = 0$ and 0 otherwise, we have the recursion formula

$$(3)\quad A(n,k+1) = (-1)^{k+1}\binom{n}{k+1} + \sum_{i=1}^{n} \binom{n}{i} \sum_{p=1}^{i} (-1)^{i-p}\binom{i}{p} A(p,k+p-i).$$

Now we can use induction on k to prove the main result.

THEOREM.

$$(4)\qquad A(n,k) = \sum_{r=0}^{k} (-1)^{r}\binom{n+1}{r}(k+1-r)^{n}.$$

Proof. We know that $A(n,0) = 1$ for all n and we see our formula agrees. Now suppose the equation holds for all n and second index at most k. Reverse the order of summation in (3), note that

$$\binom{n}{i}\binom{i}{p} = \binom{n}{p}\binom{n-p}{i-p}$$

and replace i by letting $s = p - i + k$, so

$$A(n,\, k+1) = (-1)^{k+1}\binom{n}{k+1} + \sum_{p=1}^{n} \binom{n}{p} \sum_{s=p-n+k}^{k} (-1)^{k-s}\binom{n-p}{k-s} A(p,s).$$

If $p - n + k \leqq s < 0$, then $A(p,s) = 0$ so we may drop these terms from the sum. Further we may introduce new terms for $0 \leqq s < p - n + k$ since $k - s > n - p$, so $\binom{n-p}{k-s} = 0$. Hence we may perform the sum above with s running from 0 to k. We see that $s \leqq k$, so we can apply the induction hypothesis and substitute the expression for $A(p,s)$ from (4), using summation index q. If we reverse the order of the two

innermost sums, we get

$$A(n,k+1) = (-1)^{k+1}\binom{n}{k+1} + \sum_{p=1}^{n}\binom{n}{p}\sum_{q=0}^{k}\sum_{s=q}^{k}(-1)^{k-s+q}\binom{n-p}{k-s}\binom{p+1}{q}(s+1-q)^p.$$

We replace s by letting $r = k - s + q$ and reverse again:

$$A(n,k+1) = (-1)^{k+1}\binom{n}{k+1} + \sum_{p=1}^{n}\binom{n}{p}\sum_{r=0}^{k}(-1)^r(k+1-r)^p\sum_{q=0}^{r}\binom{n-p}{r-q}\binom{p+1}{q}.$$

The innermost sum is a Vandermonde convolution and has the value $\binom{n+1}{r}$. We substitute and again reverse the order of the sums to get

$$A(n,k+1) = (-1)^{k+1}\binom{n}{k+1} + \sum_{r=0}^{k}(-1)^r\binom{n+1}{r}\sum_{p=1}^{n}\binom{n}{p}(k+1-r)^p.$$

The binomial theorem applies to the inner sum, so we have

$$A(n,k+1) = (-1)^{k+1}\binom{n}{k+1} + \sum_{r=0}^{k}(-1)^r\binom{n+1}{r}[(k+2-r)^n - 1].$$

One more identity

$$\sum_{r=0}^{k}(-1)^r\binom{n+1}{r} = (-1)^k\binom{n}{k}$$

together with Pascal's formula allows us to gather the extra bits and pieces to form the $r = k+1$ term we lack. This completes the proof of the theorem.

Finally, the probability asked for is $A(n,k)/(n!)$.

Also solved by J. P. Hoyt & David Alberth, and by L. M. Sonneborn.

Editor's comment. Sonneborn points out that the $A(n,k)$ are essentially the Eulerian numbers and hence satisfy the recurrence relation

$$A(n,k) = (k+1)A(n-1,k) + (n-k)A(n-1,k-1).$$

He also relates this problem to one in John Riordan, *An Introduction to Combinatorial Analysis*, Wiley, New York, 1958, pp. 214–215, and provides a second solution which can be used to give an elementary proof of Riordan's rook problem. D. M. Bloom correctly conjectures both the recurrence formula above and West's equation (4) from a table of $A(n,k)$ for small n. Hoyt and Alberth include a table of $A(n,k)$ for $n \leqq 13$. The proposer comments that $A(n,k)$ is exactly the number of permutations $\sigma \in S_n$ such that $\sigma(j) < j$ for precisely k values of j; he also observes that $A(n,1) = 2^n - n - 1$ for all $n > 0$.

Acton's Arctan Integral

E 2405 [1973, 316]. *Proposed by R. E. Shafer, Lawrence Radiation Laboratory*

Forman S. Acton in his book *Numerical Methods that (almost) Work* (Harper-Row, New York, 1970, pp. 29–40), proposed several numerical methods for the evaluation of

$$F(b) = \int_0^\infty \frac{\tan^{-1} bx}{1 + x^2}\,dx.$$

Derive the following additional form:

$$F(b) = -\tfrac{1}{2}\log b \log\frac{1+b}{1-b} + \sum_{n=0}^{\infty} \frac{b^{2n+1}}{(2n+1)^2}, \qquad 0 < b < 1.$$

I. *Solution by O. G. Ruehr, Michigan Technological University.* Differentiation under the integral sign (which can be justified) yields (for $0 < b < 1$)

$$F'(b) = \int_0^\infty \frac{x\,dx}{(1+b^2x^2)(1+x^2)} = \int_0^\infty \frac{x}{(1-b^2)}\left[\frac{1}{1+x^2} - \frac{b^2}{1+b^2x^2}\right]dx$$
$$= \frac{-\log b}{1-b^2}.$$

Since $F(0) = 0$, we have, with an integration by parts,

$$F(b) = -\int_0^b \frac{\log t}{1-t^2}\,dt = -\tfrac{1}{2}\log b \log\frac{1+b}{1-b} + \tfrac{1}{2}\int_0^b \log\left(\frac{1+t}{1-t}\right)\frac{dt}{t}.$$

By expanding the integrand in a Maclaurin series and integrating term-by-term (which can also be justified), we obtain the indicated result.

We note that since

$$F(b) = \int_0^\infty \frac{1}{1+x^2}\left\{\int_0^{bx} \frac{dt}{1+t^2}\right\}dx,$$

we can invert the order of integration and evaluate the inner integral to show

$$F(b) = \int_0^\infty \frac{1}{1+t^2}\left\{\frac{\pi}{2} - \tan^{-1}\left(\frac{t}{b}\right)\right\}dt$$

from which it follows that $F(b) = (\pi^2/4) - F(1/b)$. Setting $b = 1$ in the above yields $F(1) = \pi^2/8$, (a result which can be obtained directly from the integral for $F(b)$). By Abel's theorem we see that

$$\frac{\pi^2}{8} = F(1) = \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2},$$

by purely elementary methods. (From this one can easily derive the familiar identity $\sum n^{-2} = \pi^2/6$ — Ed.)

II. *Comment by the proposer.* Consider for $|x| < 1$

$$G(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^2} = -\int_0^x \frac{\log(1-t)}{t}dt = -\int_{1-x}^1 \frac{\log v}{1-v}dv.$$

Integrating by parts, expanding the new integrand in a Maclaurin series and integrating term-by-term gives

$$G(x) = \frac{\pi^2}{6} - \log x \log(1-x) - \sum_{n=1}^{\infty} \frac{(1-x)^n}{n^2}.$$

Since

$$\sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)^2} = G(x) - \tfrac{1}{4}G(x^2),$$

we have a useful formula for calculating $F(b)$ when b is near 1.

Also solved by Günter Bach (Germany), Paul Bugl, A. R. DiDonato, W. O. Egerland, H. E. Fettis, Ralph Garfield, Irving Gerst, Richard Groeneveld, M. S. Klamkin, Václav Konečný, O. P. Lossers (Netherlands), St. Olaf Problem Group, F. G. Schmitt, Jr., Allen Stenger, D. C. Stocks (England), E. Trost (Switzerland), P. H. Young, and the proposer.

Alpha-max, Beta-min, and a Limit for e

E 2406 [1973, 316]. *Proposed by Erwin Just and Norman Schaumberger Bronx Community College*

What is the maximum value of α and the minimum value of β for which

$$\left(1+\frac{1}{n}\right)^{n+\alpha} \leqq e \leqq \left(1+\frac{1}{n}\right)^{n+\beta}$$

for all positive integers n?

Solution by M. S. Klamkin, Ford Motor Company. On taking logarithms we obtain

$$\alpha_{\max} = \inf_n \left\{\frac{1}{\log(1+1/n)} - n\right\}, \beta_{\min} = \sup_n \left\{\frac{1}{\log(1+1/n)} - n\right\}.$$

We now show that the function

$$F(x) = \frac{1}{\log(1+1/x)} - x$$

is monotonically increasing for $x > 0$ by showing its derivative is positive:

$$F'(x) = \frac{1}{x(x+1)[\log(1+1/x)]^2} - 1 = \frac{\sinh^2 u}{u^2} - 1 > 0$$

where $e^{2u} = 1 + 1/x$. Thus, $\alpha_{\max} = 1/\log 2 - 1 = 0.4426950$ and $\beta_{\min} = \lim_{n\to\infty} F(n)$. By expanding $\log(1 + x)$ in a Maclaurin series, we have

$$F(n) = \left[\frac{1}{n} - \frac{1}{2n^2} + O\left(\frac{1}{n^3}\right)\right]^{-1} - n,$$

from which it follows that $\beta_{\min} = \lim_{m\to\infty} F(n) = \frac{1}{2}$.

Also solved by Günter Bach (Germany), M. T. Bird, J. R. Case, L. E. Dor (Israel), D. C. Doss, W. O. Egerland, Ellen Hertz, P. G. Kirmser, Václav Konečný, L. Kuipers, V. Linis, O. P. Lossers (Netherlands), Carolyn MacDonald, Raymond McCartney & Donald Coscia, M. R. Murty & V. K. Murty, Kenneth Rosen, O. G. Ruehr, Steven Russ, St. Olaf Problem Group, F. G. Schmitt, Jr., R. Shantaram, P. J. Short, D. A. Smith, Paul Strong, Phil Tracy, Philip Young, and the proposers.

ADVANCED PROBLEMS

All solutions of Advanced Problems should be sent to J. Barlaz, Rutgers — The State University, New Brunswick, N.J. 08903. Solutions of Advanced Problems in this issue should be typed (with double spacing) on separate, signed sheets and should be mailed before June 30, 1974.

An asterisk (*) means neither the proposer nor the editors supplied a solution.

5958. *Proposed by Alexander Abian, Iowa State University*

Let $r \neq 0$ be an interior point of the interval of convergence of the Taylor expansion $\sum_{n=0}^{\infty} a_n x^n$ of an analytic function f from reals to reals. Prove or disprove that for every interval $I(r)$ with center at r there exists a real number t and a natural number k such that $t \in I(r)$ and $f(r) = a_0 + a_1 t + \cdots + a_k t^k$.

5959. *Proposed by E. J. Howard, California State University, San Diego*

T. Husain in *Introduction to Topological Groups* (Saunders, Phila. 1966), p. 69, proved that: If E is a locally compact topological group and F is any Hausdorff topological group and $f: E \to F$ is a continuous almost open homomorphism, then F is locally compact.

Establish the same conclusion for any topological group F. (Note: f is almost open at $x \in E$ if for each open neighborhood U of x, $\overline{f(U)}$, the closure of $f(U)$, is a neighborhood of $f(x)$.)

5960. *Proposed by J. R. Higgins, The Cambridgeshire College of Arts and Technology, England*

Let $r_k(t)$ be the Rademacher "square sine wave" functions defined by $r_k(t) = \operatorname{sgn}(\sin 2^k \pi t)$, $k = 1, 2, \cdots$, $t \in [0, 1]$, and put

$$I(m, n, k) = \int_0^{m/2^n} r_k(t)\, dt$$

where n is any positive integer, and m is an odd positive integer less than 2^n. Show that

$$\sum_{k=1}^{n} 2^{k-1}[I(m,n,k)]^2 = \frac{m}{2^n}\left(1 - \frac{m}{2^n}\right).$$

5961*. *Proposed by S. Zaidman, University of Montreal*

Let X be the space of continuous functions on $[0,1]$ which are zero at 0 and 1; let $A = d^2/dt^2$ be the operator defined on the set $\mathscr{D}(A)$ of functions $\phi(t) \in X$ which are in $C^2[0, 1]$, such that also $d^2\phi/dt^2 \in X$. Then prove:

(1) A is a linear closed operator with dense domain in X.

(2) For any $\phi \in \mathscr{D}(A)$, there exists a linear continuous functional on X, F_ϕ, such that $F_\phi(\phi) = \|\phi\|^2$ and $F_\phi(A\phi) \leqq 0$, where, as usual, $\|\phi\| = \max_{0 \leqq t \leqq 1}|\phi(t)|$.

5962. *Proposed by Lee Erlebach, University of Arizona*

Find an elementary solution to the "unsolved problem" of finding a separable locally compact Hausdorff space which is not σ-compact, proposed on page v of Steen and Seebach, *Counterexamples in Topology*.

5963. *Proposed by A. de Falguerolles and G. Letac, Université de Clermont, France*

X and Y are two positive random variables such that the density $f(x, y)$ of their joint probability distribution is decreasing in each variable x and y. Prove or disprove: If $E(X)$ and $E(Y)$ are finite, then

$$E(X) + E(Y) \leqq 3\ E(|X - Y|);$$

and if X and Y are less than 1, then

$$4E(XY) \leqq 3\ E(|X - Y|).$$

SOLUTIONS OF ADVANCED PROBLEMS

Minimal Intersection in a Collection of Overlapping Sets

5883 [1972, 1044]. *Proposed by Frank Bernhart, Kansas State University*

Given a collection X of subsets of S, no one containing another, let $C(X)$ consist of all minimal subsets of S which intersect every member of X. (1) Show that $C(C(X)) = X$. (2) Characterize collections X such that $C(X) = X$.

Solution by Neal Felsinger, Yale University. To start with we must have $C(X) \neq \varnothing$ when $X \neq \varnothing$, but this is not always true; if S is infinite and X is a nonprincipal untrafilter on S, a subset $u \subseteq S$ intersects every member of X if and only if $u \in X$, whence $C(X) = \varnothing$. To guarantee that $C(X)$ is nonempty, we assume X is a nonempty finite collection of subsets of S. In this case there is a finite subset of S intersecting every element of X, hence a minimal such subset.

Let $a \in X$. Then a intersects every element of $C(X)$. Let $s \in a$ and for each $b \in X$, $b \neq a$, let $t_b \in b - a$. Then there is $u \subseteq \{s\} \cup \{t_b \mid b \in X,\ b \neq a\}$, $u \in C(X)$ and $(a - \{s\}) \cap u = \varnothing$. Since $s \in a$ is arbitrary, a is minimal. Thus $X \subset C(C(X))$.

Conversely, suppose $a \notin X$. If there is $b \in X$, $b \subseteq a$, since $b \in C(C(X))$, a is not minimal; so $a \notin C(C(X))$. But if this is not the case, for each $b \in X$, there is $t_b \in b - a$ and $u \subseteq \{t_b \mid b \in X\}$, $u \in C(X)$. Hence, $a \cap u \neq \varnothing$: so $a \notin C(C(X))$. Therefore $C(C(X)) = X$.

For the second part, consider the property $X \subset C(X)$. Certainly (i) for all $a, b \in X$, $a \cap b \neq \varnothing$. We also have (ii) for all $a \in X$ and all $s \in a$ there is $b \in X$, $a \cap b = \{s\}$. For, if whenever $s \in a \cap b$, $|a \cap b| > 1$, then $a - \{s\}$ intersects every element of X, contradicting the minimality of a. Conversely, (i) and (ii) imply $X \subset C(X)$. We characterize $C(X) \subset X$ by dualizing (i) and (ii), say, by

(iii) any two sets, each intersecting every element of X have a non-empty intersection, and

(iv) if a is a minimal subset intersecting every element of X and $s \in a$ there is b intersecting every element of X and $a \cap b = \{s\}$. Therefore (i), (ii), (iii) and (iv) characterize the collections X such that $X = C(X)$.

Also solved by R. O. Davies, D. J. Kleitman, Albert Leisinger, and Milan Lustig (Czechoslovakia).

Editor's Notes. (1) Lustig observes that (i) If $C(X) = X$, then card $X \neq 2$. (ii) For all integer $k \neq 0,2$, there exists X, for which $C(X) = X$ and card $X = k$. (iii) $\cup X = \cup C(X)$.

(2) Kleitman and the proposer offer another condition for the second part of the problem: for every A in S, either but not both of A, $S - A$ contain at least one member of X.

(3) Kleitman suggests the following problem: A collection of seven sets each having three elements for which $C(X) = X$ is $\{1,2,3\}$, $\{1,4,5\}$, $\{1,6,7\}$, $\{2,4,6,\}$, $\{2,5,7\}$, $\{3,4,7\}$, $\{3,5,6\}$, What is the size of the smallest collection X of four element sets having $C(X) = X$?

Inserting a Three Dimensional Cube in a Tesseract

5886 [1972, 1140]. *Proposed by G. de Josselin de Jong, New Mexico Institute of Mining and Technology*

What is the maximal edge of a cube that can be placed inside a tesseract of edge 1?

Partial solution by the proposer. An example may be given for which this inscribed cube has edge $1.007435\cdots$, a solution of the equation

$$4x^8 - 16\sqrt{2}x^7 + 36x^6 - 39x^4 + 20\sqrt{2}x^3 - 16\sqrt{2}x + 16 = 0.$$

Editorial Note. No other contributions have been received. It may be of some interest to note that the largest square which is inscribable in a unit cube has side equal to $3\sqrt{2}/4 = 1.061\cdots$.

Largest Disk in a Range

5887 [1972, 1140]. *Proposed by E. H. Umberger, Pennsylvania State University*

Find the radius ρ of the largest disk D for which there exists a continuous rec-

tifiable curve C of unit length such that every point of D is within unit distance of some point of C.

Solution by the proposer. We invert the problem and consider a shortest curve which passes within ε-distance ($\varepsilon < 1$) of every point on the unit circle K. Such a curve consists of the two tangents drawn from an arbitrary point P on K to a concentric circle of radius $1 - \varepsilon$, plus the greater arc between the points of tangency, less two segments of length ε measured from P along the tangents. The length $\lambda = \lambda(\varepsilon)$ of this curve is given by

$$\lambda/2 = (2\varepsilon - \varepsilon^2)^{1/2} + (1 - \varepsilon)(\pi - \arccos(1 - \varepsilon)) - \varepsilon.$$

Setting $\lambda = \varepsilon$ and solving, we obtain $\varepsilon = 0.8413\cdots$, and also observe that the corresponding curve C is a shortest curve passing within ε-distance of every point of the unit disk D. Using the appropriate re-inverting homothety which takes C into a curve of length 1 and D into the corresponding disk, we find $\rho = 1/\varepsilon = 1.1886\cdots$,

Minimum of $|a^4 + b^4 - c^4|$; a, b, c Integers

5890 [1973, 82]. *Proposed by H. D. Ruderman, Hunter College Campus School*

Prove or disprove that the minimum of $|a^4 + b^4 - c^4|$ is equal to 64, where a, b, c are integers with $1 \leqq a < b < c$.

Editor's comment. No solution has been received. We offer two notes.

Joel Brenner notes that the question of the problem and more general questions related to $\lim_n |a^n + b^n - c^n|$ were raised by I. A. Barnett.

Bernardo Recamán S. (Colombia) attacks the question by seeking solutions (or showing that none exist) for the equation

$$a^4 + b^4 = c^4 + T, \quad -63 \leqq T \leqq 63.$$

It is then possible to show that if $\min|a^4 + b^4 - c^4| \neq 64$, the minimum is in the set $\{1, 2, 14, 15, 16, 17, 30, 31, 32, 33, 34, 47, 48, 49, 50, 63\}$.

"Uniform" Connected Sets in the Plane

5892 [1973, 83]. *Proposed by John Myhill, University of Leeds, England*

Let A be a subset of the plane. For $p \in A$, $\varepsilon > 0$, let $N_\varepsilon(p) \equiv \{q \in A: d(p, q) < \varepsilon\}$. A is called *uniform* if for any p, $q \in A$ and any $\varepsilon > 0$, there is an isometry of $N_\varepsilon(p)$ onto $N_\varepsilon(q)$ taking p into q. Are there any uniform, closed connected sets other than the straight line, the circle, and the entire plane (and the empty set and a single point)?

Solution by P. R. Chernoff, University of California at Berkeley. There are no uniform, closed, connected subsets of the plane other than the obvious ones. Indeed,

et A be such a set. If A is contained in a line, it is an interval (by connectedness) and therefore is empty, a single point, or the whole line (by uniformity). Thus we may assume that A contains three non-collinear points p, q, r. Let s be any point of A; let f_n be an isometry of $N_n(p)$ onto $N_n(s)$, where we suppose n large enough so that q, $r \in N_n(p)$. Since f_n maps p, q, r to the vertices of a congruent triangle, and since any point is determined by its distances from three non-collinear points, it follows that f_n is the restriction to $N_n(p)$ of an isometry of the plane, which we continue to denote by f_n. $\{f_n\}$ is then a sequence in the group E_2 of rigid motions of the plane, and since $f_n(p) = s$ for all n, there is a convergent subsequence with limit f in E_2. It is evident that the isometry f maps A onto itself, and $f(p) = s$. To summarize: we have shown that A is the orbit of some subgroup of E_2.

Let $G = \{f \in E_2 : f(A) = A\}$. Because A is closed, G is a closed subgroup of E_2, hence a Lie subgroup. Let G_0 be the connected component of G. If $G_0 = \{e\}$ then G, hence A, is countable, and connectedness implies that A reduces to one point. If G_0 is at least two dimensional it is easy to see that it acts transitively on the plane, hence A is the whole plane. If G_0 is one dimensional, it consists of either (a) all translations in a given direction, or (b) all rotations about a given point c. In case (a), because G is a union of an at most countable number of cosets of G_0, the set A consists of at most a countable number of parallel lines; but connectedness implies that A is a single line. In case (b), the fact that G_0 is normal in G means that G consists of rotations and reflections about the point c, so A is a circle with center c.

Also solved by E. A. Herman.

REVIEWS

EDITED BY J. ARTHUR SEEBACH, JR. AND LYNN A. STEEN

with the assistance of the mathematics departments of St. Olaf and Carleton Colleges

COLLABORATING EDITOR FOR FILMS: SEYMOUR SCHUSTER, CARLETON COLLEGE

We invite readers to submit reviews of significant recent college-level mathematics books. We especially encourage reviews based on classroom use, or comparative reviews of several related books. Reviews should ordinarily not exceed two pages (per book) typed double spaced. Manuscripts of reviews as well as books submitted for review should be sent to: Book Review Editor, American Mathematical Monthly, St. Olaf College, Northfield, MN 55057.

Boundary Value Problems. By David L. Powers. Academic Press, New York, 1972. x + 238 pp. \$9.95. (Telegraphic Review, January 1973.)

Occasionally one comes upon a book that is just what is needed for a particular course. Such was the case in the following instance. I was in need of a text for a semes-

ter course in the application of mathematics which would be taken by physics majors, engineers and some math majors who don't enjoy proving a lot of theorems. Material covered by these students prior to this course included three semesters of calculus and two semesters of course work that contained elements of linear algebra, differential equations and vector analysis. I am familiar with most of the textbooks that are used for a year course of this type at the undergraduate level and a number of them which I have used in the past are quite satisfactory (Kreyszig, etc.). The special feature of this book is that it contains exactly those topics I wanted to include and they are presented in a clear and concise manner. It was easy and fun for the students to learn, and at the same time perceive how the material presented all nicely tied together.

The topics covered include Fourier series and integrals; heat, wave, and potential equations for one and two dimensions; Laplace transform; and numerical solutions for some of the previously stated equations. My students worked most of the 300 exercises, for many of which there were hints and answers. The exercises range from drill and verification of details to the development of new material; a few of them are poorly stated. I omitted no sections and supplemented the book only with a small amount of material on the calculus of variations and the numerical solutions of partial differential equations. It should be noted that topics such as characteristics, distributions, Green's functions, and all questions of existence and uniqueness have been omitted.

I think Powers' presentations on the derivation of each equation, the boundary conditions and regions which allow the method of separation of variables to work, and the formulation of problems in terms of dimensionless units are excellent. The physical interpretation of mathematical results helped build up the students' intuition about how the solutions of a problem should behave.

The sections on miscellaneous comments and references for further study were enthusiastically received. At times one wishes he would worry a little more about some of the mathematical difficulties encountered (he readily admits he doesn't). Nevertheless, Powers' treatment of the solution of boundary value problems is such that his book deserves the utmost consideration when choosing a book for a course similar to that described in the first paragraph.

MARVIN G. MUNDT, Valparaiso University

Elementary Linear Algebra. By Howard Anton. Wiley, New York, 1973. xii + 330 pp. $10.25. (Telegraphic Review, March 1973.)

Elementary Linear Algebra. By Bernard Kolman. Macmillan, New York, 1970. x + 255 pp. $8.95. (Telegraphic Review, January, 1971.)

Elementary Linear Algebra, Second Edition. By Paul C. Shields. Worth, New York, 1973. x + 393 pp. $9.95.

These three books are part of the "flood" of linear algebra textbooks aimed at college sophomores. (Cf. the review by D. E. Christie, this MONTHLY, June–July,

1973.) The reviewer has used the books by Kolman and Shields successfully with students who have studied one year of calculus.

Besides having identical titles, these books have very similar tables of contents. Each begins with a chapter on linear equations and matrices. This is followed by chapters on vector spaces over the real numbers, linear transformations, and eigenvalues and eigenvectors — in that order. Some variations do occur. For example, Anton discusses determinants before taking up vector spaces, while the other two authors introduce them after linear transformations have been thoroughly studied. Kolman and Shields have final chapters on linear differential equations, while Anton's final chapter deals with an introduction to numerical methods of linear algebra. None of the three books says much about applications of linear algebra to other subjects. This is a defect which is all too common among books in this category.

While the contents of these books are all similar, there are important differences in the ways of presenting the contents. Consider, for example, the balance between intuition and rigor. Shields has adopted a policy of proving many theorems for the 2×2 case only, and of putting some proofs in an appendix. This improves the readability of the text, but it tends to give a false assurance that these proofs are easy or unimportant. Kolman, on the other hand, attempts to prove nearly every result in its full generality. This leads to a plethora of subscripts and sigma notation which makes the book harder to read. Anton has tried to strike a good balance by including proofs in the text when they are especially elegant or simple, while omitting some other proofs which are too complicated to be enlightening at this level of mathematical maturity. Sometimes he gives an informal discussion of the idea, followed by a formal statement of the theorem.

Another point of comparison would be the use of broad mathematical structures to give some unity to the various parts of linear algebra. Shields emphasizes geometric interpretations of the algebraic concepts. This is very effective in tying together such concepts as solution spaces, null spaces, and eigenspaces. However, he pays little attention to algebraic structure, and there is a tendency to overemphasize matrix manipulation. Kolman does a better job of making the various individual problems fit into an overall algebraic framework. On the other hand, geometric interpretations are conspicuously absent from this book. Anton, like Shields, employs geometric ideas to good advantage, but he leaves out the algebra of linear transformations and the notion of isomorphism.

With respect to exercises, all three texts contain an adequate number and variety. In Shields, especially, many exercises are patterned after the examples which precede them. This lends itself nicely to self-study. Shields and Anton have included the answers to nearly all exercises in the back of their books. This is a mixed blessing, as it enables the students to check their own work, but it frustrates the teacher who wants to grade some homework on a regular basis. Kolman has answers to selected exercises in the back of the book, and an answer book is available for the others.

Each textbook has its own particular strengths and weaknesses. Anton has an exceptionally good presentation of Gauss-Jordan elimination. It is made clear to the reader that there is a *systematic* procedure for reducing any matrix. A judicious use of shading on these particular pages helps. The chapter on numerical methods is an important one, since it makes students aware of potential problems that can arise in computation. Anton's treatment of determinants, early in the course, makes them available for solving systems of equations and computing matrix inverses when the appropriate time comes. Weaknesses of Anton include his failure to exploit the concept of rank until late in the book. This is another unifying concept which could be used in a number of places. His definition of eigenvalues as roots of the characteristic equation seems to this reviewer to be an unnatural one.

Kolman provides the reader with a large number of examples of vector spaces and subspaces. In general, he probably does the best job of presenting linear algebra as a unified structure. His weaknesses include the introduction of too many new ideas all at once, such as the ten special types of matrices in Section 1.4. It would be better to introduce some of these later when the students can better appreciate their significance. In trying to be complete, Kolman treats elementary column operations and elementary row operations simultaneously. Students at this level find it more confusing than helpful. Finally, he does not introduce dot products until after eigenvectors have been discussed, so he misses out on a good example of a linear transformation.

The most significant change in the second edition of Shields is his use of *rank* as a unifying concept, beginning early in chapter one. Shields also does a good job of developing the algebra of matrices and the algebra of linear transformations simultaneously, with each motivating and reinforcing the other. His discussion of the relationships between homogeneous and non-homogeneous systems of linear equations is very illuminating. The section on determinants remains one of Shields' weak points. The inductive definition, beginning with the 2×2 case, is easy to understand, but it does not lend itself well to proofs. Also, the topics of the classical adjoint and Cramer's rule deserve to be treated as more than optional exercises. When it comes to the notion of a vector space, Shields considers only n-tuple spaces and spaces of real-valued functions. This leaves many students with too narrow a concept of vector spaces, not to mention the lack of any notion of isomorphism.

When asked to evaluate textbooks, our students have rated Shields easier to read and study on their own than Kolman. Anton has not yet been used in our classes. Each of the three books reviewed here will serve adequately as a sophomore level linear algebra textbook. Instructors in this subject would do well to examine all three, along with some of those reviewed recently by Professor Christie.

DAVID E. KULLMAN, Miami University

Foundations of Mathematics. By William S. Hatcher. Saunders, Philadelphia, Pennsylvania, 1968. xiii + 327 pp. \$12.75. (Telegraphic Review, January 1970.)

The offering of a foundations course generally involves more problems than is the

case for other courses in that the content may not easily be agreed upon. In addition there are questions as to how a course entitled "foundations" may best serve the students; the answers depend in part on the level of the course. A now classic answer to these questions was provided by Wilder's *The Foundations of Mathematics* (1952). One senses that in many instances Wilder's book dictated the direction of foundations courses. Now, in my judgment, Hatcher has made a historic step in giving another better — certainly more modern — answer, an answer so natural as to be a compelling reason for the offering of new foundations courses. Hatcher's view is that the foundations of mathematics, rather than being a collection of fundamental concepts and facts useful to developing students, is a *bona fide* mathematical subject in which one studies certain theories, called foundational theories. Briefly, these are formal deductive theories in which large portions of mathematics can be generated.

First order logic and preliminaries are treated in the first two chapters. The remaining six present the foundational theories of Frege, Russell (type theory), Zermelo-Fraenkel set theory and the variant of von Neumann-Bernays-Gödel, Quine, and categorical algebra. Hatcher suggests the book for a one semester course, following, perhaps, a course in logic. If the logic course covered the completeness theorem, the student may be able to start with Chapter 2. But the book can well be used without a logic prerequisite because of the topics covered in Chapter 1. In my own graduate class, it was even a necessary review for those (roughly half) who had taken such a course, not necessarily recently. However, the inclusion of predicate logic takes some time, so that one must select from the remaining parts of the book. My choice was to consider the theories of Frege, Zermelo-Fraenkel, and von Neumann-Bernays-Gödel, plus a little of type theory.

Chapter 1 is a good introduction to first order predicate logic, successfully generating an understanding of model versus proof-theoretic aspects of deductive theories, an essential understanding in my opinion. The completeness theorem is a climax to this study in that it justifies the passage to first order formal theories. While it may be regrettable that a proof of completeness does not appear, it could easily use up precious time. Also, I see no value in giving both Hilbert-Ackerman type and natural deduction treatments of predicate logic, since only the rules of natural deduction are used throughout the book and the inclusion of both suggests the time consuming question of their equivalence, only half of which is carried out in the text.

Chapter 3 on Frege's system is lovely. Actually Frege's tedious system is paraphrased here with modern notation and resulting simplicity. It is a great classroom plan to develop arithmetic in this theory before springing the inconsistency theorems. The story of the inconsistency of early set theory is an exciting one and the all too usual treatment is to discuss it first as a kind of acknowledgment of its existence but then to ignore it or to dismiss its importance with a casual remark of a program which if carried out would solve the difficulty. Here the message is clear. Frege's theory is inconsistent and cannot serve as a foundational theory.

The presentation of Zermelo-Fraenkel set theory is straightforward: this is followed by a comparison with von Neumann-Bernays-Gödel. That the development of real numbers from Peano arithmetic is not done in the book is not objectionable to me; one must stop somewhere in grinding out ordinary mathematics. Hatcher does treat the axioms of infinity and choice, and in a separate chapter the incompleteness theorem.

The chapter on type theory is a palatable introduction although I essentially omitted it in favor of the more standard set theories. Quine's theories of *New Foundations* and *Mathematical Logic* and Lawvere's reduction of mathematics to category theory were also omitted. If the students have a background in axiomatic set theory, these topics could be substituted for the set theory. In any case, the entire book cannot be covered in one semester and selection is necessary.

Foundational theories such as intuitionism or Errett Bishop's constructive treatment of analysis, which are outside the context of classical logic, do not appear; however, they clearly fall under the scope of Hatcher's view. These theories could make an appropriate sequel to the book.

In short, I submit that this book makes history with its presentation of foundations. Rather than skirting axiomatics, Hatcher's book enables one in the course of a semester to present a variety of foundational theories in correct mathematical form and detail. My students reported liking the book; I believe it is suitable for either graduates or undergraduates.

J. BARR, State University College at Buffalo

Elements of Abstract and Linear Algebra. By Hiram Paley and Paul M. Weichsel. Holt, Rinehart and Winston, New York, 1972. xii + 498 pp. $10.50.

An instructor teaching a first course in abstract algebra faces two major challenges. First, he has to connect the abstract material with the more concrete mathematics his students have previously studied, and, secondly, he must demonstrate the value of the abstract approach through appropriate applications. The book by Paley and Weichsel can serve as a model of how to meet both these tasks, at least in the area of group and ring theory.

The first difficulty, that of relating the abstract to the concrete, is handled by leading the students through elementary number theory in the spirit of the axiomatic approach without immediately generalizing to ring theory. There is a long chapter (60 pages) taking up the properties of the integers, divisibility, congruences, and residue classes. Essentially, the integers are studied as an ordered integral domain in which the positive elements are well-ordered and the residue class systems are treated as rings, but no formal ring theory is used. Therefore, the students are studying concrete examples of rings before taking up ring theory. I found this to be very successful. My students showed none of the usual mystification about abstract techniques and quickly learned to construct proofs.

Another interesting feature of this book is a short chapter on permutations before any abstract group theory is introduced. This material covers all the techniques for working with permutations and a proof of the uniqueness of the parity of a permutation. Hence, permutation groups are available as examples of groups when the latter are defined. I recommend that this approach be used by anyone teaching a first course in abstract algebra; it was well received by my students.

After the above material the authors have a chapter on group theory and one on ring theory. The group theory includes the first isomorphism theorem, but (wisely, I think) does not take up the Sylow theorems. Instead, the authors develop the class equation of a group and use it to determine all groups of order less than or equal to ten. This is a nice demonstration of the use of some of the abstract techniques, which, as I mentioned, is an essential part of a first course in abstract algebra. The ring theory includes the construction of polynomial rings, fields of quotients, some material on prime and maximal ideals, and a little on Euclidean rings. I do not think that the authors are as successful here as in the group theory in tying together the ideas that were covered because they attempt to do too much. Nevertheless, the value of the abstract method is clearly demonstrated.

The second half of the book covers elementary linear algebra. Of course, the fact that the same terminology and notations are used here as in the first part made the transition to this material very easy for the students. However, I was disappointed with this portion of the book. First, the material is too elementary, covering little beyond that found in books intended to be used in the calculus sequence. For example, there is nothing on bilinear or quadratic forms or on the canonical forms of matrices. Also, there is little evidence of the usefulness of linear algebra. For example, since quadratic forms are not mentioned, the study of symmetric matrices is quite unmotivated. And finally the material on infinite dimensional spaces was too difficult, while the treatment of rank and determinants is occasionally awkward.

The book is carefully written. My students had no difficulty in reading it and in covering some sections on their own. The exercises are numerous and well-chosen, although I would like to see those exercises that are needed later marked in some way. There is more in the book than can be covered in a year's course, but with a little care some material can be omitted without causing trouble later.

In conclusion, I found this to be an excellent book with several outstanding features. Its one major defect is the lack of sufficient material in linear algebra, so a potential user should be prepared to supplement this part. I would recommend that anyone teaching a year's course in algebra look carefully at this book.

ROBERT B. REISEL, Loyola University, Chicago

TELEGRAPHIC REVIEWS

Telegraphic reviews are designed to give prompt notice of new books with sufficient information to assist our readers in deciding whether to order an examination copy or to suggest library purchase. Possible uses are indicated as follows:

T = textbook P = professional reading
S = supplementary reading L = undergraduate library purchase
13 to 18 = freshman to second year graduate level usage
1 to 4 = appropriate time in semesters to cover text

Asterisks (*) or question marks (?) denote special positive or negative emphasis, respectively. Publishers are denoted by standard abbreviations; complete addresses may be found in *Books in Print*.

GENERAL, T**(13-14: 1), S(13-16), P, L**, *Mathematics in Civilization*. H.L. Resnikoff, R.O. Wells, Jr. HR&W, 1973, x + 372 pp, $11.50. An exceptionally good "liberal arts math" text. Treats through time two themes: the ability to compute, the geometrical nature of space. Unusual choice of topics includes Rhind papyrus, Babylonian tablets, Greek astronomy, marine navigation, cartography, logarithms, analytic geometry, calculus, Taylor polynomials, differential geometry, relativity theory. Healthy emphasis on historical applied mathematics, particularly astronomy. No treatment of electronic computers. Avowedly, does not teach mathematical techniques nor prepare the student to use mathematics as a tool in any way. Sparing in its demand on students' prior knowledge; but less plodding, more exciting than most books of the genre. PJC

GENERAL, T(13), *Mathematics and the Liberal Arts*. Jack C. Gill. Merrill, 1973, xxi + 377 pp, $9.95 (P). This book covers logic, high school algebra, probability and statistics, decimal and other bases, and a review of computational arithmetic. It is difficult to review completely, since it is designed to be used with a set of audio tutorial tapes which were not available. Two objections arise. First, the subjects are broken up into exercises with "behavioral objectives." This sort of fragmentation may make it easier for the student, but makes any sort of unified approach difficult. Second, most of the examples and exercises are of a standard sort having little to do with liberal arts. An index is provided, but no references to other sources of information. PJM

GENERAL, T(13-14), S, L, *Mathematical Ideas, An Introduction, Second Edition*. Charles D. Miller, Vern E. Heeren. Scott F, 1973, 372 pp, $8.95. Math for non-majors. Numeration methods, sets, mathematical systems, symbolic logic, relations and functions, probability and statistics, matrices, computers and transformational geometry are the chapter headings. Each chapter has a list of suggested references for further reading. The exercises are good and plentiful. Marginal pictures are very good. PJM

GENERAL, S, L*, *The MAA Problem Book III*. Charles T. Salkind, James M. Earl. New Math. Lib., V. 25. Random House, 1973, vi + 186 pp, $2.12 (P). Problems with solutions from MAA annual high school contests, 1966-72. LCL

GENERAL, T(13: 2), *A Survey of College Mathematics, Second Edition.* Donald R. Horner. Rinehart Pr, 1973, ix + 335 pp, $9.95. Significantly changed from the 1967 edition. Out are chapters on trig, analytic geometry and calculus, as well as some of the more taxing exercises. In are an expanded treatment of probability, flow charts throughout and an introduction to computer mathematics. Still more content oriented than most surveys. Contains lots of good mathematics in an attractive presentation, Numerous exercises (90% answered). Possible for self-paced setting. TAV

GENERAL, S, L, *Recreational Problems in Geometric Dissections and How to Solve Them.* Harry Lindgren. Revised and enlarged by Greg Frederickson. Dover, 1972, viii + 184 pp, $2 (P). Basically the 1964 edition with appendix bringing the subject up to date. Based on Bolyai-Gerwin Theorem (any rectilinear plane figure can be dissected into any other of same area by cutting it into a finite number of pieces), the main interest of dissections is to find how to dissect one figure into another in the least number of pieces. As the author remarks, it is heartening to learn there is some method to it. "The subject is nowhere near exhaustion", as the results since 1964 attest. Unfortunately, still no index. PJC

GENERAL, S*(13-18), P*, L**, *Companion to Concrete Mathematics: Mathematical Techniques and Various Applications.* Z.A. Melzak. Wiley, 1973, xiii + 270 pp, $14.95. A delightful miscellany of problems organized into over sixty sections together with historical notes, solutions, observations and remarks. Problems feature concreteness, intuitive appeal, ingenuity. Mathematicians at all levels, especially college teachers of mathematics, will enjoy sampling and using this book. High level of mathematical content. LCL

GENERAL, S, P*, L, *The AMRC Papers: An Indictment of the Army Mathematics Research Center.* Science for the People. 1973, 119 pp, $1.25 (P). Details of the operation of an "overtly violent" institution which is the chief mechanism by which mathematicians participate in the Indochina War effort. Documentation from government sources of consulting by the AMRC staff with researchers at Army installations, together with direction of their research: viz., design and testing of weapons (chemical, biological, conventional), and creation of military and political strategy (including counter-insurgency). Thorough, informative, and damning. Down-to-earth explanation of the process of mathematical modelling included. Closes with a sketch of how mathematics could better serve society's needs through a People's Mathematics Research Center. PJC

GENERAL, S(13), *Aufgaben mit Lösungen aus Olympiaden Junger Mathematiker der DDR, Band I.* W. Engel, U. Pirl. Volk und Wissen, 1972, 173 pp. A collection of problems used in the Mathematics Olympiad for secondary school students in the German Democratic Republic. Fields: number theory, equations, inequalities, functions (especially trigonometric), logic and combinatorics. Less interesting than the similar collection of Russian problems published by Freeman in 1962. JD-B

BASIC, T(13), *Beginning Algebra for Mature Students.* James J. Meadowcroft Il. P-H, 1971, 270 pp, $6.35 (P). Workbook--examples, exercises with answers, reviews, tests. Quadratic equations, first degree equations in two unknowns. Fractional exponents not included. Large, easy to read format. LCL

PRECALCULUS, T(13: 1), L, *Elementary Functions; Pre-Calculus Mathematics.* D. Franklin Wright, Kenneth E. Lindgren. Heath, 1973, xi + 338 pp, $8.95; *Instructor's Manual*, 107 pp, free (P). Here is a pre-calculus book that crosses the line and sneaks in some calculus concepts, e.g., a "slope function" for use in graphing polynomials, intuitive notions of continuity and limits, some (inappropriately) sophisticated theorems about existence of maxima and minima. Exponentials, logarithms, trig functions and a short section on permutations, combinations and probability are also included, along with a section on sequences and series. Limits of sequences are referred back to a previous definition (?) of limits for functions, the previous definition consisting merely of two examples. The concept of this book is good; however, it is carried out sloppily and the student will either be confused or (mistakenly) bored with calculus after going through it. PJM

PRECALCULUS, T(13: 2), *Essentials of Mathematics, Third Edition.* Russell V. Person. Wiley, 1973, xiii + 738 pp, $11.75. A large text made larger (in this third edition) with the addition of a section on calculus--a non-rigorous brief treatment for technical students. Second edition reviewed in February, 1969. LLK

PRECALCULUS, T(13: 1, 2), L, *College Algebra, 4th Edition.* Moses Richardson, Leonard F. Richardson. P-H, 1973, xii + 468 pp, $10.95. Any book which is in its 4th edition must be doing something right. This text is a souped up high-school algebra book, designed for college students, and makes an excellent pre-calculus course. It is well organized (complex numbers before quadratic equations) and has many topics not included in the usual pre-calculus text (solution of non-quadratic equations, a fairly sophisticated development of the real numbers). Other topics include probability, an introduction to linear algebra, mathematical induction and some results on progressions and series. An excellent book, reasonably priced. Exercises and an index but no collected references. References occur in footnotes, as well as frequent historical comments. PJM

EDUCATION, *Teaching Mathematics: Psychological Foundations.* F. Joe Crosswhite, *et al.* Jones Pub., 1973, viii + 268 pp, $6.95 (P). Collection of reprinted articles, mostly from *Mathematics Teacher* and *Arithmetic Teacher* on cognitive aspects of mathematics learning at elementary level. Editors append valuable exercises (questions and activities) after selections. Good text for discussion seminar for mathematics education students. PJC

EDUCATION, S, P, L, *Geometry in the Mathematics Curriculum: Thirty-sixth Yearbook.* NCTM, 1973, viii + 472 pp, $9. Explores approaches to and content of informal (K-14) and formal (10-12) geometry courses; presents three views of contemporary geometry; reviews and recommends programs for the education of teachers. JNC

EDUCATION, P, L, *The Process of Learning Mathematics.* Ed: L.R. Chapman. Pergamon Pr, 1972, xiii + 392 pp, $24. Lectures to education students at the University of London, including several on developments and experiments in British mathematics education. Particularly good: "The Role of Intuition" and "Motivation, Emotional, and Interpersonal Factors." PJC

HISTORY, S, *De L'Infini Mathématique*. Louis Couturat. Blanchard, 1973, xxiv + 667 pp, 56F (P). Reprint of 1896? work. *("Tous droits réservés"*--still??) Leisurely presentation of the extension of the integers to the real and complex number systems, successively in arithmetic, algebraic, and geometrical fashions. Mathematical infinity as a geometric and as an analytic object, together with philosophical reflections. The mathematics of quantity, subsequently related to number. Of historical interest; but still of value to students, especially for its integrated overall picture. PJC

HISTORY, S, P, L, *Ernst Mach: His Work, Life, and Influence*. John T. Blackmore. U of Calif Pr, 1972, xx + 414 pp, $16.95. First full-scale biography of the famous physicist, who feuded with Boltzmann, Planck, Einstein and others, opposing the atomic theory, the theory of relativity, and theoretical physics in general. He was particularly offended by the use of higher-dimensional geometry to explain the three-dimensional world of sensations. PJC

HISTORY, P, *Leibniz and Dynamics: The Texts of 1692*. Pierre Costabel. Transl: R.E.W. Maddison. Cornell U Pr, 1973, 141 pp, $8.50. Two texts by Leibniz on dynamics, from recently-discovered copies of the original manuscripts (themselves also extant). Textual comparison and intricate detective work give a glimpse of the politics of science in France in 1692. Of historical interest only. PJC

HISTORY, P, *Gesammelte Abhandlungen*. Issai Schur. Springer-Verlag, 1973. *Band I*, xv + 491 pp; *Band II*, iv + 494 pp; *Band III*, iv + 480 pp, $89.10 set. The collected research papers prefaced by a memorial address by Alfred Brauer and with an appendix containing some posthumous papers, published results, and results cited by other mathematicians. JAS

HISTORY, P, L**, *Oeuvres Scientifiques, V. I-V*. Henri Lebesgue. L'Enseignement Mathématique. *V. I*, 1972, 339 pp, 60F; *V. II*, 1972, viii + 443 pp, 60F; *V. III*, 1972, 405 pp, 60F; *V. IV*, 1973, 391 pp, 60F; *V. V*, 1973, 429 pp, 60F. Includes all major papers, one entire book--the original (1904) edition of *Leçons sur l'intégration*--, obituaries by Montel, Denjoy and Felix, various documents associated with Lebesgue's appointment and tenure at the Collège de France, and complete chronological lists of his works including those not published here. LAS

HISTORY, P, *Gesammelte Mathematische Abhandlungen*. Felix Klein. Springer-Verlag, 1973. *V. I*, xxi + 612 pp; *V. II*, vi + 713 pp; *V. III*, ix + 809 pp, $77.70 set. A reprint of the work originally published by Springer in 1921-23. Research papers and historical notes on his academic activity. JAS

FOUNDATIONS, T?(15-16: 1), L, *Mathematical Logic*. Daniel Ponasse. Gordon, 1973, x + 126 pp, $10.50. Efficient treatment of propositional and predicate calculi, demanding some algebra and topology. Standard axiom schemata briskly and neatly yield consistency and results on sets of sentences. Boolean rings and spaces are developed to derive completeness. Forest, trees: both visible. No incompleteness results, no exercises, no index, thin motivation. Profitable use of symbol $A{>}\sigma{<}$ to denote expression A which does not contain symbol σ. Translated from the French. PJC

FOUNDATIONS, T*(1, 2), *Logic and Philosophy: A Modern Introduction, Second Edition.* Howard Kahane. Wadsworth, 1973, 440 pp, $8.95; *Study Guide for Logic and Philosophy, Second Edition,* Warner Morse, 215 pp, (P). Good modern text for the traditional philosophy department course in logic. Standard axiomatic treatment of sentential and predicate calculus, plus other topics: traditional logic, induction and science, logic and philosophy, axiomatic systems. (Footnote mention of Gödel.) Broad range, but it can't all be done in one semester. Changes from first edition: more exercises; new section on modal, epistemic, deontic logics; revised (and very good) chapter on fallacies. Surprise: despite improvements, no increase in price from first edition--and a new feature, a complimentary study guide, besides! PJC

FOUNDATIONS, T(16-18), S, P, L, *Algebraic Systems.* A.I. Mal'cev. Grund. math. Wissenschaften, B. 192. Transl: B.D. Seckler, A.P. Doohovskoy. Springer-Verlag, 1973, xii + 317 pp, $39.60. Finely reasoned, completely detailed presentation from one of the originators of this subject which lies between algebra and mathematical logic. Classical algebras, first and second order languages, products and complete classes, quasivarieties and varieties. Note price!! LCL

FOUNDATIONS, T(13-16: 1), S, L, *Introduction to Modern Mathematics.* Helena Rasiowa. North-Holland, 1973, xii + 339 pp, $18.25. Accurate, complete introduction to the language and proof techniques of modern abstract mathematics. Text divided equally between set theory and mathematical logic, with some abstract algebra. Written by a distinguished author for freshman at Warsaw University. Scholarly, yet elementary. Could be used as foundations text; better as a student supplement and guide to "mathematical maturity." (Pages 117-132 were missing in the review copy.) LCL

LINEAR ALGEBRA, T(14: 1), *Linear Algebra.* Burton W. Jones. Holden-Day, 1973, xi + 315 pp, $12.95. A mathematically comprehensive treatment of introductory linear algebra. The emphasis is on development of an appreciation for more abstract mathematics. From this point of view, a well written text with an appropriate choice of problems. One is left with the distinct impression that there are no applications for linear algebra beyond 3 by 3 Markov chains. Incredible! TAV

LINEAR ALGEBRA, T?(13-14: 1), L?, *Elementary Matrix Algebra for Psychologists and Social Scientists.* A.G. Hammer. Pergamon Pr, 1971, ix + 212 pp, $5.95 (P). Written by an applied psychologist who sees himself as a go-between. Cookbook; sadly incomplete. Evokes little understanding. Main goal--factor analysis. Here is an application of interest to nearly everyone. Why not include this in our sophomore linear algebra texts? Should a go-between really be necessary? LCL

LINEAR ALGEBRA, T(14: 1), *Elementary Linear Algebra.* Evar D. Nering. Saunders, 1974, ix + 375 pp, $11. An excellent text, carefully presented for a sophomore level course. Builds to orthogonal and unitary spaces. LLK

ALGEBRA, P, *Rings, Modules, and Radicals.* Ed: A. Kertész. North-Holland, 1973 520 pp, $35.10. Papers and problems presented at the International Colloquium at Lake Balaton, Hungary, in August 1971. JAS

ALGEBRA, P. *Introduction to the Theory of Formal Groups*. J. Dieudonné. Dekker, 1973, xii + 265 pp, $18.75. First studies general $\underline{C}$-groups, where $\underline{C}$ is the category of cogebras over a field k. Then requires k to be perfect and develops the theory of infinitesimal formal groups (ending with reduced ones, where k is assumed algebraically closed). Applications to the theory of algebraic groups are not given. DFA

ALGEBRA, T(18: 1), P. *Symmetric Bilinear Forms*. J. Milnor, D. Husemoller. Ergebnisse der Math., B. 73. Springer-Verlag, 1973, viii + 147 pp, $15.60. Concentrates on recent developments, particularly the work of A. Pfister and M. Knebusch. LCL

ALGEBRA, T(17-18), L. *The Theory of Groups, An Introduction, Second Edition*. Joseph J. Rotman. Allyn, 1973, x + 342 pp, $14.50. Reset in more attractive type and format. Improved exercise sets. New sections on Grothendieck groups, edgepath groups, covering complexes and the Nielsen-Schreier theorem. A rewritten last chapter on the word problem includes Post's theorem and an imbedding theorem of G. Higman. DFA

ALGEBRA, T(16-18: 2), S, P*, L. *Algebra: Rings, Modules and Categories I*. Carl Faith. Grund. math. Wissenschaften, B. 190. Springer-Verlag, 1973, xxiii + 565 pp, $22.20. A survey of aspects of ring theory since Jacobson's Colloquium volume (1956, revised 1964) and a rapid escort to the frontiers of research. An encyclopedic undertaking. The first volume starts from scratch (parts are quite suitable for undergraduates). Basic propositions drawn evenly from classical and homological-categorical influences. An important addition to the literature. Many exercises, notes, references. LCL

ALGEBRA, T**(16-18: 2), S, L. *Category Theory: An Introduction*. Horst Herrlich, George E. Strecker. Allyn, 1973, xi + 400 pp, $18.95. Lots of problems, a real attention to pedagogy, excellent indices and bibliographies make this probably the best (if not only) textbook at its level really designed for the student who might become a mathematician. The emphasis is on the basic ideas of general category theory, limits and adjointness. It uses set theory, algebra, and topology at the advanced undergraduate or beginning graduate level but does only category theory. JAS

CALCULUS, T**(13: 3), *Honors Calculus*. Robert G. Bartle, C. Ionescu Tulcea. Scott F, 1970, 889 pp, $12.95. Possibly the most complete, rigorous text available. Clearly intended for the honors student. Logically sound and carefully constructed. One unusual feature: complex numbers and complex functions are studied leading to the function e^z from which the properties of e^x, $\log_a(x)$, the trig and inverse trig functions, hyperbolic and inverse hyperbolic functions are derived. Treatment through series in detail; introduction to multivariable calculus. TAV

CALCULUS, T(15-16: 2), *Advanced Calculus of Several Variables*. C.H. Edwards, Jr. Acad Pr, 1973, x + 457 pp, $15.50. A sophisticated, comprehensive treatment. Gives more than lip service to the calculus of R^n, $n > 3$. Introductory treatment of linear algebra, differential and integral calculus, vector calculus, calculus of variations. Extensive problem sets. The audience needs significantly more mathematical maturity than for most recent multivariable calculus texts. TAV

CALCULUS, T??(13: 2), *Introductory Calculus for Business and Economics*. William E. Beatty, R. William Gage. GLP, 1973, 371 pp, $10.95. Uses calculus to solve elementary (textbook) problems in business, economics, and management. Mathematically atrocious. For example, the student's first acquaintance with the definite integral is just after the statement of the Fundamental Theorem, where it's referred to as the (previously defined) indefinite integral where "we have specified the limits of integration." And how about this? "In mathematical terms, if x is a variable, a is the numerical value, and δ is *any number*, then we say *x approaches a* if $|x - a| < \delta$ for *all values of* δ." DFA

CALCULUS, T(13: 2, 3), *Calculus*. Leonard Gillman, Robert H. McDowell. Norton, 1973, 674 pp, $12.95. Carefully written text through multiple integrals. Theorems are stated and proved, but the tone is not overly rigorous. An unusual feature: the integral is defined, following Howard Levi, in terms of "additivity and betweenness" eliminating Riemann sums, etc. Abundant exercises. TAV

CALCULUS, T(13), S, *The Indefinite Integral*. G.M. Fichtenholz. Transl: Richard A. Silverman. Gordon, 1971, vii + 140 pp, $12.50. Covers standard methods of anti-differentiation, not unlike the treatment in most standard calculus texts. Many exercises and hints, but hardly unique enough to justify the price. TAV

CALCULUS, S(13), *The Calculus*. C.O. Oakley. B&N, 1957, viii + 221 pp, $1.75 (P). A short outline mainly for self study. Minimal textual material. Perhaps useful as a review of formulae through partials. Reprinting of 1944 edition. TAV

CALCULUS, T?(13), S, *The Definite Integral*. G.M. Fichtenholz. Transl: Richard A. Silverman. Gordon, 1973, vii + 90 pp, $8.75. Similar in content and style to the treatment found in most calculus texts. Darboux sums, mean value theorems, some techniques including integration by parts. A couple of dollars more would buy a complete text. TAV

REAL ANALYSIS, T(14-15: 1), *Introduction to Real Analysis*. Bevan K. Youse. Allyn, 1972, ix + 227 pp, $10.95. For an intermediate level course. Responsibility for proofs shifts to the student as text develops. Graded exercises, some integrated with text. Functions, sequences, limits and continuity, derivative, Riemann integral, infinite series, Riemann-Stieltjes integral. Appendices on axiomatic development of real numbers (all proofs are exercises) and project problems. RBK

REAL ANALYSIS, T(17: 1, 2), S(16-18), L, *Real Analysis*. Gabriel Klambauer. Am Elsev, 1973, xi + 436 pp, $15.95. An important addition to the collection of texts for first year graduate real variable courses. Problems in outline form give classical examples, applications, extensions of the theory. Many references to literature. Excellent for reference. Lebesgue measure, measurable functions, and integral on the real line; function spaces; differentiation and absolute continuity; abstract measure and integration; topological and metric spaces; Stone-Daniell integral; normed linear spaces. RBK

REAL ANALYSIS, P, *Lecture Notes in Mathematics-232: Tauberian Remainder Theorems.* Tord H. Ganelius. Springer-Verlag, 1971, vi + 75 pp, \$4.50 (P). Application of Wiener's general method to different kinds of tauberian problems. Classical hard analysis and distributional methods. RBK

REAL ANALYSIS, P, *Approximation Methods in Analysis.* Friedrich Stummel. Lect. Notes Series, No. 35. Aarhus U, 1973, 115 pp, (P). Generalizations of the notion of continuous convergence ($x_n \to x_o \Rightarrow f(x_n) \to f(x_o)$) with applications to approximations of differential and integral equations. Does not appear deep. Translation of known approximation methods into author's terminology. RBK

REAL ANALYSIS, T(14: 1), *Introduction to Mathematical Analysis.* C.R.J. Clapham. Routledge & Kegan, 1973, vii + 83 pp, \$2.25 (P). A nice short well written treatment of the first principles of real analysis, from a clear treatment of the completeness axioms to sequences, power series, continuity, derivatives and integrals. Appropriate to follow "intuitive calculus". 15-20 lectures worth. Attractive price as well. TAV

COMPLEX ANALYSIS, T**(17: 2), *Functions of One Complex Variable.* John B. Conway. Grad. Texts in Math-11. Springer-Verlag, 1973, xi + 313 pp, \$15.20 (P). An important text. Requires ε-δ background and knowledge of partials only. A nice feature: much of the theory of analytic functions is developed in an easy fashion from f(z) analytic if continuously differentiable. Covers through Picard's theorems. A full, intensive treatment with little motivational material and a minimum of exercises. The author views complex analysis as "an introduction to mathematics." TAV

COMPLEX ANALYSIS, T(18), P*, *Conformal Invariants: Topics in Geometric Function Theory.* Lars V. Ahlfors. McGraw, 1973, vii + 157 pp, \$10.95. Covers non-Euclidean metric, capacity, harmonic measure, extremal length (in detail), univalent functions, Schiffer's variational methods. As is his style, the author writes in 157 pages what others might in 250. The last two chapters on Riemann surfaces demonstrate the utility of this approach in development of the uniformisation theorem. TAV

COMPLEX ANALYSIS, P, *Elliptic Functions and Elliptic Curves.* Patrick DuVal. London Math. Soc. Lect. Notes, V. 9. Cambridge U Pr, 1973, v + 248 pp, \$9.95 (P). Theory of elliptic functions beginning with series for Weierstrass functions. Classical identities and properties are developed. Invariants and the modular function, Jacobi functions, inversions of integrals of first and second kind, trigonometric expansions, Theta functions, and more. Second part (chapters 11-13) gives general and explicit information on parameterizations of elliptic curves, the plane cubic, elliptic and quartic curves. Classical algebraic geometry is assumed.. RBK

COMPLEX ANALYSIS, T(18: 1), P, *Topics in Complex Function Theory, V. III.* C.L. Siegel. Transl: E. Gottschling, M. Tretkoff. Tracts in Pure and Appl. Math., V. 25. Wiley, 1973, ix + 224 pp, \$17.50. The third and final volume based on lectures given at Göttingen (TR, V.I, January 1970; V. II, February 1972). Topics: Abelian functions and modular functions on several variables. Such functions constitute the most important and best understood classes of analytic functions

of several complex variables. An extensive bibliography (800 entries) includes recent research and historically relevant works. TAV

DIFFERENTIAL EQUATIONS, P, *Fourier Integral Operators.* J.J. Duistermaat. Courant Inst, 1973, 190 pp, \$4.75 (P). Fourier integral operators generalize the Fourier integral representation of fundamental solutions to partial differential equations. Assumes theory of distributions. Review of classical theory of symplectic differential geometry which is regarded as basic for understanding of nonlinear partial differential equations, variational calculus, and classical mechanics. Serves as preliminary to invariant theory of Fourier integral operators. RBK

DIFFERENTIAL EQUATIONS, P, *Nonlinear Almost Periodic Oscillations.* M.A. Krasnosel'skii, *et al.* Transl: A. Libin. Wiley, 1973, x + 326 pp, \$36. Uses the methods of integral equations with monotone and concave operators to study global problems concerning nonlinear almost periodic oscillations and the relevant branching theory. An excellent 46-page appendix summarizes the book's methods and results, and should be read first. Note price. DFA

DIFFERENTIAL EQUATIONS, P, *Linear Differential Equations in Banach Space.* S.G. Krein. Transl. Math. Mono., V. 29. AMS, 1971, v + 390 pp, \$24.20. The theory of linear differential equations in Banach space with unbounded coefficients as developed by Hille, Yosida, Feller, Phillips, Lax, and Kato. Investigates correct statement of problem and asymptotic and approximate methods for first order equations with constant and nonconstant operators, and for certain second order equations. Bibliography of 270 references. RBK

DIFFERENTIAL EQUATIONS, P, *Introduction to the Theory and Application of Differential Equations with Deviating Arguments.* L.E. El'sgol'ts, S.B. Norkin. Transl: John L. Casti. Acad Pr, 1973, xiv + 357 pp, \$14.50. Updated version of the first author's 1964 work of the same title. Basic concepts and existence theorems, linear equations, stability theory, periodic solutions, stochastic differential equations with retarded argument, approximate methods. Over 650 references. For mathematicians, physicists, engineers. DFA

DIFFERENTIAL EQUATIONS, T*(15-16: 1, 2), L, *The Analysis and Solution of Partial Differential Equations.* Robert L. Street. Brooks/Cole, 1973, xi + 458 pp, \$14.95. Useful for many types of first courses. Later chapters concern Green's functions, integral transforms, characteristics, numerical finite difference methods. Rigorous. Quantitative and qualitative. Multitude of problems, including many requiring formulation. An attractive, carefully written book which requires students to work hard. DFA

DIFFERENTIAL EQUATIONS, T(17-18: 1, 2), L, *Linear Methods of Applied Analysis.* Allan M. Krall. A-W, 1973, xiv + 706 pp, \$9.50 (P). For a variety of "applicable mathematics" courses. Theory of ordinary differential equations (using the contraction mapping theorem), the Stone-Weierstrass Theorem, regular and singular Sturm-Liouville problems (in a Hilbert space setting), classical second-order partial differential equations of mathematical physics (using distributions). Applications supply motivation, but the concern both in text and problems sets is with the mathematics. DFA

DIFFERENTIAL EQUATIONS, P. *Hyperbolic Systems of Conservation Laws and the Mathematical Theory of Shock Waves*. Peter D. Lax. CBMS Reg. Conf. in Math., No. 11. SIAM, 1973, v + 48 pp, $4.80 (P). First order quasi-linear hyperbolic systems. Studies existence and uniqueness of generalized solutions to an initial value problem subject to entropy conditions, and the resulting dissipation. Brief introduction to numerical methods, but no numerical results. DFA

DIFFERENTIAL EQUATIONS, S(14). *Differential Equations and Circuits, Draft Edition*. SMP Further Mathematics, V. III. Cambridge U Pr, 1971, x + 178 pp, $3.95 (P). The third of a series on applications of mathematics: Boolean algebra, applications to computer circuitry, analogue computing, network analysis, analytical methods for differential equations, complex number methods, and the Laplace transform. LLK

DIFFERENTIAL EQUATIONS, T(14: 2). *Ordinary Differential Equations: A First Course, Second Edition*. Fred Brauer, John A. Nohel. Benjamin, 1973, ix + 470 pp, $11.95. Chapters 1 and 2 of the first edition (TR, June-July 1970) have been completely rewritten. Chapters 4 and 5 are new, with the arrangement such that Chapter 3 or 4 and 5 can be used depending upon backgrounds with or without linear algebra. LLK

DIFFERENTIAL EQUATIONS, P. *Partial Differential Equations*. Ed: D.C. Spencer. Proc. of Symp. in Pure Math., V. XXIII. AMS, 1973, vii + 505 pp, $33.20. Proceedings (revised papers) of the AMS summer research institute at Berkeley in August 1971. JAS

NUMERICAL ANALYSIS, S(16-17), P*. *Approximate Linear Algebraic Equations*. I.B. Kuperman. Van-N-Rein, 1971, xi + 225 pp, $18.95. Several different methods for linear systems in which each coefficient and constant is given by a midpoint and interval half-length. Solutions are obtained in terms of the solution at the midpoint values and an uncertainty for each variable, i.e., the maximum deviation from the midpoint solution. Methods use norms, linearizations, modulus analysis, interval arithmetic, linear programming and statistics. Many original results. Very readable style. Few references. RWN

NUMERICAL ANALYSIS, T(16: 1). *Numerical Analysis*. A.M. Cohen, *et al.* Wiley, 1973, xi + 341 pp, $10.75 (P). Practical, not necessarily simple, methods for interpolation, approximation, root-finding, integration, linear equations, eigenvalues, ode's, pde's and optimization. Written by several authors, the book varies in style, rigor and level although the development from simple examples to important methods is consistent. Some concern for errors. Very good problems. RWN

NUMERICAL ANALYSIS, P. *The Mathematics of Finite Elements and Applications*. Ed: J.R. Whiteman. Acad Pr, 1973, xiii + 520 pp, $35. Papers from an April 1972 conference at Brumel U. in England. LAS

NUMERICAL ANALYSIS, P. *On Approximation Theory*. Ed: P.L. Butzer, J. Korevaar. Birkhauser, 1972, xvi + 261 pp, $13. A second printing of the proceedings of the August 1963 conference at Oberwolfach commemorating de La Vallée Poussin, who died March 2, 1962. Contains an essay on his life and work written specially for this volume by J. Favard. JAS

NUMERICAL ANALYSIS, P, *Numerische Methoden bei Optimierungsaufgaben.* L. Collatz, W. Wetterling. Birkhauser, 1973, 136 pp. Eleven papers presented at the November 1971 conference at Oberwolfach. JAS

NUMERICAL ANALYSIS, P, *Spline Functions and Approximation Theory.* Ed: A. Meir, A. Sharma. Birkhauser, 1973, 386 pp. Proceedings of a symposium held at the University of Alberta, Edmonton, Canada, May 29 to June 1, 1972. JAS

FUNCTIONAL ANALYSIS, T(17: 1), S(16-18), L, *Spectral Theory of Operators in Hilbert Space.* K.O. Friedrichs. Appl. Math. Sci., V. 9. Springer-Verlag, 1973, viii + 244 pp, $6.50 (P). An expository work by a master of the theory. The spectral analysis of a linear operator is introduced by generalizing from three problems: diagonalization of quadratic forms, representation by Fourier series, and representation by Fourier integrals. Spectral Representation; Norm and Inner Product; Hilbert Space; Bounded Operators; Operators with Discrete Spectra; Non-Bounded Operators; Differential Operators; Perturbation of Spectra. RBK

FUNCTIONAL ANALYSIS, P, *Meromorphic Operator Valued Functions.* H. Bart. Math. Centre Tracts, No. 44. Mathematisch Centrum, 1973, 126 pp, Dfl. 14,00 (P). Results on the Laurent coefficients and global relative inverses of degenerate and semi-Fredholm meromorphic operator valued functions, and a characterization of poles of the resolvent of an arbitrary holomorphic operator valued function. RBK

FUNCTIONAL ANALYSIS, T*(18), L, *Integral Equations.* Harry Hochstadt. Wiley, 1973, viii + 282 pp, $16.95. Mixes classical and functional analytic techniques, using the latter in Hilbert space. Basic existence theorems, equations with L_2 kernels (via compact operators), applications to partial differential equations, Fourier and other transforms and the projection method, Fredholm theory, the Schauder fixed-point theorem and nonlinear equations. Many examples and exercises. DFA

FUNCTIONAL ANALYSIS, P, *Numerical Ranges II.* F.F. Bonsall, J. Duncan. London Math. Society Lect. Note Ser., No. 10. Cambridge U Pr, 1973, vii + 179 pp, $7.50 (P). Sequel to *Numerical Ranges of Operators on Normed Spaces and of Elements of Normed Algebras* (TR, December 1971), also in this series. Reflects recent developments. Largely concerns spatial and algebraic numerical ranges, but touches on essential and joint numerical ranges and matrix ranges. Applications to initial value problems are not treated. DFA

FUNCTIONAL ANALYSIS, S(16-18), P, L, *Lecture Notes in Mathematics-338: Classical Banach Spaces.* Joram Lindenstrauss, Lior Tzafriri. Springer-Verlag, 1973, ix + 243 pp, $9.10 (P). A valuable addition to the literature on the structure of particular Banach spaces. Contains new results and open questions for research. Bibliography of 214 classical and recent references. Part I Sequence Spaces: Shauder bases; the spaces l_p and c_0; symmetric bases; Orlitz sequence spaces. Part II Function Spaces: Banach lattices; lattice characterizations; the L_p-spaces; the C(K)-spaces and preduals of L_1-spaces; the $\mathcal{L}_p$-spaces. References. RBK

OPTIMIZATION, T(16-17), P, *Applied Nonlinear Programming.* David M. Himmelblau. McGraw, 1972, xi + 498 pp, $18.50. A good senior or first year graduate text. Two main sections: unconstrained maximization, and constrained maximization. Unconstrained covers many different methods, two main subsections with and without derivatives. Exercises, good references (both by footnote and collected at the end of each chapter). Geometric interpretations encouraged where applicable. PJM

OPTIMIZATION, P*, *Introduction to Optimization Practice.* Lucas Pun. Wiley, 1969, x + 309 pp, $16. An excellent book for self study. The only lack is exercises. After an introductory chapter, the subject is divided into three topics: static and dynamic problems, where the model is known (Chapters 2 and 4) and not fully known (Chapter 3). Chapter 5 discusses methods of optimizing computational tasks along the way. Plentiful examples, many taken from real life situations. Recommended for teaching yourself the subject. For a course, a supplementary list of exercises may be needed. PJM

OPTIMIZATION, P, *Lecture Notes in Economics and Mathematical Systems-79: Cones, Matrices and Mathematical Programming.* A. Berman. Springer-Verlag, 1973, v + 96 pp, $6 (P). A brief survey of results pertaining to recent applications of the theory of cones, particularly the complementarity problem and iterative methods for singular systems. LCL

OPTIMIZATION, T(15: 1), *Dynamic Programming (with Management Applications).* N.A.J. Hastings. Crane, Russak, 1973, 173 pp, $11.50. Intended for the beginner, this text develops techniques for solving a variety of problems. Emphasis is given to Markov finite and infinite programming with some semi-Markov cases as well. The style is a bit terse, but a useful book for the careful reader. TAV

OPTIMIZATION, T(15: 1), S, *Queueing Theory in OR.* E. Page. Crane, Russak, 1972, 187 pp, $11. An introductory treatment aimed at the OR student. Some calculus and probability required. Extreme situations treated first, then more general queues. A brief discussion of economic considerations in server design. Many examples, few exercises. TAV

OPTIMIZATION, T*(16), S, L, *Communication Nets, Stochastic Message, Flow and Delay.* Leonard Kleinrock. Dover, 1972, xi + 209 pp, $3 (P). A very readable treatment of the optimization methods used in message handling in a communications net. Topics include topology of nets, priority handling, queue behavior, random routing procedures. A worthwhile overview of the area, with extensive bibliographic references for further study. TAV

ANALYSIS, L, *Tables of Laplace Transforms.* Fritz Oberhettinger, Larry Badii. Springer-Verlag, 1973, vii + 428 pp, $16 (P). An extensive list of transforms and inverse transforms including most standard functions. Appendices on notations and lists of functions. TAV

ANALYSIS, L, *Fourier Expansions; A Collection of Formulas.* Fritz Oberhettinger. Acad Pr, 1973, xi + 64 pp, $11. After a very brief (6 pp) introduction to Fourier and Fourier-Bessel series and transforms, an extensive list of series is given for increasingly complex functions and coefficients. TAV

ANALYSIS, T(18), L. *Summability Theory and Its Applications.* R.E. Powell, S.M. Shah. Van-N-Rein, 1972, ix + 178 pp, $9.95. Classical approach. Includes general theory of matrix transformations (and the Silverman-Toeplitz theorem), standard methods (Nörlund, Hölder, Cesàro, Euler, Borel, Taylor, Hausdorff means) and Tauberian theorems (but not those of Wiener). Applications to Fourier series and Fourier transforms and to analytic continuation. More exercises would be welcome. DFA

ANALYSIS, T(17: 1), S(16-17), L. *An Introduction to the Theory of Distributions.* José Barros-Neto. Dekker, 1973, ix + 221 pp, $14.50. Clearly written and thorough introduction to the theory of distributions--prerequisite topological vector space theory, definitions, structure theory, Fourier transforms (including Paley-Wiener-Schwarz Theorem), K^2 theory of Sobolev spaces, applications (including Malgrange's original proof of the existence of fundamental solutions of partial differential equations with constant coefficients). Advanced calculus, metric space topology, Lebesgue integration, and some Banach space theory is assumed. Basically for graduate study, but much is accessible to advanced undergraduates. Good problems.RBK

ANALYSIS, P. *Asymptotic Expansions: Their Derivation and Interpretation.* R.B. Dingle. Acad Pr, 1973, xv + 521 pp, £13.50. Origin, nature, derivation, expression for the general late term, and interpretation beyond their least term of asymptotic expansions of various types--power, large-order, transitional and uniform--which can be derived from convergent series, integral representations, and second-order linear ordinary differential equations. Exercises, tables. Attempts primarily to be heuristic and descriptive. DFA

ANALYSIS, T(17-18: 2), S, P**, L. *Linear Analysis and Representation Theory.* Steven A. Gaal. Grund. math. Wissenschaften, B. 198. Springer-Verlag, 1973, ix + 688 pp, $50.90. A highly readable reference volume as well as a systematic introduction to topics in modern functional analysis, harmonic analysis and representation theory in Hilbert space. Will become a classic in its field. LCL

GEOMETRY, P. *Strong Rigidity of Locally Symmetric Spaces.* G.D. Mostow. Princeton U Pr, 1973, v + 195 pp, $7 (P). Two compact Riemann surfaces can have isomorphic fundamental groups and not be analytically equivalent. Essentially, except for factors of this type, a semi-simple analytic group or equivalently, a locally symmetric space is determined by its fundamental group. Monograph gives proof along with several formulations of the result. RBK

GEOMETRY, T*(14), S*, L. *An Introduction to Non-Euclidean Geometry.* David Gans. Acad Pr, 1973, xii + 274 pp, $10.95. A much needed and very readable presentation for the beginner. Hyperbolic geometry is developed on the basis of Euclid's first four postulates plus Saccheri's hypothesis of the acute angle. A brief discussion of the geometry of a sphere and modified hemisphere introduces the axiomatic development of the two elliptic geometries. JNC

GEOMETRY, S(18), P. *Proceedings of the International Conference on Projective Planes.* Ed: M.J. Kallaher, T.G. Ostrom. Washington St U Pr, 1973, vii + 287 pp, $8 (P).

TOPOLOGY, P, *Canonical Differential Operators and Lower-order Symbols*. Robert John Victor Jackson. Mem. of AMS, No. 135. AMS, 1973, viii + 235 pp, \$3.90 (P). Study of global invariants associated to classical pseudo-differential operators that act upon sections of the complexified 1/2-density bundle over a manifold, and of various canonical differential operators on this bundle. Uses the algebraic relations arising among the jet bundles of the 1/2-density bundle and the tangent bundle. DFA

TOPOLOGY, P, *The Theory of Analytic Spaces*. J. Hoffmann-Jørgensen. Various Pub. Series, No. 10. Aarhus U, 1970, vi + 314 pp, \$5.75 (P). Analytic spaces are Hausdorff images of Polish spaces, spaces which have a compatible metric with respect to which they are separable and complete. They include more important examples of topological measure spaces and exclude most pathological examples. Prerequisites: measure theory, functional analysis of locally convex spaces. RBK

PROBABILITY, P, *Probabilistic Methods in Applied Mathematics, V. 3*. Ed: A.T. Bharucha-Reid. Acad Pr, 1973, xi + 346 pp, \$32. The third of an irregularly appearing series of volumes; the applications are principally to physics, engineering and game theory. JAS

PROBABILITY, T(18: 2), P, *Stochastic Processes and the Wiener Integral*. J. Yeh. Dekker, 1973, viii + 551 pp, \$24.75. Requires a strong background in real analysis and probability theory. Much is derived from the fact that a stochastic process on an arbitrary probability space induces a probability measure in the space of all real valued functions. Depends heavily on the works of R.H. Cameron and J.L. Doob. TAV

PROBABILITY, P, *Stochastic Differential Equations*. Ed: Joseph B. Keller, Henry P. McKean. SIAM-AMS Proc., V. VI. AMS, 1973, v + 209 pp, \$17.90. Eight "extremely varied" papers from the March, 1972 symposium on applied mathematics held in New York City. LAS

PROBABILITY, P, *Selected Translations in Mathematical Statistics and Probability. V. 13*. AMS, 1973, v + 298 pp, \$26.40. 20 papers mostly on stochastic, Markov and renewal processes. LAS

PROBABILITY, P, L, *Stochastic Analysis: A Tribute to the Memory of Rollo Davidson*. Ed: D.G. Kendall, E.F. Harding. Wiley, 1973, xiii + 465 pp, \$29.95. Following a specially written *Introduction to Stochastic Analysis* by D.G. Kendall are 25 sections--some reprints, some original--providing a well organized introduction to the field. LAS

PROBABILITY, P, *Lecture Notes in Economics and Mathematical Systems-87: Approximate Stochastic Behavior of n-Server Service Systems with Large n*. G.F. Newell. Springer-Verlag, 1973, v + 118 pp, \$7.20 (P). An account, basically descriptive, of the behavior of queues in the presence of a large number of servers. The author shows how this differs dramatically from the case where the number is small compared to the queue lengths. Asymptotic results and transition estimates are presented. TAV

STATISTICS, P, *Asymptotic Theory of Rank Tests for Independence*. F. H. Ruymgaart. Math. Centre Tracts-43. Mathematisch Centrum, 1973, iii + 115 pp, Dfl. 12 (P).

STATISTICS, T(13-14: 1), *Elementary Statistics in Social Research.* Jack Levin. Har-Row, 1973, viii + 279 pp, $7.95. An introductory text for social science students. No calculus. Begins with descriptive statistics, proceeds to hypothesis testing, chi-square, some analysis of variance, regression. Numerous appropriate examples and exercises taken primarily from sociology and psychology. TAV

STATISTICS, T(15-17: 2), *Social Statistics, Second Edition.* Hubert M. Blalock, Jr. McGraw, 1972, xiv + 583 pp, $10.95. For the advanced social science student. No calculus prerequisite. Both parametric and non-parametric tests. A very comprehensive text covering most methods used by applied statisticians, through multiple regression and analysis of covariance. Examples restricted to sociology. Extensive tables. TAV

STATISTICS, S, *Errors of Observation and Their Treatment, Third Edition.* J. Topping. Chapman, 1971, 119 pp, $2.50 (P). An odd little book. Discusses the treatment of experimental errors, effects of averaging, significant digits, some statistical methods. TAV

STATISTICS, T(13: 1), *Significance Tests.* Evelyn Caulcott. Routledge, 1973, vii + 145 pp, $8.75. Presupposes high school algebra and descriptive statistics. Compact, non-mathematical and somewhat imprecise treatment of standard topics. No Bayesian methods. FLW

STATISTICS, T(13: 1), *Understanding Statistics.* William Mendenhall, Lyman Ott. Duxbury Pr, 1972, viii + 310 pp, $9. Designed for an "appreciation of statistics" course. Presents the concepts of statistical inference, including the analysis of variance, without any of the mathematics of probability. Concludes with a short chapter on "Lying with Statistics." RSK

STATISTICS, T*(13: 2), *Modern Elementary Statistics, Fourth Edition.* John E. Freund. P-H, 1973, xii + 532 pp, $11.95. Expanded version of the 1967 *Third Edition*, with many new illustrations and exercises. First half is devoted to descriptive methods and basic theory; last half to statistical inference. A sound well-written text. RSK

STATISTICS, T(14: 1, 2), *Elementary Statistical Methods.* G. Barrie Wetherill. Chapman, 1972, xiii + 346 pp, $6.50 (P). Soft cover edition of the 1967 British text (TR, June-July 1968). Coverage is standard, through multiple regression and analysis of variance. RSK

STATISTICS, T*(13: 1), S, *Introduction to Biostatistics.* Robert R. Sokal, F. James Rohlf. Freeman, 1973, xiii + 368 pp, $9.50. Briefer version of the authors' comprehensive 1969 book *Biometry* (TR, May 1970). Introduces analysis of variance early and devotes much space to it. Classical approach; no Bayesian methods. Strongest feature is its use of realistic examples and problems to illustrate practical applications to biology. Tables are included. RSK

STATISTICS, S(17-18), P, *Mathematische Theorie statistischer Experimente.* H. Heyer. Springer-Verlag, 1973, xxii + 209 pp, $8.20 (P). Mathematical statistics for students who have already been introduced to the subject, and who also know some probability and measure theory. Concentrates on the theory of estimation and hypothesis testing in finite spaces. No problems. JD-B

STATISTICS, T(13-14: 1, 2), S. *Introductory Statistics with FORTRAN.* Allan M. Kirch. HR&W, 1973, xv + 458 pp, $11.50. Presupposes only high school algebra. Includes a gradual introduction to FORTRAN and self-teaching appendices on computing. No F-distribution techniques and no Bayesian methods. FLW

STATISTICS, S(16-17), P. *Sequential Analysis and Optimal Design.* Herman Chernoff. CBMS Reg. Conf. in Appl. Math., No. 8. SIAM, 1972, v + 119 pp, $7.20 (P). Lectures given at the Regional Conference sponsored by C.B.M.S. at New Mexico State University in 1972. FLW

STATISTICS, T(13: 1). *Elements of Statistical Inference, Third Edition.* David V. Huntsberger, Patrick Billingsley. Allyn, 1973, ix + 349 pp, $11.50. An attractive and sound treatment presupposing only high school algebra. This edition has a chapter on non-parametric methods. FLW

STATISTICS, T(13: 1, 2). *Fundamental Statistics in Psychology and Education, Fifth Edition.* J.P. Guilford, Benjamin Fruchter. McGraw, 1973, xii + 546 pp, $11.95. Covers topics of interest in the indicated fields, presupposing only high school algebra and building on a cursory and somewhat misleading treatment of probability. Includes some non-parametric methods and some experimental design, but no Bayesian methods. (Pages 53 to 84 were missing in the reviewer's copy.) FLW

STATISTICS, T(15-16: 1-3), S. *Statistical Methods for the Process Industries.* Maurice H. Belz. Wiley, 1973, xxiv + 706 pp, $27.50. Presupposes calculus. Many examples from industry. Includes distribution-free methods, quality control, factorical designs, and response surface methodologies. No Bayesian methods. FLW

STATISTICS, T(13-14: 1). *Introductory Statistics: A Service Course.* A.H. Pollard. Pergamon Pr, 1972, 262 pp, $3.95 (P). Presupposes only high school algebra. Little development of probability. No Bayesian methods. FLW

STATISTICS, S(14-16), P. *Methods of Multivariate Analysis: With Handbook of Multivariate Methods Programmed in Atlas Autocode.* Keith Hope. Gordon, 1968, 288 pp, $24.50. Discusses, without proofs, the interpretations of standard multivariate methods that assume normality and homogeneity of variances. Emphasizes geometric interpretations of the results of matrix operations. The last 100 pages describe the use of various canned programs. FLW

STATISTICS, P. *Multivariate Analysis-III.* Ed: Paruchuri R. Krishnaiah. Acad Pr, 1973, xvii + 410 pp, $36. 27 invited papers from the third International Symposium at Wright State U., Dayton, Ohio, June 1972. (TR, V. I, December 1967; V. II, January 1970.) LAS

COMPUTER SCIENCE, T(15: 1), S, L. *Digital Interface Design.* D. Zissos, F.G. Duncan. Oxford U Pr, 1973, ix + 174 pp, $13. Background in hardware, Boolean algebra and logical design. Basic computer operations and machine level programming. Interfacing to peripherals of several different types. Use of flags, flag sorters, and programmed and autonomous data transfers. Emphasizes simplicity, reliability and economy. RWN

COMPUTER SCIENCE, S(13). *Problem Solving and Flowcharting.* Ronald E. Elliott. Reston, 1972, vii + 98 pp, $4.95; $2.95 (P). Most students would pick this up within a couple of class periods, but if not, here it is in eight detailed chapters. LCL

COMPUTER SCIENCE, T(16-17: 1, 2), S, L. *Formal Languages.* Arto Salomaa. Acad Pr, 1973, xiii + 322 pp, $19. Self-contained and reasonably complete. Starts with rewriting systems and grammars, develops languages, then considers solvability and complexity. Automata included as recognition devices. Fairly mathematical, without particular consideration for natural and programming languages. Good examples and exercises. RWN

COMPUTER SCIENCE, S, P*. *Computer System Organization: The B5700/B6700 Series.* Elliott I. Organick. Acad Pr, 1973, ix + 132 pp, $8.95. An objective study of one of the most innovative systems. Explains the design and use of stacks, virtual memory, tasking, paging, interrupts, and storage control strategies. Recommended reading for anyone interested in systems. RWN

COMPUTER SCIENCE, T(13: 1). *Introduction to Computer Science Mathematics.* Robert V. Jamison. McGraw, 1973, viii + 273 pp, $7.95. Introductory. Prerequisite: high school algebra. Discusses decimal and binary numbers; linear, polynomial, exponential and log functions; Fortran, excluding subroutines; and boolean algebra. Exercises and answers. Index. RWN

COMPUTER SCIENCE, S(16-18), P, L. *Discrete Models.* Donald Greenspan. A-W, 1973, xv + 165 pp, $8.50 (P). Discrete models for phenomena in Newtonian mechanics: oscillation, nonlinear string vibrations, planetary motion, 3- and n-body problems, motion of fluids. Also: discrete special relativity. Nonspecialist will find it very readable. Concludes with a list of research problems. DFA

SYSTEMS THEORY, P. *Dynamical Systems.* Ed: M.M. Peixoto. Acad Pr, 1973, xiv + 745 pp, $39. Proceedings of an August, 1971 symposium held at U. Bahia, Salvador, Brasil with series of special lectures by J. Mather, S. Smale, R. Thom and E.C. Zeeman. Includes several excellent applications of contemporary mathematics. LAS

APPLICATIONS (BIOLOGY), P*, L. *Mathematical Problems in the Biological Sciences.* Sol. I. Rubinow. CBMS Reg. Conf. in Appl. Math., No. 10. SIAM, 1973, vii + 90 pp, $5.50 (P). Contributions by a theoretical physicist/applied mathematician. "Because of its soft character, biological data cannot support sophisticated mathematical models...My own view is that such attempts must be considered at the present time to be exercises in mathematical speculation." (Introduction, p. vii.) Two lectures each: circulatory system and flow of blood, tracer analysis of physiological systems, enzyme kinetics, cell populations, diffusions in biology. PJC

APPLICATIONS (BIOLOGY), P. *Quantitative Cellular Biology: An Approach to the Quantitative Analysis of Life Processes.* F. Heinmets. Dekker, 1970, ix + 327 pp, $17.50. Introduction to methods of designing models of processes in cell biology, and analyzing them via analog computers. Main topics: enzyme synthesis, other cellular processes (including growth, reproduction). Lots of graphs and figures. PJC

APPLICATIONS (BIOLOGY), P, *Automaton Theory and Modeling of Biological Systems*. M.L. Tsetlin. Math. in Sci. and Eng., V. 102. Acad Pr, 1973, xvi + 288 pp, $23.50. Collection of the author's papers. Cooperative interaction of collections of automata--closely akin to Western theories of games, learning machines, perceptrons. Games automata play (no-information, zero-sum), featuring the Goore game. Bioelectric control, motor control, operation of the central nervous system. Includes author's thesis and excellent popular talk to physiologists based on it ("Mathematical Modeling of the Simplest Forms of [Collective] Behavior"). PJC

APPLICATIONS (INFORMATION THEORY), P, *Lecture Notes in Economics and Mathematical Systems-78: Gesellschaft für Informatik e. V. 2. Jahrestagung*. Ed: M. Beckmann, *et al*. Springer-Verlag, 1973, xi + 576 pp, $11.50 (P). Proceedings of the conference in Karlsruhe, October 1972. A wide variety of papers varing from Turing-machines to dynamic document rooms. JAS

APPLICATIONS (SIMULATION), T(15-16), S, P, L, *Simulation of Discrete Stochastic Systems*. Herbert Maisel, Giuliano Gnugnoli. SRA, 1972, xiv + 465 pp, $12.50. Modelling. Statistical review. GPSS. A very detailed case study of a simulation of the operation of social security district offices. RWN

APPLICATIONS (SIMULATION), S**, L*, *The Settlement of Polynesia: A Computer Simulation*. Michael Levison, *et al*. U Minn Pr, 1973, viii + 137 pp, $10.75. Intriguing, easy to understand, and thoroughly well-done demonstration of the techniques and value of mathematical modelling. The problem: how could such a far-flung area come to be settled by a single people? Among the results: the settlement of Hawaii, Easter Island, and New Zealand cannot be satisfactorily explained except by intentionally navigated (as opposed to drift) voyages. PJC

APPLICATIONS (SOCIAL SCIENCE), S(15-17), P*, L**, *The Theory of Social Choice*. Peter C. Fishburn. Princeton U Pr, 1972, xii + 264 pp, $13.50. A thorough, well-organized survey of the literature generated by Kenneth Arrow's now-classic theorem on voting--that reasonable desiderata for procedures to convert individual preferences into transitive social preferences are inconsistent. An excellent application of mathematical techniques to a fundamental problem of democracy. LAS

APPLICATIONS (TRAFFIC FLOW), P, *Traffic Flow and Transportation*. Ed: Gordon F. Newell. Am Elsev, 1972, xiv + 453 pp, $20. Proceedings of the fifth international symposium held at Berkeley in June, 1971. 30 papers ranging widely from theory to application. LAS

Reviewers Whose Initials Appear Above

David F. Appleyard, Carleton; Paul J. Campbell, St. Olaf; Judith N. Cederberg, St. Olaf; John Dyer-Bennet, Carleton; Lorraine L. Keller, St. Olaf; Roger B. Kirchner, Carleton; Richard S. Kleber, St. Olaf; Loren C. Larson, St. Olaf; Pierre J. Malraison, Carleton; R.W. Nau, Carleton; J. Arthur Seebach, Jr., St. Olaf; Lynn A. Steen, St. Olaf; T.A. Vessey, St. Olaf; Frank L. Wolf, Carleton.

NEWS AND NOTICES

EDITED BY RAOUL HAILPERN, SUNY at Buffalo

Readers are invited to contribute to the general interst of this department by sending news items to Mathematical Association of America, 1225 Connecticut Avenue, NW, Washington, D. C. 20036. Items must be submitted at least two months before publication can take place.

PERSONAL ITEMS

Vanderbilt University: Drs. R. E. Heisey, Cornell University, and J. B. Nation, Caltech, have been appointed Visiting Assistant Professors; Associate Professor T. P. Whaley, Southwestern at Memphis, has been appointed Visiting Associate Professor; Assistant Professor G. F. Webb has been promoted to Associate Professor.

Assistant Professor A. K. Agarwal, Grambling College, has been promoted to Associate Professor.

Dr. W. B. Caton, Illinois Institute of Technology, has been named Associate Professor Emeritus.

Dr. K. K. Gorowara, Wright State University, has been appointed Chairman of the Mathematics Department.

Dr. Marshall Hall, Jr., California Institute of Technology, has been designated as the first IBM Professor of Mathematics.

Professor G. J. Minty, Indiana University, Bloomington, will be Senior Fellow of the Alexander von Humboldt Foundation and Guest Professor at the Universität Hamburg, West Germany, from September 1973 through September 1974.

Assistant Professor P. J. Murray, Westminster College, has been appointed Assistant Professor and Chairman of the Mathematics Department at St. Martin's College.

SYMPOSIUM ON FINITE ELEMENTS AND PARTIAL DIFFERENTIAL EQUATIONS

The Mathematics Research Center at the University of Wisconsin, Madison, will hold a symposium on finite elements and partial differential equations on April 1–3, 1974. The symposium will consist of 14 invited lectures dealing with the mathematical aspects of the use of finite elements in the numerical solution of partial differential equations.

The members of the program committee are C. de Boor (Chairman), J. Bramble, J. Douglas, Jr., J. Nitsche, and B. Noble. A detailed program of the symposium and information on registration and accommodations is available. Requests for the program and all related inquiries should be directed to Professor C. de Boor, Mathematics Research Center, University of Wisconsin-Madison, 610 Walnut Street, Madison, Wisconsin 53706.

MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

Erratum. The title of paper 17, presented by Professor J. T. B. Beard, Jr., at the March Meeting of the Southeastern Section (this MONTHLY, 80 (1973) 970) should read: *A Representation of* $\overline{GF}(p)$ *in* $(GF(p))_\infty$.

CALENDAR OF FUTURE MEETINGS

Summer Meeting 1974: There will be no joint summer meeting in 1974, in order that mathematicians may attend the International Congress of Mathematicians to be held in Vancouver, British Columbia, August 21–29, 1974.

Fifty-eighth Annual Meeting, Washington, D. C., January 25–27, 1975.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Editorial Director.

ALLEGHENY MOUNTAIN, Allegheny College, Meadville, Pennsylvania, May 3–4, 1974.

FLORIDA

ILLINOIS, Knox College, Galesburg, May 10–11, 1974.

INDIANA, Rose-Hulman Institute of Technology, Terre Haute, April 27, 1974.

IOWA, Upper Iowa College, Fayette, April 19, 1974.

KANSAS, Ottawa University, Ottawa, Spring 1974.

KENTUCKY

LOUISIANA-MISSISSIPPI

MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA

METROPOLITAN NEW YORK, College of Mount St. Vincent, Riverdale, April 28, 1974.

MICHIGAN, Central Michigan University, Mount Pleasant, May 3–4, 1974.

MISSOURI

NEBRASKA, University of South Dakota, Vermillion, April 19–20, 1974.

NEW JERSEY

NORTH CENTRAL, South Dakota State University, Brookings, April 27, 1974.

NORTHEASTERN, Lowell Technical Institute, Lowell, Massachusetts, November 30, 1974.

NORTHERN CALIFORNIA, Chabot College, Hayward, February 1975.

OHIO, Muskingum College, New Concord, May 3–4, 1974.

OKLAHOMA-ARKANSAS, University of Arkansas, Little Rock, April 5–6, 1974.

PACIFIC NORTHWEST, University of British Columbia, Vancouver, August 21–24, 1974 (business meeting only — no general meeting).

PHILADELPHIA

ROCKY MOUNTAIN, Colorado School of Mines, Golden, April 26–27, 1974.

SEAWAY, Union College, Schenectady, April 27, 1974.

SOUTHEASTERN

SOUTHERN CALIFORNIA

SOUTHWESTERN, New Mexico State University, Las Cruces, April 5–6, 1974.

TEXAS, University of Texas, Austin, April 5–6, 1974.

WISCONSIN, Marquette University, Milwaukee, May 3–4, 1974.

FUTURE MEETINGS OF OTHER ORGANIZATIONS

AMERICAN ASSOCIATION FOR THE ADVANCEMENT OF SCIENCE

AMERICAN MATHEMATICAL SOCIETY, Washington, D. C., January 23–26, 1975.

AMERICAN SOCIETY FOR ENGINEERING EDUCATION, Rensselaer Polytechnic Institute, Troy, June 17–20, 1974.

ASSOCIATION FOR COMPUTING MACHINERY, San Diego, California, November 11–13, 1974.

ASSOCIATION FOR SYMBOLIC LOGIC, Biltmore Hotel, New York City, April 12–13, 1974.

FIBONACCI ASSOCIATION

INSTITUTE OF MATHEMATICAL STATISTICS

MU ALPHA THETA, University of Arkansas, Fayetteville, August 4–7, 1974.

NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS, Atlantic City, New Jersey, April 17–20, 1974.

OPERATIONS RESEARCH SOCIETY OF AMERICA, Boston, April 22–24, 1974.

PI MU EPSILON, Western Michigan University, Kalamazoo, August 19–20, 1975.

SCHOOL SCIENCE AND MATHEMATICS ASSOCIATION, Sheraton-Gibson Hotel, Cincinnati, November 7–9, 1974.

SOCIETY FOR INDUSTRIAL AND APPLIED MATHEMATICS, Montana State University, Bozeman, June 24–26, 1974.


Let's Talk
About
Your Course

THE AMERICAN MATHEMATICAL MONTHLY

(FOUNDED IN 1894 BY BENJAMIN F. FINKEL)

THE OFFICIAL JOURNAL OF

THE MATHEMATICAL ASSOCIATION OF AMERICA

VOLUME 81 **NUMBER 4**

CODEN: AMMYAE

CONTENTS

(*Continued on inside cover*)

APRIL **1974**

NOTICE TO AUTHORS

Research papers per se are unsuitable; see statement of policy (Vol. 81, p. 1).

Please follow the format in current issues of the MONTHLY. Manuscripts must be legibly typewritten or reproduced from typewritten copy, double spaced with wide margins. Three copies should be submitted to the appropriate editor and one kept by the author as protection against loss. The author's full address *must* appear at the end of the manuscript.

Backlog: Main Articles 12 months, Math. Notes 27 months, Research Problems 15 months, Classroom Notes 23 months, Math. Education 17 months.

EDITORIAL CORRESPONDENCE AND MAIN ARTICLES: to ALEX ROSENBERG, Department of Mathematics, Cornell University, Ithaca, N.Y. 14850; NOTES, etc.: to the corresponding Associate Editor; ADVERTISING CORRESPONDENCE: to RAOUL HAILPERN, Mathematical Association of America, SUNY at Buffalo, Buffalo, N. Y. 14214; CHANGE OF ADDRESS and SUBSCRIPTIONS: to A. B WILLCOX, Mathematical Association of America, 1225 Connecticut Ave., N.W., Washington, D.C. 20036.

Annual dues for members of the Association (including a subscription to the American Mathematical Monthly) are $14.00 for each of the first two years of membership and $18.00 thereafter. Student Membership is available with annual dues of $9.00. For nonmembers the subscription price is $18.00.

PUBLISHED BY THE ASSOCIATION at Washington, D. C., and Menasha, Wisconsin, during the months of January, February, March, April, May, June–July, August–September, October, November, December.

Second-class postage paid at Washington, D. C., and additional mailing offices.

PRINTED IN THE UNITED STATES OF AMERICA

COMPUTER SCIENCE AND ITS RELATION TO MATHEMATICS

DONALD E. KNUTH

A new discipline called Computer Science has recently arrived on the scene at most of the world's universities. The present article gives a personal view of how this subject interacts with Mathematics, by discussing the similarities and differences between the two fields, and by examining some of the ways in which they help each other. A typical nontrivial problem is worked out in order to illustrate these interactions.

1. What is Computer Science? Since Computer Science is relatively new, I must begin by explaining what it is all about. At least, my wife tells me that she has to explain it whenever anyone asks her what I do, and I suppose most people today have a somewhat different perception of the field than mine. In fact, no two computer scientists will probably give the same definition; this is not surprising, since it is just as hard to find two mathematicians who give the same definition of Mathematics. Fortunately it has been fashionable in recent years to have an "identity crisis," so computer scientists have been right in style.

My favorite way to describe computer science is to say that it is the study of *algorithms*. An algorithm is a precisely-defined sequence of rules telling how to produce specified output information from given input information in a finite number of steps. A particular representation of an algorithm is called a program, just as we use the word "data" to stand for a particular representation of "information" [**14**]. Perhaps the most significant discovery generated by the advent of computers will turn out to be that algorithms, as objects of study, are extraordinarily rich in interesting properties; and furthermore, that an algorithmic point of view is a useful way to organize knowledge in general. G. E. Forsythe has observed that "the question 'What can be automated?' is one of the most inspiring philosophical and practical questions of contemporary civilization" [**8**].

From these remarks we might conclude that Computer Science should have existed long before the advent of computers. In a sense, it did; the subject is deeply rooted in history. For example, I recently found it interesting to study ancient manuscripts, learning to what extent the Babylonians of 3500 years ago were computer scientists [**16**]. But computers are really necessary before we can learn much about the general properties of algorithms; human beings are not precise enough nor fast enough to carry out any but the simplest procedures. Therefore the potential richness of algorithmic studies was not fully realized until general-purpose computing machines became available.

I should point out that computing machines (and algorithms) do not only compute with *numbers*; they can deal with information of any kind, once it is represented in a precise way. We used to say that a sequence of symbols, such as a name, is re-

presented inside a computer as if it were a number; but it is really more correct to say that a number is represented inside a computer as a sequence of symbols.

The French word for computer science is *Informatique*; the German is *Informatik*; and in Danish, the word is *Datalogi* [**21**]. All of these terms wisely imply that computer science deals with many things besides the solution to numerical equations. However, these names emphasize the "stuff" that algorithms manipulate (the information or data), instead of the algorithms themselves. The Norwegians at the University of Oslo have chosen a somewhat more appropriate designation for computer science, namely *Databehandling*; its English equivalent, "Data Processing" has unfortunately been used in America only in connection with business applications, while "Information Processing" tends to connote library applications. Several people have suggested the term "Computing Science" as superior to "Computer Science."

Of course, the search for a perfect name is somewhat pointless, since the underlying concepts are much more important than the name. It is perhaps significant, however, that these other names for computer science all de-emphasize the role of computing machines themselves, apparently in order to make the field more "legitimate" and respectable. Many people's opinion of a computing machine is, at best, that it is a necessary evil: a difficult tool to be used if other methods fail. Why should we give so much emphasis to teaching how to use computers, if they are merely valuable tools like (say) electron microscopes?

Computer scientists, knowing that computers are more than this, instinctively underplay the machine aspect when they are defending their new discipline. However, it is not necessary to be so self-conscious about machines; this has been aptly pointed out by Newell, Perlis, and Simon [**22**], who define computer science simply as the study of computers, just as botany is the study of plants, astronomy the study of stars, and so on. The phenomena surrounding computers are immensely varied and complex, requiring description and explanation; and, like electricity, these phenomena belong both to engineering and to science.

When I say that computer science is the study of algorithms, I am singling out only one of the "phenomena surrounding computers," so computer science actually includes more. I have emphasized algorithms because they are really the central core of the subject, the common denominator which underlies and unifies the different branches. It might happen that technology someday settles down, so that in say 25 years computing machines will be changing very little. There are no indications of such a stable technology in the near future, quite the contrary, but I believe that the study of algorithms will remain challenging and important even if the other phenomena of computers might someday be fully explored.

The reader interested in further discussions of the nature of computer science is referred to [**17**] and [**29**], in addition to the references cited above.

2. Is Computer Science Part of Mathematics? Certainly there are phenomena

about computers which are now being actively studied by computer scientists, and which are hardly mathematical. But if we restrict our attention to the study of algorithms, isn't this merely a branch of mathematics? After all, algorithms were studied primarily by mathematicians, if by anyone, before the days of computer science. Therefore one could argue that this central aspect of computer science is really part of mathematics.

However, I believe that a similar argument can be made for the proposition that mathematics is a part of computer science! Thus, by the definition of set equality, the subjects would be proved equal; or at least, by the Schröder-Bernstein theorem, they would be equipotent.

My own feeling is that neither of these set inclusions is valid. It is alwasy difficult to establish precise boundary lines between disciplines (compare, for example, the subjects of "physical chemistry" and "chemical physics"); but it is possible to distinguish essentially different points of view between mathematics and computer science.

The following true story is perhaps the best way to explain the distinction I have in mind. Some years ago I had just learned a mathematical theorem which implied that any two $n \times n$ matrices A and B of integers have a "greatest common right divisor" D. This means that D is a right divisor of A and of B, i.e., $A = A'D$ and $B = B'D$ for some integer matrices A' and B'; and that every common right divisor of A and B is a right divisor of D. So I wondered how to calculate the greatest common right divisor of two given matrices. A few days later I happened to be attending a conference where I met the mathematician H. B. Mann, and I felt that he would know how to solve this problem. I asked him, and he did indeed know the correct answer; but it was a mathematician's answer, not a computer scientist's answer! He said, "Let $\mathscr{R}$ be the ring of $n \times n$ integer matrices; in this ring, the sum of two principal left ideals is principal, so let D be such that

$$\mathscr{R}A + \mathscr{R}B = \mathscr{R}D.$$

Then D is the greatest common right divisor of A and B." This formula is certainly the simplest possible one, we need only eight symbols to write it down; and it relies on rigorously-proved theorems of mathematical algebra. But from the standpoint of a computer scientist, it is worthless, since it involves constructing the infinite sets $\mathscr{R}A$ and $\mathscr{R}B$, taking their sum, then searching through infinitely many matrices D such that this sum matches the infinite set $\mathscr{R}D$. I could not determine the greatest common divisor of $\left(\begin{smallmatrix}1 & 2\\ 3 & 4\end{smallmatrix}\right)$ and $\left(\begin{smallmatrix}4 & 3\\ 2 & 1\end{smallmatrix}\right)$ by doing such infinite operations. (Incidentally, a computer scientist's answer to this question was later supplied by my student Michael Fredman; see [**15**, p. 380].)

One of my mathematician friends told me he would be willing to recognize computer science as a worthwhile field of study, as soon as it contains 1000 deep theorems. This criterion should obviously be changed to include algorithms as

well as theorems, say 500 deep theorems and 500 deep algorithms. But even so it is clear that computer science today does not measure up to such a test, if "deep" means that a brilliant person would need many months to discover the theorem or the algorithm. Computer science is still too young for this; I can claim youth as a handicap. We still do not know the best way to describe algorithms, to understand them or to prove them correct, to invent them, or to analyze their behavior, although considerable progress is being made on all these fronts. The potential for "1000 deep results" is there, but only perhaps 50 have been discovered so far.

In order to describe the mutual impact of computer science and mathematics on each other, and their relative roles, I am therefore looking somewhat to the future, to the time when computer science is a bit more mature and sure of itself. Recent trends have made it possible to envision a day when computer science and mathematics will both exist as respected disciplines, serving analogous but different roles in a person's education. To quote George Forsythe again, "The most valuable acquisitions in a scientific or technical education are the general-purpose mental tools which remain serviceable for a lifetime. I rate natural language and mathematics as the most important of these tools, and computer science as a third" [**9**].

Like mathematics, computer science will be a subject which is considered basic to a general education. Like mathematics and other sciences, computer science will continue to be vaguely divided into two areas, which might be called "theoretical" and "applied." Like mathematics, computer science will be somewhat different from the other sciences, in that it deals with man-made laws which can be proved, instead of natural laws which are never known with certainty. Thus, the two subjects will be like each other in many ways. The difference is in the subject matter and approach — mathematics dealing more or less with theorems, infinite processes, static relationships, and computer science dealing more or less with algorithms, finitary constructions, dynamic relationships.

Many computer scientists have been doing mathematics, but many more mathematicians have been doing computer science in disguise. I have been impressed by numerous instances of mathematical theories which are really about particular algorithms; these theories are typically formulated in mathematical terms that are much more cumbersome and less natural than the equivalent algorithmic formulation today's computer scientist would use. For example, most of the content of a 35-page paper by Abraham Wald can be presented in about two pages when it is recast into algorithmic terms [**15**, pp. 142–144]; and numerous other examples can be given. But that is a subject for another paper.

3. Educational side-effects. A person well-trained in computer science knows how to deal with algorithms: how to construct them, manipulate them, understand them, analyze them. This knowledge prepares him for much more than writing good computer programs; it is a general-purpose mental tool which will be a definite aid to his understanding of other subjects, whether they be chemistry, linguistics,

or music, etc. The reason for this may be understood in the following way: It has often been said that a person does not really understand something until he teaches it to someone else. Actually a person does not *really* understand something until he can teach it to a *computer*, i.e., express it as an algorithm. "The automatic computer really *forces* that precision of thinking which is alleged to be a product of any study of mathematics" [7]. The attempt to formalize things as algorithms leads to a much deeper understanding than if we simply try to comprehend things in the traditional way.

Linguists thought they understood languages, until they tried to explain languages to computers; they soon discovered how much more remains to be learned. Many people have set up computer models of things, and have discovered that they learned more while setting up the model than while actually looking at the output of the eventual program.

For three years I taught a sophomore course in abstract algebra, for mathematics majors at Caltech, and the most difficult topic was always the study of "Jordan canonical form" for matrices. The third year I tried a new approach, by looking at the subject algorithmically, and suddenly it became quite clear. The same thing happened with the discussion of finite groups defined by generators and relations; and in another course, with the reduction theory of binary quadratic forms. By presenting the subject in terms of algorithms, the purpose and meaning of the mathematical theorems became transparent.

Later, while writing a book on computer arithmetic [**15**], I found that virtually every theorem in elementary number theory arises in a natural, motivated way in connection with the problem of making computers do high-speed numerical calculations. Therefore I believe that the traditional courses in elementary number theory might well be changed to adopt this point of view, adding a practical motivation to the already beautiful theory.

These examples and many more have convinced me of the pedagogic value of an algorithmic approach; it aids in the understanding of concepts of all kinds. I believe that a student who is properly trained in computer science is learning something which will implicitly help him cope with many other subjects; and therefore there will soon be good reason for saying that undergraduate computer science majors have received a good general education, just as we now believe this of undergraduate math majors. On the other hand, the present-day undergraduate courses in computer science are not yet fulfilling this goal; at least, I find that many beginning graduate students with an undergraduate degree in computer science have been more narrowly educated than I would like. Computer scientists are of course working to correct this present deficiency, which I believe is probably due to an overemphasis on computer languages instead of algorithms.

4. Some interactions. Computer science has been affecting mathematics in many ways, and I shall try to list the good ones here. In the first place, of course, computers

can be used to compute, and they have frequently been applied in mathematical research when hand computations are too difficult; they generate data which suggests or demolishes conjectures. For example, Gauss said [**10**] that he first thought of the prime number theorem by looking at a table of the primes less than one million. In my own Ph.D. thesis, I was able to resolve a conjecture concerning infinitely many cases by looking closely at computer calculations of the smallest case [**13**]. An example of another kind is Marshall Hall's recent progress in the determination of all simple groups of orders up to one million.

Secondly, there are obvious connections between computer science and mathematics in the areas of numerical analysis [**30**], logic, and number theory; I need not dwell on these here, since they are so widely known. However, I should mention especially the work of D. H. Lehmer, who has combined computing with classical mathematics in several remarkable ways; for example, he has proved that every set of six consecutive integers > 285 contains a multiple of a prime $\geqq 43$.

Another impact of computer science has been an increased emphasis on constructions in all branches of mathematics. Replacing existence proofs by algorithms which construct mathematical objects has often led to improvements in an abstract theory. For example, E. C. Dade and H. Zassenhaus remarked, at the close of a paper written in 1963, "This concept of genus has already proved of importance in the theory of modules over orders. So a mathematical idea introduced solely with a view to computability has turned out to have an intrinsic theoretical value of its own." Furthermore, as mentioned above, the constructive algorithmic approach often has pedagogic value.

Another way in which the algorithmic approach affects mathematical theories is in the construction of one-to-one correspondences. Quite often there have been indirect proofs that certain types of mathematical objects are equinumerous; then a direct construction of a one-to-one correspondence shows that in fact even more is true.

Discrete mathematics, especially combinatorial theory, has been given an added boost by the rise of computer science, in addition to all the other fields in which discrete mathematics is currently being extensively applied.

For references to these influences of computing on mathematics, and for many more examples, the reader is referred to the following sampling of books, each of which contains quite a few relevant papers: [**1**], [**2**], [**4**], [**5**], [**20**], [**24**], [**27**]. Peter Lax's article [**19**] discusses the effect computing has had on mathematical physics.

But actually the most important impact of computer science on mathematics, in my opinion, is somewhat different from all of the above. To me, the most significant thing is that the study of algorithms themselves has opened up a fertile vein of interesting new mathematical problems; it provides a breath of life for many areas of mathematics which had been suffering from a lack of new ideas. Charles Babbage, one of the "fathers" of computing machines, predicted this already in

1864: "As soon as an Analytical Engine [i.e., a general-purpose computer] exists, it will necessarily guide the future course of the science. Whenever any result is sought by its aid, the question will then arise — By what course of calculation can these results be arrived at by the machine in the shortest time?" [3]. And again, George Forsythe in 1958: "The use of practically any computing technique itself raises a number of mathematical problems. There is thus a very considerable impact of computation on mathematics itself, and this may be expected to influence mathematical research to an increasing degree" [**26**]. Garrett Birkhoff [**4**, p. 2] has observed that such influences are not a new phenomenon, they were already significant in the early Greek development of mathematics.

I have found that a great many intriguing mathematical problems arise when we try to analyze an algorithm quantitatively, to see how fast it will run on a computer; a typical example of such a problem is worked out below. Another class of problems of great interest concerns the search for best possible algorithms in a given class; see, for example, the recent survey by Reingold [**25**]. And one of the first mathematical theories to be inspired by computer science is the theory of languages, which by now includes many beautiful results; see [**11**] and [**12**]. The excitement of these new theories is the reason I became a computer scientist.

Conversely, mathematics has of course a profound influence on computer science; nearly every branch of mathematical knowledge has been brought to bear somewhere. I recently worked on a problem dealing with discrete objects called "binary trees," which arise frequently in computer representations of things, and the solution to the problem actually involved the complex gamma function times the square of Riemann's zeta function [**6**]. Thus the results of classical mathematics often turn out to be useful in rather amazing places.

The most surprising thing to me, in my own experiences with applications of mathematics to computer science, has been the fact that so much of the mathematics has been of a particular discrete type, examples of which are discussed below. Such mathematics was almost entirely absent from my own training, although I had a reasonably good undergraduate and graduate education in mathematics. Nearly all of my encounters with such techniques during my student days occurred when working problems from this MONTHLY. I have naturally been wondering whether or not the traditional curriculum (the calculus courses, etc.) should be revised in order to include more of these discrete mathematical manipulations, or whether computer science is exceptional in its frequent application of them.

5. A detailed example. In order to clarify some of the vague generalizations and assertions made above, I believe it is best to discuss a typical computer-science problem in some depth. The particular example I have chosen is the one which first led me personally to realize that computer algorithms suggest interesting mathematical problems. This happened in 1962, when I was a graduate student in mathematics; computer programming was a hobby of mine, and a part time job, but I had never

really ever worn my mathematician's cloak and my computing cap at the same time. A friend of mine remarked that "some good mathematicians at IBM" had been unable to determine how fast a certain well-known computer method works, and I thought it might be an interesting problem to look at.

Here is the problem: Many computer applications involve the retrieval of information by its "name"; for example, we might imagine a Russian-English dictionary, in which we want to look up a Russian word in order to find its English equivalent. A standard computer method, called *hashing*, retrieves information by its name as follows. A rather large number, m, of memory positions within the computer is used to hold the names; let us call these positions $T_1, T_2, \cdots, T_m$. Each of these positions is big enough to contain one name. The number m is always larger than the total number of names present, so at least one of the T_i is empty. The names are distributed among the T_i's in a certain way described below, designed to facilitate retrieval. Another set of memory positions $E_1, E_2, \cdots, E_m$ is used for the information corresponding to the names; thus if T_i is not empty, E_i contains the information corresponding to the name stored in T_i.

The ideal way to retrieve information using such a table would be to take a given name x, and to compute some function $f(x)$, which lies between 1 and m; then the name x could be placed in position $T_{f(x)}$, and the corresponding information in $E_{f(x)}$. Such a function $f(x)$ would make the retrieval problem trivial, if $f(x)$ were easy to compute and if $f(x) \neq f(y)$ for all distinct names $x \neq y$. In practice, however, these latter two requirements are hardly ever satisfied simultaneously; if $f(x)$ is easy to compute, we have $f(x) = f(y)$ for some distinct names. Furthermore, we do not usually know in advance just which names will occur in the table, and the function f must be chosen to work for all names in a very large set U of potential names, where U has many more than m elements. For example, if U contains all sequences of seven letters, there are $26^7 = 8{,}031{,}810{,}176$ potential names; it is inevitable that $f(x) = f(y)$ will occur.

Therefore we try to choose a function $f(x)$, from U into $\{1, 2, \cdots, m\}$, so that $f(x) = f(y)$ will occur with the approximate probability $1/m$, when x and y are distinct names. Such a function f is called a **hash function**. In practice, $f(x)$ is often computed by regarding x as a number and taking its remainder modulo m, plus one; the number m in this case is usually chosen to be prime, since this can be shown to give better results for the sets of names that generally arise in practice. When $f(x) = f(y)$ for distinct x and y, a "collision" is said to occur; collisions are resolved by searching through positions numbered $f(x) + 1$, $f(x) + 2$, etc.

The following algorithm expresses exactly how a hash function $f(x)$ can be used to retrieve the information corresponding to a given name x in U. The algorithm makes use of a variable i which takes on integer values.

STEP 1. Set the value of i equal to $f(x)$.

STEP 2. If memory position T_i contains the given name x, stop; the derived information is located in memory position E_i.

STEP 3. If memory position T_i is empty, stop; the given name x is not present.

STEP 4. Increase the value of i by one. (Or, if i was equal to m, set i equal to one.) Return to step 2.

We still haven't said how the names get into $T_1, \cdots, T_m$ in the first place; but that is really not difficult. We start with all the T_i empty. Then to insert a new name x, we "look for" x using the above algorithm; it will stop in step 3 because x is not there. Then we set T_i equal to x, and put the corresponding information in E_i. From now on, it will be possible to retrieve this information, whenever the name x is given, since the above algorithm will find position T_i by repeating the actions which took it to that place when x was inserted.

The mathematical problem is to determine how much searching we should expect to make, on the average; how many times must step 2 be repeated before x is found?

This same problem can be stated in other ways, for example in terms of a modified game of "musical chairs." Consider a set of m empty chairs arranged in a circle. A person appears at a random spot just outside the circle and dashes (in a clockwise direction) to the first available chair. This is repeated m times, until all chairs are full. How far, on the average, does the nth person have to run before he finds a seat?

For example, let $m = 10$ and suppose there are ten players: $A, B, C, D, E, F, G, H, I, J$. To get a random sequence, let us assume that the players successively start looking for their seats beginning at chairs numbered according to the first digits of π, namely 3, 1, 4, 1 5, 9, 2, 6, 5, 3. Figure 1 shows the situation after the first six have been seated.

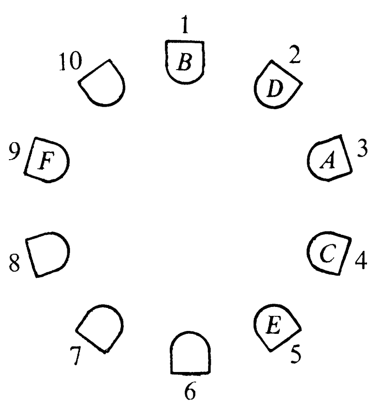


FIG. 1.

A "musical chairs" game which corresponds to an important computer method.

(Thus player A takes chair 3, then player B takes chair $1, \cdots$, player F takes chair 9.) Now player G starts at chair number 2, and eventually he sits down in number 6. Finally, players H, I and J will go into chairs 7, 8, and 10. In this example, the distances travelled by the ten players are respectively 0, 0, 0, 1, 0, 0, 4, 1, 3, 7.

It is not trivial to analyze this problem, because congestion tends to occur; one or more long runs of consecutive occupied chairs will usually be present. In order to see why this is true, we may consider Figure 1 again, supposing that the next player H starts in a random place; then he will land in chair number 6 with probability 0.6, but he will wind up in chair number 7 with probability only 0.1. Long runs tend to get even longer. Therefore we cannot simply assume that the configuration of occupied vs. empty chairs is random at each stage; the piling-up phenomenon must be reckoned with.

Let the starting places of the m players be $a_1 a_2 \cdots a_m$; we shall call this a **hash sequence**. For example, the above hash sequence is 3 1 4 1 5 9 2 6 5 3. Assuming that each of the m^n possible hash sequences is equally likely, our problem is to determine the average distance traveled by the nth player, for each n, in units of "chairs passed." Let us call this distance $d(m,n)$. Obviously $d(m,1) = 0$, since the first player always finds an unoccupied place; furthermore $d(m,2) = 1/m$, since the second player has to go at most one space, and that is necessary only if he starts at the same spot as the first player. It is also easy to see that $d(m,m) = (0+1+\cdots+(m-1))/m = \frac{1}{2}(m-1)$, since all chairs but one will be occupied when the last player starts out. Unfortunately the in-between values of $d(m,n)$ are more complicated.

Let $u_k(m,n)$ be the number of partial hash sequences $a_1 a_2 \cdots a_n$ such that chair k will be unoccupied after the first n players are seated. This is easy to determine, by cyclic symmetry, since chair k is just as likely to be occupied as any other particular chair; in other words, $u_1(m,n) = u_2(m,n) = \cdots = u_m(m,n)$. Let $u(m,n)$ be this common value. Furthermore, $mu(m,n) = u_1(m,n) + u_2(m,n) + \cdots + u_m(m,n) = (m-n)m^n$, since each of the m^n partial hash sequences $a_1 a_2 \cdots a_n$ leaves $m-n$ chairs empty, so it contributes one to exactly $m-n$ of the numbers $u_k(m,n)$. Therefore

$$u_k(m,n) = (m-n)m^{n-1}.$$

Let $v(m,n,k)$ be the number of partial hash sequences $a_1 a_2 \cdots a_n$ such that, after the n players are seated, chairs 1 through k will be occupied, while chairs m and $k+1$ will not. This number is slightly harder to determine, but not really difficult. If we look at the numbers a_i which are $\leqq k+1$ in such a partial hash sequence, and if we cross out the other numbers, the k values which are left form one of the sequences enumerated by $u(k+1,k)$. Furthermore the $n-k$ values crossed out form one of the sequences enumerated by $u(m-1-k, n-k)$, if we subtract $k+1$ from each of them. Conversely, if we take any partial hash sequence $a_1 \cdots a_k$ enumerated by $u(k+1,k)$, and another one $b_1 \cdots b_{n-k}$ enumerated by $u(m-1-k, n-k)$,

and if we intermix $a_1 \cdots a_k$ with $(b_1 + k + 1) \cdots (b_{n-k} + k + 1)$ in any of the $\binom{n}{k}$ possible ways, we obtain one of the sequences enumerated by $v(m,n,k)$. Here

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

is the number of ways to choose k positions out of n. For example, let $m = 10$, $n = 6$, $k = 3$; one of the partial hash sequences enumerated by $v(10, 6, 3)$ is 2 7 1 8 2 8. This sequence splits into $a_1a_2a_3 = 2\ 1\ 2$ and $(b_1 + 4)(b_2 + 4)(b_3 + 4) = 7\ 8\ 8$, intermixed in the pattern *ababab*. From each of the $u(4,3) = 16$ sequences $a_1a_2a_3$ that fill positions 1, 2, 3, together with each of the $u(6,3) = 108$ sequences $(b_1 + 4)(b_2 + 4)(b_3 + 4)$ that fill three of positions 5, 6, 7, 8, 9, we obtain $\binom{6}{3} = 20$ sequences that fill positions 1, 2, 3, and which leave positions 4 and 10 unoccupied, by intermixing the a's and b's in all possible ways. This correspondence shows that

$$v(m,n,k) = \binom{n}{k} u(k+1,k)u(m-k-1,n-k),$$

and our formula for $u(m,n)$ tells us that

$$v(m,n,k) = \binom{n}{k}(k+1)^{k-1}(m-n-1)(m-k-1)^{n-k-1}.$$

This is not a simple formula; but since it is correct, we cannot do any better. If $k = n = m-1$, the last two factors in the formula give 0/0, which should be interpreted as 1 in this case.

Now we are ready to compute the desired average distance $d(m,n)$. The nth player must move k steps if and only if the preceding partial hash sequence $a_1 \cdots a_{n-1}$ has left chairs a_n through $a_n + k - 1$ occupied and chair $a_n + k$ empty. The number of such partial hash sequences is

$$v(m,n-1,k) + v(m,n-1,k+1) + v(m,n-1,k+2) + \cdots,$$

since circular symmetry shows that $v(m,n-1,k+r)$ is the number of partial hash sequences $a_1 \cdots a_{n-1}$ leaving chairs $a_n + k$ and $a_n - r - 1$ empty while the $k + r$ chairs between them are filled. Therefore the probability $p_k(m,n)$ that the nth player goes exactly k steps is

$$p_k(m,n) = \left(\sum_{r \geq k} v(m,n-1,r)\right)\Big/m^{n-1};$$

and the average distance is

$$d(m,n) = \sum_{k \geq 0} kp_k(m,n) = (m-n)m^{1-n} \sum_{r \geq k \geq 0} k\binom{n-1}{r}(r+1)^{r-1}(m-r-1)^{n-r-2}$$

$$= \frac{(m-n)m^{1-n}}{2} \sum_{r \geq 0} r\binom{n-1}{r}(r+1)^{r}(m-r-1)^{n-r-2}.$$

At this point, a person with a typical mathematical upbringing will probably stop; the answer is a horrible-looking summation. Yet, if more attention were paid during our mathematical training to finite sums, instead of concentrating so heavily on integrals, we would instinctively recognize that a sum like this can be considerably simplified. When I first looked at this sum, I had never seen one like it before; but I suspected that something could be done to it, since for example, the sum over k of $p_k(m,n)$ must be 1. Later I learned of the extensive literature of such sums. I do not wish to go into the details, but I do want to point out that such sums arise repeatedly in the study of algorithms. By now I have seen literally hundreds of examples in which finite sums involving binomial coefficients and related functions appear in connection with computer science studies; so I have introduced a course called "Concrete Mathematics" at Stanford University, in which this kind of mathematics is taught.

Let $\delta(m,n)$ be the average number of chairs skipped past by the first n players:

$$\delta(m,n) = (d(m,1) + d(m,2) + \cdots + d(m,n))/n.$$

This corresponds to the average amount of time needed for the hashing algorithm to find an item when n items have been stored. The value of $d(m,n)$ derived above can be simplified to obtain the following formulas:

$$d(m,n) = \frac{1}{2}\left(2\frac{n-1}{m} + 3\frac{n-1}{m}\,\frac{n-2}{m} + 4\frac{n-1}{m}\,\frac{n-2}{m}\,\frac{n-3}{m} + \cdots\right),$$

$$\delta(m,n) = \frac{1}{2}\left(\frac{n-1}{m} + \frac{n-1}{m}\,\frac{n-2}{m} + \frac{n-1}{m}\,\frac{n-2}{m}\,\frac{n-3}{m} + \cdots\right).$$

These formulas can be used to see the behavior for large m and n: for example, if $\alpha = n/m$ is the ratio of filled positions to the total number of positions, and if we hold α fixed while m approaches infinity, then $\delta(m,\alpha m)$ increases to the limiting value $\frac{1}{2}\alpha/(1-\alpha)$.

The formula for $\delta(m,n)$ also tells us another surprising thing:

$$\delta(m,n) = \frac{n-1}{2m} + \frac{n-1}{m}\delta(m,n-1).$$

If somebody could discover a simple trick by which this simple relation could be proved directly, it would lead to a much more elegant analysis of the hashing algorithm and it might provide further insights. Unfortunately, I have been unable to think of any direct way to prove this relation.

When $n = m$ (i.e., when all players are seated and all chairs are occupied), the average distance traveled per player is

$$\delta(m,m) = \frac{1}{2}\left(\frac{m-1}{m} + \frac{m-1}{m}\,\frac{m-2}{m} + \frac{m-1}{m}\,\frac{m-2}{m}\,\frac{m-3}{m} + \cdots\right).$$

It is interesting to study this function, which can be shown to have the approximate value

$$\delta(m, m) \approx \sqrt{\frac{\pi m}{8}} - \frac{2}{3}$$

for large m. Thus, the number π, which entered Figure 1 so artificially, is actually present naturally in the problem as well! Such asymptotic calculations, combined with discrete summations as above, are typical of what arises when we study algorithms; classical mathematical analysis and discrete mathematics both play important roles.

6. Extensions. We have now solved the musical chairs problem, so the analysis of hashing is complete. But many more problems are suggested by this one. For example, what happens if each of the hash table positions T_i is able to hold two names instead of one, i.e., if we allow two people per chair in the musical chairs game? Nobody has yet found the exact formulas for this case, although some approximate formulas are known.

We might also ask what happens if each player in the musical chairs game starts *simultaneously* to look for a free chair (still always moving clockwise), starting at independently random points. The answer is that each player will move past $\delta(m, n)$ chairs on the average, where $\delta(m, n)$ is the same as above. This follows from an interesting theorem of W. W. Peterson **[23]**, who was the first to study the properties of the hashing problem described above. Peterson proved that the total displacement of the n players, for any partial hash sequence $a_1 a_2 \cdots a_n$, is independent of the order of the a_i's; thus, 3 1 4 1 5 9 2 leads to the same total displacement as 1 1 2 3 4 5 9 and 2 9 5 1 4 1 3. His theorem shows that the average time $\delta(m, n)$ per player is the same for all arrangements of the a_i, and therefore it is also unchanged when all players start simultaneously.

On the other hand, the average amount of time required until all n players are seated has not been determined, to my knowledge, for the simultaneous case. In fact, I just thought of this problem while writing this paper. New problems flow out of computer science studies at a great rate!

We might also ask what happens if the players can choose to go either clockwise or counterclockwise, whichever is shorter. In the non-simultaneous case, the above analysis can be extended without difficulty to show that each player will then have to go about half as far. (We require everyone to go all the way around the circle to the nearest seat, not taking a short cut through the middle.)

Another variant of the hashing problem arises when we change the cyclic order of probing, in order to counteract the "piling up" phenomenon. This interesting variation is of practical importance, since the congestion due to long stretches of occupied positions slows things down considerably when the memory gets full. Since the analysis of this practical problem is largely unresolved, and since it has

several interesting mathematical aspects, I shall discuss it in detail in the remainder of this article.

A generalized hashing technique which for technical reasons is called **single hashing** is defined by any $m \times m$ matrix Q of integers for which

(i) Each row contains all the numbers from 1 to m in some order;

(ii) The first column contains the numbers from 1 to m in order.

The other columns are unrestricted. For example, one such matrix for $m = 4$, selected more or less at random, is

$$Q_1 = \begin{bmatrix} 1 & 3 & 2 & 4 \\ 2 & 1 & 3 & 4 \\ 3 & 4 & 1 & 2 \\ 4 & 3 & 2 & 1 \end{bmatrix}.$$

The idea is to use a hash function $f(x)$ to select a row of Q and then to probe the memory positions in the order dictated by that row. The same algorithm for looking through memory is used as before, except that step 4 becomes

STEP 4′. Advance i to the next value in row $f(x)$ of the matrix, and return to step 2.

Thus, the cyclic hashing scheme described earlier is a special case of single hashing, using a cyclic matrix like

$$Q_2 = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \\ 3 & 4 & 1 & 2 \\ 4 & 1 & 2 & 3 \end{bmatrix}.$$

In the musical chair analogy, the players no longer are required to move clockwise; different players will in general visit the chairs in different sequences. However, if two players start in the same place, they must both follow the same chair-visiting sequence. This latter condition will produce a slight congestion, which is noticeable but not nearly as significant as in the cyclic case.

As before, we can define the measures $d'(m,n)$ and $\delta'(m,n)$, corresponding to the number of times step 4′ is performed. The central problem is to find matrices Q which are **best possible**, in the sense that $\delta'(m,m)$ is minimized. This problem is not really a practical one, since the matrix with smallest $\delta'(m,m)$ might require a great deal of computation per execution of step 4′. Yet it is very interesting to establish absolute limits on how good a single-hashing method could possibly be, as a yardstick by which to measure particular cases.

One of the most difficult problems in algorithmic analysis that I have had the

pleasure of solving is the determination of $d'(m,n)$ for single hashing when the matrix Q is chosen at random, i.e., to find the value of $d'(m,n)$, averaged over all $((m-1)!)^m$ possible matrices Q. The resulting formula is

$$d_r'(m,n) = m - \frac{m-n+1}{m-n+2}\left(1+\left(m+\sum_{j=1}^{n-1}\frac{1-1/(m+2-j)}{m\prod_{i=1}^{j}(1-1/(m(m+2-i)))}\right)\right.$$
$$\left.\times\prod_{j=1}^{n-1}\left(1-\frac{1}{m(m+2-j)}\right)\right).$$

This one I do not know how to simplify at the present time. However, it is possible to study the asymptotic behavior of $d_r'(m,n)$, and to show that

$$\delta_r'(m,m) \approx \ln m + \gamma - 1.5$$

for large m, plus a correction term of order $(\log m)/m$. (Here γ is Euler's constant.) This order of growth is substantially better than the cyclic method, where $\delta(m,m)$ grows like the square root of m; and we know that some single-hashing matrices must have an even lower value for $\delta'(m,m)$ than this average value $\delta_r'(m,m)$. Table 1 shows the exact values of $\delta(m,m)$ and $\delta_r'(m,m)$ for comparatively small values of m; note that cyclic hashing is superior for $m \leqq 11$, but it eventually becomes much worse.

Proofs of the above statements, together with additional facts about hashing, appear in [**18**].

No satisfactory lower bounds for the value of $\delta'(m,m)$ in the best single-hashing scheme are known, although I believe that none will have $\delta'(m,m)$ lower than

$$\left(1+\frac{1}{m}\right)\left(1+\frac{1}{2}+\cdots+\frac{1}{m}\right)-2;$$

this is the value which arises in the musical chairs game if each player follows a random path independently of all the others. J. D. Ullman [**28**] has given a more general conjecture from which this statement would follow. If Ullman's conjecture is true, then a *random* Q comes within $\frac{1}{2}$ of the best possible value, and a large number of matrices will therefore yield values near the optimum. Therefore it is an interesting practical problem to construct a family of matrices for various m, having provably good behavior near the optimum, and also with the property that they are easy to compute in step 4′.

It does not appear to be easy to compute $\delta'(m,m)$ for a given matrix M. The best method I know requires on the order of $m\cdot 2^m$ steps, so I have been able to experiment on this problem only for small values of m. (Incidentally, such experiments represent an application of computer science to solve a mathematical problem suggested by computer science.) Here is a way to compute $\delta'(m,m)$ for a given matrix $Q=(q_{ij})$: If A is any subset of $\{1,2,\cdots,m\}$, let $\|A\|$ be the number of ele-

ments in A, and let $p(A)$ be the probability that the first $\| A \|$ players occupy the chairs designated by A. Then it is not difficult to show that

$$p(A) = \frac{1}{m} \sum_{(i,j) \in s(A)} p(A - \{q_{ij}\})$$

when A is nonempty, where $s(A)$ is the set of all pairs (i,j) such that $q_{ik} \in A$ for $1 \leqq k \leqq j$; consequently

$$d'(m,n) = \frac{1}{m} \sum_{\|A\| = n-1} \| s(A) \| \, p(A),$$

$$\delta'(m,m) = \frac{1}{m^2} \sum_{A} \| s(A) \| \, p(A).$$

For example, in the 4×4 matrix Q_1 considered earlier, we have

A	$p(A)$	$\|s(A)\|$	A	$p(A)$	$\|s(A)\|$
$\varnothing$	1	0	$\{4\}$	1/4	1
$\{1\}$	1/4	1	$\{1,4\}$	2/16	2
$\{2\}$	1/4	1	$\{2,4\}$	2/16	2
$\{1,2\}$	3/16	3	$\{1,2,4\}$	9/64	4
$\{3\}$	1/4	1	$\{3,4\}$	4/16	4
$\{1,3\}$	3/16	3	$\{1,3,4\}$	20/64	7
$\{2,3\}$	2/16	2	$\{2,3,4\}$	16/64	6
$\{1,2,3\}$	19/64	7	$\{1,2,3,4\}$	1	16

The first three chairs occupied will most probably be $\{1,3,4\}$; the set of chairs $\{1,2,4\}$ is much less likely. The "score" $\delta'(m,m)$ for this matrix comes to 653/1024, which in this case is worse than the score 624/1024 for cyclic hashing. In fact, cyclic hashing turns out to be the *best* single hashing scheme when $m = 4$.

When $m = 5$, the best single hashing scheme turns out to be obtained from the matrix

$$Q_5 = \begin{bmatrix} 1 & 2 & 4 & 5 & 3 \\ 2 & 3 & 5 & 1 & 4 \\ 3 & 4 & 1 & 2 & 5 \\ 4 & 5 & 2 & 3 & 1 \\ 5 & 1 & 3 & 4 & 2 \end{bmatrix}$$

whose score is 0.7440, compared to 0.7552 for cyclic hashing. Note that Q_5 is very much like cyclic hashing, since cyclic symmetry is present: each row is obtained

from the preceding row by adding 1 modulo 5, so that the probing pattern is essentially the same for all rows. We may call this **generalized cyclic hashing**; it is a special case of practical importance, because it requires knowing only one row of Q instead of all m^2 entries.

When $m > 5$, an exhaustive search for the best single hashing scheme would be too difficult to do by machine, unless some new breakthrough is made in the theory. Therefore I have resorted to "heuristic" search procedures. For all $m \leqq 11$, the best single hashing matrices I have been able to find actually have turned out to be generalized cyclic hashing schemes, and I am tempted to conjecture that this will be true in general. It would be extremely nice if this conjecture were true, since it would follow that the potentially expensive generality of a non-cyclic scheme would never be useful. However, the evidence for my guess is comparatively weak;

TABLE 1. Cyclic hashing versus random single hashing

m	$\delta(m, m)$	$\delta_r'(m, m)$
1	0.0000	0.0000
2	0.2500	0.2500
3	0.4444	0.4630
4	0.6094	0.6426
5	0.7552	0.7973
6	0.8874	0.9330
7	1.0091	1.0538
8	1.1225	1.1626
9	1.2292	1.2616
10	1.3301	1.3523
11	1.4262	1.4360
12	1.5180	1.5138
15	1.7729	1.7183
20	2.1468	1.9911
30	2.7747	2.3888
40	3.3046	2.6774
50	3.7716	2.9037
75	4.7662	3.3181
100	5.6050	3.6135

it is simply that (i) the conjecture holds for $m \leqq 5$; (ii) I have seen no counterexamples in experiments for $m \leqq 11$; (iii) the best generalized cyclic hashing schemes for $m \leqq 9$ are "locally optimum" single hashing schemes, in the sense that all possible interchanges of two elements in any row of the matrix lead to a matrix that is no better; (iv) the latter statement is *not* true for the standard (ungeneralized) cyclic hashing scheme, so the fact that it holds for the best ones may be significant.

Even if this conjecture is false, the practical significance of generalized cyclic hashing makes it a suitable object for further study, especially in view of its additional

mathematical structure. One immediate consequence of the cyclic property is that $p(A) = p(A + k)$ for all sets A, in the above formulas for computing $d'(m,n)$, where "$A + k$" means the set obtained from A by adding k to each element, modulo m. This observation makes the calculation of scores almost m times faster. Another, not quite so obvious property, is the fact that the generalized cyclic hashing scheme generated by the permutation $q_1 q_2 \cdots q_m$ has the same score as that generated by the "reflected" permutation $q'_1 q'_2 \cdots q'_m$ where $q'_j = m + 1 - q_j$. (It is convenient to say that a generalized cyclic hashing scheme is "generated" by any of its rows.) This equivalence under reflection can be proved by showing that $p(A)$ is equal to $p'(m + 1 - A)$.

I programmed a computer to find the scores for all generalized cyclic hashing schemes when $m = 6$, and the results of this computation suggested that two further simplifications might be valid:

(i) $q_1 q_2 q_3 \cdots q_m$ and $q_2 q_1 q_3 \cdots q_m$ generate equally good generalized cyclic hashing schemes.

(ii) $q_1 \cdots q_{m-2} q_{m-1} q_m$ and $q_1 \cdots q_{m-2} q_m q_{m-1}$ generate equally good generalized cyclic hashing schemes.

In fact, both of these statements are true; here is a typical instance where computing in a particular case has led to new mathematical theorems.

In fact, the above results made me suspect that $q_1 \cdots q_m$ and

$$(m + 1 - q_1) \cdots (m + 1 - q_k) q_{k+1} \cdots q_m$$

will always generate equally good schemes, whenever both of these sequences are permutations. If this statement were true, it would include the three previous results as special cases, for $k = 2$, $m - 2$ and m. Unfortunately, I could not prove it; and I eventually found a counterexample (by hand), namely $q_1 \cdots q_m = 1\ 3\ 8\ 6\ 2\ 7\ 5\ 4$ and $k = 4$. However, this mistaken conjecture did lead to an interesting purely mathematical question, namely to determine how many inequivalent permutations of m objects there are, when $q_1 \cdots q_m$ is postulated to be equivalent to $(\varepsilon q_1 + j) \cdots (\varepsilon q_k + j) q_{k+1} \cdots q_m$, for $\varepsilon = \pm 1$ and $1 \leqq j, k \leqq m$ (whenever these are both permutations, modulo m). We might call these "necklace permutations," by analogy with another well-known combinatorial problem, since they represent the number of different orders in which a person could change the beads of a necklace from all white to all black, ignoring the operation of rotating and/or flipping the necklace over whenever such an operation preserves the current black/white pattern. The total number of different necklace permutations for $m = 1, 2, 3, 4, 5, 6, 7$ is 1, 1, 1, 2, 4, 14, 62, respectively, and I wonder what can be said for general m.

Returning to the hashing problem, the theorems mentioned above make it possible to study all of the generalized cyclic hashing schemes for $m \leqq 9$, by computer; and the following turn out to be the best:

best permutation	$\delta'_{\min}(m, m)$	$\delta'_{\text{ave}}(m, m)$
1 2 3 4	0.6094	0.6146
1 2 4 5 3	0.7440	0.7514
1 2 5 3 4 6	0.8650	0.8819
1 4 2 3 6 5 7	0.9713	0.9866
1 3 4 8 7 2 6 5	1.0676	1.0919
1 5 2 3 8 4 6 7 9	1.1568	1.1790

The righthand column gives the average $\delta'(m, m)$ over all $m!$ schemes. For $m = 10$ and 11 the best permutations I have found so far are 1 2 8 6 4 9 3 10 7 5 and 1 3 4 8 9 7 11 2 10 6 5, with respective scores of 1.2362 and 1.3103. The *worst* such schemes for $m \leqq 9$ are

worst permutation	$\delta'_{\max}(m,m)$
1 3 2 4	0.6250
1 2 3 4 5	0.7552
1 3 5 2 4 6	0.9132
1 2 3 4 5 6 7	1.0091
1 5 3 7 4 8 2 6	1.1719
1 4 7 2 5 8 3 6 9	1.2638

(This table suggests that the form of the worst cyclic scheme might be obtainable in a simple way from the prime factors of m.)

Finally I have tried to find the worst possible Q matrices, *without* the cyclic constraint. Such matrices can be very bad indeed; the worst I know, for any m, occur when $q_{ij} < q_{i(j+1)}$ for all $j \geqq 2$, e.g.

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 1 & 3 & 4 & 5 \\ 3 & 1 & 2 & 4 & 5 \\ 4 & 1 & 2 & 3 & 5 \\ 5 & 1 & 2 & 3 & 4 \end{bmatrix}$$

when $m = 5$. Using discrete mathematical techniques like those illustrated above, I have proved that the score for such matrices is

$$\delta'(m, m) = \left(m + 3 + \frac{2}{m}\right)\left(1 + \frac{1}{m}\right)^m - 2.5m - 7 - \frac{2.5}{m},$$

which is approximately $(e - 2.5)m + 3e - 8$ when m is large. We certainly would not want to retrieve information in this way, and perhaps it is the worst possible single hashing scheme.

Thus, the example of hashing illustrates the typical interplay between computer science and mathematics.

I wish to thank Garrett Birkhoff for his comments on the first draft of this paper.

References

1. Amer. Math. Society and Math. Assoc. of America, co-sponsors of conference, The Influence of Computing on Mathematical Research and Education, August 1973.

2. A. O. L. Atkin and B. J. Birch, eds., Computers in Number Theory, Academic Press, New York, 1971.

3. Charles Babbage, Passages from the Life of a Philosopher, (London, 1864). Reprinted in *Charles Babbage and His Calculating Engines*, by Philip and Emily Morrison, Dover, New York 1961; esp. p. 69.

4. Garrett Birkhoff and Marshall Hall, Jr., eds., Computers in Algebra and Number Theory, SIAM-AMS Proceedings, 4 (Amer. Math. Soc., 1971).

5. R. F. Churchhouse and J. -C. Herz, eds., Computers in Mathematical Research, North-Holland, Amsterdam, 1968.

6. N. G. de Bruijn, Donald E. Knuth, and S. O. Rice, The average height of planted plane trees, in *Graph Theory and Computing*, ed. by Ronald C. Read, Academic Press, New York, 1972, 15–22.

7. George E. Forsythe, The role of numerical analysis in an undergraduate program, this MONTHLY, 66 (1959) 651–662.

8. ———, Computer Science and Education, Information Processing 68, 1025–1039.

9. ———, What to do till the computer scientist comes, this MONTHLY, 75 (1968) 454–462.

10. K. F. Gauss, Letter to Enke, *Werke*, vol. 2, 444–447.

11. Seymour Ginsburg, The Mathematical Theory of Context Free Languages, McGraw-Hill, New York; 1966.

12. ———, Sheila Greibach, and John Hopcroft, Studies in abstract families of languages, Amer. Math. Society Memoirs, 87 (1969) 51 pp.

13. Donald E. Knuth, A class of projective planes, Trans. Amer. Math. Soc., 115 (1965) 541–549.

14. ———, Algorithm and program; information and data, Comm. ACM, 9 (1966), 654.

15. ———, Seminumerical Algorithms, Addison-Wesley, Reading, Mass., 1969.

16. ———, Ancient Babylonian algorithms, Comm. ACM, 15 (1972) 671–677.

17. ———, George Forsythe and the development of Computer Science, Comm. ACM, 15 (1972) 721–726.

18. ———, Sorting and Searching, Addison-Wesley, Reading, Mass., 1973.

19. Peter D. Lax, The impact of computers on mathematics, Chapter 10 of *Computers and Their Role in the Physical Sciences*, ed. by S. Fernbach and A. Taub, Gordon and Breach, New York, 1970, 219–226.

20. John Leech, ed., Computational Problems in Abstract Algebra, Pergamon, Long Island City, 1970.

21. Peter Naur, 'Datalogy', the science of data and data processes, and its place in education, Information Processing 68, vol. 2, 1383–1387.

22. Allen Newell, Alan J. Perlis, and Herbert A. Simon, Computer Science, Science, 157 (1967) 1373–1374.

23. W. W. Peterson, Addressing for random-access storage, IBM Journal of Res. and Devel., 1 (1957) 130–146.

24. Proc. Symp. Applied Math 15, Experimental Arithmetic, High-Speed Computing, and Mathematics, Amer. Math. Soc., 1963.

25. E. Reingold, Establishing lower bounds on algorithms — A survey, AFIPS Conference Proceedings, 40 (1972) 471–481.

26. Paul C. Rosenbloom and George E. Forsythe, Numerical Analysis and Partial Differential Equations, Surveys in Applied Math 5, Wiley, New York, 1958.

27. Computers and Computing, Slaught Memorial Monograph No. 10, supplement to this MONTHLY, 72 (February 1965) 156 pp.

28. J. D. Ullman, A note on the efficiency of hashing functions, J. ACM, 19 (1972) 569–575.

29. Peter Wegner, Three computer cultures, Advances in Computers, 10 (1970) 7–78.

30. J. H. Wilkinson, Some comments from a numerical analyst, J. ACM, 18 (1971) 137–147.

COMPUTER SCIENCE DEPARTMENT, STANFORD UNIVERSITY, STANFORD, CALIFORNIA 94305.

MAXWELL'S EQUATIONS

THEODORE FRANKEL

1. Introduction. We shall consider Maxwell's equations

$$(1) \cdots \operatorname{div} \mathbf{B} = 0 \qquad (2) \cdots \operatorname{div} \mathbf{D} = \sigma$$

$$(3) \cdots \operatorname{curl} \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \qquad (4) \cdots \operatorname{curl} \mathbf{H} = \mathbf{j} + \frac{\partial \mathbf{D}}{\partial t}$$

in a "non-inductive" medium; i.e., $\mathbf{E} = \mathbf{D}$ is the electric field vector, $\mathbf{B} = \mathbf{H}$ is the magnetic field vector, σ is the charge density, and $\mathbf{j}$ is the current density vector.

These equations are usually taken as axioms in electromagnetic field theory. (1) says that there are no magnetic charges. (2) is Gauss' law, stating that one can compute the total charge inside a closed surface by integrating the normal component of $\mathbf{D}$ or $\mathbf{E}$ over the surface. (3) is Faraday's law; a changing magnetic field produces an electric field. Finally, (4) is Ampere's law curl $\mathbf{H} = \mathbf{j}$ modified by Maxwell's term $\partial \mathbf{D}/\partial t$, stating that currents and changing electric fields produce magnetic fields. Equations (1) and (2) are relatively simple and easily understood while (3) and (4) seem much more sophisticated. It is comforting to know then, that **in a certain sense, Faraday's law** (3) **is a consequence of** (1), **while the Ampere-Maxwell law** (4) **is a consequence of Gauss' law** (2). The precise statement will be found in Section 4. This apparently is a "folk-theorem" of physics; I first ran across the statement of it in an article of J. A. Wheeler ([3], p. 84). The precise statement involves only the simplest notions of special relativity and the proof of the statement is an extremely simple application of the formalism of exterior differential forms and could be written down in a few lines. I prefer to preface the proof with a very brief summary of special relativity and of how electromagnetism fits into special relativity, mainly because most (but not all) treatments of this subject motivate their constructions by means of Maxwell's equations; from our view point this would be circular and far less appealing than the approach via the Lorentz force.

2. The Minkowski Space of Special Relativity. Space-time is a 4-dimensional

manifold M^4 that is topologically just R^4. A point in space-time is called an "event" and a curve of events is called a "world line". There are given admissible coordinate systems for M^4 corresponding to "inertial observers". In terms of such a coordinate system we may write $M^4 = R \times R^3$, and an event has coordinates (t, x, y, z) or briefly $(t, \mathbf{r})$; sometimes we shall write instead (x^0, x^1, x^2, x^3). Let (t', x', y', z') be coordinates set up by another inertial observer (physically, an observer moving uniformly with respect to the first observer). The basic assumption of special relativity is the following: if both observers look at the same world line, then in general

$$dt^2 \neq dt'^2 \text{ and } dx^2 + dy^2 + dz^2 \neq dx'^2 + dy'^2 + dz'^2,$$

but they will agree that (for simplicity we shall put the velocity of light $= 1$ in this article)

$$dt^2 - (dx^2 + dy^2 + dz^2) = dt'^2 - (dx'^2 + dy'^2 + dz'^2).$$

That is

$$d\tau^2 = dt^2 - d\mathbf{r} \cdot d\mathbf{r} = dt^2 - (dx^2 + dy^2 + dz^2)$$

defines an "invariant line element" in M^4.

The world line representing the history of a particle always has $d\tau^2 > 0$ (that is, the particle must travel at a speed less than the speed of light). For such a world line we can introduce τ as a new parameter (it is called "proper time"; the "chronometric hypothesis" states that τ is the time kept by an atomic clock moving with the particle) and we then have the unit tangent vector (the "velocity 4-vector")

$$V = \frac{dx}{d\tau}, \quad \text{i.e., } V^i = \frac{dx^i}{d\tau} \quad i = 0, 1, 2, 3.$$

Since

$$\frac{d\tau}{dt} = \left(1 - \frac{d\mathbf{r}}{dt} \cdot \frac{d\mathbf{r}}{dt}\right)^{\frac{1}{2}} = (1 - v^2)^{\frac{1}{2}},$$

where $\mathbf{v} = d\mathbf{r}/dt$ is the classical velocity vector, we may write the vector V as the 4 tuple

$$V = \left(\frac{dt}{d\tau}, \frac{d\mathbf{r}}{d\tau}\right) = \gamma(1, \mathbf{v}) \text{ with } \gamma = (1 - v^2)^{-\frac{1}{2}}.$$

Each particle has a "rest mass" m_0 (an invariant, independent of coordinates), and one then forms the momentum 4 vector

$$P = m_0 V = m_0 \gamma(1, \mathbf{v}) = (m, m\mathbf{v}),$$

where $m = m_0 \gamma$ is called the mass of the particle as viewed from the given coordinate system. As $v \to 1$ (speed of light) we have that $m = m_0 \gamma \to \infty$. Note that the spatial part $m\mathbf{v}$ of P is the classical momentum of a particle whose mass is m.

Finally, one has the notion of the Minkowski force 4-vector

$$f = \frac{dP}{d\tau} = \left(\frac{dm}{d\tau}, \frac{d}{d\tau}(m\mathbf{v})\right) = \left(\frac{dm}{d\tau}, \gamma\frac{d}{dt}(m\mathbf{v})\right) = (f^0, \gamma\mathbf{f}_c).$$

The term $(d/dt)(m\mathbf{v})$ is nothing but the classical force $\mathbf{f}_c$. Because $P = m_0 V$ is a vector of constant length m_0, we must have that $f = dP/d\tau$ is orthogonal to V in the Lorentz metric defined by $d\tau^2$. Now the scalar product of two 4 vectors $(A^0, \mathbf{A})$ and $(B^0, \mathbf{B})$ is given by

$$(A^0, \mathbf{A}) \cdot (B^0, \mathbf{B}) = A^0 B^0 - \mathbf{A} \cdot \mathbf{B}.$$

Since $f \cdot V = 0$ we conclude $\gamma f^0 - \gamma^2 \mathbf{f}_c \cdot \mathbf{v} = 0$, or $f^0 = \gamma \mathbf{f}_c \cdot \mathbf{v}$. (Incidentally, if we then equate f^0 with $dm/d\tau = \gamma\, dm/dt$ we get $dm = \mathbf{f}_c \cdot \mathbf{v}dt = \mathbf{f}_c \cdot d\mathbf{r}$, the element of work done by the force; this is Einstein's equivalence of mass and energy.)

We have written f as a contravariant vector $f^i = dP^i/d\tau$ since P is a tangent vector to a world line of a particle. One defines the covariant form of f, written $\mathfrak{f}$, by means of the usual process

$$\mathfrak{f}_i = \sum_{j=0}^{3} g_{ij} f^j, \text{ i.e., } \mathfrak{f} = (f^0, -\gamma\mathbf{f}_c) = \gamma(\mathbf{f}_c \cdot \mathbf{v}, -\mathbf{f}_c), \tag{5}$$

where (g_{ij}) is the metric tensor for $d\tau^2$, a diagonal matrix with $(1, -1, -1, -1)$ down the diagonal.

3. The electromagnetic field. The electromagnetic field vectors **E** and **B** in ordinary 3-space are measured via the Heaviside-Lorentz force formula; the force on a unit charge test particle is given by

$$\mathbf{f}_L = \mathbf{E} + \mathbf{v} \times \mathbf{B}, \tag{6}$$

where $\mathbf{v}$ is the velocity vector of the test charge. One determines **E** by measuring the force when the test particle is at rest, $\mathbf{v} = 0$. Having determined **E**, one then measures the force when the test charge is given velocities in several directions, and so **B** is determined. *Thus the Heaviside-Lorentz force formula actually serves to define the fields* **E** *and* **B**.

Einstein's study of special relativity apparently originated in essentially looking at equation (6) when one uses a coordinate system in 3-space that is moving uniformly with respect to the given system. For example, if one has $\mathbf{E} = 0$ and **B** is a constant field parallel to the z axis in a "fixed system", then when one passes to a system moving with speed v in the x direction, a unit test charge "at rest" in the moving system will suffer a force $\mathbf{v} \times \mathbf{B}$, and hence the moving observer will claim the existence of an electric field $\mathbf{E}' \neq 0$. Thus **E** and **B** individually may not be conceived as being vectors when one uses moving coordinate systems. To see how **E** and **B** should be joined together we can proceed as follows.

Suppose the unit test charge is moving along a curve in 3-space under the influence

of a purely electromagnetic field. The classical force is then the Heaviside-Lorentz force, $\mathbf{f}_c = \mathbf{f}_L = \mathbf{E} + \mathbf{v} \times \mathbf{B}$. The covariant Minkowski 4 force is then, from equation (5)

$$\mathfrak{f} = (f^0, -\gamma\mathbf{f}_L) = \gamma(\mathbf{f}_L \cdot \mathbf{v}, -(\mathbf{E} + \mathbf{v} \times \mathbf{B}))$$
$$= \gamma(\mathbf{E} \cdot \mathbf{v}, -(\mathbf{E} + \mathbf{v} \times \mathbf{B})).$$

Writing out the components in full, one sees that there is a unique matrix $(\mathscr{F}_{ij})$ such that

$$\mathfrak{f}_i = -\sum_{j=0}^{3} \mathscr{F}_{ij} V^j,$$

namely

$$(\mathscr{F}_{ij}) = \begin{bmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & B_z & -B_y \\ E_y & -B_z & 0 & B_x \\ E_z & B_y & -B_x & 0 \end{bmatrix},$$

which is called the Minkowski electromagnetic field tensor. Since this matrix is skew-symmetric it defines an exterior form (apparently first considered by Hargreaves [see **4**, p. 250]).

$$\mathscr{F} = \sum_{i<j} \mathscr{F}_{ij}\, dx^i \wedge dx^j = \mathscr{E} \wedge dt + \mathscr{B}, \tag{7}$$

where $\mathscr{E} = E_x(t, \mathbf{r})\, dx + E_y(t, \mathbf{r})\, dy + E_z(t, \mathbf{r})\, dz = \mathbf{E} \cdot d\mathbf{r}$ and

$$\mathscr{B} = B_x(t, \mathbf{r})\, dy \wedge dz + B_y(t, \mathbf{r})\, dz \wedge dx + B_z(t, \mathbf{r}) dx \wedge dy.$$

We shall refer to the book of Flanders [**2**] and the book of Y. Choquet-Bruhat [**1**] for any questions concerning differential forms and also for the uses of the form $\mathscr{F}$ in electromagnetic situations.

The 2-form $\mathscr{F} = \mathscr{E} \wedge dt + \mathscr{B}$ has been constructed using a particular inertial coordinate system (t, x, y, z). However, since all inertial coordinate systems will agree to use the Heaviside-Lorentz force formula to define the fields $\mathbf{E}$ and $\mathbf{B}$ in their coordinate systems we conclude that, in fact, *$\mathscr{F}$ is a well-defined exterior form on the Minkowski space M^4*. This has profound consequences. For instance, if (t', x', y', z') are coordinates used by another inertial coordinate system moving uniformly with respect to the first along the x axis, their coordinates are (as is well known) related by a Lorentz transformation

$$x' = \gamma(x - vt),\ y' = y,\ z' = z,\ t' = \gamma(t - vx). \tag{8}$$

Then, from the fact that $\mathscr{F}$ is well defined, we have $\mathscr{F} = \mathscr{E} \wedge dt + \mathscr{B} = \mathscr{E}' \wedge dt'$

$+ \mathscr{B}'$. Substituting equations (8) into $\mathscr{E}' \wedge dt' + \mathscr{B}'$ and comparing with $\mathscr{E} \wedge dt + \mathscr{B}$, one gets the well-known transformation laws for the fields **E** and **B**

$$\begin{aligned} E_x &= E'_x & B_x &= B'_x \\ E_y &= \gamma(E'_y + vB'_z) & B_y &= \gamma(B'_y - vE'_z) \\ E_z &= \gamma(E'_z - vB'_y) & B_z &= \gamma(B'_z + vE'_y). \end{aligned}$$

4. Maxwell's equations. Maxwell's source-free equations (1) and (3) are equivalent to the equation

$$d\mathscr{F} = 0,$$

where d is the exterior differential operator on Minkowski space M^4 (see [**1**] and [**2**] for details; we indicate briefly the essential ideas below). Symbolically, the exterior differential $\mathbf{d}$ for R^3 is

$$d = \mathbf{d} = dx \wedge \frac{\partial}{\partial x} + dy \wedge \frac{\partial}{\partial y} + dz \wedge \frac{\partial}{\partial z}$$

and the exterior differential d for M^4 is then $d = \mathbf{d} + dt \wedge (\partial/\partial t)$, all in terms of coordinates. Then

$$d\mathscr{F} = \left(\mathbf{d} + dt \wedge \frac{\partial}{\partial t}\right)(\mathscr{E} \wedge dt + \mathscr{B}) = \left(\mathbf{d}\mathscr{E} + \frac{\partial \mathscr{B}}{\partial t}\right) \wedge dt + \mathbf{d}\mathscr{B},$$

where by $\partial\mathscr{B}/\partial t$ we mean the 2-form $(\partial B_x/\partial t)\, dy \wedge dz + \cdots$. Thus

$$d\mathscr{F} = 0 \text{ iff } \begin{cases} \mathbf{d}\mathscr{B} = 0, \text{ i.e., } \operatorname{div} \mathbf{B} = 0 \\ \qquad \text{and} \\ \mathbf{d}\mathscr{E} + \dfrac{\partial \mathscr{B}}{\partial t} = 0, \text{ i.e., } \operatorname{curl} \mathbf{E} + \dfrac{\partial \mathbf{B}}{\partial t} = 0. \end{cases}$$

THEOREM. *Suppose that each inertial observer observes that* $\operatorname{div} \mathbf{B} = 0$. *Then* $d\mathscr{F} = 0$ *and hence each observer also observes that* $\operatorname{curl} \mathbf{E} = -(\partial \mathbf{B}/\partial t)$.

Proof. An inertial observer gives rise to a coordinate system (t, x, y, z) for Minkowski space, i.e., M^4 becomes the product $R \times R^3$. For fixed t_0 we have the inclusion map $\phi\colon R^3 \to R \times R^3$ sending R^3 onto the spatial section, $(x, y, z) \to (t_0, x, y, z)$. The observer puts $\mathscr{F} = \mathscr{E} \wedge dt + \mathscr{B}$. The restriction of $\mathscr{F}$ to his spatial section is simply $\phi^*\mathscr{F}$ (see [**2**], p. 23). But ϕ^* applied to forms merely says, put $t = t_0$ and $dt = 0$, and so

$$\phi^*\mathscr{F} = \phi^*\mathscr{B} = B_x dy \wedge dz + B_y dz \wedge dx + B_z dx \wedge dy.$$

By hypothesis $d(\phi^*\mathscr{B}) = \mathbf{d}(\phi^*\mathscr{B}) = (\operatorname{div} \mathbf{B})\, dx \wedge dy \wedge dz = 0$, and so $d(\phi^*\mathscr{F}) = 0$ for each inertial observer. But d is a "natural" operation, i.e., $d(\phi^*\mathscr{F}) = \phi^*(d\mathscr{F})$. Thus the exterior 3-form $d\mathscr{F}$ has the property that its restriction to the spatial section

(t_0, x, y, z) is the zero-form, and this is true for all t_0 and for all inertial observers. We claim that this implies $d\mathscr{F} = 0$.

Being a 3-form in 4-space, $d\mathscr{F}$ has the form

$$d\mathscr{F} = N^0 dx \wedge dy \wedge dz - N^1 dt \wedge dy \wedge dz + N^2 dt \wedge dx \wedge dz - N^3 dt \wedge dx \wedge dy$$

for some coefficients N^0, N^1, N^2, N^3. In terms of the 4-vector N whose components are these coefficients, we can write $d\mathscr{F}$ as the "interior product" (see [**1**], p. 149)

$$d\mathscr{F} = i_N dt \wedge dx \wedge dy \wedge dz.$$

We claim that $N = 0$, hence $d\mathscr{F} = 0$. Suppose $N \neq 0$; then one can easily find three vectors e_1, e_2, and e_3 such that N, e_1, e_2, e_3 are linearly independent on M^4 and e_1, e_2, and e_3 are tangent to the spatial section of some inertial observer. (If N is not in the x, y, z plane, choose e_1, e_2, e_3 in this plane. If N is in the x, y, z plane we may assume it is along the x axis; then e_1, e_2, e_3 can be chosen along the x', y', z' axes of the coordinate system defined by the Lorentz transformation (8).) But $d\mathscr{F}$, as a multilinear function on vector triples, has value

$$d\mathscr{F}(e_1, e_2, e_3) = dt \wedge dx \wedge dy \wedge dz(N, e_1, e_2, e_3) \neq 0$$

since N, e_1, e_2, e_3 are linearly independent. This contradicts the fact that $d\mathscr{F} = 0$ on any spatial section of any inertial observer. This concludes the proof that $d\mathscr{F} = 0$.

THEOREM. *Suppose that each inertial observer observes that* div $\mathbf{E} = \sigma$. *Then each observer also observes that* curl $\mathbf{B} = \mathbf{j} + \partial\mathbf{E}/\partial t$.

Proof. Let $*$ denote the Hodge star operator taking p forms on M^4 into $(4 - p)$ forms (see [**1**] or [**2**]). Since $\mathscr{F} = \mathscr{E} \wedge dt + \mathscr{B}$ is a well defined 2-form on M^4, so is

$$\mathscr{G} = * \mathscr{F} = -\mathscr{H} \wedge dt + \mathscr{D},$$

where $\mathscr{H} = B_x dx + B_y dy + B_z dz = \mathbf{H} \cdot d\mathbf{r}$, and $\mathscr{D} = E_x dy \wedge dz + E_y dz \wedge dx + E_z dx \wedge dy$. (Here we are glossing over the question of orientations; however both the forms $\mathscr{G}$ and $\mathscr{S}$ defined below are in fact forms of "odd kind" in the sense of de Rham, and so no difficulties arise.)

There is also the notion of rest charge density σ_0, an invariant for all observers. The 4-vector $J = \sigma_0 V = (\sigma_0\gamma, \sigma_0\gamma\mathbf{v}) = (\sigma, \mathbf{j})$ is called the current 4-vector density, $\sigma = \sigma_0\gamma$ is the charge density, and $\mathbf{j} = \sigma\mathbf{v}$ is the classical current density vector. The exterior 3-form

$$\mathscr{S} = i_J dt \wedge dx \wedge dy \wedge dz = \sigma dx \wedge dy \wedge dz - (j_x dy \wedge dz + j_y dz \wedge dx + j_z dx \wedge dy) \wedge dt$$

is called the current 3-form. A simple calculation (see [**1**], [**2**]) as was done in our first Theorem shows that

$$d\mathscr{G} = \mathscr{S} \text{ iff } \begin{cases} \text{div } \mathbf{E} = \sigma \\ \quad\text{and} \\ \text{curl } \mathbf{B} = \mathbf{j} + (\partial\mathbf{E}/\partial t). \end{cases}$$

Again let $\phi: R^3 \to R \times R^3$ be defined by taking the spatial section of an inertial observer. Then

$$\phi^*(d\mathscr{G} - \mathscr{S}) = d\phi^*\mathscr{G} - \phi^*\mathscr{S} = d\phi^*\mathscr{D} - \phi^*\mathscr{S} = (\operatorname{div}\mathrm{E} - \sigma)dx \wedge dy \wedge dz$$

vanishes, by hypothesis, for each inertial observer. By the same reasoning as before one concludes that $d\mathscr{G} - \mathscr{S} = 0$, and we are finished.

Work supported in part by N. S. F. grant G. P. -27670.

References

1. Y. Choquet-Bruhat, Géométrie différentielle et systèmes extérieurs, Dunod, Paris, 1968.

2. Harley Flanders, Differential forms, Academic Press, New York, 1963.

3. J. A. Wheeler, Gravitation as geometry-II, a chapter in the book *Gravitation and Relativity*, edited by H. Y. Chiu and W. F. Hoffmann, Benjamin, New York, 1964.

4. E. T. Whittaker, A history of the theories of aether and electricity, Vol. 2, Harper and Brothers, New York, 1960.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA, SAN DIEGO, LA JOLLA, CALIFORNIA 92037

EVERYWHERE DIFFERENTIABLE, NOWHERE MONOTONE, FUNCTIONS

Y. KATZNELSON AND KARL STROMBERG

The purpose of this paper is to construct an example of a real-valued function on R that is everywhere differentiable but is monotone on no interval and to examine further peculiar properties of any such function. Examples of such functions are seldom given, or even mentioned, in books on real analysis. The first explicit construction of such a function was given by Köpcke (1889). An example due to Pereno (1897) is reproduced in **[1]**, pp. 412–421. We believe that the present construction, which grew out of a question asked of the first named author by the second, is shorter, more elementary, and easier to understand than any that we have seen. We proceed through a sequence of easy lemmas.

LEMMA 1. *Let r and s be real numbers.*
(i) *If $r > s > 0$, then $(r-s)/(r^2-s^2) < 2/r$.*
(ii) *If $r > 1$ and $s > 1$, then $(r+s-2)/(r^2+s^2-2) < 2/s$.*

Proof. Assertion (i) is obvious. Inequality (ii) is equivalent to

$$(r-s)^2 + (r-1)(s-1) + r^2 + r + 3s > 5.$$

But this too is obvious when $r > 1$ and $s > 1$.

LEMMA 2. *Let* $\phi(x) = (1 + |x|)^{-\frac{1}{2}}$ *for* $x \in R$. *Then* $1/(b-a)\int_a^b \phi(x)dx < 4\min\{\phi(a), \phi(b)\}$ *whenever a and b are distinct real numbers.*

Proof. We may obviously suppose that $a < b$. In case $0 \leqq a$, we have by Lemma 1 that

$$\frac{1}{b-a}\int_a^b \phi(x)dx = \frac{2(\sqrt{1+b} - \sqrt{1+a})}{(1+b)-(1+a)}$$

$$< \frac{4}{\sqrt{1+b}} = 4\min\{\phi(a), \phi(b)\}.$$

The case that $b \leqq 0$ follows from the evenness of ϕ. Thus we suppose that $a < 0 < b$. Then Lemma 1 yields

$$\frac{1}{b-a}\int_a^b \phi(x)dx = \frac{2(\sqrt{1+b} + \sqrt{1-a} - 2)}{(1+b)+(1-a)-2}$$

$$< 4\min\{\phi(a), \phi(b)\}.$$

LEMMA 3. *If ϕ is as in Lemma 2 and ψ is any function of the form*

$$\psi(x) = \sum_{j=1}^{n} c_j\phi(\lambda_j(x-\alpha_j)),$$

where $c_1, \cdots, c_n$ and $\lambda_1, \cdots, \lambda_n$ are positive real numbers and $\alpha_1, \cdots, \alpha_n$ are any real numbers, then

$$\frac{1}{b-a}\int_a^b \psi(x)dx < 4\min\{\psi(a), \psi(b)\},$$

whenever a and b are distinct real numbers.

Proof. This follows at once from Lemma 2 and the fact that

$$\frac{1}{b-a}\int_a^b \phi(\lambda(x-\alpha))dx = \frac{1}{\lambda(b-\alpha)-\lambda(a-\alpha)}\int_{\lambda(a-\alpha)}^{\lambda(b-\alpha)} \phi(t)dt.$$

LEMMA 4. *Let $(\psi_n)_{n=1}^{\infty}$ be any sequence of functions as in Lemma 3. For $x \in R$ and each n define*

$$\Psi_n(x) = \int_0^x \psi_n(t)dt.$$

Suppose that $\sum_{n=1}^{\infty}\psi_n(a) = s < \infty$ for some $a \in R$. Then the series $F(x) = \sum_{n=1}^{\infty}\Psi_n(x)$ converges uniformly on every bounded subset of R, the function F is differentiable at a, and $F'(a) = s$. In particular, if $\sum_{n=1}^{\infty}\psi_n(t) = f(t) < \infty$ for all $t \in R$, then F is differentiable everywhere on R and $F' = f$.

Proof. Let $b \in R$ satisfy $b \geqq |a|$. Then, using Lemma 3, $-b \leqq x \leqq b$ implies

$$\begin{aligned} |\Psi_n(x)| &\leqq \left|\int_0^a \psi_n(t)dt\right| + \left|\int_a^x \psi_n(t)dt\right| \\ &\leqq 4|a|\psi_n(a) + 4|x-a|\psi_n(a) \\ &\leqq 12b\psi_n(a). \end{aligned}$$

Thus, uniform convergence on $[-b, b]$ follows from the Weierstrass M-test.

To prove that $F'(a) = s$, let $\varepsilon > 0$ be given. Choose N such that $10 \cdot \sum_{n=N+1}^{\infty} \psi_n(a) < \varepsilon$. Since each ψ_n is continuous at a, there exists some $\delta > 0$ such that

$$\left|\frac{1}{h}\int_a^{a+h} \psi_n(t)dt - \psi_n(a)\right| < \frac{\varepsilon}{2N},$$

whenever $0 < |h| < \delta$ and $1 \leqq n \leqq N$. Therefore, using Lemma 3 again, $0 < |h| < \delta$ implies that

$$\begin{aligned} \left|\frac{F(a+h)-F(a)}{h} - s\right| &= \left|\sum_{n=1}^{\infty}\left\{\frac{1}{h}\int_a^{a+h}\psi_n(t)dt - \psi_n(a)\right\}\right| \\ &\leqq \sum_{n=1}^{N}\left|\frac{1}{h}\int_a^{a+h}\psi_n(t)dt - \psi_n(a)\right| \\ &\quad + \sum_{n=N+1}^{\infty}\left\{\frac{1}{h}\int_a^{a+h}\psi_n(t)dt + \psi_n(a)\right\} \\ &< \frac{\varepsilon}{2} + \sum_{n=N+1}^{\infty} 5\psi_n(a) < \varepsilon. \end{aligned}$$

LEMMA 5. *Let $I_1, \cdots, I_n$ be disjoint open intervals, let α_j be the midpoint of I_j, and let ε and $y_1, \cdots, y_n$ be positive real numbers. Then there exists a function ψ as in Lemma 3 such that for each j*

(i) $\psi(\alpha_j) > y_j$,

(ii) $\psi(x) < y_j + \varepsilon$ *if* $x \in I_j$,

(iii) $\psi(x) < \varepsilon$ *if* $x \notin I_1 \cup \cdots \cup I_n$.

Proof. Choose $c_j = y_j + \varepsilon/2$ and write $\phi_j(x) = c_j\phi(\lambda_j(x - \alpha_j))$, where λ_j is chosen so large that $\phi_j(x) < \varepsilon/2n$ if $x \notin I_j$ (one needs only to check this inequality at an endpoint of I_j). Take $\psi = \phi_1 + \cdots + \phi_n$. Since the I_j's are disjoint and since ϕ_j takes its maximum value c_j at α_j, properties (i), (ii), and (iii) are evident.

THEOREM. *Let $\{\alpha_j\}_{j=1}^{\infty}$ and $\{\beta_j\}_{j=1}^{\infty}$ be disjoint countable subsets of R. Then there exists a real-valued, everywhere differentiable, function F on R satisfying $F'(\alpha_j) = 1$, $F'(\beta_j) < 1$ for all j, and $0 < F'(x) \leqq 1$ for all x.*

Proof. We obtain F as in Lemma 4 by first constructing $F' = f = \sum_{n=1}^{\infty} \psi_n$, or, more precisely, the partial sums $f_n = \sum_{k=1}^{n} \psi_k$, in such a way that

$$A_n\colon\ f_n(\alpha_j) > 1 - \frac{1}{n} \qquad (1 \leqq j \leqq n),$$

$$B_n\colon\ f_n(x) < 1 - \frac{1}{n+1} \qquad (x \in R),$$

$$C_n\colon\ \psi_n(\beta_j) < \frac{1}{2n \cdot 2^n} \qquad (1 \leqq j \leqq n).$$

Supposing that this were done we would have

$$F'(\alpha_j) = \lim_{n \to \infty} f_n(\alpha_j) = 1,$$

$$0 < F'(x) = \lim_{n \to \infty} f_n(x) \leqq 1,$$

and, choosing $n > j$,

$$\begin{aligned} F'(\beta_j) &= f_{n-1}(\beta_j) + \sum_{k=n}^{\infty} \psi_k(\beta_j) \\ &< 1 - \frac{1}{n} + \sum_{k=n}^{\infty} \frac{1}{2k \cdot 2^k} \\ &< 1 - \frac{1}{n} + \frac{1}{2n} \cdot 1 \\ &= 1 - \frac{1}{2n} < 1 \end{aligned}$$

and thus we would have the desired F.

We proceed inductively. First choose an open interval I with midpoint α_1 such that $\beta_1 \notin I$. Then apply Lemma 5 with $\varepsilon = y_1 = \frac{1}{4}$ to obtain $f_1 = \psi_1$ that satisfies A_1, B_1, C_1.

Suppose that $n > 1$ and that f_{n-1} and ψ_{n-1} have been chosen which satisfy A_{n-1}, B_{n-1}, C_{n-1}. Select disjoint open intervals $I_1, \cdots, I_n$ such that, for each $j \in \{1, \cdots, n\}$, α_j is the midpoint of I_j, $I_j \cap \{\beta_1, \cdots, \beta_n\} = \varnothing$ and $f_{n-1}(x) < f_{n-1}(\alpha_j) + \delta$ for $x \in I_j$, where

$$\delta = \frac{1}{n(n+1)} - \frac{1}{2n \cdot 2^n} > 0.$$

Now apply Lemma 5, with $\varepsilon = 1/(2n \cdot 2^n)$ and $y_j = 1 - (1/n) - f_{n-1}(\alpha_j)$ $(1 \leqq j \leqq n)$, to obtain ψ_n. Plainly C_n obtains. Also

$$f_n(\alpha_j) = f_{n-1}(\alpha_j) + \psi_n(\alpha_j) > f_{n-1}(\alpha_j) + y_j = 1 - \frac{1}{n}$$

$(1 \leqq j \leqq n)$, and so A_n obtains. To check B_n, notice that if $x \in I_j$, then

$$\begin{aligned} f_n(x) &= f_{n-1}(x) + \psi_n(x) \\ &< f_{n-1}(\alpha_j) + \delta + y_j + \varepsilon \\ &= 1 - \frac{1}{n} + \frac{1}{n(n+1)} = 1 - \frac{1}{n+1}; \end{aligned}$$

while if $x \notin \bigcup_{j=1}^{n} I_j$, then

$$f_n(x) = f_{n-1}(x) + \psi_n(x) < 1 - \frac{1}{n} + \varepsilon < 1 - \frac{1}{n+1}.$$

This completes the proof.

COROLLARY. *There exists a real-valued, everywhere differentiable, function H on R such that H is monotone on* **no** *subinterval of R and H' is bounded.*

Proof. Let $\{\alpha_j\}_{j=1}^{\infty}$ and $\{\beta_j\}_{j=1}^{\infty}$ be disjoint dense subsets of R. Apply the preceding theorem to obtain everywhere differentiable functions F and G on R such that

$$F'(\alpha_j) = G'(\beta_j) = 1, \qquad G'(\alpha_j) < 1,\ F'(\beta_j) < 1,$$
$$0 < F'(x) \leqq 1,\ 0 < G'(x) \leqq 1$$

for all j and x. Now write $H = F - G$. Then

$$H'(\alpha_j) > 0,\ H'(\beta_j) < 0$$
$$-1 < H'(x) < 1$$

for all j and x. Since $\{\alpha_j\}_{j=1}^{\infty}$ and $\{\beta_j\}_{j=1}^{\infty}$ are both dense, H cannot be monotone on an interval.

REMARKS. Let H be any function as in the preceding Corollary.

(a) H has a local maximum and a local minimum in every interval of R. In fact, if $a < b$, we can choose $\alpha < \beta$ in $[a, b]$ such that $H'(\alpha) > 0$ and $H'(\beta) < 0$. Then H takes an absolute maximum value on $[\alpha, \beta]$ at some $\gamma \in [\alpha, \beta]$. Clearly $\alpha < \gamma < \beta$, and so $H(\gamma)$ is a local maximum value for H. Similar reasoning produces a local minimum in $[a, b]$.

(b) Since H' is bounded, it is clear from the mean value theorem that H is absolutely continuous on every closed interval of R. Thus H' is Lebesgue integrable on each such interval and

$$H(x) = H(0) + \int_0^x H'(t)dt$$

for all $x \in R$.

(c) H' is **not** Riemann integrable on any closed interval $[a,b]$, for assume that it is. Then H' is continuous a.e. on $[a,b]$. But it is clear that $H'(t) = 0$ if H' is continuous at t, and so $H' = 0$ a.e. on $[a,b]$. It follows from (b) that H is a constant on $[a,b]$—a palpable contradiction.

(d) H' is of Baire class one, being the pointwise limit of the continuous functions $H_n(x) = n[H(x+1/n) - H(x)]$, and so the set of points at which H' is continuous is residual; i.e., its complement is of first category.

(e) Write $A = \{x: H'(x) > 0\}$ and $B = \{x: H'(x) < 0\}$. Thus $A \cap I$ and $B \cap I$ both have positive Lebesgue measure for every interval I. In fact, assuming that there exists some interval $I = [a,b]$ such that $B \cap I$ has measure zero, it follows that $H' \geqq 0$ a.e. on I. Therefore, since

$$H(x) = \int_a^x H'(t)dt + H(a)$$

for all $x \in I$, we conclude that H is nondecreasing on I—a contradiction. Similarly, if $A \cap I$ had measure zero, then H would be nonincreasing on I.

Reference

1. E. W. Hobson, Theory of Functions of a Real Variable II, Dover, New York, 1957.

DEPARTMENT OF MATHEMATICS, THE HEBREW UNIVERSITY OF JERUSALEM, ISRAEL.
DEPARTMENT OF MATHEMATICS, KANSAS STATE UNIVERSITY, MANHATTAN, KS 66502.

IS MATHEMATICAL TRUTH TIME-DEPENDENT?

JUDITH V. GRABINER

1. Introduction. Is mathematical truth time-dependent? Our immediate impulse is to answer no. To be sure, we acknowledge that standards of truth in the natural sciences have undergone change; there was a Copernican revolution in astronomy, a Darwinian revolution in biology, an Einsteinian revolution in physics. But do scientific revolutions like these occur in mathematics? Mathematicians have most often answered this question as did the nineteenth-century mathematician Hermann Hankel, who said, "In most sciences, one generation tears down what another has built, and what one has established, the next undoes. In mathematics alone, each generation builds a new story to the old structure." [**20**, p. 25.]

Hankel's view is not, however, completely valid. There have been several major upheavals in mathematics. For example, consider the axiomatization of geometry in ancient Greece, which transformed mathematics from an experimental science into a wholly intellectual one. Again, consider the discovery of non-Euclidean geometries

and non-commutative algebras in the nineteenth century; these developments led to the realization that mathematics is not about anything in particular; it is instead the logically connected study of abstract systems. These were revolutions in thought which changed mathematicians' views about the nature of mathematical truth, and about what could or should be proved.

Another such mathematical revolution occurred between the eighteenth and nineteenth centuries, and was focussed primarily on the calculus. This change was a rejection of the mathematics of powerful techniques and novel results in favor of the mathematics of clear definitions and rigorous proofs. Because this change, however important it may have been for mathematicians themselves, is not often discussed by historians and philosophers, its revolutionary character is not widely understood. In this paper, I shall first try to show that this major change did occur. Then, I shall investigate what brought it about. Once we have done this, we can return to the question asked in the title of this paper.

2. Eighteenth-century analysis: practice and theory. To establish what eighteenth-century mathematical practice was like, let us first look at a brilliant derivation of a now well-known result. Here is how Leonhard Euler derived the infinite series for the cosine of an angle. He began with the identity

$$(\cos z + i \sin z)^n = \cos nz + i \sin nz.$$

He then expanded the left-hand side of the equation according to the binomial theorem. Taking the real part of that binomial expansion and equating it to $\cos nz$, he obtained

$$\cos nz = (\cos z)^n - \frac{n(n-1)}{2!}(\cos z)^{n-2}(\sin z)^2 + \frac{n(n-1)(n-2)(n-3)}{4!}(\cos z)^{n-4}(\sin z)^4 - \cdots.$$

Let z be an infinitely small arc, and let n be infinitely large. Then:

$$\cos z = 1,\ \sin z = z,\ n(n-1) = n^2,\ n(n-1)(n-2)(n-3) = n^4,\ \text{etc.}$$

The equation now becomes recognizable:

$$\cos nz = 1 - \frac{n^2z^2}{2!} + \frac{n^4z^4}{4!} - \cdots.$$

But since z is infinitely small and n infinitely large, Euler concludes that nz is a finite quantity. So let $nz = v$. The modern reader may be left slightly breathless; still, we have

$$\cos v = 1 - \frac{v^2}{2!} + \frac{v^4}{4!} - \cdots.$$

(See [**16**, sections 133–4] and [**32**, pp. 348–9].)

Now that we have worked through one example, we shall be able to appreciate some generalizations about the way many eighteenth-century mathematicians worked. First, the primary emphasis was on getting results. All mathematicians know many of the results from this period, results which bear the names of Leibniz, Bernoulli, L'Hospital, Taylor, Euler, and Laplace. But the chances are good that these results were originally obtained in ways utterly different from the ways we prove them today. It is doubtful that Euler and his contemporaries would have been able to derive their results if they had been burdened with our standards of rigor. Here, then, is one major difference between the eighteenth-century way of doing mathematics and our way.

What led eighteenth-century mathematicians to think that results might be more important than rigorous proofs? One reason is that mathematics participated in the great explosion in science known as the Scientific Revolution [**19**]. Since the Renaissance, finding new knowledge had been a major goal of all the sciences. In mathematics, ever since the first major new result — the solution to the cubic equation published in 1545 — increasing mathematical knowledge had meant finding new results. The invention of the calculus at the end of the seventeenth century intensified the drive for results; here was a powerful new method which promised vast new worlds to conquer. One can imagine few more exciting tasks than trying to solve the equations of motion for the whole solar system. The calculus was an ideal instrument for deriving new results, even though many mathematicians were unable to explain exactly why this instrument worked.

If the overriding goal of most eighteenth-century mathematics was to get results, we would expect mathematicians of the period to use those methods which produced results. For eighteenth-century mathematicians, the end justified the means. And the successes were many. New subjects arose in the eighteenth century, each with its own range of methods and its own domain of results: the calculus of variations, descriptive geometry, and partial differential equations, for instance. Also, much greater sophistication was achieved in existing subjects, like mathematical physics and probability theory.

The second generalization we shall make about eighteenth-century mathematics and its drive for results is that mathematicians placed great reliance on the power of symbols. Sometimes it seems to have been assumed that if one could just write down something which was symbolically coherent, the truth of the statement was guaranteed. And this assumption was not applied to finite formulas only. Finite methods were routinely extended to infinite processes. Many important facts about infinite power series were discovered by treating the series as very long polynomials [**30**].

This trust in symbolism in the eighteenth century is somewhat anomalous in the history of mathematics, and needs to be accounted for. It came both from the success of algebra and the success of the calculus. Let us first consider algebra.

General symbolic notation of the type we now take for granted was introduced in 1591 by the French mathematician François Viète [**6**, pp. 59–65] and [**32**, pp. 74–81]. This notation proved to be the greatest instrument of discovery in the history of mathematics. Let us illustrate its power by one example. Consider the equation

$$(x-a)(x-b)(x-c) = x^3 - (a+b+c)x^2 + (ab+ac+bc)x - abc. \tag{2.1}$$

Symbolic notation lets you discover what dozens of numerical examples may not: the relation between the roots and the coefficients of any polynomial equation of any degree. Equation (2.1), furthermore, has degree three, and has three roots. Relying on results like (2.1), Albert Girard in 1629 stated that an nth degree equation had n roots — the first formulation of what Gauss later called the Fundamental Theorem of Algebra.

But why are algebraic formulas like (2.1) considered true by eighteenth-century mathematicians? Because, as Newton put it, algebra is just a "universal arithmetic" [**29**]. Equation (2.1) is valid because it is a generalization about valid arithmetical statements. What, then, about infinite arguments, like the one of Euler's we examined earlier? The answer is analogous. Just as there is an arithmetic of infinite decimal fractions, we may generalize and create an algebra of infinite series [**28**, p. 6]. Infinite processes are like finite ones — except that they take longer.

The faith in symbolism nourished by algebra was enhanced further by the success of the calculus. Leibniz had invented the notations dy/dx and $\int y dx$ expressly to help us do our thinking. The notation serves this function well; we owe a debt to Leibniz every time we change variables under the integral sign. Or, suppose y is a function of x and that x is a function of t; we want to know dy/dt. It is not Leibniz, but Leibniz's notation that discovers the chain rule:

$$dy/dt = (dy/dx)\,(dx/dt).$$

The success of Leibniz's notation for the calculus reinforced mathematicians' belief in the power of symbolic arguments to give true conclusions.

In the eighteenth century, belief in the power of good notation extended beyond mathematics. For instance, it led the chemist Lavoisier to foresee a "chemical algebra," in the spirit of which Berzelius in 1813 devised chemical symbols essentially like those we use today. Anybody who has balanced chemical equations knows how the symbols do some of the thinking for us. The fact that the idea of the validity of purely symbolic arguments spread from mathematics to other areas shows us how prevalent an idea it must have been.

What has been said so far should not lead the reader to believe that eighteenth-century mathematicians were completely indifferent to the foundations of analysis. They certainly discussed the subject, and at length. I shall not here summarize the diverse eighteenth-century attempts to explain the nature of dy/dx, of limits, of the infinite, and of integrals, during a century that Carl Boyer has rightly called "the period of indecision" as far as foundations were concerned [**7**, Chapter VI]. What

must be emphasized for our present purposes is that discussions of foundations were not the basic concern of eighteenth-century mathematicians. That is, discussions of foundations do not generally appear in research papers in scientific journals; instead, they are relegated to Chapter I of textbooks, or found in popularizations. More important, the practice of mathematics did not depend on a perfect understanding of the basic concepts used. But this was no longer the situation in nineteenth-century mathematics, and, of course, is not the situation today.

Nineteenth-century analysts, beginning with Cauchy and Bolzano, gave rigorous, inequality-based treatments of limit, convergence, and continuity, and demanded rigorous proofs of the theorems about these concepts. We know what these proofs were like; we still use them. This new direction in nineteenth-century analysis is not just a matter of differences in technique. It is a major change in the way mathematics was looked at and done. Now that we have sketched the eighteenth-century approach, we are ready to deal with what are—from the historical point of view—the most interesting questions of this paper. What made the change between the old and new views occur? How did mathematics get to be the way it is now?

Two things were necessary for the change. Most obviously, the techniques needed for rigorous proofs had to be developed. We shall discuss the history of some major techniques in Section 4, below. But also, there had to be a change in attitude. Without the techniques, of course, the change in attitude could never have borne fruit. But the change in attitude, though not sufficient, was a necessary condition for the establishment of rigor. Our next task, accordingly, will be to explain the change in attitude toward the foundations of the calculus between the eighteenth and nineteenth centuries. Did the very nature of mathematics force this change? Or was it motivated by factors outside of mathematics? Let us investigate various possibilities.

3. Why did standards of mathematical truth change? The first explanation which may occur to us is like the one we use to justify rigor to our students today: the calculus was made rigorous to avoid errors, and to correct errors already made. But this is not quite what happened. In fact, there are surprisingly few mistakes in eighteenth-century mathematics. There are two main reasons for this. First, some results could be verified numerically, or even experimentally; thus, their validity could be checked without a rigorous basis. Second, and even more important, eighteenth-century mathematicians had an almost unerring intuition. Though they were not guided by rigorous definitions, they nevertheless had a deep understanding of the properties of the basic concepts of analysis. This conclusion is supported by the fact that many apparently shaky eighteenth-century arguments can be salvaged, and made rigorous by properly specifying hypotheses. Nevertheless, we must point out that the need to avoid errors became more important near the end of the eighteenth century, when there was increasing interest among mathematicians in complex functions, in functions of several variables, and in trigonometric series. In these subjects, there are many plausible conjectures whose truth is relatively difficult to

evaluate intuitively. Increased interest in such results may have helped draw attention to the question of foundations.

A second possible explanation which may occur to us is that the calculus was made rigorous in a spirit of generalization. The eighteenth century had produced a mass of results. The need to unify such a mass of results could have led automatically to a rigorous, axiomatic basis. But there had been large numbers of results for a hundred years before Cauchy's work. Besides, unifying results does not always make them rigorous; moreover, the function of rigor is not just to unify, but to prove. Still, there is something to be said for the hypothesis that the calculus became rigorous partly to unify the wealth of existing results. At the end of the eighteenth century, several mathematicians thought that the pace of getting new results was decreasing. This feeling had some basis in fact; most of the results obtainable by the routine application of eighteenth-century methods had been obtained. Perhaps, if progress was slowing, it was time to sit back and reflect about what had been done [**31**, pp. 136–7]. This feeling helped get some mathematicians interested in the question of rigor.

A third possible explanation depends on the prior existence of rigor in geometry. Everybody from the Greeks on knew that mathematics was supposed to be rigorous. One might thus assume that mathematicians' consciences began to trouble them, and that as a result analysts returned their new methods to the old standards. In fact, Euclidean geometry did provide a model for the new rigor. But the old ideas of rigor were not enough in themselves to make mathematicians strive to make the calculus rigorous—as the hundred and fifty years from Newton to Cauchy shows. This is true even though the discrepancy between Euclidean standards and the actual practice of eighteenth-century mathematicians did not go unnoticed. George Berkeley, Bishop of Cloyne, attacked the calculus in 1734, on the perfectly valid grounds that it was not rigorous the way mathematics was supposed to be. Berkeley wanted to defend religion against the attacks of unreasonableness levelled against it by eighteenth-century scientists and mathematicians. Berkeley said that his opponents did not even reason well about mathematics. He conceded that the results of the calculus were valid, but attacked its methods. Berkeley's attack, *The Analyst*, is a masterpiece of polemics [**32**, pp. 333–338] and [**3**]. He said of the "vanishing increments" that played so crucial a role in Newton's calculus, "And what are these ...vanishing increments? They are neither finite quantities, nor quantities infinitely small, nor yet nothing. May we not call them the ghosts of departed quantities?" Berkeley's attack—which included point-by-point mathematical criticisms of some basic arguments of Newton's calculus—provoked a number of mathematicians to write refutations. However, neither Berkeley's attack nor the replies to it produced the change in attitude toward rigor which we are trying to explain. First of all, the replies are not very convincing [**8**]. Besides, the subject of foundations was still not considered serious mathematics. Berkeley did get people thinking, more than they would have without him, about the problem of foundations. The discussions of

foundations by Maclaurin, D'Alembert, and Lagrange were all at least somewhat influenced by Berkeley's work. Nevertheless, Berkeley's attack in itself was not enough to cause foundations to become a major mathematical concern.

In bringing about the change, there is one other factor which, though seldom mentioned in this connection, was important: the mathematician's need to teach. Near the end of the eighteenth century, a major social change occurred. Before the last decades of the century, mathematicians were often attached to royal courts; their job was to do mathematics and thus add to the glory, or edification, of their patron. But almost all mathematicians since the French Revolution have made their living by teaching [**31**, p. 140] [**2**, p. 95,108].

This change in the economic circumstances of mathematicians had other causes than the decline of particular royal courts. In the eighteenth century, science was expanding. This was the "age of Newton" and the success of Newtonian science. Governments and businessmen felt that science was important and could be useful; scientists encouraged them in these beliefs. So governments founded educational institutions to promote science. Military schools were founded to provide prospective officers with knowledge of applied science. New scientific chairs were endowed in existing universities. By far the most important new institution for scientific instruction, one which served as a model to several nations in the nineteenth century, was the *École polytechnique* in Paris, founded in 1795 by the revolutionary government in France.

Why might the new economic circumstances of mathematicians—the need to teach—have helped promote rigor? Teaching always makes the teacher think carefully about the basis for the subject. A mathematician could understand enough about a concept to use it, and could rely on the insight he had gained through his experience. But this does not work with freshmen, even in the eighteenth century. Beginners will not accept being told, "After you have worked with this concept for three years, you'll understand it."

What is the evidence that teaching helped motivate eighteenth and nineteenth century mathematicians to make analysis rigorous? First, until the end of the eighteenth century, most work on foundations did not appear in scientific journals, apparently because foundations were not considered to pose major mathematical (as opposed to philosophical) questions. Instead, such work appeared in courses of lectures, in textbooks, or in popularizations. Even in the nineteenth century, when foundations had been established as essential to mathematics, their origin was often in teaching. The work on foundations of analysis of Lagrange [**23, 26**], of Cauchy [**10, 11**], of Weierstrass [**21**, pp. 283–4] [**7**, pp. 284–7], and of Dedekind [**14**, p. 1], all originated in courses of lectures.

Each of the points we have made so far helps explain what motivated mathematicians to shift from the result-oriented view of the eighteenth century to the more rigorous standards of the nineteenth. One more catalyst of the change should be identified: Joseph-Louis Lagrange. Lagrange's own interest in the problem of

foundations was first engaged by having to teach the calculus at the military school in Turin [**24**]. In 1784, by proposing the foundations of the calculus as a prize problem for the Berlin Academy of Sciences, he stimulated the first major book-length contributions to foundations of the calculus written on the Continent. (see [**27**] [**9**] [**7**, p. 254–255] and [**18**, pp. 149–150]). Above all, Lagrange's lectures at the *École polytechnique*, published in two widely influential books, attempted to give a general and algebraic framework for the calculus [**26**] [**23**]. Lagrange did not correctly solve the problem of foundations — we can no longer accept his *definition* of $f'(x)$ as the coefficient of h in the Taylor series expansion of $f(x + h)$. Nevertheless, his vision of reducing the calculus to algebra decisively influenced the work of Bolzano [**5**] and — as we shall see — of Cauchy.

The change in attitude we have been discussing was not enough in itself to establish rigor in the calculus — as the example of Lagrange shows. Having decided that we want to make a subject rigorous, what else do we need? Two more things are required: the right definitions, and techniques of proof to derive the known results from the definitions. We must now answer another question: where did the required definitions and proofs come from?

Eighteenth-century mathematicians themselves had developed many of the techniques, and isolated many of the basic defining properties — even though they did not know that this is what they were doing. It is amazing that so many of the techniques used by Cauchy in rigorous arguments had been around for so long. This fact shows that a real change in point of view was required for the rigorization of analysis; it was not an automatic development out of eighteenth-century mathematics.

4. The eighteenth-century origins of nineteenth-century rigor. We shall illustrate the eighteenth-century origins of nineteenth-century rigor by giving several examples of eighteenth-century work which was transformed into nineteenth-century definitions and proofs. The principal area of eighteenth-century mathematics we shall investigate is the study of approximations. Eighteenth-century mathematicians, whether solving algebraic equations or differential equations, developed many useful approximation methods. When the goal is results, an approximate result is better than nothing. Paradoxically, eighteenth-century mathematicians were most exact when they were being approximate; their work with inequalities in approximations later became the basis for rigorous analysis.

We shall discuss two classes of eighteenth-century approximation work: the actual working out of approximation procedures, and the computation of error estimates. Let us see what use nineteenth-century analysts made of these.

One new way in which nineteenth-century mathematicians looked at eighteenth-century approximations was to see the approximate solution as a construction of that solution, and therefore as a proof of its existence. For instance, Cauchy did this in developing what is now called the Cauchy-Lipschitz method of proving the

existence of the solution to a differential equation; the proof is based on an approximation method developed by Euler [**15**, pp. 424–5] [**12**, p. 399 ff]. Similarly Cauchy's elegant proof of the intermediate-value theorem for continuous functions was based on an eighteenth-century approximation method [**22**, pp. 260–1] [**25**, sections 2,6] [**10**, pp. 378–80]. For a continuous function $f(x)$, Cauchy took $f(a)$ and $f(b)$ of opposite sign, divided the interval $[a, b]$ into n parts, and concluded that there were at least two values of x on $[a, b]$, differing by $(b-a)/n$, which yielded opposite sign for $f(x)$. He then repeated the procedure on the interval between these two new values, on an interval of length $(b-a)/n$, which gives two more values, differing by $(b-a)/n^2$, and so on. Where Lagrange had used this technique to approximate to the root ξ of a polynomial included between $x=a$ and $x=b$, Cauchy used it to argue for the existence of the number ξ as the common limit of the sequences of values of x which gave positive sign for f, and negative sign for f. The origin of Cauchy's proof in algebraic approximations is further demonstrated by the context in which he gave it: a "*Note*" devoted to discussing the approximate solution of algebraic equations [**10**, p. 378 ff].

Another example of the conversion of approximations into existence proofs is given by Cauchy's theory of the definite integral. In the eighteenth century, it was customary to define the integral as the inverse of the derivative. It was known, however, that the value of the integral could be approximated by a sum. Cauchy took Euler's work on approximating the values of definite integrals by sums [**15**, pp. 184–7], and looked at it from an entirely new point of view. Cauchy *defined* the definite integral as the limit of a sum, proved the existence of the definite integral of a continuous (actually, uniformly continuous) function, and then used his definition to prove the Fundamental Theorem of Calculus [**11**, pp. 122–5, 151–2].

Now let us consider another type of result in eighteenth-century approximations: approximations given along with an error estimate. These results took a form like this: given some n, the mathematician could compute an upper bound on the error made in taking the nth approximation for the true value. Near the end of the eighteenth century, the algebra of inequalities was exploited with great skill in computing such error estimates [**13**, pp. 171–183] and [**25**, pp. 46–7, p. 163]. Cauchy, Abel, and their followers turned the approximating process around. Instead of being given n and finding the greatest possible error, we are *given* what is in effect the "error"—epsilon—and, provided that the process converges, we can always find n such that the error of the nth approximation is less than epsilon. (This seems to be the reason for the use of the letter "epsilon" in its usual modern sense by Cauchy [**10**, pp. 64–5 *et passim*].) [**1**] [**10**, pp. 400–415]. Cauchy's definition of convergence—which is essentially ours—is based on this principle [**10**, Chapter VI].

Another way in which nineteenth-century mathematicians changed eighteenth-century views of results using inequalities was to take facts known to eighteenth-century mathematicians in special cases and to make them legitimate in general. For instance, D'Alembert and others had shown that some particular series con-

verged by showing that they were, term-by-term, less than a convergent geometric progression [**13**]. Gauss in 1813 used this criterion to investigate, in a rigorous manner, the convergence of the hypergeometric series [**17**]. Cauchy used the comparison of a given series with a geometric one to derive and to prove some general tests for the convergence of any series; the ratio test, the logarithm test, and the root test [**10**, pp. 121–127].

Let us look at one last example — a very important one — of an eighteenth-century result which became something different in the nineteenth century: the property of the derivative expressed by

$$f(x+h) = f(x) + hf'(x) + hV, \tag{4.1}$$

where V goes to zero with h. As we have remarked, Lagrange had defined $f'(x)$ as the coefficient of h in the Taylor expansion of $f(x+h)$. He then "derived" (4.1) from that Taylor series expansion, considering V to be a convergent infinite series in h. Lagrange used (4.1) to investigate many properties of the derivative. To do this, he interpreted "V goes to zero with h" to mean that, for any given quantity D, we can find h sufficiently small so that $f(x+h) - f(x)$ "will be included between" $h[f'(x) - D]$ and $h[f'(x) + D]$ [**23**, p. 87]. First Cauchy, and then Bolzano and Weierstrass, made (4.1) and its associated inequalities into the *definition* of $f'(x)$. (Cauchy's definition was actually verbal, but he translated it into the language of inequalities in proofs.) [**11**, pp. 44–5; 122–3], [**4**, Chapter 2] and [**7**, pp. 285–7]. This definition made legitimate the results about $f'(x)$ that Lagrange had derived from (4.1) — for instance, the mean-value theorem for derivatives. (Except, we must note, for a few errors, especially the confusion between convergence and uniform convergence, which was not cleared up until the 1840's.)

Of course, we do not mean to imply that Gauss, Cauchy, Bolzano, Abel, and Weierstrass were not original, creative mathematicians. They were. To show that major changes in point of view occur in mathematics, we have concentrated in this section on what these men owed to eighteenth-century techniques. But, besides transforming what they borrowed, they contributed much of their own that was new. Cauchy, in particular, devised beautiful proofs about convergent power series in real and complex variables, about real and complex integrals, and, of course, contributed to a variety of subjects besides analysis. Nevertheless, for our present purposes, we need the biassed sample we have chosen — things accomplished either by taking what the eightenth century knew for particular cases and making it general, or by taking what the eighteenth century had derived for one purpose and putting it to a more profound use.

Much effort was needed to transform eighteenth-century techniques in the ways we have discussed. But it was more than just a matter of effort. It took asking the right questions *first*; and then using — and expanding — the already existing techniques to answer them. It took — and was — a major change in point of view. The reawakening of interest in rigor was just as necessary as the availability of techniques to produce

the point of view of Bolzano and Cauchy—the point of view which has been with us ever since. Mathematics requires not only results, but clear definitions and rigorous proofs. Individual mathematicians may still concentrate on the creation of fruitful methods and ideas to be exploited, but the mathematical community as a whole can no longer be indifferent to rigor.

5. Conclusion. We began by asking whether mathematical truth was time-dependent. Perhaps mathematical truth is eternal, but our knowledge of it is not. We have now seen an example of how attitudes toward mathematical truth have changed in time. After such a revolution in thought, earlier work is re-evaluated. Some is considered worth more; some, worth less.

What should a mathematician do, knowing that such re-evaluations occur?

Three courses of action suggest themselves. First, we can adopt a sort of relativism which has been expressed in the phrase "Sufficient unto the day is the rigor thereof." Mathematical truth is just what the editors of the *Transactions* say it is. This is a useful view at times. But this view, if universally adopted, would mean that Cauchy and Weierstrass would never have come along. Unless there were the prior appearance of major errors, standards could never improve in any important way. So the attitude of relativism, which would have counselled Cauchy to leave foundations alone, will not suffice for us.

Second, we can attempt to set the highest conceivable standard: never use an argument in which we do not completely understand what is going on, dotting all the *i*'s and crossing all the *t*'s. But this is even worse. Euler, after all, knew that there were problems in dealing with infinitely large and infinitely small quantities. According to this high standard, which textbooks sometimes urge on students, Euler would never have written a line. There would have been no mathematical structure for Cauchy and Weierstrass to make rigorous.

So I suggest a third possibility: a recognition that the problem I have raised is just the existential situation mathematicians find themselves in. Mathematics grows in two ways: not only by successive increments, but also by occasional revolutions. Only if we accept the possibility of present error can we hope that the future will bring a fundamental improvement in our knowledge. We can be consoled that most of the old bricks will find places somewhere in the new structure. Mathematics is *not* the unique science without revolutions. Rather, mathematics is that area of human activity which has at once the least destructive and still the most fundamental revolutions.

This paper was originally delivered at the Mathematical Association of America, Southern California Section, March 1972. The author wishes to thank Elmer Tolsted for encouragement and suggestions.

References

1. N. H. Abel, Recherches sur la série $1+(m/1)x+[m(m-1)/1.2]x^2+[m(m-1)(m-2)/1.2.3]x^3 + \cdots$, Oeuvres complètes, Vol. I, Christiania, 1881.

2. J. Ben David, The Scientist's Role in Society, Prentice-Hall, Englewood Cliffs, 1971.

3. G. Berkeley, The Works of George Berkeley, Vol. IV, ed. A. A. Luce and T. E. Jessop, Edinburgh, 1948–1957.

4. B. Bolzano, Functionenlehre, Schriften, Band I, Prague, 1930.

5. ———, Rein analytischer Beweis des Lehrsatzes, dass zwischen je zwei Werthen, die ein entgegengesetztes Resultat gewaehren, wenigstens eine reelle Wurzel der Gleichung liege, 1817, Englemann, Leipzig, 1905.

6. Carl Boyer, History of Analytic Geometry, Scripta Mathematica, New York, 1956.

7. ———History of the Calculus and its Conceptual Development, Dover, New York, 1959.

8. F. Cajori, A History of the Conceptions of Limits and Fluxions in Great Britain from Newton to Woodhouse, Open Court, Chicago, 1931.

9. L. N. M. Carnot, Réflexions sur la métaphysique du calcul infinitésimal, Duprat, Paris, 1797.

10. A.-L. Cauchy, Cours d'analyse de l'école royale polytechnique, Imprimerie royale, Paris, 1821, in Oeuvres Complètes, Series 2, Vol. III, Gauthier-Villars, Paris, 1897.

11. A.-L. Cauchy, Résumé des leçons données à l'école royale polytechnique sur le calcul infinitésimal, Imprimerie royale, Paris, 1823, in Oeuvres Complètes, Series 2, Vol. IV, Gauthier-Villars, Paris, 1899.

12. A.-L. Cauchy, Exercices d'analyse, 1840, in Oeuvres, Series 2, Vol. XI.

13. Jean D'Alembert, Réflexions sur les suites et sur les racines imaginaires, Opuscules mathématiques, vol. V, Paris 1768, pp. 171–215.

14. Richard Dedekind, Essays on the theory of numbers, Dover, New York, 1963.

15. Leonhard Euler, Institutiones calculi integralis 1768, Opera Omnia, Series 1, vol. XI, Teubner, Leipzig and Berlin, 1911.

16. ———, Introductio in analysin infinitorum 1748, Opera Omnia, Series 1, vols. 8–9.

17. K. F. Gauss, Disquisitio generales circa seriem infinitam

$$1 + \frac{\alpha \cdot \beta}{1 \cdot \gamma}x + \frac{\alpha(\alpha+1)\beta(\beta+1)}{1 \cdot 2 \cdot \gamma(\gamma+1)}x^2 + \frac{\alpha(\alpha+1)(\alpha+2)\beta(\beta+1)(\beta+2)}{1 \cdot 2 \cdot 3 \cdot \gamma(\gamma+1)(\gamma+2)}x^3 + \cdots, [1813],$$

Werke, Vol. 3, pp. 123–162; German translation, Berlin, 1888.

18. C. C. Gillispie, Lazare Carnot Savant, Princeton, 1971.

19. A. R. Hall, The Scientific Revolution, 1500–1800, Beacon, Boston, 1966.

20. H. Hankel, Die Entwicklung der Mathematik im letzten Jahrhundert, 1884, quoted by M. Moritz, On Mathematics and Mathematicians, Dover, New York, 1942, p. 14.

21. F. Klein, Vorlesungen über die Entwicklung der Mathematik im 19. Jahrhundert, 1926, reprinted by Chelsea, New York, 1967.

22. J.-L. Lagrange, Leçons élémentaires sur les mathématiques, données à l'école normale en 1795, Oeuvres, VIII, Gauthier-Villars, Paris, 1867–1892, pp. 181–288.

23. ———, Leçons sur le calcul des fonctions, 2d edition, 1806, Oeuvres, X.

24. ———, Letter to Euler, 24 November 1759, Oeuvres, XIV, pp. 170–174.

25. ———, Traité de la résolution des équations numériques de tous les degrés, 1808, Oeuvres, VIII.

26. ———, Théorie des fonctions analytiques, 2d. edition, 1813, Oeuvres, IX.

27. S. L'Huilier, Exposition élémentaire des principes des calculs supérieurs, Decker, Berlin, 1787.

28. Isaac Newton, On the analysis by equations of an infinite number of terms, 1669, in D. T. Whiteside, ed., The Mathematical Works of Isaac Newton, Johnson Reprint, London and New York, 1964, vol. I.

29. Isaac Newton, Universal Arithmetic, 1707, in D. T. Whiteside ed., The Mathematical Works of Isaac Newton, vol. II, Johnson, London and New York, 1970.

30. R. Reiff, Geschichte der unendlichen Reihen, Tübingen, 1889.

31. D. J. Struik, Concise History of Mathematics, Dover, New York, 1967.

32. D. J. Struik, ed., A Source Book in Mathematics, 1200–1800, Harvard, Cambridge, 1967.

Small College, California State College, Dominigues Hills, CA 90505.

QUERIES

EDITED BY A. C. ZITRONENBAUM

This Department welcomes queries from readers about mathematics at the collegiate level, such as sources for exposition of a particular topic from a special point of view, references to vaguely remembered articles, descriptions of special kinds of courses or teaching methods, and methods for constructing illustrative examples for exercises of particular kinds (questions on research topics should, in general, be addressed to the "Queries Department" of the Notices of the American Mathematical Society). Replies will be forwarded to the questioner and may also be edited into a composite answer for publication in this Department. Consequently all items submitted for consideration for possible publication should include the name and complete mailing address of the person who is to receive the reply. Queries and answers should be sent to A. C. Zitronenbaum, Department of Mathematics, Cornell University, Ithaca, NY 14850.

7. D. E. Christie. Are there in print mathematical discussions, suitable for sophomores who have had a first introduction to linear algebra, of

i) the lens formulas of Newton and Gauss?
ii) the circuit theory theorems of Thévenin and Norton?
iii) the Buckingham pi theorem of dimensional analysis?

8. D. E. Christie. Teachers of mathematics courses for non-majors need as bait expository articles by **social scientists** using (respectably) simple mathematical models. Does anyone have a favorite article of this type in political science, psychology, sociology, economics?

MATHEMATICAL NOTES

EDITED BY DAVID ROSELLE

Material for this Department should be sent to David Roselle, Department of Mathematics, Louisiana State University, Baton Rouge, LA 70803.

A PROBABILISTIC PROOF OF STIRLING'S FORMULA

RASUL A. KHAN

1. Introduction. The object of this note is to give a probabilistic proof of Stirling's formula using essentially the central limit theorem. It is worthwhile to point out the motivation in writing this paper. In order to follow adequate historical developments (Bernoulli, DeMoivre and Laplace, etc.), a number of probability texts prove certain limit theorems for coin-tossing random variables by making use of

Stirling's formula. Now that one uses the powerful tools of characteristic functions to prove the general limit theorems, one wonders if it is possible to deduce Stirling's formula from any limit theorem. The present note shows this connection.

2. Two simple theorems. In this section we mention two well-known results (the first being the classical central limit theorem and the second a special case of a moment convergence theorem), which are easily proved by the methods of characteristic functions and Helly-Bray lemma, etc. In what follows E denotes the expectation operator.

THEOREM 1. *Let $X_1, X_2, \cdots$ be iid (independent and identically distributed) random variables with $EX_1 = 0$, $EX_1^2 = 1$, and set $S_n = X_1 + X_2 + \cdots + X_n$. Then*

$$\lim_{n\to\infty} P\left(\frac{S_n}{\sqrt{n}} \leqq x\right) = (2\pi)^{-\frac{1}{2}} \int_{-\infty}^{x} e^{-u^2/2} du. \tag{1}$$

A moment convergence theorem (see for example, Loève: pp. 183–184, or Breiman: pp. 163–164) gives the following special case.

THEOREM 2. *Let $\{X_n, n \geqq 1\}$ be a sequence of random variables such that X_n converges to X in distribution, and that* $\limsup_{n\to\infty} EX_n^2 < \infty$, *then*

$$\lim_{n\to\infty} E|X_n|^r = E|X|^r, \quad 0 \leqq r < 2. \tag{2}$$

Hence, under the conditions and notation of Theorem 1 we have

$$\lim_{n\to\infty} E\frac{|S_n|}{\sqrt{n}} = (2/\pi)^{\frac{1}{2}}. \tag{3}$$

3. Proof of Stirling's formula. Stirling's formula states that

$$\lim_{n\to\infty} \frac{\sqrt{2n\pi}\, n^n e^{-n}}{\Gamma(n+1)} = 1. \tag{4}$$

To deduce (4) from (3) is an easy matter. To this end, consider iid random variables $X_1, X_2 \cdots$ having probability density function (with respect to Lebesgue measure) as

$$\begin{aligned} f(x) &= e^{-x}, \quad && x > 0 \\ &= 0 && \text{otherwise.} \end{aligned}$$

If we set $Y_i = X_i - 1$, $i = 1,2 \cdots$, then $Y_1, Y_2, \cdots$ are iid random variables with $EY_1 = 0$, and $EY_1^2 = 1$. Setting $S_n = X_1 + X_2 + \cdots + X_n$, it follows from (3) that

$$\lim_{n\to\infty} E\frac{|S_n - n|}{\sqrt{n}} = (2/\pi)^{\frac{1}{2}}. \tag{5}$$

Or equivalently we have

$$\lim_{n\to\infty} \sqrt{2\pi}\, E\frac{|S_n - n|}{\sqrt{n}} = 2. \tag{6}$$

We now evaluate $E|S_n - n|/\sqrt{n}$. Since the probability density function of S_n is given by

$$\begin{aligned} f_n(x) &= \frac{1}{\Gamma(n)} x^{n-1} e^{-x}, \quad x > 0 \\ &= 0 \qquad \text{otherwise,} \end{aligned} \tag{7}$$

it follows that

$$\begin{aligned} \sqrt{2\pi}\, E\frac{|S_n - n|}{\sqrt{n}} &= \frac{\sqrt{2\pi}}{\sqrt{n}\Gamma(n)} \int_0^\infty |x - n| x^{n-1} e^{-x} dx \\ &= \frac{\sqrt{2\pi n}}{\Gamma(n)} \int_0^\infty \left| \frac{x}{n} - 1 \right| x^{n-1} e^{-x} dx \end{aligned}$$

$$= \frac{\sqrt{2n\pi}}{\Gamma(n)} \left[\int_0^n \left| \frac{x}{n} - 1 \right| x^{n-1} e^{-x} dx + \int_n^\infty \left| \frac{x}{n} - 1 \right| x^{n-1} e^{-x} dx \right].$$

Letting $x/n = u$, we have

$$\begin{aligned} \sqrt{2\pi}\, E\frac{|S_n - n|}{\sqrt{n}} &= \frac{n^n\sqrt{2n\pi}}{\Gamma(n)} \left[\int_0^1 (1 - u) u^{n-1} e^{-nu} du \right. \\ &\qquad \left. + \int_1^\infty (u - 1) u^{n-1} e^{-nu} du \right] \\ &= \frac{n^n\sqrt{2n\pi}}{\Gamma(n)} \left[\int_0^1 u^{n-1} e^{-nu} du - \int_0^1 u^n e^{-nu} du \right. \\ &\qquad \left. + \int_1^\infty u^n e^{-nu} du - \int_1^\infty u^{n-1} e^{-nu} du \right]. \end{aligned}$$

Integrating by parts the first and the last integral, we obtain

$$\begin{aligned} \sqrt{2\pi}\, E\frac{|S_n - n|}{\sqrt{n}} &= \frac{n^n\sqrt{2n\pi}}{\Gamma(n)} \left[\frac{u^n e^{-nu}}{n} \Big|_0^1 - \frac{u^n e^{-nu}}{n} \Big|_1^\infty \right] \\ &= \frac{2\sqrt{2n\pi}\; n^n e^{-n}}{\Gamma(n+1)}. \end{aligned}$$

Hence it follows from (6) that

$$\lim_{n\to\infty} \frac{\sqrt{2n\pi}\, n^n e^{-n}}{\Gamma(n+1)} = 1.$$

This research was initiated at the Mathematics Research Center, U.S. Army, University of Wisconsin.

References

1. Leo Breiman, Probability, Addison-Wesley, Reading, Mass., 1968.

2. K. L. Chung, A course in probability theory, Harcourt, Brace & World, New York, 1968.

3. William Feller, An introduction to probability theory and its applications, Vol. 1 (2nd ed.), Wiley, New York, 1957.

4. Michel Loève, Probability theory, 3rd ed., Van Nostrand, Princeton, N. J., 1963.

DEPARTMENT OF MATHEMATICS, CASE-WESTERN RESERVE UNIVERSITY, CLEVELAND, OHIO, 44106.

ON THE SELF-DUALITY OF Q_p

LAWRENCE WASHINGTON

An important result in modern analytic number theory is that the additive group Q_p^+ of p-adic numbers is isomorphic to its character group (i.e., the group of all continuous homomorphisms from Q_p^+ to the circle group). Most authors (for example, Goldstein [**1**, p. 115]) obtain this theorem by resorting to the theory of Pontryagin duality, with which many students are unfamiliar. Therefore, it seems desirable to give an "elementary" proof, based only on the definition of the character group. The purpose of this note is to present such a proof.

Let $x = \sum_{j=k}^{\infty} x_j p^j$ be an arbitrary element of Q_p, where $x_j \in \{0, 1, \cdots, p-1\}$ for all $j \geqq k$ (k possibly negative). We define

$$\Lambda(x) = \sum_{j=k}^{-1} x_j p^j \text{ (a rational number!).}$$

An easy calculation shows that

$$\Lambda(x+y) \equiv \Lambda(x) + \Lambda(y) \pmod 1, \text{ for } x, y \in Q_p$$

(by $a \equiv b \pmod 1$ we mean $a - b$ is an integer). Fix y in Q_p and let

$$\chi_y(x) = \exp(2\pi i \Lambda(xy)) \text{ for all } x \in Q_p.$$

It is well known (and easily proved) that χ_y is a character of Q_p^+ and that the map $y \to \chi_y$ is a topological isomorphism of Q_p^+ onto its image in the character group (see Goldstein [**1**, pp. 114–116]). The interesting question is the following: Are all characters of Q_p^+ of the form χ_y? (Here is where Pontryagin duality theory is normally used.) In order to give an answer, we first need the following:

LEMMA. *Let z be a complex number of absolute value 1 and suppose $\lim_{n \to \infty} z^{p^n} = 1$. Then z is a p^m-th root of 1 for some non-negative integer m.*

Proof. Since $\lim z^{p^n} = 1$, there exists m such that $|\arg(z^{p^k})| < \pi/p$ for all $k \geqq m$.

Suppose $c = \arg(z^{p^m}) \neq 0$. Let t be the smallest positive integer satisfying $|p^t c| \geqq \pi/p$. Since $|p^{t-1}c| < \pi/p$, it follows that $\pi > |p^t c| \geqq \pi/p$. But

$$|p^t c| = |p^t \arg(z^{p^m})| = |\arg((z^{p^m})^{p^t})| = |\arg(z^{p^{m+t}})|.$$

Since $m + t > m$, this contradicts the choice of m. Therefore $\arg(z^{p^m}) = 0$, so $z^{p^m} = 1$, which implies the lemma.

Now let χ be an arbitrary character of Q_p^+. We shall construct y in Q_p such that $\chi = \chi_y$. Since χ is a homomorphism from the *additive* group Q_p^+ to the *multiplicative* circle group, $\chi(x_1 + x_2) = \chi(x_1)\chi(x_2)$ for all x_1 and x_2 in Q_p. Since χ is continuous and $\lim_{n\to\infty} p^n = 0$ in the p-adic topology,

$$\lim_{n\to\infty} \chi(1)^{p^n} = \lim_{n\to\infty} \chi(p^n) = \chi(\lim_{n\to\infty} p^n) = \chi(0) = 1.$$

Therefore, by the lemma, there exist non-negative integers b and m, with $0 \leqq b < p^m$ such that $\chi(1) = \exp(2\pi i b p^{-m})$. Since $\chi(p^{-1})^p = \chi(1)$, it follows that $\chi(p^{-1}) = \exp(2\pi i p^{-1}(bp^{-m} + b_0))$ for some integer b_0 satisfying $0 \leqq b_0 < p$. Similarly using the fact that $\chi(p^{-n})^p = \chi(p^{-(n-1)})$, we obtain by induction a sequence $\{b_n\}_{n=0}^{\infty}$ of integers, $0 \leqq b_n < p$ for all n, such that

$$\chi(p^{-n}) = \exp(2\pi i p^{-n}(bp^{-m} + b_0 + b_1 p + \cdots + b_{n-1}p^{n-1})) \text{ for } n > 0. \tag{1}$$

We let $y = bp^{-m} + \sum_{n=0}^{\infty} b_n p^n$ and claim that $\chi = \chi_y$.

Since $p^{-n}y = bp^{-m-n} + b_0 p^{-n} + \cdots + b_{n-1}p^{-1} + b_n + b_{n+1}p + \cdots$, we have

$$\Lambda(p^{-n}y) = p^{-n}(bp^{-m} + b_0 + \cdots + b_{n-1}p^{n-1}) \text{ for } n > 0.$$

Therefore, from equation (1),

$$\chi(p^{-n}) = \exp(2\pi i \Lambda(p^{-n}y)) = \chi_y(p^{-n}) \text{ for } n > 0.$$

Consequently,

$$\chi(p^n) = \chi(p^{-1})^{p^{n+1}} = \chi_y(p^{-1})^{p^{n+1}} = \chi_y(p^n) \text{ for } n \geqq 0,$$

so $\chi(p^n) = \chi_y(p^n)$ for all n. It follows that $\chi(ap^n) = \chi_y(ap^n)$ for all integers a and n Now let $x = \sum_{j=k}^{\infty} x_j p^j = \lim_{n\to\infty} \sum_{j=k}^{n} x_j p^j$ be an arbitrary element of Q_p. Then

$$\begin{aligned} \chi(x) &= \lim_{n\to\infty} \chi\left(\sum_{j=k}^{n} x_j p^j\right) = \lim_{n\to\infty} \prod_{j=k}^{n} \chi(x_j p^j) \\ &= \lim_{n\to\infty} \prod_{=k}^{n} \chi_y(x_j p^j) = \lim_{n\to\infty} \chi_y\left(\sum_{j=k}^{n} x_j p^j\right) = \chi_y(x). \end{aligned}$$

Therefore $\chi = \chi_y$, so the map $y \to \chi_y$ is actually a topological isomorphism of Q_p^+ *onto* its character group, as desired.

The above considerations may easily be extended to give an elementary proof of the self-duality of any finite algebraic extension of Q_p, since such a field must have the product topology when regarded as a vector space over Q_p [**1**, p. 48].

This work was supported by an NSF Graduate Fellowship.

Reference

1. L. J. Goldstein, Analytic Number Theory, Prentice-Hall, Englewood Cliffs, N. J., 1971.

DEPARTMENT OF MATHEMATICS, PRINCETON UNIVERSITY, PRINCETON, N. J. 08540.

CATALAN'S EQUATION IN $K(t)$

MELVYN B. NATHANSON

In 1842, Eugène Catalan [**1**] conjectured that the only consecutive integral powers were 8 and 9, that is, that the only solution in integers x, y, m, n greater than one of the equation $x^m - y^n = 1$ is $x = n = 3$ and $y = m = 2$. Like the Fermat conjecture, this conjecture of Catalan has never been proved. Shanks [**3**] and Greenleaf [**2**] have shown that Fermat's equation $x^n + y^n + z^n = 0$, for $n > 2$, has no nontrivial solution in the field $C(t)$ of rational functions with complex coefficients. (A trivial solution of Fermat's equation is a solution in which x, y, z are constant multiples of the same rational function.) Indeed, if K is any field whose characteristic does not divide n, then Greenleaf's proof shows that Fermat's equation has no nontrivial solution in the rational function field $K(t)$. In this note we consider Catalan's equation over $K(t)$.

THEOREM. *Let K be a field. If m and n are integers greater than two and not divisible by the characteristic of K, then Catalan's equation*

$$x^m - y^n = 1 \tag{1}$$

has no nonconstant solution in the rational function field $K(t)$. If m and n are integers greater than one and not divisible by the characteristic of K, then Catalan's equation (1) *has no nonconstant solution in the polynomial ring $K[t]$.*

Proof. It is enough to consider only algebraically closed fields K. Let m and n be integers greater than two and not divisible by the characteristic of K. If Catalan's equation (1) has a nonconstant solution in $K(t)$ then there are polynomials f, g, h, k in $K[t]$ such that $(f/g)^m - (h/k)^n = 1$, where $(f,g) = (h,k) = 1$, the polynomials f, g, h, k are not all constant, and the denominators g and k are monic. Then

$$f^m k^n - h^n g^m = g^m k^n. \tag{2}$$

Let $\theta \in K$. If $g(\theta) = 0$, then $f(\theta)k(\theta) = 0$. Since $(f,g) = 1$, it follows that $f(\theta) \neq 0$, and so $k(\theta) = 0$. Similarly, if $k(\theta) = 0$, then $g(\theta) = 0$. Therefore, g and k have the same zeros in K. The polynomials g and k are both monic, and so $g(x) = \prod (x - \theta)^{m_\theta}$

and $k(x) = \prod(x-\theta)^{n_\theta}$, where the m_θ and n_θ are positive integers and the products range over the zeros of g and k. Since θ is a zero of the polynomials $f^m k^n$, $h^n g^m$, and $g^m k^n$ of orders $n_\theta n$, $m_\theta m$, and $n_\theta n + m_\theta m$, respectively, it follows from (2) that $m_\theta m = n_\theta n$. Therefore, $g^m = k^n$, and so $f^m - h^n = g^m$.

Since K is algebraically closed and the characteristic of K does not divide m, there is a primitive mth root of unity ζ in K. Then

$$h^n = f^m - g^m = \prod_{i=0}^{m-1} (f - \zeta^i g). \tag{3}$$

Because $(f,g) = 1$, the factors $f - \zeta^i g$ for $i = 0, 1, \cdots, m-1$ are pairwise relatively prime nonzero polynomials, and at most one of these polynomials is constant. Therefore, there are pairwise relatively prime polynomials h_0, h_1, h_2 in $K[t]$ not all constant and nonzero elements β_0, β_1, β_2 in K such that $f - \zeta^i g = \beta_i h_i^n$ for $i = 0, 1, 2$.

Now consider $K[t]$ as a vector space over the field K. The three polynomials $f-g$, $f-\zeta g$, and $f-\zeta^2 g$ are contained in the two-dimensional vector subspace of $K[t]$ spanned by f and g, and so are linearly dependent. Therefore, there are nonzero constants α_0, α_1, α_2 in K such that

$$\begin{aligned}\alpha_0(f-g) + \alpha_1(f-\zeta g) + \alpha_2(f-\zeta^2 g) &\\ = \alpha_0\beta_0 h_0^n + \alpha_1\beta_1 h_1^n + \alpha_2\beta_2 h_2^n &\\ = (\alpha_0^{1/n}\beta_0^{1/n}h_0)^n + (\alpha_1^{1/n}\beta_1^{1/n}h_1)^n + (\alpha_2^{1/n}\beta_2^{1/n}h_2)^n &\\ = 0. &\end{aligned}$$

But this is a nontrivial solution of Fermat's theorem in $K(t)$, which is impossible. Therefore, Catalan's equation has no nonconstant solution in $K(t)$ if the exponents m and n of equation (1) are greater than two and not divisible by the characteristic of K.

Suppose $m = 2$ and $n > 2$ in (1). If f and g are nonconstant polynomials in $K[t]$ such that $f^2 - g^n = 1$, then $g^n = f^2 - 1 = (f+1)(f-1)$. Since $(f+1, f-1) = 1$, there are relatively prime nonconstant polynomials h and k in $K[t]$ such that $f+1 = h^n$ and $f-1 = k^n$. Then $h^n - k^n = 2$, and $(2^{1/n})^n + k^n + ((-1)^{1/n}h)^n = 0$, which is a nontrivial solution of the Fermat equation in $K[t]$. But this is impossible.

Similarly, if $m > 2$ and $n = 2$, then Catalan's equation has no nonconstant solution in $K[t]$.

If $m = n = 2$ and f, g in $K[t]$ satisfy $1 = f^2 - g^2 = (f+g)(f-g)$, then $f+g$ and $f-g$ are both constants, and so f and g are both constants.

Problem. If $m = n = 2$, then (1) does have nonconstant solutions in $K(t)$, for example,

$$\left(\frac{t^2-1}{t^2+1}\right)^2 - \left(\frac{2\sqrt{-1}\,t}{t^2+1}\right)^2 = 1.$$

Is there a nonconstant solution of Catalan's equation in $K(t)$ if $m = 2$ and $n > 2$?

References

1. Eugène Catalan, Problème 48, Nouvelles Annales de Mathématiques, 1 (1842) 520.
2. Newcomb Greenleaf, On Fermat's equation in $C(t)$, this MONTHLY, 76 (1969) 808–809.
3. Daniel Shanks, Solved and Unsolved Problems in Number Theory, vol. 1, Spartan Books, Washington, D. C., 1942, pp. 144–147.

DEPARTMENT OF MATHEMATICS, SOUTHERN ILLINOIS UNIVERSITY, CARBONDALE, ILLINOIS 62901.

TRIANGLES WITH INTEGER-VALUED SIDES

MARTIN J. MARSDEN

1. Introduction. We begin with some remarks on the number of factorings of an integer as well as some remarks on the smallest integer which can be represented as the difference of two squares in a given number of ways. While we do not completely develop this theory, we find that our partial results are useful in a discussion of triangles having integer-valued sides.

Our objective in that discussion is to further characterize $N(\lambda)$, the number of such triangles whose perimeter equals λ times their area. Subbarao [**1**], see also [**2**], has shown that $N(\lambda)$ is finite for each λ. We show that $N(\lambda)$ is larger than any integer infinitely often as λ tends to zero and discuss the cluster points of the sets $\{\lambda: N(\lambda) = L\}$.

2. Representing an integer as the difference of squares. Let $M = 2^{k_0}3^{k_1}5^{k_2}7^{k_3}11^{k_4}\cdots$ be the prime factorization of the positive integer M.

LEMMA 2.1. *The number of positive integers which divide* M *is* $\prod_{i=0}^{\infty}(k_i + 1)$.

DEFINITION 2.2. Set $d(M)$ equal to the number of ways M can be represented as $M = K^2 - J^2$ with $K > J \geqq 0$ integers.

LEMMA 2.3. *Let* L *and* M *be integers with* $d(M) = L$.
If M *is an odd square, then* $2L-1 = (k_1 + 1)(k_2 + 1)\cdots$.
If M *is odd, but not a square, then* $2L = (k_1 + 1)(k_2 + 1)\cdots$.
If M *is an even square, then* $2L-1 = (k_0-1)(k_1 + 1)(k_2 + 1)\cdots$.
If M *is even, but not a square, then* $2L = (k_0-1)(k_1 + 1)(k_2 + 1)\cdots$.

DEFINITION 2.4. For integer $L > 0$, set $s(L)$ equal to the smallest integer M such that $d(M) = L$.

LEMMA 2.5. *Let* $M = s(L)$. *Then* $k_1 \geqq k_2 \geqq \cdots$. *If* M *is even, then* $k_0 > 2$ *and* $k_0 - 1 \geqq k_1 + 1$.

Proof. That $k_1 \geqq k_2 \geqq \cdots$ is clear. If M is even, then $k_0 \geqq 2$. Since

$d(8j+4) = d(2j+1)$, $k_0 = 2$ is untenable. Finally, if M is even and $i = k_0 - k_1 - 2$ then $d(M)$ and $d(3^i M/2^i)$ are equal so that $i \geqq 0$.

THEOREM 2.6. *If $L > 3$, then $s(L)$ is even and either $2L = (k_0-1)(k_1+1)(k_2+1)\cdots$ or $2L-1 = (k_0-1)(k_1+1)(k_2+1)\cdots$ with $k_0 > 2$ and $k_0 - 1 \geqq k_1 + 1 \geqq k_2 + 1 \geqq \cdots$.*

Proof. Let M be odd and such that $d(M) = L$. Then $M = 3^{k_1}5^{k_2}\cdots p_m^{k_m}$ where p_m is the mth odd prime. Set the remaining k_i equal to zero and consider $M_i = 2^{k_i+2}M/p_i^{k_i}$. Then $d(M_i) = d(M)$ for all i. Suppose that $M = s(L)$ so that $M < M_i$ for all i. Then $11^{k_4} < 2^{k_4+2}$, implying $k_4 = 0$ or $m \leqq 3$. Moreover, $5^{k_2} < 2^{k_2+2}$implies $k_2 \leqq 1$. Similarly, $k_1 \leqq 3$. Since $d(32\cdot 3^{k_1}) = d(35\cdot 3^{k_1})$, $k_3 = 0$. A case-by-case examination of the remaining seven cases completes the proof.

Since $s(7) = 768$ and $s(8) = 480$, $s(L)$ is not an increasing function of L.

DEFINITION 2.7. For integer $L > 0$, set $\sigma(L) = \min\{s(L'): L' \geqq L\}$.

If we fix $N = 2L$or $2L-1$ and also fix m and then permit M and the k_i to assume noninteger real values, then M is minimized by the choice

$$2^{k_0-1} = 3^{k_1+1} = 5^{k_2+1} = \cdots = p_m^{k_m+1} = e^{\mu_m}$$

with $(\mu_m)^{m+1} = N\ln 2 \ln 3 \cdots \ln p_m$. This suggests that one should choose the k_i as integers so that these inequalities are nearly satisfied when seeking $s(L)$.

While there are still some problems connected with the $s(L)$ and $\sigma(L)$ characterizations, we now have sufficient material for a useful application to the theory of triangles with integer-valued sides.

3. Triangles having integer-valued sides.

DEFINITION 3.1. For $0 < a \leqq b \leqq c < a+b$, set $r(a,b\ c) = P/A$ where P and A are the perimeter and area, respectively, of the triangle having sides whose lengths are a, b, c.

DEFINITION 3.2. For $\lambda > 0$, set $N(\lambda)$ equal to the number of triangles having integer-valued sides a, b, c for which $r(a,b,c) = \lambda$.

LEMMA 3.3. *The number of integer triples (a,b,c) such that*

$$0 < a \leqq b \leqq c < a+b \tag{3.4}$$

and,

$$\lambda^2 = 16(a+b+c)/(a+b-c)(a+c-b)(b+c-a) \tag{3.5}$$

is precisely $N(\lambda)$. If $a = K$ and $c - b = J$ are fixed, then for all integers $b \geqq K$

$$1 < (K^2 - J^2)\lambda^2/16 = (2b+J+K)/(2b+J-K) \leqq 3. \tag{3.6}$$

THEOREM 3.7 (Subbarao [1]). *For each λ, $N(\lambda)$ is finite.*

Proof. From (3.6), $K^2 - J^2 \leqq 48/\lambda^2$ so that $c - b$ and a are restricted to a finite range.

One may deduce that $N(\lambda) \leqq 24(48 - \lambda^2)/\lambda^4$ by counting the pairs $c - b$ and a. We now place a lower bound, of sorts, on $N(\lambda)$.

THEOREM 3.8. *Let M be an integer of the form $4j$, $4j+1$, $4j+2$, or $8j+3$ and let $L = d(M)$. Then, for infinitely many $\lambda > 0$ satisfying $16/M < \lambda^2 < 16/(M-1)$, $N(\lambda) = L$. For only finitely many of these λ is $N(\lambda) > L$.*

Proof. For irrational λ^2, $N(\lambda) = 0 = d(4j+2)$. Thus, the first assertion is true if $L = 0$.

For $L > 0$ let M be represented as

$$M = K_1^2 - J_1^2 = K_2^2 - J_2^2 = \cdots = K_L^2 - J_L^2 \tag{3.9}$$

with K_i, J_i nonnegative integers and all K_i distinct. For triangles having sides a_i, b_i, c_i with

$$a_i = K_i \leqq b_i, \; c_i = b_i + J_i, \; b_i \text{ an integer}; \tag{3.10}$$

we have

$$1 < Mr^2(a_i, b_i, c_i)/16 = (2b_i + J_i + K_i)/(2b_i + J_i - K_i). \tag{3.11}$$

Now,

$$r(a_1, b_1, c_1) = r(a_2, b_2, c_2) = \cdots = r(a_L, b_L, c_L) \tag{3.12}$$

has integer solutions satisfying (3.9) and (3.10) if and only if

$$(2b_1 + J_1)/K_1 = (2b_2 + J_2)/K_2 = \cdots = (2b_L + J_L)/K_L$$

has large positive integer solutions $b_1, b_2, \cdots, b_L$.

If M is of the form $4j$, the choice

$$b_i = b_i(t) = (tK_i - J_i)/2 \quad (i = 1, 2, \cdots, L), \tag{3.13}$$

with $t \geqq 3$ an odd integer, gives infinitely many integer solutions of (3.9), (3.10), and (3.12) since, for these t, the parity of tK_i is the same as that of J_i for all $i = 1, 2, \cdots, L$.

If M is of the form $4j+1$, the K_i are odd and the J_i are even since both $K_i - J_i$ and $K_i + J_i$ are of the (same) form $4j \pm 1$. Thus, the choice of $t \geqq 4$ as an even integer in (3.13) gives the same result.

If M is of the form $8j + 3 = (4j' - 1)(4j'' + 1)$ with $j' + j''$ odd, the choice of $t = 7/2, 9/2, 11/2, \cdots$ forces tK_i and J_i to have the same parity so that (3.9), (3.10), and (3.12) have infinitely many integer solutions.

In all of these cases, infinitely many values of t produce λ in the desired interval since the right member of (3.11) decreases to 1 for each i as t increases.

We have thus far shown that $N(\lambda) \geqq L$ for infinitely many $\lambda > 4/\sqrt{M}$. To show

that $N(\lambda) > L$ only finitely often in the given interval, it suffices to note that there are only finitely many triangles having integer-valued sides a, b, c satisfying $16/M < r^2(a,b,c) < 16/(M-1)$ which are not of the forms given by (3.9) and (3.10). This is vacuously true when M is of the form $4j + 2$ as well. Thus, the proof is complete.

In this theorem, we have omitted M of the form $8j + 7$. For such M satisfying $d(M) > 1$, the first assertion of this theorem is true if and only if each of the K_i of (3.9) have precisely the same power of two in their prime factorization. The smallest such M for which this is the case is $M = 119$. A simple characterization of such M eludes the author at the present time.

Since $s(L)$ is not a strictly decreasing function of L, it is not always true that $\lambda_0 = 4/\sqrt{s(L)}$ is the largest λ_0 for which the statement: "For infinitely many $\lambda > \lambda_0$, $N(\lambda) \geqq L$," is valid. However, the following is true:

THEOREM 3.14. *For integer $L > 0$ set $\lambda_0 = 4/\sqrt{\sigma(L)}$ and let $\lambda_1 > \lambda_0$ be arbitrary, but fixed. Then, for infinitely many $\lambda > \lambda_0$, $N(\lambda) \geqq L$ and for only finitely many $\lambda > \lambda_1$ is $N(\lambda) \geqq L$.*

Since, for example, $s(5) = \sigma(5) = 144$, this theorem says that for infinitely many $\lambda > 1/3$, $N(\lambda) = 5$ and that $\lambda_0 = 1/3$ is the greatest cluster point of $\{\lambda : N(\lambda) = 5\}$. The largest λ for which $N(\lambda) = 5$ is $\lambda = 4/\sqrt{7}$ corresponding to the triangles $(3,12,12)$ $(3,22,23)$ $(4,5,6)$ $(5,18,22)$ $(6,11,16)$.

A final remark: Subbarao asks in [1] for an extension of the theory to quadrilaterals with integer-valued sides. Since the area of a quadrilateral is not specified by its sides, there must be an additional restriction. One possible restriction would be to "cyclic" quadrilaterals, whose vertices lie on a circle. For such quadrilaterals, a formula by the 7th Century Hindu mathematician Brahmagupta yields: For each $\lambda > 0$, the number of integer solutions of

$$0 < a \leqq b \leqq c \leqq d < a + b + c$$

and

$$\lambda^2 = \frac{16(a+b+c+d)^2}{(a+b+c-d)(a+b+d-c)(a+c+d-b)(b+c+d-a)}$$

is the number of cyclic quadrilaterals having integer-valued sides whose perimeter is λ times their area. For each λ, this number is finite. In this enumeration, only one permutation is counted for each 4-tuple (a, b, c, d).

References

1. M. V. Subbarao, Perfect triangles, this MONTHLY, 78 (1971) 384–5.
2. D. Drasin and R. Gilmer, Complements and Comments, this MONTHLY, 78 (1971) 1104–6.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF PITTSBURGH, PITTSBURGH, PA 15213.

AN ELEMENTARY PROOF OF SHARP UNIQUENESS THEOREMS FOR SECOND-ORDER INITIAL-VALUE PROBLEMS

RAY REDHEFFER

On a given interval $a \leqq x < b$ let u and v be real-valued functions satisfying

$$u'' = f(x,u,u'),\ v'' = f(x,v,v'),\ a < x < b, \tag{1}$$

together with the initial conditions $u(a) = v(a)$, $u'(a) = v'(a)$. A uniqueness theorem asserts that $u = v$ for $a \leqq x < b$.

If $w = u - v$ then $w'' = f(x, v + w, v' + w') - f(x,v,v')$. Hence, under suitable hypotheses on f,

$$w(a) = w'(a) = 0,\ |w''| \leqq p|w'| + q|w|,\ a < x < b, \tag{2}$$

where p and q are nonnegative functions of x, depending on f and perhaps on v, and even on w. The condition $u = v$ in (1) is equivalent to $w = 0$ in (2), and we consider (2) henceforward.

LEMMA 1. *Let w satisfy (2). For each $r \in [a,b)$ suppose there exists $t > r$ and a function W such that*

$$W(r) = 0,\ W'(r) > 0,\ W'' > pW' + qW,\ r < x < t.$$

Then $w(x) = 0,\ a \leqq x < b$.

For proof suppose the conclusion fails and let $a \leqq x \leqq r$ denote the largest interval containing a throughout which $w(x) = 0$. Then for every $t > r$ we have $|w(x)| > 0$ at some point of (r,t); hence $|w(x)| > \delta W(x)$ at some point of this interval, if the positive constant δ is sufficiently small. It is easily checked that $w(r) = w'(r) = 0$, whereas by hypothesis $W(r) = 0$, $W'(r) > 0$. Hence $|w(x)| < \delta W(x)$ for x near $r+$, that is, for $x > r$ and $x - r$ sufficiently small.

Let s denote the first point beyond r where the graph of $y = |w(x)|$ crosses the graph of $y = \delta W(x)$. The above discussion shows s exists, $r < s < t$. Changing the sign of w, if necessary, we can suppose $w(s) = \delta W(s)$ rather than $-w(s) = \delta W(s)$.

The function $w(x) - \delta W(x)$ is 0 at r and s but is negative for x near $r+$. Hence it has a minimum at some point x, $r < x < s$. At that point

$$|w(x)| < \delta W(x),\ w'(x) = \delta W'(x),\ w''(x) \geqq \delta W''(x)$$

and hence

$$|w''| - p|w'| - q|w| \geqq \delta(W'' - pW' - qW) > 0.$$

This contradiction establishes Lemma 1.

If p and q are constant we can take $W(x) = e^{\alpha x} - e^{\alpha r}$ where $\alpha^2 > p\alpha + q$. Thus, Lemma 1 gives uniqueness when f satisfies a Lipschitz condition. However, Lemma 1

applies when neither p nor q is bounded, and even when q is not integrable. Furthermore, $w''(x)$ can fail to exist on a dense subset of $[a, b)$, such as the set of all rationals.

We say that (2) holds $\operatorname{mod} E$ (for enumerable) if $w'(x)$ exists, $a \leqq x < b$, and if (2) holds except for x in a countable subset of $[a, b)$. Similarly, "p is continuous $\operatorname{mod} E$" means that the set of discontinuities of p is countable. The conclusion of Lemma 1 remains valid if the differential inequalities are given only $\operatorname{mod} E$, because $w'(x) = \delta W'(x)$ at the minimum point, and the exceptional set can be avoided by suitable choice of δ. Upon setting $W' = \exp(\phi)$ we are led to the following theorem:

THEOREM 1. *Let* (2) *hold* $\operatorname{mod} E$ *and let* p *and* q *be continuous* $\operatorname{mod} E$. *For each* $r \in [a, b)$ *suppose there exists* $t > r$ *such that*

$$\int_r^t [p(x) + q(x)(x - r)]dx < \infty. \tag{3}$$

Then $w(x) = 0$, $a \leqq x < b$.

The only use of the two-sided inequality for w'' in (2) is to permit replacement of w by $-w$ in the proof of Lemma 1. Hence, the conclusion of Theorem 1 remains valid if (2) is replaced by

$$w(a) = w'(a) = 0, \; w'' \leqq pw' + qw \operatorname{mod} E, \; w \geqq 0.$$

Example 1. Suppose for each $r \in [a, b)$ there exists a constant $\theta < 1$ such that

$$\limsup_{x \to r+} \, [(x - r)^{\theta} p(x) + (x - r)^{1+\theta} q(x)] < \infty. \tag{4}$$

Then (3) holds, since $(x-r)^{-\theta}$ is integrable near $r+$, and uniqueness follows. On the other hand the condition

$$\limsup_{x \to r+} \, [(x - r)p(x) + (x - r)^2 q(x)] = 0 \tag{5}$$

is not sufficient for uniqueness, as seen by taking $W(x) = x(\log x)^{-1}$, $r = a = 0$.

Example 2. If the differential inequalities for w and W hold at the point r in Lemma 1 then one can allow $W'(r) = 0$. (The proof under this new hypothesis is virtually unchanged.) The choice $W(x) = (x-r)^2 + (x-r)^3$ shows that (4) can be replaced by

$$p(x)(x - r) + q(x)\frac{(x - r)^2}{2} \leqq 1, \quad r < x < t, \tag{6}$$

which is even weaker than (5), if (2) holds at $x = r$. Considering $w = (x-r)^2$, we see that this latter proviso is essential; in other words, the point r itself must not belong to the exceptional set E, even if $w \in C^{(2)}$, $a \leqq x < b$. The example $w(x) = x^2(\log x)^{-1}$ shows that (6) cannot be replaced by

$$\limsup_{x \to r+} \left[p(x)(x - r) + q(x)\frac{(x - r)^2}{2}\right] \leqq 1.$$

Historical note. I have not happened to come across any prior statement of Theorem 1, though the result can be deduced (even for nth order equations) from the general theory of differential inequalities [**1**, **3**, **6**, **7**]. Equation (6) is the case $n = 2$ of a well-known uniqueness theorem of Nagumo, as extended to nth order equations by Hartman [**2**].

The proof here does not follow the pattern [**1**, **2**, **3**, **6**, **7**] but is more akin to the familiar proof of Hopf's maximum principle [**5**]. (One can consider the interval $r < x < s$ to be a one-dimensional Hopf sphere with $x = r$ as the Hopf point.) The fact that Hopf's procedure applies to initial-value problems was suggested to me by [**4**], and the results of this note generalize [**4**], Theorems 1, 2, 3, 4, 6.

This paper was written while the author was Guest Professor at the University of Karlsruhe.

References

1. Earl Coddington and Norman Levinson, Theory of Ordinary Differential Equations, McGraw-Hill, New York, 1955, Chaps. 1, 2.

2. Philip Hartman, Ordinary Differential Equations, Wiley, New York, 1964, especially p. 32.

3. V. Lakshmikantham and S. Leela, Differential and Integral Inequalities, Vol. 1., Academic Press, New York, 1961.

4. M. H. Protter and Hans Weinberger, Maximum Principles in Differential Equations, Chapter 1.

5. ———, Chapter 2, Section 3.

6. J. Szarski, Differential Inequalities, Pol. Akad. Nauk, 1965, Chapters IV and V.

7. W. Walter, Differential and Integral Inequalities, Springer-Verlag, New York, 1970.

Department of Mathematics, UCLA, Los Angeles, California 90024.

FOURIER UNIQUENESS VIA COMPLEX VARIABLES

D. J. Newman

The uniqueness theorem referred to in our title is the statement that *if* $f(t)$ *is a summable function for which* $\int_{-\infty}^{\infty} f(t)e^{ixt}dt = 0$ *for all* (*real*) x *then* $f(t) = 0$ *a.e.* The usual proof relies on the lucky fact that an explicit inversion formula can be given which recovers $f(t)$ from its "transform", $\int_{-\infty}^{\infty} f(t)e^{itx}dt$. There is, however, a pleasant little proof which does not at all need an inversion formula and it is this which we should like to give here. This proof has perhaps more than aesthetic value in that it allows generalization to some cases of kernels for which there is no explicit inverse, but it is mainly interesting as a *tour de force* in complex variables.

So suppose, as above, that $\int_{-\infty}^{\infty} f(t)e^{ixt}dt = 0$, pick an arbitrary real number a, write this as

$$\int_{-\infty}^{a} f(t)e^{ix(t-a)}dt = -\int_{a}^{\infty} f(t)e^{ix(t-a)}dt$$

and call their common value $F(x)$.

The left side may be regarded throughout the entire lower half x-plane, $\operatorname{Im} x \leqq 0$. It is easily seen to be continuous, bounded, (by $\int_{-\infty}^{a} |f(t)|\, dt$), and analytic throughout the interior ($\operatorname{Im} x < 0$). Similarly the right side is a function continuous and bounded in $\operatorname{Im} x \geqq 0$ and analytic throughout $\operatorname{Im} x > 0$.

Thus $F(x)$ is analytic in $\operatorname{Im} x > 0$ and $\operatorname{Im} x < 0$ and continuous everywhere. An application of Morera's theorem shows that $F(x)$ is entire. The aforementioned boundedness of $F(x)$ therefore allows an application of Liouville's theorem and permits the conclusion that $F(x)$ is a constant.

If we now let x go to ∞ along the upper imaginary axis then a glance at the right side shows that this constant must be 0.

Conclusion: $\int_{-\infty}^{a} f(t) e^{ix(t-a)} dt \equiv 0$. Setting $x = 0$ gives $\int_{-\infty}^{a} f(t)dt = 0$, and, this holding for all real a, gives, by differentiation, that $f(a) = 0$ almost everywhere.

BELFER GRADUATE SCHOOL OF SCIENCE, YESHIVA UNIVERSITY, 2495 AMSTERDAM AVE, NEW YORK, N.Y. 10033.

A NEW CHARACTERIZATION OF THE EXPONENTIAL FUNCTION

G. P. KAPOOR

A function $f: \mathscr{C} \to \mathscr{C}$, where $\mathscr{C}$ is the complex plane is said to be entire if it is analytic for every $z = re^{i\theta}$ belonging to $\mathscr{C}$. Besides polynomials, the simplest example of an entire function is the exponential function $\exp(z)$ defined as

$$\exp(z) = \sum_{n=0}^{\infty} \frac{1}{n!} z^n. \tag{1}$$

Various characterizations of the exponential function are known (see, e.g., [**1**, pp. 138–142]).

Let

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \tag{2}$$

be an entire function. For $|z| = r$ set $\mu(r) = \max_{n \geqq 0}\{|a_n| r^n\}$ and $\nu(r) = \max\{n: \mu(r) = |a_n| r^n\}$. We call $\mu(r)$ the maximum term of $f(z)$ and $\nu(r)$ the rank of the maximum term.

For the entire function $f(z)$, given by (2), Valiron [**2**, p. 31] has found the following widely used relation between the maximum term $\mu(r)$ and its rank $\nu(r)$:

$$\log \mu(r) = \log \mu(r_0) + \int_{r_0}^{r} \frac{\nu(t)}{t} dt; \quad 0 \leqq r_0 < r < \infty. \tag{3}$$

With the help of this relation we shall show that the rank of the maximum term

characterizes the exponential function completely. Using the notation $[r]$ for the greatest integer not exceeding r, we have the following theorem:

THEOREM. *Let* $f(z) = 1 + \sum_{n=1}^{\infty} a_n z^n$ *be an entire function and* $v(r)$ *be the rank of its maximum term, then* $v(r) = [r]$ *implies* $|a_n| = 1/n!$. *In particular, if* $a_n \geqq 0$ *for all* n, *then* $v(r) = [r]$, *if and only if* $f(z) = \exp(z)$.

Proof. Let $v(r) = [r]$. Then, $v(r) = n$ for $n \leqq r < n + 1$. By (3), we have

$$\begin{aligned}\log \mu(r) &= \log \mu(n) + \int_n^r \frac{v(t)}{t} dt \\ &= n \log r - \log n - (n-1)\log(n-1) + \log \mu(n-1) \\ &= n \log r - \log n - \log(n-1) - (n-2)\log(n-2) + \log \mu(n-2).\end{aligned}$$

Continuing this process, we ultimately get

$$\log \mu(r) = n \log r - \log n - \log(n-1) - \cdots - \log 2 + \log \mu(1).$$

Since $\mu(1) = 1$, we have $\mu(r) = r^n/n!$ for $n \leqq r < n+1$. But $\mu(r) = |a_n| r^n$ for $n \leqq r < n + 1$, and therefore $|a_n| = 1/n!$. This proves the first part of the theorem.

Now, let $a_n \geqq 0$ for all n, so that $|a_n|$ can be replaced by a_n. If $v(r) = [r]$, then from the first part we get

$$f(z) = 1 + \sum_{n=1}^{\infty} \frac{1}{n!} z^n = \exp(z).$$

Conversely, if $f(z) = \exp(z)$, then $v(r) = [r]$ is trivially true.

References

1. E. Hille, Analytic Function Theory, Vol. 1, Ginn, Boston, 1959.
2. G. Valiron, Lectures on the General Theory of Integral Functions, Chelsea, New York, 1949.

DEPARTMENT OF MATHEMATICS, INDIAN INSTITUTE OF TECHNOLOGY, KANPUR — 208016, INDIA.

If convexity is not required, the constant $\frac{1}{2}$ in Bender's theorem is no longer valid; for example, let G be the unit square lattice and consider the figure bounded by semicircular arcs (Figure 2). A simple computation shows that $A(F)/P(F)$ can be made arbitrarily close to $(1/\pi)+(1/4)\approx 0.56831$.

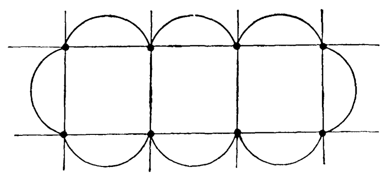


FIG. 2

PROBLEM 3: If the condition of convexity is deleted from Bender's theorem, what is the best replacement for the constant $\frac{1}{2}$?

In a private communication, Professor J. Marik has shown that $7/4$ is an upper bound. It is conjectured that the "best possible" constant is less than one, perhaps even $1/\pi + 1/4$.

In [3], Hadwiger generalized Bender's theorem to n dimensions. Nothing seems to be known about the analogous extremal bodies in higher dimensions.

References

1. E. Bender, Area-perimeter relations for two dimensional lattices, this MONTHLY, 69 (1962) 742–744.
2. M. Silver, On extremal figures admissible relative to rectangular lattices, Pacific J. Math., 40 (1972) 451–457.
3. H. Hadwiger, Volumen und Oberfläche eines Eikörpers, Math. Z., 116 (1970) 191–196.

DEPARTMENT OF MATHEMATICS, BOWDOIN COLLEGE, BRUNSWICK, MAINE 04011.

A MATRIX PROBLEM

E. A. MOLNAR

First we detour.

Given an abelian semigroup A, one can associate with A a unique abelian group $G(A)$, called the Grothendieck group of A (see [**2**, p. 58]). Now base-pointed topological spaces can be added, where the addition of spaces X and Y is their disjoint union with base points identified (called the "wedge" of X and Y and denoted $X \vee Y$). In particular, the set P of homotopy types of base-pointed finite CW-complexes has the structure of an abelian semigroup with zero. Thus one can ask about the Grothendieck group $G(P)$, and especially whether or not $G(P)$ has torsion. In order to find torsion elements, one needs to find spaces X and Y (in P) so that $nX \vee Z \simeq nY \vee Z$ for some Z (and $n > 1$), but $X \vee W \simeq Y \vee W$ does not hold for any space W (where $\simeq$ denotes homotopy equivalence).

It is not too hard to show that for special types of spaces cancellation sometimes holds: that is, if $\bar{X}$ and $\bar{Y}$ are "special", then for certain spaces Z, $\bar{X} \vee Z \simeq \bar{Y} \vee Z$ will imply $\bar{X} \simeq \bar{Y}$ [3]. This suggests that cancellation may always hold for these "special" spaces, and if this could be shown, one could get torsion in $G(P)$ simply by finding an example of "special" spaces where $n\bar{X} \simeq n\bar{Y}$, but $\bar{X} \not\simeq \bar{Y}$ (examples of this kind are plentiful for non-special spaces [1, 3]).

"Special" spaces are built using spheres; a map between two m-dimensional spheres can be characterized by its degree, an integer, and a map between wedges of m-dimensional spheres can be described by an integral matrix. Homotopy equivalences give matrices of determinant ± 1. Assuming that "special" spaces would not yield a torsion example, one tries to prove that $n\bar{X} \simeq n\bar{Y}$ implies $\bar{X} \simeq \bar{Y}$. Using a commutative diagram and comparing maps, one can prove the assertion if the following is true: there cannot be a unimodular (i.e., determinant $= \pm 1$) integral matrix (a_{ij}), with no $a_{ij} = \pm 1$, so that (a_{ij}^2) is also unimodular. Finding such a matrix would not only indicate the assertion is false, but it could also be a big step in finding a torsion example.

So now there is a number theoretic problem to solve; it can be restated as follows: does there exist an $n \times n$ matrix (a_{ij}) with integral entries, but no $a_{ij} = \pm 1$, so that the determinant of (a_{ij}) is 1, and the determinant of (a_{ij}^2) is ± 1? (Requiring the determinant of (a_{ij}) to be 1 instead of ± 1 does not change the problem; also, requiring that no a_{ij} equal ± 1 is a topological restriction, but it rules out trivial cases in the number theoretic problem. This restriction could of course be eased.) I know of no previous reference to this kind of problem. For $n = 2$, there cannot be such a matrix since the determinant of (a_{ij}) divides that of (a_{ij}^2). However, for $n > 2$, the problem seems to become much harder.

Number theoretic problems are not rare in algebraic topology. Sieradski [4] recently gave an example of spaces X, Y where even products (instead of wedges) of X were homotopic to even products of Y, but odd products of X were not homotopic to odd products of Y. The example depended mainly on the lemma: there exists an invertible $n \times n$ matrix (a_{ij}) over Z with $(a_{ij}) \equiv (7\delta_{ij})$ mod 12 if and only if n is even. Our matrix problem is somewhat different from that of Sieradski, and does not guarantee to give any firm topological result, although it could. Even without topological applications, the problem is interesting in itself.

References

1. P. J. Hilton, On the Grothendieck group of compact polyhedra, Fund. Math., 61 (1967) 199–214.

2. ———, General Cohomology Theory and K-Theory, Cambridge University Press, London, 1971 (London Math. Soc. Lecture Note Series No. 1).

3. E. A. Molnar, Relation between wedge cancellation and localization for complexes with two cells, J. Pure Appl. Algebra, 3 (1973) 141–158.

4. A. J. Sieradski, Square roots up to homotopy type, Amer. J. Math., 94 (1972) 73–81.

U.S. Army Institute of Administration, Fort Benjamin Harrison, Indiana 46226.

EDITED BY DAVID ROSELLE

Material for this Department should be sent to David Roselle, Department of Mathematics, Louisiana State University, Baton Rouge, LA 70803.

UPPER BOUNDS ON ARC LENGTH

RICHARD T. BUMBY

When we feel rigorous about the subject of arc length, we usually insist that the length of a parameterized curve be defined as the least upper bound of the lengths of inscribed polygons. One easily argues that the curve must be longer than the inscribed polygon, for whatever reasonable notion we take of 'length'. The question of why the least upper bound, and not some larger quantity, is never raised. This definition of arc length essentially tells us that we believe that the lengths of curves will be measured with some sort of flexible ruler. But if we use such a tool to estimate distances on a winding road from a road map, we may be in for a surprise.

It is easy to write a program which computes the length of the inscribed polygon of 2^n vertices. Let L_n denote the length with 2^n points equally spaced with respect to the parameter. Then we find that $L_{n+1} - L_n \approx \frac{1}{4}(L_n - L_{n-1})$ experimentally. This suggests that the error in estimating arc length by this method is inversely proportional to the square of the number of points. In order to prove such a result we must find some way of obtaining an upper bound on arc length. Such upper bounds also give further philosophical justification of our definition.

We now consider convex curves and we assert the following axiom:

If a closed convex curve C_1 is contained in an arbitrary closed curve C_2, then the length of C_1 is less than or equal to the length of C_2.

The intuitive justification of this is given by representing C_1 by a block of wood and C_2 by a loop of string. If we pull the loop tight, it will fit snugly around the block, measuring its perimeter. At the same time, the original loop measured the length of C_2. It was made shorter in measuring C_1, completing the justification. From the axiom we easily get that the length of a convex arc is bounded above by the length of a circumscribed polygon. This is, in fact, equivalent to our axiom.

We first assume that we have a convex arc. Even if the arc is not smooth, it can be parameterized by the angle of inclination α of the support (tangent) line. Any other parameter which moved along the curve without backtracking would be a monotonic function of α. We obtain our first upper bounds using the parameterization by α. We choose points $p_0, \cdots, p_n$ on the arc such that p_0 and p_n are the endpoints and the remaining points are determined by given increments of α. These points will be the vertices of the inscribed polygon. From them we construct $p_0^*, \cdots, p_{n+1}^*$, which

will be the vertices of the circumscribed polygon, by setting p_i^* equal to the point of intersection of the support lines at p_i and p_{i-1}, with $p_0^* = p_0$, $p_{n+1}^* = p_n$. Note that the direction of the support line at p_i is given by the value of α, which is our basic quantity, and so is well defined here even if there is no well-defined tangent at that point of the curve. For use in the arc length formula we also construct the point p_i' on the arc p_{i-1}, p_i where the support line is parallel to the chord p_{i-1}, p_i. These points are illustrated in Figure 1.

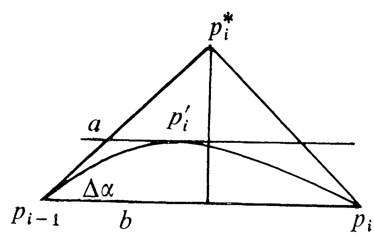


FIG. 1

The difference between the lengths of the inscribed and circumscribed polygons is a sum of quantities of the form $a - b = a(1 - \cos\Delta\alpha) < ca(\Delta\alpha)^2$, for some constant c. Thus, the difference between the lengths of the two polygons is bounded by c times the length of the circumscribed polygon times the square of the largest value of $\Delta\alpha$. Note that $\Delta\alpha$ is the difference in α between p_{i-1} and p_i', or between p_i' and p_i, each of which is bounded by the difference in α between p_{i-1} and p_i. If the points are chosen so that the $\Delta\alpha$ are equal, then the difference between upper and lower approximations to arc length is inversely proportional to the square of the number of points. More generally, if the points are equally spaced with respect to some parameter t such that $d\alpha/dt$ is bounded, the same conclusion holds. For any C^2 parameterization, $d\alpha/dt$ will be continuous. On an interval on which it is never zero, the curve will be (locally) convex. Thus, our empirical observation can be verified in this case.

These upper bounds can also be related to the arc length integral. We illustrate with the special case in which x is the parameter. We use x_i, x_i^*, x_i' for the x coordinates of p_i, p_i^*, p_i'. The length of the inscribed polygon is then

$$\Sigma \sqrt{1 + y'(x_i')^2} \quad (x_i - x_{i-1});$$

and the length of the circumscribed polygon is

$$\Sigma \sqrt{1 + y'(x_i)^2} \quad (x_{i+1}^* - x_i^*).$$

Each of these is a Riemann sum for $\int \sqrt{1 + y'(x)^2}\, dx$. The assumption of convexity is equivalent to assuming that $y'(x)$, and hence the integrand is a monotonic function of x. The convergence of the Riemann sums in this case is easily established.

There are other related uses of convexity in elementary calculus. In curve sketching, we know that tangents lie on one side and chords on the other side of a convex curve. The author has used an $x - y$ plotter driven by a desk top computer to draw very close inscribed and circumscribed polygons to various curves. The circumscribed polygons were obtained by computing the points of intersection of consecutive tangents.

This bracketing of the curve between tangent and chord can also be applied to Newton's method for finding zeroes of functions. The desired value is shown to lie in a series of nested intervals. This gives estimates on the rate of convergence of the process as part of the calculation. Of course, prior knowledge of convexity is a strong assumption, but one which can frequently be demonstrated. We hope to have shown that this information is valuable in analyzing limiting processes in which convergence is usually from one side.

DEPARTMENT OF MATHEMATICS, RUTGERS COLLEGE, NEW BRUNSWICK, N. J. 08903.

THE BOUNDED CONVERGENCE THEOREM

W. R. WADE

If f is a Lebesgue measurable function defined on a closed, bounded interval $[a,b]$ we shall define its **L^p norm** for $0 < p < \infty$ by

$$\|f\|_p = \left(\int_a^b |f(x)|^p\,dx\right)^{1/p} \tag{1}$$

and **L^∞ norm** by

$$\|f\|_\infty = \text{ess. sup.}\,\{|f(x)| : a \leqq x \leqq b\}.$$

The Bounded Convergence Theorem is a useful result from Lebesgue integration theory which, using the L^p norm notation, can be stated in the following form.

THEOREM 1. *If $f, f_1, f_2, \cdots$ is a sequence of measurable functions and M is a positive constant such that*

$$\|f_n\|_\infty \leqq M \quad n = 1,2,\cdots$$

and $\lim_{n\to\infty} f_n(x) = f(x)$ a.e. $x \in [a,b]$ then

$$\lim_{n\to\infty} \|f_n - f\|_p = 0$$

for any p satisfying $0 < p < \infty$.

Stated in this form the Bounded Convergence Theorem can be extended (see Theorem 2 or 3) to apply to sequences of functions which are uniformly bounded in the L^r norm, $0 < r \leqq \infty$. This result is extremely powerful yet its proof depends

upon three well-known results of real variables: Fatou's lemma, Egoroff's theorem, and Hölder's inequality.

THEOREM 2. *Let M be a fixed positive constant. Suppose* $0 < r \leqq \infty$, *and that* $f, f_1, f_2, \cdots$ *is a sequence of measurable functions such that*

$$\| f_n \|_r \leqq M \qquad n = 1, 2, \cdots \tag{2}$$

and $\lim_{n\to\infty} f_n(x) = f(x)$ a.e. $x \in [a,b]$. *Then*

$$\lim_{n\to\infty} \| f_n - f \|_p = 0$$

for any p satisfying $0 < p < r$.

Proof. By Theorem 1 we may suppose that $r < \infty$. Using Fatou's lemma and (2),

$$\| f \|_r \leqq \liminf_{n\to\infty} \| f_n \|_r \leqq M. \tag{3}$$

We can then conclude that f^p is integrable for any $0 < p \leqq r$.

Fix $p \in (0, r)$. Then $r/p > 1$, so we choose a real number $q \in (1, \infty)$ such that

$$p/r + 1/q = 1.$$

Let $\varepsilon > 0$, and choose a set $E \subseteq [a,b]$ by Egoroff's theorem so that

$$m(E) < [\varepsilon/(2M)^p]^q \tag{4}$$

and

$$f_n \to f \text{ uniformly in } [a,b] \sim E. \tag{5}$$

In particular, by (5)

$$\lim_{n\to\infty} \int_{[a\ b]-E} |f_n(x) - f(x)|^p dx = 0.$$

Hence it suffices to show

$$\int_E |f_n(x) - f(x)|^p dx < \varepsilon \tag{6}$$

for $n = 1, 2, \cdots$.

We apply Hölder's inequality to the integral displayed in (6) obtaining,

$$\int_E |f_n(x) - f(x)|^p dx \leqq \left\{ \int_E |f_n(x) - f(x)|^r dx \right\}^{p/r} \cdot \left\{ \int_E |1|^q dx \right\}^{1/q} \leqq (2M)^p \cdot \left\{ m(E) \right\}^{1/q},$$

by (3). By (4) this is $< \varepsilon$.

Statement (6) is thereby established and the proof of the theorem is complete.

Is Theorem 2 true if we replace $([a,b], m)$ *by a complete finite measure space*

(X,μ)? Yes. If we define

$$\|f\|_p = \left\{\int_X |f|^p d\mu\right\}^{1/p}$$

and similarly extend the symbol $\|f\|_\infty$ the proof to Theorem 2 goes through without a change.

Is Theorem 2 true if we allow p to equal r? No. Indeed, the sequence

$$f_n(x) = \begin{cases} \sqrt[r]{n} & \text{of } 0 < x < 1/n \\ 0 & \text{otherwise,} \end{cases}$$

converges to zero everywhere, satisfies (2) for $M = 1$, but does *not* converge to zero in the L^r norm.

Is Theorem 2 true if we replace $([a,b],m)$ *with a nonfinite measure space* (X,μ)? No. Indeed, with $X = [0,\infty)$ and μ being Lebesgue measure, the sequence

$$f_n(x) = \begin{cases} 1 & \text{if } n < x \leqq n+1 \\ 0 & \text{otherwise} \end{cases}$$

converges to zero everywhere, is uniformly bounded in the L^2 norm but fails to converge to zero in the L^1 norm.

However, one can modify the conclusion of Theorem 2 to take care of the case when μ is not finite.

THEOREM 3. *Let M be a positive constant and* (X,μ) *be a complete measure space. Suppose that* $0 < r \leqq \infty$ *and that* $f, f_1, f_2, \cdots$ *is a sequence of* μ*-measurable functions, such that*

$$\|f_n\|_r \leqq M, \quad n = 1,2,\cdots$$

and

$$\lim_{n\to\infty} f_n(x) = f(x) \text{ a.e. } [\mu] \ x \in X.$$

Then $\lim_{n\to\infty} \|f_n g - fg\|_p = 0$ *for every* $p \in (0,r)$ *and every* $g \in L^q_\mu(X)$, *where* $1/q + p/r = 1$.

The proof of Theorem 3 involves retracing the proof of Theorem 2 with g in place of 1. It uses two additional consequences of the integrability of $|g|^q$:

(7), $\{x : x \in X \text{ and } |g(x)| > 0\}$ is σ-finite

(8) given $\varepsilon > 0$ there is an $\delta > 0$ such that $\mu(E) < \delta$ implies $\int_E |g|^q d\mu < \varepsilon$.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF TENNESSEE, KNOXVILLE, TN 37916.

A PROOF OF THE BINOMIAL THEOREM

DAVID SHELUPSKY

Proofs of the binomial theorem for arbitrary real exponents usually given in advanced calculus texts depend on Taylor's theorem, [**1**, p. 572] or on other computations involving the derivatives of the function $(1 + x)^a$ with respect to x. The proof offered here is elementary in the sense that it uses only ideas actually needed to define the two sides of the equation

$$(1) \qquad (1 + x)^a = \sum_{k=0}^{\infty} (a)_k \frac{x^k}{k!}$$

(where, as usual, $(a)_0 = 1$ and $(a)_k = a(a-1)(a-2)\cdots(a-k+1)$ for $k = 1,2,3,\cdots$). The idea of the proof is not new. It is given by Knopp [**2**, p. 208] and a variation of it is an exercise in Hardy [**3**, p. 432] however, the treatment of the details given here is simplified (in particular the proof of **B** below).

To define the left-hand side of (1) we need the properties of the real numbers and of continuous functions which permit us to define c^a, a real, $c > 0$ real, as the extension of the mapping $r \to c^r$ defined for the rationals; these ideas also permit the discussion of logarithms. To define the left hand side we need these facts about infinite series: the ratio test for the interval of convergence of a power series, and the fact that two real power series about the origin can be multiplied together by Cauchy's method in the interior of the interval of convergence common to both [**2**, Theorem 91, p. 146].

Finally we shall make use of the theorem that the only continuous solution of Cauchy's functional equation

$$(2) \qquad \phi(a)\,\phi(b) = \phi(a + b)$$

is of the form $\phi(a) = c^a$ for some $c \geqq 0$. This is standard fare in many calculus books, usually as a problem, and sometimes for the additive version of (2),

$$(3) \qquad \Psi(a) + \Psi(b) = \Psi(a + b),$$

[**1**, p. 105, ex. 2,3 and p. 107, ex. 22].

To motivate the proof we begin with the binomial theorem for positive integer exponents, usually proved by induction on n,

$$(4) \qquad (1 + x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k,$$

which we assume known, and which is an easy exercise in any case. Since

$$\binom{n}{k} = n!/k!\,(n - k)! = (n)_k/k!$$

this suggests that we consider the series which is the right-hand side of (1). For x

such that $-1 < x < +1$ we find by an application of the ratio test that

$$\sum_{k=0}^{\infty} (a)_k \frac{x^k}{k!} \tag{5}$$

converges absolutely for any real a. We fix x at such a value and denote (5) by $\phi(a)$. If n is a positive integer $\phi(n) = (1 + x)^n$ by (4). To prove the binomial theorem for arbitrary real exponent we show that

(A) $\phi(a)$ *satisfies the functional equation* (2).

(B) $\phi(a)$ *is a continuous function of* a *for all real* a. *It will then follow that* $\phi(a) = c^a$, *where* $c = \phi(1) = 1 + x$.

Proof of A: Because these series are absolutely convergent we can multiply the series for $\phi(a)$ and $\phi(b)$ together using Cauchy's method to get

$$\phi(a)\,\phi(b) = \sum_{n=0}^{\infty} \left(\sum_{k=0} \binom{n}{k} (a)_k (b)_{n-k}\right) \frac{x^n}{n!}.$$

To prove A, we have to prove the identity

$$(a + b)_n = \sum_{k=0}^{n} \binom{n}{k} (a)_k (b)_{n-k}. \tag{6}$$

A proof of(6) by induction on n is similar to the corresponding proof of (4).

Proving (6) was a problem on a Putnam Examination some years ago and the published proof, Bush [4], was based on the binomial theorem for arbitrary real exponent.

An induction proof of (6) is as follows: for $n = 0$, (6) is true by definition. If it is true for some n then we have

$$\begin{aligned}
(a + b)_{n+1} &= (a + b - n)\,(a + b)_n \\
&= (a + b - n) \sum_{k=0}^{n} \binom{n}{k} (a)_k (b)_{n-k} \\
&= \sum_{k=0}^{n} \binom{n}{k} (a)_k (b)_{n-k} [(a - k) + (b - (n - k))] \\
&= \sum_{k=0}^{n} \binom{n}{k} [(a)_{k+1} (b)_{n-k} + (a)_k (b)_{n+1-k}] \\
&= (a)_{n+1} + (b)_{n+1} + \sum_{k=1}^{n} \left[\binom{n}{k} + \binom{n}{k-1}\right] (a)_k (b)_{n+1-k} \\
&= (a)_{n+1} + (b)_{n+1} + \sum_{k=1}^{n} \binom{n+1}{k} (a)_k (b)_{n+1-k} \\
&= \sum_{k=0}^{n+1} \binom{n+1}{k} (a)_k (b)_{n+1-k}.
\end{aligned}$$

This proves (6) for $n+1$ and so in general.

Proof of B: We remark that from A and (4) we have $\phi(a+n) = \phi(a)\phi(n) = (1+x)^n\phi(a)$ if n is a non-negative integer, so that it is sufficient to prove that $\phi(a)$ is a continuous function of a for $a < 0$. If $u > v > 0$ then

$$
\begin{aligned}
&|\phi(-u) - \phi(-v)| \leqq \sum_{k=0}^{\infty} |((-u)_k - (-v)_k)| \frac{|x|^k}{k!} \\
(7) \qquad & = \sum_{k=0}^{\infty} |u(u+1)(u+2)\cdots(u+k-1) - v(v+1)(v+2)\cdots(v+k-1)| \frac{|x|^k}{k!}.
\end{aligned}
$$

To estimate the difference $u(u+1)(u+2)\cdots(u+k-1) - v(v+1)(v+2)\cdots(v+k-1)$ in a simple way (without differential calculus in the form of the mean-value theorem for example) we remark that any polynomial $p(u)$ of degree $d \leqq k$ with non-negative coefficients satisfies the inequalities

$$
(8) \qquad 0 \leqq [p(u) - p(v)]/(u-v) \leqq kp(u)/u
$$

if $u > v > 0$. To prove this we note that for the polynomials, u^d, $0 < d \leqq k$.

$$
\begin{aligned}
0 \leqq (u^d - v^d)/(u-v) &= (u^{d-1} + u^{d-2}v + \cdots + v^{d-1}) \\
&= du^{d-1} \leqq ku^d/u
\end{aligned}
$$

while for $d = 0$ the equality holds; thus (8) follows from multiplying these inequalities for various d by the non-negative coefficients of $p(u)$ and summation. Applying (8) to the polynomial $p(u) = u(u+1)\cdots(u-k+1)$ which appears in (7) we get

$$
|\phi(-u) - \phi(-v)| \leqq \left[|x| + \sum_{k=2}^{\infty} (u+1)(u+2)\cdots(u+k-1)\frac{|x|^k}{(k-1)!}\right] |u-v|.
$$

Consider the infinite series in the square brackets on the right hand side of this inequality; call it $m(u)$. An application of the ratio test shows that $m(u)$ converges for any u since $-1 < x < +1$, and since $m(u)$ is clearly an increasing function of u for u positive we have $|\phi(-u) - \phi(-u)| \leqq M|u-v|$, where $M = m(w)$ and w is any number larger than u and v. It follows that $\phi(a)$ is continuous for $a < 0$ and so for all a. Thus we have proved B and so the binomial theorem: (1) is true for all real a and all x such that $-1 < x < +1$.

NOTE: It is known (Asplund and Bungart, [3], p. 141, ex. 3, 6.37) that the only measurable solution of (3) are linear functions. Using this and elementary measure theory, we may dispense with the proof of B; indeed $\phi(a)$ is the point-wise limit of the partial sums of (5), which are polynomials in a and hence continuous, thus $\phi(a)$ is measurable.

References

1. A. E. Taylor, Advanced Calculus, Ginn and Company, Boston, 1955.

2. K. Knopp, Theory and Application of Infinite Series, Blackie and Son, London and Glasgow, 1951.

3. G. H. Hardy, A Course in Pure Mathematics, 9th Edition, Cambridge University Press, New York, 1945.

4. L. E. Bush, The 1962 William Lowell Putnam Mathematical Competition, this MONTHLY, 70 (1963), 714 (problem 1, Part II).

5. E. Asplund and L. Bungart, A First Course in Integration, Holt Rinehard and Winston, New York, 1966.

DEPARTMENT OF PHYSICS, THE CITY COLLEGE, NEW YORK, N.Y. 10031.

MATHEMATICAL EDUCATION

EDITED BY SHIRLEY HILL AND PAUL MIELKE

Material for this Department should be sent to Shirley Hill, Department of Mathematics, University of Missouri, Kansas City, MO 64110, or to Paul Mielke, Department of Mathematics, Wabash College, Crawfordsville, IN 47933.

THE COMPUTER ILLITERACY PROBLEM: A PARTIAL SOLUTION

PETER G. LYKOS

1. How the computer is affecting education. The computer is transforming the ways in which the problem solvers and the decision makers of our society go about doing their jobs. Not only are old problem-solving techniques being speeded up and scaled up, but new techniques are being invented and developed which would not have been considered seriously before the invention and proliferation of the computer. And the proliferation of the computer is being accelerated as a direct consequence of two technological developments [1]:

1. The low cost and increasing flexibility of *minicomputers*, and
2. The increasing flexibility and ease of use of *tele-communications* systems whereby users can use a typewriter-like or keyboard plus TV-like terminal to access a variety of computers remotely located.

Although the first widespread use of the computer in education, other than as a management tool, was to support graduate research in the hard and soft sciences, gradually computer-based elements have come to be developed and incorporated into undergraduate curricula as well. Not only have B.S., M.S., and Ph.D. academic programs in Computer Science evolved in many of our nation's universities, but a large amount of experimentation and ferment in other disciplines is taking place as well, particularly in Accounting, Business, and Management. Indeed, as the major

impact of the computer on our society is, and continues for some time to be, in the general area of management [**2**], we can anticipate a large and continuing growth and proliferation of new curricular emphases on accounting [**3**], on management information systems [**4**], on simulation or modeling [**5**], and on gaming [**6**].

2. The computer and the curriculum. Three and a half years ago the American Institute for Research issued a report on an NSF-supported survey made of all 23,000 public high schools in the country. At that time 13% of the schools reported use of the computer as part of the instructional process, primarily in business and accounting, or in mathematics. That percentage has, very likely, increased significantly since that time.

The NSF has sponsored numerous efforts in computers-in-education. Computer-based curricular material at the college level is being developed. Faculty in the several disciplines are being trained in the use of computer hardware and software. Cost-effective systems of delivering computer service are being sought, primarily through regional cooperative networks whereby research-oriented university computer centers have attempted to provide cost-effective computer service appropriate for classroom use in colleges. Approximately 10% of the nation's universities and 10% of the nation's colleges have been involved in 30 regional cooperatives. The NSF is also supporting major projects attempting to collect and adapt "for export" computer programs developed in college environments. A case in point is CONDUIT, a consortium of five university-based remote terminal-accessible computer services organized to study and evaluate the transportability and dissemination of computer related curriculum materials. Much of that material will initially prove suitable for use in secondary schools in honors courses and will then diffuse into the other programs.

In addition to the many NSF sponsored efforts, there are other computer-in-education activities which are having, or will have, a direct influence on secondary schools and community colleges.

An interesting and revealing example is that of Wilbur F. Pillsbury, Chairman of the Department of Economics and Business Administration, Knox College, Galesburg, Illinois. He used a sabbatical leave to learn more about the role of the computer in his discipline. Using an elementary subset of a standard computer language, he developed about 60 short computer programs to augment the teaching of accounting and business. Having had many years of experience in teaching those concepts before computer augmentation, by comparison he was able to demonstrate the increased effectiveness of the computer-augmented approach in the classroom. The South-Western Publishing Company worked with Professor Pillsbury to develop corresponding textbooks under the title, "Computer Augmented Accounting." As of a year ago, over 200 institutions across the country were using his material based on FORTRAN programs running on 10 different computers. Thus, a viable approach to the preparation and dissemination of *usable* computer-based curricular materials is one where materials are developed by an *experienced* teacher, designed to

augment teaching in the classroom, based on a number of *simple* programs in a *standard* language, and edited and distributed by a textbook *publisher*.

Gradually Computer Science has come to be recognized as a separate and distinct discipline. The publication of the report "Curriculum 68" [7] has contributed to the design of computer science courses, and graduate programs in Computer Science have begun to produce M.S. and Ph.D.'s whose primary training has been in Computer Science. Although most professionals who are identified as computer scientists have had their formal training and degrees in Electrical Engineering, Physics, or Applied Mathematics, gradually the field will be dominated and defined by professionals trained as computer scientists just as is the case with other disciplines. The whole field of Computer Science, and of its interface with other disciplines, has become too important to leave to the *ad hoc*-ists.

For those teachers (or those preparing to be teachers) who wish to work at the computer science—other discipline interface (or in Computer Science itself) there needs to be a coordinated set of courses designed to display and develop what Computer Science is. In addition they need to discover the important ways the computer is affecting *what* they teach as well as *how* they teach.

3. The secondary school: pressure from the computer. Several pressures are coming to be felt in the secondary school and community college environments. These include:

1. Computer awareness and experience on the part of incoming students due to proliferation of the computer and ease of its cost-effective use by pre senior high school students.
2. Substantial and growing computer access at the secondary school and community college levels.
3. Substantial and growing development of computer-based curricular materials in the colleges with concomitant developments following at the secondary school level.
4. Lack of training of teachers and administrators regarding computer hardware, software and courseware selection and use. What training there is is usually an elementary computer programming experience; i.e. a vocational skill.
5. Considerable and increasing confusion about the distinction between computer-assisted pedagogy, computer-augmented discipline-oriented techniques of problem-solving and decision-making, vocational training in data processing, computer science and engineering, and use of the computer in the management of the educational enterprise itself.
6. Difficulty of acquiring computer service as a new expense in the face of cost of education rising faster than the gross national product.

Secondary schools attempting to react to these pressures find the difficulty of the task compounded by the layman's very limited view of the computer as an accounting machine on one hand or a super desk calculator on the other. The difficulty is further aggravated by a corresponding mistrust and even antagonism on the part of the average citizen faced by invasion of privacy on one hand and the irritation of having to deal with unresponsive machine-generated billing and accounting statements and amazingly individualized mass mailings on the other.

In the greater Chicago area the pressure of the computer in the secondary schools

became particularly acute because of the massive Secondary School Computer Science Education program [8] which, over the past 10 years, brought over 15,000 high school students and over 1200 high school teachers, from over 300 high schools, to the IIT campus to take courses and workshops in computer programming and in computer applications. That IIT-supported program evolved further when, in 1966, IIT installed an IBM 360/40 computer and augmented the IBM operating system with the IIT Remote Job Entry system. (A close copy of that system survives on the UNIVAC 9400 in use by the Montreal Public School System.) High schools and colleges were then able to send computer programs to, and receive computer output from, the IIT computer over ordinary telephone lines, from ordinary teletypewriters. For $2,000 for the academic year a school in Chicago was able to rent a teletypewriter, dataphone, and telephone line from Illinois Bell Telephone Company, and purchase enough computer time on the highly student-oriented IIT computer system so that 50 students, each submitting three programs per week, could be supported for the entire academic year.

In the Chicago Public School system, for example, extensive computer use began when Lane Technical High School and South Shore High School used the IIT computer from teletypewriters [9]. The rapid growth was further facilitated as about 200 Chicago Public High School teachers had received training in the IIT Saturday Teacher's Computer Workshops. By 1971 the Chicago Public School system had installed its own computer and was supporting terminals in all of its 58 secondary schools, as well as in several of its elementary schools.

4. A master of science for teachers in computer science. As a consequence of all the pressures on the secondary school and the community college, particularly in the greater Chicago area, it seemed appropriate to design and to implement a degree program, "Master of Science for Teachers in Computer Science."

Through acquisition of such a degree teachers and administrators could have both the training and credibility to provide competent leadership in addressing the difficult question of what should be happening with computers in secondary and elementary schools.

The IIT MST/CS provides an integrated and coherent program of professional training based on extensive experience in both Computer Science education at the university level, and a large and varied program of long standing of computer training and computer use in many high schools. In addition, with its inception, the IIT MST/CS provided a beginning for the setting of standards of qualification for teachers and administrators charged with responsibility in the use of computers in their educational programs.

The primary purpose of the IIT MST/CS program is to strengthen the teacher's academic background in the emerging discipline of Computer Science. While flexibility is desirable and exists within the program, substantive course work in the core of Computer Science is required.

The 32 semester hour MST/CS program involves a *core curriculum* required of all degree candidates, complemented by an *elective program* which is designed and adapted to meet the needs and career goals of each individual degree candidate.

As originally conceived, the *core curriculum* involved 17 semester hours of work allocated as follows:

3 Sem. Hr. Computers and Society — a lecture and term paper discussion course concerned with the effect of computer/communications technology on academia, industry, and government on one hand, and the life of the individual on the other.

3 Sem. Hr. Computer Languages — a lecture and laboratory course concerned with a comparative study of computer languages and applications programs.

3 Sem Hr. Computer and Curriculum Content — primarily a laboratory course with discussion sessions including the preparation and organization of computer-based curricular elements and a concern with the problems of incorporating such materials in the educational process.

3 Sem. Hr. Computer-Assisted Instruction — a lecture and laboratory course concerned with techniques such as drill and practice, tutorials, author languages, particular CAI systems, and the general problem of computer-assisted pedagogy.

4 Sem. Hr. Special Project — a unique computer-based project done by the degree candidate with a faculty advisor.

1 Sem. Hr. Computer Science Departmental Seminar. Participation is required by all graduate degree candidates.

The *elective program*, comprising 15 Sem. Hrs., was made up of a coordinated set of courses in Computer Science designed to build on the core program and, where appropriate, in other disciplines as well.

Recently the IIT MST/CS core program was revised so that the courses, "Computers and Society," and "Computer Languages" are no longer required; "Computer-Assisted Instruction" has become an alternative to "Practicum in the Application of Computers to Education," a new graduate course "Computer Science in the Classroom" has been added, and four regular senior courses in Computer Science dealing with the structure of algorithms and with programming languages and translators have been added as well. The net result is an increase in the core program (including an MST project) to 26 semester hours, leaving 6 semester hours for electives. Thus the IIT MST/CS program can accommodate teachers from any discipline, although it is expected that the interest will come primarily from teachers of Physics, Chemistry, Biology, Mathematics, and Business.

5. The teacher of mathematics. The teacher of mathematics is in a curious position regarding computers. Research in mathematics, as a discipline, has not been affected significantly by the advent of the computer **[10]**. Other than as an aid in exhaustive proofs in Number Theory and some work in Group Theory, not much new in mathematics has been discovered because of the computer. Thus, insofar as the teacher of mathematics is preparing students for careers as "pure" mathematicians, the computer is not particularly important at this time. However, to the extent that the teacher of mathematics, as a service to other disciplines, is preparing students to develop or

to use mathematics as a language and as an analytic tool in problem-solving and decision-making, considerable attention needs to be given to how the needs of problem solvers and decision makers *have changed* as a consequence of the invention and proliferation of the computer. Accordingly, the priorities must be adjusted regarding which established areas within mathematics need to be taught.

Secondly, an unfortunate problem exists to which the teacher of mathematics needs to be particularly sensitive. The layman regards the computer as somehow "being mathematics." This erroneous concept has an unfortunate consequence in that those administrators, faculty, parents, and students who feel they have no aptitude for mathematics, and hence avoid it, shy away from the computer in the same way. That misconception is probably the single largest factor inhibiting the infusion, and diffusion, of informational technology (of which the computer is only a part) throughout our society. Yet, that misconception is reinforced again and again in part because computer programming courses are usually offered by teachers of mathematics.

Presented in summary form at the Fiftieth Annual Meeting, National Council of Teachers of Mathematics, Section on Teacher Education, April 16–19, 1972. (The opinions expressed here do not necessarily reflect NSF policy.)

References

1. P. Lykos, Hierarchical computing — a computer system cost effectively optimized for education, Computing in Higher Education 1971, EDUCOM, Princeton, N. J., (1972) 11–14.

2. C. A. Meyers, Computers in Knowledge-Based Fields, MIT Press, 1970.

3. W. F. Pillsbury, Computer Augmented Accounting CompuGuide One, Southwestern Publishing Company, 1970; CompuGuide Two, *ibid.*, 1971; FORTRAN IV Programming With Business Applications CompuGuide Three, *ibid.*, 1973.

4. D. Teichroew, Education related to use of computers in organizations, Comm. ACM, 14 (1971) 573–588; D. Teichroew, chairman and R. Ashenhurst, editor, Curriculum recommendations for graduate professional programs and information systems, Comm. ACM, 15 (1972) 363–398.

5. 1971 Winter Simulation Conference Proceedings, AFIPS Press, 1971.

6. K. Cohen, W. R. Dill, A. Kuehn, and P. R. Winters, Administrator's Manual to Accompany the Carnegie Tech Management Game, Richard D. Irwin, Inc., Homewood, Illinois, 1964.

7. ACM Curriculum Committee on Computer Science, Curriculum 68: recommendations for academic programs in computer science, Comm. ACM, 11, 3 (March 1968) 151–197.

8. A. Peluso, Answer for a dime, American Education, 4 (May 1968) 28–29.

9. Reports to the Board of Education of the City of Chicago from the General Superintendent of Schools, 74037–G (22 June 1966), and 67–754 (12 July 1967).

10. L. A. Zadeh, Newsletter, Conference Board of the Mathematical Sciences, I (May 1972) 1.

Division of Computing Research, National Science Foundation, Washington, D. C. 20550. (On leave from Chemistry Department, Illinois Institute of Technology.)

A FOURTH COURSE IN MATHEMATICS FOR ELEMENTARY SCHOOL TEACHERS: ANALYTIC GEOMETRY

D. R. Andrew

The purpose of this note is to remind the reader of the fact that one of the most exciting mathematics courses in the mathematics curricula of the 1950's has almost completely disappeared from most current mathematics curricula and to show how this course can be brought back into the curricula immediately to fill a specific need. The course is analytic geometry.

Any of several analytic geometry textbooks written in the 1950's would make an excellent text for the fourth course in the Level I mathematics four-course sequence recommended for elementary school teachers by CUPM in 1961.

The University of Southwestern Louisiana adopted the four-course sequence in 1962. Since that time we have tried various textbooks and sets of notes following CUPM guidelines for the geometry course. It is the consensus of the faculty that a great deal of improvement is needed in this course.

To date at least six textbooks have been written specifically for a geometry course for elementary school teachers. Our experience has shown that these books in the main share the following difficulties: (1) They are very boring to both the students and the instructors, (2) One would never guess that geometry has anything whatsoever to do with arithmetic or algebra from reading one of these books.

It would serve no purpose to mention the names or authors of these books. For the most part, the authors merely followed the CUPM guidelines for the geometry course.

A standard text on analytic geometry offers all of the following advantages to a prospective elementary teacher:

1. The course demands that the student *use* the algebra he has just learned.
2. It relates algebra and geometry in an interesting, if not exciting manner.
3. The teacher can adjust the level of the course over a broad range simply by carefully selecting the problems that he assigns and emphasizes.
4. The course builds nicely on not only the algebra the student has learned but also on his introduction to coordinate systems from high school and junior high school mathematics courses.
5. There are dozens of ways to use the "discovery" approach with analytic geometry problems.
6. With this course contained in his background, the elementary teacher would be comfortable with all geometrical concepts that he is supposed to teach in elementary school.
7. Having taken this course, those students who decide to continue in mathematics for a teaching minor are better prepared for the calculus courses.

The more challenging and interesting aspects of analytic geometry have almost entirely disappeared in most mathematics curricula. Calculus texts have incorporated properties of conic sections, but have dropped nearly all of the problems like the following: prove algebraically that the line segments joining the mid-points of the opposite sides of a quadrilateral bisect each other. There are hundreds of problems

of this kind of varying difficulty that really excite large numbers of students. One does not have to be a mathematician to appreciate that the following statement under appropriate assumptions is obvious when thought of geometrically: two simultaneous linear equations in two unknowns have a unique solution.

It may be possible to write exciting textbooks using a unified approach for the four Level I courses within the framework of the CUPM revised recommendations of 1971. Until that happens a prospective elementary school teacher would enjoy and profit more from a study of analytic geometry than from any of the texts currently available as geometry for elementary school teachers.

Of course the analytic geometry course would also be open to students needing a second terminal mathematics course or desiring a mathematics elective.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF SOUTHWESTERN LOUISIANA, LAFAYETTE, LA 70501.

A VISITING LECTURE PROGRAM FOR SECONDARY SCHOOLS IN MAINE

DONALD B. SMALL

This is a report of the development and the first eleven weeks of the *Maine Visiting Lecturers Program for Secondary Schools.**

1. Formulation of the Program. The idea of a high school lecture program grew primarily out of (1) a concern over what the author feels is a widening communications gap between the college or university mathematics professor on the one hand and the high school teacher and student on the other, and (2) a realization of an opportunity to provide an enrichment program for high schools. It seemed that a program similar to the Visiting Lecturer Program for Colleges sponsored by the Mathematical Association could be developed on a State basis which would help lessen this communications gap and at the same time provide enrichment for the high schools. In particular, the envisioned program has four major aims:

1. To strengthen and stimulate interest in mathematics among high school students.

2. To aid in the motivation of able high school students to continue their formal education.

3. To create and strengthen ties between high school and college programs in mathematics.

4. To provide an opportunity for personal contact between high school students and college teachers.

---

* **Editor's note.** The statewide program in Maine furnishes an illustration of the kind of program of secondary school lecturers that has been well received in a number of states and Sections of the Association. This Editor, as a member of the Association's Committee on Secondary School Lecturers, hopes that additional Section sponsored Lecturer programs as well as additional statewide programs will derive encouragement from the success of the program in Maine. (J. G. HARVEY)

The challenges in developing a lecture program fell into four categories: (1) organizing and administering the program, (2) obtaining lecturers, (3) financing the program (money for honorariums, travel, mimeographing, and mailing), (4) getting the high schools to take the initiative to request lecturers to speak at their schools. The fact that the program was a State program rather than a sectional or national program minimized the difficulties in the first three categories considerably. A standardized system and form letters for replying to requests, a telephone, and an efficient secretary took care of the first category. No difficulty was encountered in obtaining lecturers and, again, this was probably due to the program's being a State program. How long the pool of lecturers will last under the type of demand that we had the first eleven weeks remains to be seen. The difficulties in the third category were resolved by not paying honorariums, by having lecturers pay their own expenses (funds were obtained for travel expenses after six weeks of the program), by having Colby College and the Maine Department of Education absorb the telephone, mimeographing, and mailing expenses. Arnold Johnson, a State Curriculum Consultant (mathematics), took care of the fourth category. Mr. Johnson was very enthusiastic about the program and offered to act as the liaison person to the high schools and, as a State Consultant, to "talk up" the program with the high school mathematics teachers.

The involvement of a liaison person on the State level is very important to the success of such a program.

2. Organization of the Program. During December 1971, the chairmen of the Mathematics Departments in the various colleges and universities in the State of Maine were asked to ascertain which members of their departments would be interested in lecturing in high schools within a fifty mile radius, and furthermore, would be willing to forgo an honorarium and to pay their own traveling expenses. The response was tremendous. By the middle of February 1972, forty-two faculty members from Bates, Bliss, Bowdoin, Colby, and Ricker Colleges, and the University of Maine at Farmington, Orono, Portland-Gorham, and Presque Isle had volunteered. There were one hundred and four talks listed by these forty-two faculty members with most of the faculty submitting two or three titles.

The next step was to draw up a brochure that would be distributed to the high schools. Advice was sought from Arnold Johnson and Arthur Clark, a high school mathematics teacher in Winslow, Maine. It was our opinion that simplicity in the brochure, ease of requesting a lecturer, immediate response to a request, and emphasis on the State aspect of the program were important factors in obtaining high school participation. Because Maine abounds in small schools, an extra effort was made to encourage all schools to take advantage of the program regardless of their size. The brochure consisted of three parts. The first part describes the program, the second part lists the lecturers (grouped by college or university), together with their titles, and the third part is a form for requesting a lecturer. The first part appears as follows:

"*Maine Visiting Mathematics Lectures for Secondary Schools* — 1972.

The major aims of the High School Visiting Lectures Program are:

1. To strengthen and stimulate interest in mathematics among high school students.
2. To aid in the motivation of able high school students to continue their formal education.
3. To create and strengthen ties between high school and college programs in mathematics.
4. To provide an opportunity for personal contact between high school students and college teachers.

The lecturers are college or university mathematics professors who have volunteered to donate their time and energy to this program. Lecturers are available to visit schools from the *present time* through December 1972.

The program is open to every high school in the State of Maine. Schools are encouraged to make as many requests for lecturers as they choose. Since lecturers are paying their own expenses, schools are asked to request lecturers within a fifty mile radius. However, if no lecturer is available within this distance, schools are encouraged to request lecturers from farther away. There will be no expense to any school.

In order to encourage all schools to take advantage of this program, the size of the group anticipated at the lecture is of no consequence.

Requests for a lecturer and the date desired should be sent to Professor Donald Small, Colby College, Waterville, Maine 04901.

Professor Small will contact the lecturer and then contact the school telling them of the availability of the lecturer on the desired date."

Arnold Johnson volunteered to have his office print the brochures and distribute them to all the high schools in the State. The brochures were mailed on 17 March, 1972.

In requesting a lecturer, the high school teacher is asked to list an alternate as well as alternative dates and times. Requests are transmitted to the author who then calls the requested lecturer to determine his or her availability. This procedure enables the author to notify the high school teacher by return mail of the availability of the lecturer. He also asks the teacher to contact the lecturer directly as to the level of the class, the length of the period, etc. Also included in this letter is a new Request Form and a Report Form which the teacher is asked to fill out and return to the author after the talk. A similar Report Form is sent to the lecturer. These forms ask for the title of the talk, name of the lecturer, number of students and faculty in attendance, comments on the visit, and solicit suggestions for improving the program. These reports provide the feedback for the program.

3. Results, 17 March – 6 June, 1972. There were forty-five requests for lecturers received (including six for the fall of 1972). Thirty-eight lectures were given and reports were received on thirty-six of these. The following statistics were taken from these reports:

There were 9 lectures given in April, 26 in May, and 3 in June.
20 high schools requested 1 lecturer each
4 high schools requested 2 lecturers each
2 high schools requested 3 lecturers each

1 high school requested 4 lecturers
1 high school requested 6 lecturers
There were requests for 19 different lecturers.
7 lecturers gave 1 lecture
6 lecturers gave 2 lectures
5 lecturers gave 3 lectures
1 lecturer gave 4 lectures

A total of 2166 students and 81 teachers attended the lectures. The total distance traveled by the lecturers was 3150 miles.

An indication of audience sizes is given by the following table:

Number of Students:	7–20	21–40	41–70	73–90	100–130	300–325
Numbers of Lecturers Presented:	6	14	8	4	2	2

During May two additional guidelines were adopted: (1) lecturers would be limited to three requests (although one person did speak four times), (2) schools requesting a lecturer outside of their fifty mile neighborhood when there were other lecturers available within the neighborhood would be asked to pay the lecturer's expenses. One school made such a request and agreed to pay the expenses.

On one occasion when the author knew that a lecturer was going to be in an area that was outside the neighborhoods of all of the lecturers, he asked Arnold Johnson to contact a school in that area and suggest that it invite the lecturer to speak at that school. This he did with the result that an additional lecture was given without an extended trip. We expect to do more of this type of arranging next year.

Early in May Arnold Johnson was able to allocate through his office $300 for travel expenses.

The comments from both the high school teachers and lecturers have been very favorable. Even when travel funds were available, a number of lecturers did not ask to be reimbursed for their expenses saying that they enjoyed the opportunity to speak in a high school and were willing to pay their own expenses.

DEPARTMENT OF MATHEMATICS, COLBY COLLEGE, WATERVILLE, MAINE 04901.

A COMMENT ON M. F. DACEY'S "MATHEMATICS FOR THE UNDERGRADUATE IN THE SOCIAL SCIENCES"

WILLIAM D. SPEARS

In a recent article Dacey [2] reviewed the need for and problems of offering satisfactory preparation in mathematics for students in the biological and social sciences. He pointed out that the CUPM report [1] which outlined an undergraduate program in mathematics for these students has been largely ignored by the social scientists. He is correct in stating that this report "has many fine attributes and identifies course contents that are suited to numerous social science programs...."

However, a question can be raised concerning the general acceptability of the major thrusts of the CUPM recommendations.

Specifically, the CUPM report appears to emphasize mathematics necessary for what many biological and social scientists are doing now rather than what at least some have done and aspire to in the future. The focus is upon probability, statistics, and linear algebra. Basic traditional topics in analysis are included in the recommendations, but one feels from the discussion that a major reason for their inclusion is to provide a foundation for better statistics courses.

There is no doubt that we need better statistics courses as well as a more sophisticated approach to probability and linear algebra. On the other hand, all sciences are concerned with change — "motion" to use Frank's [3] term — and mastery of the language of change is essential if causal conceptions are to be developed.

Many psychologists still prefer the standard analytic approach using differential equations that was so common in the nineteenth century [e.g., **4**] and still prevalent among some experimentalists [e.g., **5**, **6**]. Furthermore, to the extent that biological sciences have developed mathematical models, standard analysis has been emphasized by a number of them [**7**, **8**, **9**, **10**].

There is no assurance that biological and social science data can be treated most profitably with analysis. However, it has hardly been tried; but when it has, success has been the rule when data were obtained in functional rather than descriptive contexts.

The CUPM report [**1**] and Dacey's comments [**2**] are certainly relevant in view of much, perhaps most, present methodology in biological and social sciences. On the other hand, some of us retain hope for a different sort of science.

References

1. CUPM, Tentative Recommendations for the Undergraduate Mathematics Program of Students in the Biological, Management and Social Sciences, Berkeley, California, January, 1964.

2. M. F. Dacey, Mathematics for the undergraduate in the social sciences, this MONTHLY, 78 (1971) 784–788.

3. Philipp Frank, Philosophy of Science, Prentice-Hall, Englewood Cliffs, N. J. 1957.

4. Hermann L. F. Helmholtz, Sensations of Tone, Dover, New York, 1954.

5. Clark L. Hull, Principles of Behavior, Appleton-Century-Crofts, New York, 1943.

6. Don Lewis, Quantitative Methods in Psychology, McGraw-Hill, New York, 1960.

7. Alfred J. Lotka, Elements of Mathematical Biology, Dover, New York, 1956.

8. N. Rashevsky, Mathematical Biophysics, Vol. I, II, Dover, New York, 1960.

9. Douglas S. Riggs, The Mathematical Approach to Physiological Problems, M. I. T. Press, Cambridge, Mass., 1963.

10. Norbert Wiener, Cybernetics, M. I. T. Press, Cambridge, Mass., 1961.

265 CARY DRIVE, AUBURN, ALABAMA 36830.

PROBLEMS AND SOLUTIONS

Edited by Emory P. Starke

All problems (both elementary and advanced) proposed for inclusion in this Department should be sent to E. P. Starke, 1000 Kensington Ave., Plainfield, NJ 07060. Proposers of problems are urged to enclose any solutions or information that will assist the editors. Ordinarily, problems in well-known textbooks and results in generally accessible sources are not appropriate for this Department. No solutions (except those accompanying proposals) should be sent to Professor Starke.

ELEMENTARY PROBLEMS

Solutions of Elementary Problems should be sent to Problems Group, Mathematics Department, University of Maine, Orono, ME 04473. To facilitate their consideration, solutions of Elementary Problems in this issue should be typed (with double spacing) and should be mailed before July 31, 1974.

An asterisk (*) means neither the proposer nor the editors supplied a solution.

E 2468*. *Proposed by Harry Ruderman, Hunter College Campus School*

Suppose that $m > n \geqq 0$ are integers such that $2^m - 2^n$ divides $3^m - 3^n$. Show that $2^m - 2^n$ divides $x^m - x^n$ for all natural numbers x.

E 2469. *Proposed by David Gale and C. Roger Glassey, University of California, Berkeley*

Two players, the Hider and the Seeker, simultaneously choose points in a closed disk of unit radius. The Hider escapes if his point is more than $\frac{1}{2}$ unit from that of the Seeker. Show that if both players play optimally in the sense of game theory, then the Hider will be caught with probability $1/7$.

E 2470. *Proposed by G. Tsintsifas, Thessalonika, Greece*

Let $A_0 A_1 \cdots A_n$ be an n-simplex in R^n and let G be its centroid. Let P be any point and let M_i be the intersection of the line $A_i G$ with the hyperplane through P parallel to the face opposite A_i. Show that

$$\sum_{i=0}^{n} \frac{GM_i}{GA_i} = 0,$$

where GM_i/GA_i denotes the directed ratio of the distances from G to M_i and G to A_i respectively. Show also that this equation characterizes the point G as the centroid; i.e., if the sum is 0 for every point P, then necessarily G is the centroid. (Cf. Problem E 2394 [1974, 283].)

E 2471. *Proposed by G. Tsintsifas, Thessalonika, Greece*

Let a, b, c denote the sides of a triangle ABC and let m_a, w_a, h_a denote the median, angle bisector, and altitude to side a respectively. Show that

$$(1)\ \frac{(b+c)^2}{4bc} \leqq \frac{m_a}{w_a}, \qquad (2)\ \frac{b^2+c^2}{2bc} \leqq \frac{m_a}{h_a}.$$

When does equality hold?

E 2472. *Proposed by David Shelupsky. City College of New York, and H. W. Gould, West Virginia University*

Let n and p be nonnegative integers. Show that

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{2^{2k}}{\binom{2k}{k}} \left\{ \frac{1}{(2k+1)} \frac{3}{(2k+3)} \cdots \frac{2p-1}{(2k+2p-1)} \right\} = \frac{2p-1}{2p-1+2n},$$

where we make the interpretation that when $p = 0$ the "empty product" in the curly brackets is unity. (The case $p = 0$ is Formula (4.12) of H. W. Gould's *Combinatorial Identities*, Morgantown, West Virginia, 1972.)

E 2473. *Proposed by Robert Brooks, Harvard University*

Let f and g be irreducible polynomials with coefficients in a field K. Prove that there exists a polynomial which is a rational function of f/g if and only if $f = ag + b$ for some a, $b \in K$.

SOLUTIONS OF ELEMENTARY PROBLEMS

An Interesting Geometric Construction

E 2407 [1973, 316]. *Proposed by A. W. Walker, Toronto, Canada*

Given the circumcenter O, orthocenter H, and incenter I of an unknown triangle T, (A) locate by Euclidean construction the Gergonne point and the Lemoine point of T (i.e., the centers of perspective of T with the triangles formed respectively by the contact points of the sides of T with its incircle and by the tangent lines at the vertices of T to its circumcircle). (B) Locate the orthocenters of the pedal triangles of H and I.

Editorial Note. This problem is interesting because triangle T cannot in general be constructed from the given points, but many points related to T, including those mentioned in this problem, can be so constructed. The two solutions received are quite involved, so we do not take the space here to print either of them. The Problems Group will be happy to send to any interested reader a copy of each solution.

Solved by M. G. Greening, University of New South Wales, Australia, and by the proposer.

Colombian Numbers

E 2408 [1973, 434]. *Proposed by Bernardo Recamán S., Colegio San Carlos, Bogotá, Colombia*

A natural number is a *decimal Colombian number* if it cannot be written as $m + s(m)$ for any natural number m, where $s(m)$ denotes the sum of the digits of m when m is expressed in decimal notation. For example, 28 is not a decimal Colombian number since $28 = 23 + 2 + 3$, whereas 9 is a decimal Colombian number. Base n Colombian numbers are defined analogously.

Prove that in any base there are infinitely many Colombian numbers.

Solution by D. W. Bange, University of Wisconsin, La Crosse. We present general formulas that generate infinitely many Colombian numbers. First we shall treat the base 10 case. Note that 9 is a Colombian number and that any number of the form

$$8 \cdot 10^j + 8, \qquad j = 1, 2, \cdots,$$

must be non-Colombian. Consequently the recursion formula

$$C_1 = 9, \quad C_k = 8 \cdot 10^{k-1} + 8 + C_{k-1}, \quad k = 2, 3, \cdots,$$

will give infinitely many decimal Colombian numbers. Note also that C_k is always a k-digit number.

In any base $n \geqq 3$, we can use similar formulas:

$$C_1 = \begin{cases} n-1, \text{ if } n \text{ is even} \\ n-2, \text{ if } n \text{ is odd} \end{cases}$$

$$C_k = (n-2)n^{k-1} + (n-2) + C_{k-1}, \quad k = 2, 3, \cdots.$$

Again C_k will be a k-digit number.

With base 2, we can no longer find Colombian numbers with any specified number of digits; for example, there are no 2-digit base 2 Colombian numbers. However, since numbers of the form $2^j + 1$ are non-Colombian (for $j \geqq 1$) in base 2, we can construct the desired numbers by using

$$C_1 = 1, \quad C_k = 2^j + 1 + C_{k-1}, \quad k = 2, 3, \cdots,$$

where j denotes the number of digits in C_{k-1}.

Also solved by D. M. Bloom, D. Ž. Djoković, R. J. Evans, Scott Forrest, M. G. Greening (Australia), K. S. Hanmantrao (India), D. Z. Kilhefner, O. P. Lossers (Netherlands), L. E. Mattics, E. H. Merryman, K. Rosen & J. Glauser, Michael Shimshoni (Israel), Alan Stein, Guy Torchinelli, Phil Tracy, M. R. Vitale, B. L. Wadha, and the proposer.

Forrest, Shimshoni, and Torchinelli observe that if the base is odd, then every odd number is Colombian.

A Binomial Coefficient Summation

E 2409 [1973, 434]. *Proposed by A. V. Boyd, University of the Witwatersrand, Johannesburg, South Africa*

Sum the series

$$\sum_{n=0}^{\infty} \binom{2n}{n}^{-1} (4x)^n.$$

I. *Solution by M. T. Bird, San Jose, California.* The series converges and defines the function $y(x)$ for $|x| < 1$. This function satisfies the differential equation

$$2(x - x^2)y' - (2x + 1)y = -1$$

and the initial condition $y'(0) = 2$. This determines the function $y(x)$ uniquely to be

$$y(x) = \begin{cases} (1-x)^{-1} + \sqrt{x}(1-x)^{-3/2}\sin^{-1}\sqrt{x}, & 0 \leqq x < 1, \\ (1-x)^{-1} + \sqrt{-x}(1-x)^{-3/2}\log(\sqrt{1-x} - \sqrt{-x}), & -1 < x \leqq 0. \end{cases}$$

II. *Solution by V. Linis, University of Ottawa.* The series converges for $|x| < 1$. (The radius of convergence is 1 by the ratio test, and the series diverges at $x = \pm 1$ since the terms do not approach 0 as $n \to \infty$; this follows from Stirling's formula.—Ed.) Since

$$\binom{2n}{n}^{-1} = \frac{n!n!}{(2n)!} = (2n+1)\int_0^1 (1-u)^n u^n du$$

(the integral is a Beta function) it follows that

$$\sum_{n=0}^{\infty} \binom{2n}{n}^{-1} (4x)^n = \sum_{n=0}^{\infty} (2n+1)\int_0^1 [4x(1-u)u]^n du.$$

Interchanging the order of summation and integration and summing the resulting series yields

$$\sum_{n=0}^{\infty} \binom{2n}{n}^{-1} (4x)^n = \int_0^1 \frac{1 + 4(1-u)ux}{[1 - 4(1-u)ux]^2} du$$

$$= \begin{cases} (1-x)^{-1}\left(1 + \sqrt{\dfrac{x}{1-x}}\sin^{-1}\sqrt{x}\right) \text{ if } 0 \leqq x < 1 \\ (1-x)^{-1}\left(1 - \sqrt{\dfrac{-x}{1-x}}\sinh^{-1}\sqrt{-x}\right) \text{ if } -1 < x \leqq 0. \end{cases}$$

III. *Comment by I. I. Kotlarski, Oklahoma State University.* This problem has an interesting interpretation in probability theory. If the random variable X has the distribution given by the density

$$g(u) = \begin{cases} 0 & |u| \geqq 1 \\ \dfrac{1}{\pi\sqrt{1-u^2}} & |u| < 1, \end{cases}$$

then the moments of X are

$$m_{2n} = \frac{1}{4^n}\binom{2n}{n}, \qquad n = 0, 1, 2, \cdots,$$

while all the odd moments are 0. Therefore the series is the generating function for the reciprocals of the even moments of X.

Also solved by Günter Bach (Germany), D. M. Bloom, R. G. Buschman, L. Carlitz, J. Case, D. Ž. Djoković, R. J. Evans, Sidney Heller, Margaret J. Hodel, Mourad Ismail, Aleksandar Ivić (Yugoslavia), M. J. Knight, I. I. Kotlarski, Charlotte Krauthamer (Austria), O. P. Lossers (Netherlands), L. E. Mattics, M. R. Modak (India), M. R. Murty & V. K. Murty, R. E. Overy (England), K. R. Penrose, E. B. Rockower, Kenneth Rosen, O. G. Ruehr, St. Olaf Problem Group, David Shelupsky, P. J. Short, Franklin Smith, Phil Tracy, Barry Wolk, L. M. Young, P. H. Young, David Zeitlin, and the proposer.

Editor's comment: Several solvers note that the series is the hypergeometric function ${}_2F_1(1, 1; \frac{1}{2}; x)$.

A Sum Related to Chebyshev Polynomials

E 2410 [1973, 434]. *Proposed by Barry Wolk, University of Manitoba*

Evaluate

$$\sum_{r=0}^{n} \frac{(-1)^r}{(n+r)(2r+1)}\binom{n+r}{2r}.$$

Solution by R. G. Buschman, University of Victoria, British Columbia. Consider

$$g(x) = \sum_{r=0}^{n} \frac{(-1)^r}{(n+r)(2r+1)}\binom{n+r}{2r}x^r$$

which is the hypergeometric polynomial $g(x) = n^{-1}{}_2F_1(-n, n; 3/2; x/4)$. From recurrence relations for contiguous functions (see *Handbook of Mathematical Functions*, National Bureau of Standards Applied Mathematics Series Number 55 (1964), p. 558), we can rewrite this in the form

$$g(x) = \frac{((2n-1)x-4n)\,{}_2F_1(-n, n; \frac{1}{2}; \frac{1}{4}x) + 4n\,{}_2F_1(-n+1, n-1; \frac{1}{2}; \frac{1}{4}x)}{nx(4n^2-1)}$$

$$= \frac{((2n-1)x-4n)T_n(1-\frac{1}{2}x) + 4nT_{n-1}(1-\frac{1}{2}x)}{nx(4n^2-1)},$$

where T_n represents a Chebyshev polynomial of the first kind (*ibid.*, Formula 15.4.3, p. 561). Since the original problem asks for $g(1)$, we substitute $x = 1$ and evaluate the polynomials (*ibid.*, Formula 22.3.15, p. 776) to obtain

$$g(1) = \frac{2n\sqrt{3}\sin(n\pi/3) - \cos(n\pi/3)}{n(4n^2 - 1)}.$$

Also solved by M. T. Bird, R. J. Evans, O. P. Lossers (Netherlands), Otto Ruehr, Allen Stenger, David Zeitlin, and the proposer.

Forcing Sets of Exponents in Groups

E 2411 [1973, 434]. *Proposed by F. W. Barnes, University of Michigan*

Let G be a group. Give sufficient conditions on a and b so that $(xy)^a = x^a y^a$ and $(xy)^b = x^b y^b$ for all x, $y \in G$, force G to be commutative. The conditions must be general enough to imply the result for $a = 8$ and $b = 11$.

Solution by Walter Stromquist, U.S. Treasury Department. If the greatest common divisor of $(a^2 - a)$ and $(b^2 - b)$ is 2, we can conclude that the group is Abelian. This condition on a and b is both necessary and sufficient, and is also valid in the case of more than two integers. A proof can be found in F. W. Levi, *Notes on group theory* I, II, J. Indian Math. Soc. 8 (1944), 1–9, and the special case of periodic groups is treated in J. R. Durbin, *Commutativity and n-Abelian groups*, Math. Zeit. 98 (1967), 89–92.

Also solved by D. Ž. Djoković, R. J. Evans, M. R. Modak (India), Carole Lutz Welch, and the proposer.

Editor's comment: The generalization to which Stromquist refers is that if $x^n y^n = (xy)^n$ for all x, y in a group G and all n in a set S of integers, then G is necessarily Abelian if and only if the greatest common divisor of the set $\{n^2 - n : n \in S\}$ is 2. Call such a set a *forcing set*. The result is also true if S has but a single element, since in this case the only forcing sets are $\{-1\}$ and $\{2\}$ — both familiar results in elementary group theory. Herstein's well-known problem that $\{n, n+1, n+2\}$ is a forcing set also follows immediately from this result (I. N. Herstein, *Topics in Algebra*, Xerox, Lexington, Mass. 1964, Problem 4, p. 31.)

The proposer and Djoković obtain independently the characterization referred to by Stromquist, and Lutz refers to Durbin's paper. Evans obtains the sufficient condition that $a + 3 = b = 6k + 5$ for some integer k, and Modak the sufficient condition that $a + 3 = b = 2 \cdot 3^m + 5$ for $m = 0, 1, \cdots$.

Minimum Track of a Line Segment

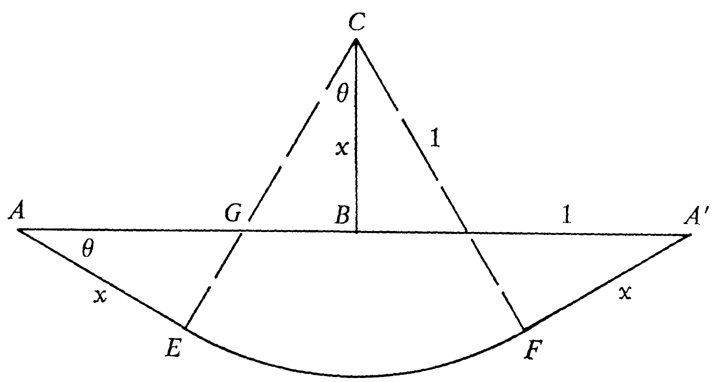
E 2412 [1973, 434]. *Proposed by Michael Goldberg, Washington, D.C.*

If a line segment AB of unit length is rotated 180° about the fixed end B to the position BA', then the end A makes a track of length π. However, if the end B is allowed to move also, then the sum of the lengths of the tracks made by A and B can be shorter. Note that if a portion of the track is retraced, then the motion is increased,

but the length of the track is not increased. What is the shortest total length of track needed to carry the line AB to BA'?

Composite solution edited from those submitted by J. D. Hiscocks, University of Lethbridge, Alberta, the proposer, and Robert Brooks and Jacob Sturm, Weizmann Institute, Israel. From the symmetry of the problem, we can assume that the paths of the ends of the segment are symmetric with respect to the perpendicular to ABA' at B. Therefore at some time the ends of the segment are simultaneously on this perpendicular.

Since end B must move and return to its original position, its track is minimized if it collapses into a straight line segment BC which, by symmetry, lies along the perpendicular. With C as center draw an arc of unit radius and draw tangents AE and $A'F$ as shown in the accompanying figure.



Between E and F, the path of end A cannot lie inside arc EF; if any part of it lies outside, then replacing that portion with (part of) arc EF shortens the path. Similarly, segments AE and $A'F$ are shorter than any replacement paths.

Letting x denote the length of BC, the total length of the track is

$$3x + 2 \arctan \frac{1 - x^2}{2x},$$

which has a minimum at $x = 1/\sqrt{3}$. Hence the shortest path has length $\sqrt{3} + \pi/3 \doteq 2.780$.

Also solved by Walter Bluger, and by C. S. Ogilvy.

Editor's comment: A related problem, concerned with minimum motion rather than minimum track, is posed by S. M. Ulam in his *Problems in Modern Mathematics*, Interscience, 1964, p. 79. For a discussion of these problems, see the proposer's paper, *The minimum path and the minimum motion of a moved line segment*, Math. Mag. 46 (1973), 31–34.

ADVANCED PROBLEMS

All solutions of Advanced Problems should be sent to J. Barlaz, Rutgers—The State University, New Brunswick, N. J. 08903. Solutions of Advanced Problems in this issue should be typed (with double spacing) on separate, signed sheets and should be mailed before July 31, 1974.
An asterisk (*) means neither the proposer nor the editors supplied a solution.

5964*. *Proposed by C. W. Anderson, University of California, Berkeley*

Define $\operatorname{pow}(n) = (\alpha_1 + \alpha_2 + \cdots + \alpha_k)/k$, where $n = p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_k^{\alpha_k}$. pow (n) measures the average power of a prime in the decomposition of n. Prove that the average value of $\operatorname{pow}(n)$ is unity, i.e., show that

$$\frac{1}{N}\sum_{n=1}^{N}\operatorname{pow}(n) = 1 + \frac{c}{\log\log N} + o\left(\frac{1}{[\log\log N]^2}\right),$$

where $c = \Sigma_p 1/p(p-1)$.

For example, the average power of a prime in the decomposition of all numbers less than a "googol", 10^{100}, is about 1.2.

5965. *Proposed by D. K. Cohoon, University of Minnesota*

Let $\kappa: N-\{0\} \to N \times N$ be a one-to-one correspondence, where $N=\{0,1,2,\cdots\}$. Let P denote the set of positive integers. Let V_1 denote the functions valued in the field F which are defined on P but vanish off a finite subset of P. Let V_0 denote the F-valued functions defined on N which vanish off a finite subset of N. Define a linear transformation $B_\kappa: V_1 \to V_0$ by the rule

$$(B_\kappa\psi)(n) = \sum_{m=0}^{\infty} \psi(\kappa^{-1}(m,n))$$

for every n in N. Find all the subspaces U of V_1 such that

$$V_1 = \ker(B_\kappa) \oplus U \qquad \text{(direct sum)}.$$

5966. *Proposed by E. F. Schmeichel, Itasca, Illinois*

Does every maximal planar graph have a Hamiltonian circuit?

5967. *Proposed by C. W. Anderson, University of California, Berkeley*

In 1521, Giardus Ruffus conjectured that most odd numbers are deficient. Show that the density of odd deficient numbers is at least

$$\frac{48-3\pi^2}{32-\pi^2} \approx 0.84.$$

5968. *Proposed by Michael Golomb, Purdue University*

Is the set of zeros of all entire functions F, for which $F^{(k)}(0)$ $(k = 0,1,\cdots)$ are integers, the field of complex numbers? Compare Problem 5898, solution in this issue of this MONTHLY.

5969*. *Proposed by Charles Small, Queen's University, Canada*

Let R be a ring; for $x, y \in R$ define $[x, y] = xy - yx$; call elements of the form $[x, y]$ commutators. Prove or disprove: the subring of R generated by all commutators is an ideal. Note that "ideal" is unambiguous because $a[x, yb] + ay[b, x] = a[x, y]b = [ax, y]b + [y, a]xb$. For an affirmative solution it would suffice for example to show that any element of the form $a[x, y]$ is a sum of products of commutators.

SOLUTIONS OF ADVANCED PROBLEMS

A Maximal Distribution with Prescribed Marginals

5894 [1973, 83]. *Proposed by J. F. Kemp, Jr., Amoco Research, Tulsa, Okla.*

If $F_1(x_1)$, $F_2(x_2), \cdots, F_n(x_n)$ are n probability distribution functions, then prove that $\min(F_1(x_1), F_2(x_2), \cdots, F_n(x_n))$ is an n-dimensional probability distribution function with marginals $F_1(x_1)$, $F_2(x_2)$, $\cdots$, $F_n(x_n)$.

Solution by C. R. Blyth, Queen's University. If we define

$$F_i^*(x) = \min\{t: F_i(t) \geqq x\},$$

then for a random variable X with uniform $(0,1)$ distribution we have

$$P\{F_i^*(X) \leqq x_i\} = P\{X \leqq F_i(x_i)\} = F_i(x_i).$$

(Here F_i^* is F_i^{-1} except for flat stretches and discontinuities of F_i; this is the standard method of using a table of random digits to generate realizations of a random variable with cumulative probability function F_i.) We also have

$$P[\{F_1^*(X) \leqq x_1\}\{F_2^*(X) \leqq x_2\} \cdots \{F_n^*(X) \leqq x_n\}]$$
$$= P[\{X \leqq F_1(x_1)\}\{X \leqq F_2(x_2)\} \cdots \{X \leqq F_n(x_n)\}] = \min_i F_i(x_i),$$

so the function given by $\min_i F_i(x_i)$ is the cumulative probability function of the random vector $F_1^*(X), \cdots, F_n^*(X)$; that its ith marginal is F_i was the first thing noted above. This n-dimensional distribution is highly singular—all the probability lies in a one-dimensional subset of the n-space.

Also solved by B. C. Arnold, C. R. Blyth (a second solution), Mark Brown, L. E. Clarke (England), A. A. Jagers (Netherlands), F. W. Steutel (Netherlands), W. R. Ugolik, J. G. Wendel, P. H. Young, and the proposer.

Editor's note. Steutel and Clarke note that the problem generalizes a question posed by Fréchet for $n = 2$. See W. Feller, *An Introduction to Probability Theory and Its Applications*, v. 2, 2nd edition, p. 166, ex. 7. *If $H(x_1, x_2, \cdots, x_n)$ is any n-dimensional distribution function with marginal distributions $F_i(x_i)$, then $H(X) \leqq F_i(x_i)$ or $H \leqq \min_i F_i(x_i)$.*

Uniformly Locally Compact Metric

5896 [1973, 209]. *Proposed by A. W. Schurle, University of North Carolina at Charlotte*

It is easy to show that the metric space (X, d) is complete if it is uniformly locally compact, i.e., if there is a positive ε such that $\{y: d(x,y) \leqq \varepsilon\}$ is compact for all x. Is the converse true for the real line, i.e., is every compact metric that yields the usual topology on the line uniformly locally compact?

Solution by D. W. Brown. The answer is no. (We shall use the notation: $\mathbb{R}$ for the real line and $\mathbb{N}$ for the positive integers.) Let $S_1, S_2, \cdots$ form a partition of $\mathbb{N}$ into infinite sets, i.e., $\mathbb{N} = \bigcup_{n=1}^{\infty} S_n$ where $n \neq m$ implies $S_n \cap S_m = \varnothing$ and each S_n is infinite.

Define $d\colon \mathbb{N} \times \mathbb{N} \to [0, \infty)$ by

$$d(p,q) = \begin{cases} 0 & \text{if } \ p = q \\ 1/n & \text{if } \ p, q \in S_n \ \text{ and } \ p \neq q \\ 1 & \text{otherwise.} \end{cases}$$

It is easy to check that d is a metric that induces the discrete topology on $\mathbb{N}$. Since any d-Cauchy sequence in $\mathbb{N}$ is eventually constant, d is complete. The counter example now depends upon the following theorem:

Let X be a metrizable topological space and let $A \subseteq X$ be a closed set. Then for every metric ρ on A which induces the relative topology on A, there is a metric $\bar{\rho}$ on X which is an extension of ρ and is compatible with the topology of X. Moreover if X is complete metrizable and ρ is a complete metric on A, then the extension $\bar{\rho}$ can be obtained as a complete metric on X.

The first part of the theorem is due to F. Hausdorff (1930) and (independently) to R. H. Bing (1947). The completeness result was added by P. Bacon (1968). See, e.g., H. Torúnczyk, *A short proof of Hausdorff's theorem on extending metrics*, Fundamenta Mathematicae, LXXVII, 2 (1972), pp. 191–193.

Let $\bar{d}\colon \mathbb{R} \times \mathbb{R} \to [0, \infty)$ be a complete metric that extends $d\colon \mathbb{N} \times \mathbb{N} \to [0, \infty)$ and induces the usual topology. Let $\varepsilon > 0$. One can choose $m, n \in \mathbb{N}$ so that $1/n < \varepsilon$ and $m \in S_n$. Then $S_n \subseteq \{y \in \mathbb{R}\colon \bar{d}(m,y) \leqq \varepsilon\}$. Since S_n is an unbounded set under the Euclidean metric, $\{y \in \mathbb{R}\colon \bar{d}(m,y) \leqq \varepsilon\}$ is not compact. Therefore $\bar{d}$ is not uniformly locally compact.

Also solved by E. A. Herman, A. A. Jagers (Netherlands), and the proposer.

The Set of Zeros of Entire Functions with Rational Coefficients

5898 [1973, 209]. *Proposed by Sylvester Reese, Baruch College, New York City*

Is the set of zeros of all entire functions with rational coefficients (for their Maclaurin series) the field of complex numbers?

Solution by D. G. Cantor, University of California, at Los Angeles. The answer is affirmative and, more generally, let $f(z)$ be any entire function with real coefficients satisfying $f(0) = 1$. Put $g(z) = \log f(z)$; $g(z)$ is regular in some neighbourhood of 0. Choose $h(z)$ entire with real coefficient such that $h(0) = 0$ and $g(z) + h(z)$ has rational coefficients. Then $f(z)e^{h(z)} = \exp\{g(z) + h(z)\}$ has rational coefficients and has the same zeros (with the same multiplicities) as $f(z)$. For a given complex number a, $z = a$ is a root of $(1-\bar{b}z)(1-\bar{b}z)$ $(= f(z))$, $b = 1/a$.

Also solved by D. W. Brown, L. J. Dickson (Australia), Michael Golomb, L. Haddad (France), P. W. Lindstrom, O. P. Lossers (Netherlands), and Melvin Rosenfeld.

Irredundant Strongly Connected Directional Graph

5899 [1973, 209]. *Proposed by Joel Spencer, University of California, Los Angeles*

Professor Södre is, once again, unprepared for his Epsilondeltopology class. He has prepared the first half of his lecture in which he proves a certain n propositions $P_1, P_2, \cdots, P_n$ equivalent. He had planned the most efficient proof, by showing $P_1 \Rightarrow P_2 \Rightarrow \cdots \Rightarrow P_n \Rightarrow P_1$ (The theorems $P_i \Rightarrow P_j$ take an equal amount of time to prove.) Then he notices he may essentially double the length of his proof (from n to $2n-2$) by showing $P_1 \Leftrightarrow P_2 \Leftrightarrow \cdots \Leftrightarrow P_n$. This method of proof is irredundant, that is, if any implication is deleted we may not deduce that $P_1, \cdots, P_n$ are equivalent. Prove that this is the longest (in terms of number of implications) irredundant method of proof.

Solution by M. J. Hoffman, University of California, Berkeley. A proof scheme may be considered as an arrangement of points representing the propositions and arrows representing the implications proved. The problem is to show that the maximal number of arrows in an irredundant proof of the equivalence of n propositions is $2n-2$. For $n = 1$ or $n = 2$ this is clear. Inductively, suppose it is true for any collection of fewer than n propositions. Let S be the irredundant proof scheme for n propositions using N arrows. Pick a starting point P_1, and let $P_1 \Rightarrow P_2 \Rightarrow \cdots \Rightarrow P_j$ be a maximal chain. That is, proceed along arrows in S until you can go no farther without encountering a P_i already met. This must happen eventually since there are finitely many points in S. There must be some arrow out of P_j since otherwise equivalence could never be proved. Thus there is a loop in S which, we may assume, is of the form $P_1 \Rightarrow P_2 \Rightarrow \cdots \Rightarrow P_j \Rightarrow P_1$; ($j$ may be as little as 2, but this does not matter). Any other arrows among only these points would be redundant. Any arrows into or out of the loop from other points of S may be moved to P_1 without changing the total number of arrows or the redundancy. The scheme now looks like this:

$$P_1 \Rightarrow P_2 \Rightarrow P_3 \Rightarrow \cdots \Rightarrow P_j \Rightarrow P_1, \ \cdots \Rightarrow P_1 \Rightarrow \cdots.$$

The loop may now be considered as a single proposition P'. This gives a new proof scheme S' among $n-j+1$ propositions still irredundant and using $N-j$ arrows. By the induction hypothesis $N-j \leqq 2(n-j+1)-2$ so that $N \leqq 2n-j \leqq 2n-2$. (N maximal requires always that $j=2$ no matter where we start.)

We note that the scheme $P_1 \Leftrightarrow P_2 \Leftrightarrow \cdots \Leftrightarrow P_n$ achieves the maximal number of arrows, $2n-2$. However, other schemes may also use $2n-2$ arrows; for example, with $n=6$,

(i) $P_1 \Leftrightarrow P_2 \Leftrightarrow P_3$, $P_4 \Leftrightarrow P_5 \Leftrightarrow P_6$, $P_2 \Leftrightarrow P_5$;

(ii) $P_1 \Leftrightarrow P_2$, $P_1 \Leftrightarrow P_3, \cdots, P_1 \Leftrightarrow P_6$.

Also solved by Anders Bager (Denmark), F. R. Bernhart, David Bienenfeld (Israel), D. G. Cantor, Kevin Compton, D. J. Kleitman, O. P. Lossers (Netherlands), R. B. Lumbart, Louis Raymon, John H. Smith, R. J. Weber, and the proposer.

REVIEWS

EDITED BY J. ARTHUR SEEBACH, JR. AND LYNN. A. STEEN

with the assistance of the mathematics departments of St. Olaf and Carleton Colleges

COLLABORATING EDITOR FOR FILMS: SEYMOUR SCHUSTER, CARLETON COLLEGE

We invite readers to submit reviews of significant recent college-level mathematics books. We especially encourage reviews based on classroom use, or comparative reviews of several related books. Reviews should ordinarily not exceed two pages (per book) typed double spaced. Manuscripts of reviews as well as books submitted for review should be sent to: Book Review Editor, American Mathematical Monthly, St. Olaf College, Northfield, MN 55057.

Statistics: A Guide to the Unknown. Edited by: Judith M. Tanur, Frederick Mosteller, *et al.* Holden-Day, San Francisco, California, 1972. xxiii + 430 pp. \$9.95. (Telegraphic Review, January 1973.)

The *Guide* is a collection of 44 essays describing applications of statistics intended for the audience of non-specialists in probability and statistics, such as people associated with secondary education. Even the editors have noted that these essays are useful in university education. I believe the professional statistician will find the *Guide* interesting and challenging. Among the many topics are: can famous people postpone their deaths until a birthday (by David P. Phillips); sampling versus auditing of waybills to increase profit (by John Neter); and the projection of future populations (by Nathan Keyfitz).

The editors worked very hard to invite outstanding authorities as authors. The

essays, which are between five and thirteen pages in length, mostly consist of a statement of the substantive problem, techniques for finding solutions, data sources, conclusions, and key references.

The table of contents, arranged by subject, includes authors, essay titles, and short abstracts. The basic divisions are man in his biological, political, social and physical worlds. The essays are classified by data sources: samples, available data, surveys and questionnaires, experiments, and quasi experiments. The essays are also classified by their tools: estimation, tables, and probability.

While reading, I often attempted to fill in technical details. If your students are studying these essays they will often wish instant guidance not involving a visit to the library.

The problem is illustrated in the seven pages of D. G. Chapman's "The Plight of the Whales," where he describes the natural history of whales, the whaling industry's problems, methods for measuring the whale populations, and policy consequences of knowledge of whale population dynamics. He gives vivid descriptions of capture-recapture, catch-per-day, and age analysis methods for measuring population sizes.

From the data Chapman says we can "infer" the population size. His presentation does not include sampling errors, so the reader should realize that "infer" means the solution for an unknown in a deterministic proportion with three terms given. The statistics instructor should be prepared to present simple models with sampling errors.

In speaking of the catch-per-day method Chapman says "the results obtained in this way must be combined with estimates obtained in other ways." The formal combining of statistical data (estimates) is a major problem that is seldom carefully developed. This might be an area for research.

The high quality and variety of the essays make them useful sources. Thus, I tried to react to the essays from my Bayesian viewpoint of the foundations of statistics: all probability statements are summaries of a (corporate) individual's beliefs about acceptable betting odds. Complete problems are in an economic setting and are solved by maximizing expected utility. Some of my reactions were the following:

Mosteller and Wallace use textual analysis for deciding authorship (between Hamilton and Madison) of the twelve *Federalist* essays which had a single but unknown author. The economic aspect of the problem is nearly absent; professors who draw conclusions which are later counter-indicated might obtain adverse (monetary) rewards. "A chief motivation" of the authors was "comparing several different statistical approaches." Hence, we have a meta-experiment which I will not attempt to analyze.

The detailed Bayesian work of Mosteller and Wallace was complex but roughly they did the following: known writings of Hamilton and Madison were examined to detect differences in frequencies in word usages. By trial and error, differences were located; for example, Hamilton preferred "while" and Madison preferred "whilst." For the disputed essays the likelihood ratios were astronomically in favor

of Madison. Thus, the prior beliefs could not change the conclusions, except under some extreme constraint such as the physical impossibility of Hamilton or Madison alone doing all of that writing.

Robert G. Miller's "The probability of rain" starts with an example of the economic use of probability forecasts. He continues with: "Probability distributions may be arrived at in various ways. The most common method is to use the human judgment of an experienced weather expert. He considers all the evidence and on the basis of his experience chooses a number that he thinks expresses the chance of rain. Another way of generating a probability distribution is to apply statistical methods to weather data stored in government archives. This essay describes a method for arriving at such a probability distribution based on the statistical evidence of past years."

Thus Miller is close to my viewpoint. He starts with the joint distributions of cause and effect rather than the more common (but technically equivalent) distribution of effect given cause.

The "Drug screening" essay by Charles W. Dunnett agrees at least formally with my viewpoint. Bayesian language appears in the writing of people who may not have thought about foundations: "thus (smoking) is likely to be its (bronchitis) cause" (page 82), "After all, educated opinion is always the weighing of probabilities" (page 94). S. James Press in "Police manpower versus crime" talks like a Bayesian: "intervals of credibility," "degree of belief," "chance was very great (95%) that the *true* decrease. . .," and "highly unlikely in the future." Donald T. Campbell speaks of "quasi experiments," where social or other constraints prevent the use of randomization and experimental versus control groups. These are situations where expert opinion should be placed in a Bayesian framework and it should be noted that a pure Bayesian would not pay the price of randomization. Robert Hooke considers decision problems with non-monetary utility, for example, runs in baseball. Such examples show the catholic nature of decision theory.

The *Guide* gives a powerful demonstration that, contrary to folk belief, statistics is a lively subject which is instrumental in solving important problems and which creates fascinating research areas.

I. R. SAVAGE, Florida State University

The Nature and Growth of Modern Mathematics. By Edna E. Kramer. Hawthorn Books, New York, 1970. xxiv + 758 pp. $24.95. (Telegraphic Review, April 1971.)

Training in most university level subjects, such as economics, biology, and sociology, usually starts with a first year survey course that is intended to give a more-or-less coherent view and at least some orientation to the student who is embarking on a serious study of the field. This has not been the practice in mathematics. Survey courses for mathematics majors are more the exception than the rule, and the price

paid is the alienation of many students. The reasons advanced for the scarcity of survey courses are numerous, one of them being the lack of suitable textbooks.

Edna Kramer's book is a possible candidate as text for such a course. Her aim is to survey some fairly sophisticated, modern mathematical ideas in the perspective of their historical development, to give a clear exposition of these ideas, and to stress that mathematics is created by human beings with all their frailties and imaginative energy.

The book succeeds mainly, I think, because the author genuinely believes that mathematics at even the highest level can be and should be presented to people who would like to hear about it. After all, this is being done in the other sciences. The lacuna in mathematics is probably more due to mathematicians' inability to write than to the nature of mathematics. What raises the book above the run-of-the-mill exposition is not only the richness of its contents, but also its high literary quality — very rare in a mathematical work, but all the more appreciated by the thoughtful reader. In Kramer's prose the human element comes through, and mathematicians appear as biological and social beings.

The contents are too diverse to list. In addition to the usual topics such as algebra, set theory, and logic, there are chapters on calculus, Newtonian mechanics, game theory, statistical decision theory, quantum mechanics, differential geometry, relativity, number theory, integral equations, topology, and more. The reviewer especially enjoyed the chapters on Riemannian geometry, the Erlangen program, and functional analysis. The last handles the concepts of integral equations, linear operators, eigenvalues, and Hilbert space in a way that would make it possible for a bright high school student with some exposure to calculus to appreciate the central ideas. Biographies of Felix Klein, Sophus Lie, Tullio Levi-Civita, and Elie Cartan are given (amongst others) together with simple, clear explanations of some of their work such as infinitesimal parallelism and fiber bundles. The book contains much that would be dismissed as mathematical physics by people of excessively pure taste, but the inclusion of such material gives the reader a deeper perspective and is historically accurate, since physical problems did, in fact, generate much mathematical development. Passages on the treatment that women mathematicians received in the past illustrate that even in mathematics people are not evaluated solely on the basis of ability and work produced.

In conclusion, it is a pleasure to find an author who can write on such a wide range of topics with so much skill, both literary and mathematical. The discouraging thing about the book is its price: $25 according to the dust jacket, higher in Canada. The book is large. A per page cost of 3.2¢ is not excessive, but is still a handicap in using it as a text for a liberal arts course. Issuing in sections and/or in paperback might soften the blow.

STEPHEN REGOCZEI, University of Toronto

TELEGRAPHIC REVIEWS

Telegraphic reviews are designed to give prompt notice of new books with sufficient information to assist our readers in deciding whether to order an examination copy or to suggest library purchase. Possible uses are indicated as follows:

T = textbook P = professional reading
S = supplementary reading L = undergraduate library purchase
13 to 18 = freshman to second year graduate level usage
1 to 4 = appropriate time in semesters to cover text

Asterisks (*) or question marks (?) denote special positive or negative emphasis, respectively. Publishers are denoted by standard abbreviations; complete addresses may be found in *Books in Print*.

GENERAL, S, L*, *Mathematical Circles Squared: A Third Collection of Mathematical Stories and Anecdotes*. Howard W. Eves. Prindle, 1972, xxi + 186 pp, $10. A third and (tentatively) final installment of miscellaneous mathematical memorabilia. A fitting sequel to the author's two earlier *Mathematical Circles*. LAS

GENERAL, S, P, *Mathematics Research Center White Paper*. R.C. Buck. U Wisc, Madison, Jan, 1974, 16 pp, free. A memo to the U. of Wisconsin faculty which attempts to refute many of the allegations made in *The AMRC Papers* (TR, March 1974). The style is explanatory, in contrast to the high blown rhetoric of "the people." Clear examples are given to show the use of half truth and innuendo in *The AMRC Papers*. Each is a political (not a mathematical) document with the goal of public persuasion. TAV

BASIC, T(13: 1), *Algebra: An Algorithmic Treatment*. Kenneth E. Iverson. A-W, 1971, 361 pp, $6 (P). Employs APL (a computer programming language) to study elementary functions, with claims of its being more natural and pedagogically superior to conventional function notation which is often inconsistent. LCL

BASIC, T(13: 1), *A First Course in Mathematics*. John Peterson. HR&W, 1973, xi + 289 pp, $8.50. Number systems, probability and statistics, equations, areas and volumes. Practical, computational, intuitive development. No prerequisites. LCL

EDUCATION, P*, *Guidelines for the Preparation of Teachers of Mathematics*. NCTM, 1973, 32 pp, $.90 (P). Consolidates and expands on other sets of guidelines. Makes recommendations in the areas of academic and professional knowledge, professional competencies and attitudes, and institutional responsibilities. RSK

EDUCATION, T(13-15: 1), *Foundations of Number Systems*. Bruce E. Meserve, Max A. Sobel. P-H, 1973, x + 292 pp, $8.95. Intended for prospective elementary school teachers. Sets, logic, modular and place value arithmetic, integers, rationals, real line. Style is pleasant with many illustrations, numerous exercises. Important feature: "Pedagogical Considerations" sprinkled generously throughout with suggested aids for classroom presentation. TAV

EDUCATION, P*, *Mathematics Tests Available in the United States*. James S. Braswell. NCTM, 1972, 32 pp, $1.10 (P). Comprehensive list of commercially available tests in the areas of arithmetic, junior

high mathematics, K-14 mathematics, general high school mathematics, algebra, geometry, trigonometry, and college-related mathematics. Includes references to reviews appearing in the *Mental Measurement Yearbook* series. RSK

FOUNDATIONS, P, *A Probabilistic Theory of Causality*. Patrick Suppes. Acta Phil. Fennica, XXIV. North-Holland, 1970, 130 pp, $10.80 (P). A detailed yet admittedly unfinished development of a variation on Hume's analysis of causality as the relation of constant conjunction. Drawing on recent research in philosophy, physics and social science, Suppes explores the hypothesis that one event is the cause of another if the second event follows the first with high probability, provided that there is no intervening third event which can factor out the probability relationship. LAS

COMBINATORICS, L**, *A Handbook of Integer Sequences*. N.J.A. Sloane. Acad Pr, 1973, xiii + 206 pp, $10. Incomparable, eccentric, yet very useful. Contains thousands of "well-defined and interesting" infinite integer sequences together with references for each. Sequences are arranged lexicographically and (to minimize errors) typeset from computer tape. If you ever wondered what comes after 1, 2, 4, 8, 17, 35, 71,..., this is the place to look it up. LAS

LINEAR ALGEBRA, T(15-17: 1), S, P, L, *Matrices in Control Theory with Applications to Linear Programming*. S. Barnett. Van N-Rein, 1971, xiv + 221 pp, $19.95. Matrices for applied mathematicians, engineers, and systems analysts. Requires general background in matrix theory (linear algebra and vector spaces are never mentioned) and some acquaintance with control theory and linear programming; otherwise the material, which has not previously appeared in book form, is self-contained, though somewhat condensed. Complete references after each chapter; exercises with worked solutions. LCL

LINEAR ALGEBRA, S(14-15), *Linear Mathematics, 25 Vols*. Open University. Har-Row, 1971, 1392 pp, $77.05 (P). Workbook modules for a multi-media self-instructional course in linear algebra and differential equations for the Open University of Great Britain. Each module is available separately ($2.75-$4.75@), as are 33 films ($125@) and 6 audio tapes ($7.50@); see Harper-Row catalogue for details. The course is based on two texts (Kreider, *et al.*, *An Introduction to Linear Analysis*, Addison Wesley, 1966; and Nering, *Linear Algebra and Matrix Theory*, Wiley, 1970) and "will not make sense without them." Each module supplements the texts with additional commentary, examples, exercises and review sections. Topics include canonical forms, linear programming, differential equations, Fourier and Laplace transforms, game theory. LAS

ALGEBRA, T*(15-17: 2), S, L, *Abstract Algebra: A First Course*. Larry Joel Goldstein. P-H, 1973, xii + 335 pp, $11.95. A smooth, flowing, balanced introduction to groups, rings, fields (suitable for first-term only students) followed by more group theory and Galois theory. Unity and cohesion are provided by extensive historical perspective, exercises, and examples which emphasize concrete computations as the ultimate goal of general theories. LCL

CALCULUS, T(13: 2), *Calculus and Analytic Geometry*. Harold Schachter. McGraw, 1972, xii + 307 pp, $8.95. A straightforward one-variable calculus text intended for students of business, social science, etc.SG

CALCULUS, T(13: 2), S, L, *Biomathematics, V. 2: Introduction to Mathematics for Life Scientists*. Edward Batschelet. Springer-Verlag, 1974, xiv + 495 pp, $9.80 (P). New paperback edition; see TR, October 1972, extended review, February 1974. LAS

CALCULUS, T(13-14: 1-3), S, *Mathematics: A Foundations Course, 32 Vols*. Open University. Har-Row, 1970, 1580 pp, $87.45 (P). Modules for a 36 unit multi-media self-instructional course for the Open University of Great Britain. Each module is available separately for $2.50-$3.25@, as are 42 related films ($125@) and 15 audio tapes ($7.50@); see Harper-Row catalogue for details. The course covers the syllabus of an enriched calculus course, including elementary linear algebra, differential equations, and brief introductions to logic, groups and topology. The units are well organized and cross-referenced so it would be possible to select a subset of the 32 modules for a course with more restricted purposes. Each module contains attractive text material, objectives, glossary, table of notation, problems and unit summaries. LAS

CALCULUS, T(13: 2), *Calculus: A Programmed Text*. David M. Merriell. Benjamin, 1974. *V. I: Techniques and Applications*, xii + 489 pp, $7.95 (P); *V. II: Theory*, xi + 214 pp, $4.95 (P). Could be subtitled "How to Read a Mathematics Book." This was to be a revision of *A Programmed Course in Calculus* but became a complete reorganization: the theory is now in a separate volume, and each section begins with an introduction and ends with a summary and review exercises which are not programed. Information is presented in small numbered frames with short answer responses--answers have been placed between arrows (not an improvement). LLK

REAL ANALYSIS, T**(16: 1), S**, P, L**, *Convex Functions*. A. Wayne Roberts, Dale E. Varberg. Acad Pr, 1973, xx + 300 pp, $19.50. The authors have drawn together the central facts about convex functions and organized them into a highly readable and informative format which can be enjoyed and studied by anyone with linear algebra and advanced calculus background. Excellent problem sections and references to the literature make it particularly appropriate for senior seminars and undergraduate research and independent study projects.LCL

COMPLEX ANALYSIS, T(16-18: 1-3), S, P, L, *Analytic Function Theory, Second Edition*. Einar Hille. Chelsea, 1973. *V. I*, xi + 308 pp, $9.95; *V. II*, xii + 496 pp, $9.95. Reprint of a contemporary classic, first published in 1959 by Ginn-Blaisdell. An attractive, reasonably priced reference which is well suited to classroom use. LAS

COMPLEX ANALYSIS, T*(16: 1, 2), *Basic Complex Analysis*. Jerrold E. Marsden. Freeman, 1973, x + 472 pp, $11.95. For a first course. In addition to usual topics, the author covers asymptotic methods and the Laplace transform. Treatment is complete and rigorous. Many proofs may be omitted without a loss of coherence. Contains many more worked examples and exercises than most other texts. An attractive choice for a text. TAV

FUNCTIONAL ANALYSIS, T(18: 1, 2), S(18), P, *Banach Algebras: An Introduction*. Ronald Larsen. Dekker, 1973, x + 345 pp, $16.75. The first half is devoted to the general theory of Banach algebras, the second to topics related to harmonic analysis and function algebras. LCL

FUNCTIONAL ANALYSIS, T(18: 1, 2), P, L. *Functional Analysis, An Introduction.* Ronald Larsen. Dekker, 1973, xi + 497 pp, $19.50. Lectures given at Wesleyan University 1970-72. Topics include the usual theorems--Hahn-Banach, Uniform Boundedness, Open Mapping, Closed-Graph--plus theorems of Krein-Mil'man, Markov-Kakutani, Eberlein-Smulian. A final chapter on Hilbert space. LCL

FUNCTIONAL ANALYSIS, P, *Invariant Subspaces.* Heydar Radjavi, Peter Rosenthal. Ergebnisse der Math., B. 77. Springer-Verlag, 1973, xi + 219 pp, $20.50. A fairly self-contained exposition of the theory of subspaces invariant under operators defined on a separable Hilbert space. Some topics: normal, compact, and shift operators; analytic functions of operators; invariant subspace lattices; von Neumann algebras; transitive operator algebras; and a discussion of unsolved problems. SG

ANALYSIS, T(16-17: 2), L, *Real and Complex Analysis, Second Edition.* Walter Rudin. McGraw, 1974, xii + 452 pp, $14.50. A minor revision of a major text. Changes from the original edition include modification of the chapter on differentiation, a simplification of the proof of the global version of Cauchy's theorem, and some new exercises. It remains an elegant, concise statement of the core of modern analysis. LAS

TOPOLOGY, P, *Lecture Notes in Mathematics-347: Homotopy Invariant Algebraic Structures on Topological Spaces.* J.M. Boardman, R.M. Vogt. Springer-Verlag, 1973, x + 257 pp, $9.10 (P). An H-space is a topological space X with a multiplication $m: X \times X \to X$ which is continuous. If m is associative, X is an associative H-space. If Y has the homotopy type of X, Y is an H-space, but not necessarily associative. These notes investigate structures on spaces which are preserved by homotopies, essentially structures which are as the original ones with diagrams commuting only up to homotopy. The results presented have been announced earlier but the present exposition is a great improvement over previously available materials. PJM

PROBABILITY, T(13: 1, 2), *A Guide to Probability Theory and Application.* Cyrus Derman, Leon J. Gleser, Ingram Olkin. HR&W, 1973, xvi + 750 pp, $14.95. Extensive treatment of basic probability assuming a background of college algebra. First third is devoted to fundamental concepts, last two-thirds to probability models, particularly special distributions. Concludes with two chapters on Markov chains. RSK

PROBABILITY, P, *Thinning of Renewal Point Processes: A Flow Graph Study.* Lennart Råde. Matematisk Statistik AB, Göteborg, 1972, 178 pp, $8 (P). After introductory chapters on thinning by binomial and Poisson processes, the author concentrates on applications to neuron firing and stochastic service systems. Includes a list of open questions and unsolved problems. TAV

STATISTICS, T**(1), S*, L, *Sturdy Statistics; Nonparametrics and Order Statistics.* Frederick Mosteller, Robert E.K. Rourke. A-W, 1973, xv + 395 pp, $11.95. Well-written text for a second course in elementary statistics emphasizing nonparametric methods. Discusses procedures based on signs, counts, ranks and order statistics. This is not just a cookbook and demands some sophistication, although formally requiring only two years of high school algebra and a term of statistics. RSK

STATISTICS, T(14: 1), S*(14), *A Computer-Assisted Approach to Elementary Statistics; Examples and Problems*. William Bulgren. Wadsworth, 1971, 162 pp, $4.50 (P). To be used as a supplement to a text in precalculus statistics. Typical problems are posed, analyzed and programs called for (in FORTRAN IV). The goal is to demonstrate usefulness of the computer in statistical analysis. Appendices include an introduction to FORTRAN IV programming, generation of pseudorandom and normal random numbers and a list of programs (student written).TAV

STATISTICS, T?(13), S?, *Experiment, Design and Statistics in Psychology*. Colin Robson. Penguin, 1973, 174 pp, $3.45 (P). A very brief treatment of statistical techniques used by social scientists. Treatment is decidedly cookbook, with little motivation. Includes t, χ^2, F, Mann-Whitney and Wilcoxon tests, and chapters on design, and writing up results. Such books aren't likely to improve the social scientist's understanding or use of statistics. TAV

STATISTICS, T?(13: 2), *Statistics and Calculus: A First Course*. James Murtha, Earl Willard. P-H, 1973, x + 592 pp, $12. Designed for use in an introductory course for social, biological or management science students. Textual material (515 pp) is about one-third calculus and two-thirds statistics, so the calculus treatment, which includes partial derivatives and infinite series, is very cursory. Statistical topics are given a mixed treatment, with some, e.g., F-test, decidedly short-changed. RSK

STATISTICS, T(14: 1), *Statistics for Business and Economics*. Jerome D. Braverman, William C. Stewart. Ronald Pr, 1973, ix + 482 pp, $11. Primarily a standard text, requiring some calculus for fullest appreciation. Contains chapters on analysis of variance, multiple regression and correlation, nonparametric statistics, time series analysis and Bayesian decision theory. RSK

COMPUTER SCIENCE, T(16-17: 1), S*, P, *Theory and Application of a Bottom-Up Syntax-Directed Translator*. Harvey Abramson. Acad Pr, 1973, x + 160 pp, $11. A smooth development from a brief, fundamental theory of context-free languages to syntax analyzers and synthesizers. The major application is an ALGOL W program which translates BASIC programs into SPS. General enough to be useful for writing similar compilers. No explicit exercises. RWN

COMPUTER SCIENCE, T(14-16), S, *Schaum's Outline of Theory and Problems of Boolean Algebra and Switching Circuits*. Elliott Mendelson. McGraw, 1970, 213 pp, $4.95 (P). Standard Schaum Outline format (includes 150 solved problems) divided into two independent but related topics--the synthesis and simplification of combinational circuits (outputs depend only on present value of the inputs), and the theory of Boolean algebra, which is quite complete (e.g., includes Stone representation theorem and m-completeness). No prerequisites. LCL

COMPUTER SCIENCE, S, P, L, *Computer Models of Thought and Language*. Ed: Roger C. Schank, Kenneth Mark Colby. Freeman, 1973, vii + 454 pp, $13.50. A selection of ten essays on the frontier of computer science, linguistics, psychology and artificial intelligence with the common aim of faithfully representing human cognitive processes. Begins with an excellent exposition by Allen Newell of AI and the concept of mind. LAS

APPLICATIONS (BIOLOGY), S(14-16), P, L. *Some Mathematical Problems in Biology.* Ed: Murray Gerstenhaber. Lect. on Math. in Life Sci., AMS, 1968, vi + 117 pp, $6.60 (P). First volume in this series; contains three papers (on Volterra's differential equations, on evolution of complex genetic systems, on biological clocks) from the first (1966) Symposium on Mathematical Biology. The first paper (by E.R. Leigh) is an excellent example of mathematical modelling, progressing in considerable detail from older, simplified models to more recent, more realistic and more complex variations. LAS

APPLICATIONS (BIOLOGY), S(14-16), P, L. *Some Mathematical Questions in Biology.* Ed: Murray Gerstenhaber. Lect. on Math. in Life Sci., V. 2. AMS, 1970, vii + 156 pp, $5.50 (P). Proceedings of the second (1967) and third (1968) Symposia on Mathematical Biology: statistical mechanics of nervous activity, graphical analysis of ecological systems, models of extinction, and temporal morphology of biological clocks. LAS

APPLICATIONS (BIOLOGY), S(16-18), P, L. *Some Mathematical Questions in Biology, II.* Ed: Jack D. Cowan. Lect. on Math. in Life Sci., V. 3. AMS, 1972, vii + 121 pp, $6 (P). Second volume of *Mathematical Questions,* third in its series, contains proceedings of the fourth (1969) Symposium on Mathematical Biology. Two papers examine models for the spatial and temporal organization of information (called maps and clocks) in developing organisms; a third paper surveys current understanding of genetic mechanisms from the viewpoint of network and control theory. LAS

APPLICATIONS (BIOLOGY), S(16-18), P, L. *Some Mathematical Questions in Biology, III.* Ed: Jack D. Cowan. Lect. on Math. in Life Sci., V. 4. AMS, 1972, vi + 151 pp, $6.80 (P). Proceedings of fifth (1970) Symposium on Mathematical Biology. Three papers continue the analysis begun in the preceding volume on the theoretical biology of developing organisms; a fourth relates standard problems of statistical mechanics, quantum mechanics and noise theory to ecological kinetics as derived from the classical Lotka-Volterra equations. LAS

APPLICATIONS (BIOLOGY), P, L. *Some Mathematical Questions in Biology, IV.* Ed: Jack D. Cowan. Lect. on Math. in Life Sci., V. 5. AMS, 1973, vi + 150 pp, $11.10 (P). Proceedings of the sixth Symposium on Mathematical Biology: four expositions on three topics--molecular biology, embryology, and neurobiology. LAS

APPLICATIONS (BIOLOGY), P. *Mathematical Physiology: Blood Flow and Electrically Active Cells.* H. Melvin Lieberstein. Am Elsev, 1973, xiv + 377 pp, $19.50. Studies blood flow and wall tension problems, and electrically active cells using the Hodgkin-Huxley system of partial differential equations with a numerical instability removed. Provides discussions of physiological principles, as no background in physiology is assumed. Numerical investigations. Suitable for a graduate seminar in analysis. DFA

APPLICATIONS (CONTROL THEORY), P. *Optimal Control of Differential and Functional Equations.* J. Warga. Acad Pr, 1972, xiii + 531 pp, $27.50. Mathematical theory of deterministic optimal control. Part I Foundations: topology, integration, functional analysis, differential, integral, functional-integral equations. Part II Optimal Control: basic

problems, concepts of original, approximate, and relaxed solutions, control problems defined by equations in Banach spaces, ordinary differential equations, functional-integral equations; conflicting control problems with relaxed and with hyperrelaxed adverse controls. RBK

APPLICATIONS (CONTROL THEORY), T(18), P. *Discrete Techniques of Parameter Estimation: The Equation Error Formulation.* Jerry M. Mendel. Dekker, 1973, xiv + 385 pp, $19.50. Volume 1 in a new series of monographs and textbooks on control theory. Presents a unified treatment of four estimation techniques: generalized least squares; unbiased minimum variance; deterministic gradient; and stochastic gradient. Deals mainly with constant parameters, but includes a chapter on extensions to time-varying parameters. RSK

APPLICATIONS (DEMOGRAPHY), T(16-18), P*, L*. *Population: Facts and Methods of Demography.* Nathan Keyfitz, Wilhelm Flieger. Freeman, 1971, xi + 613 pp, $13.50. Theory and computation (including computer programs) of fundamental population models, together with numerous tables giving data and computer parameters for seventy countries based on information from the mid-sixties. A major, invaluable reference for contemporary demography. LAS

APPLICATIONS (DEMOGRAPHY), T(17-18: 1, 2), P, L. *Mathematical Models of Conception and Birth.* Mindel C. Sheps, Jane A. Menken. U Chicago Pr, 1973, xxiii + 428 pp, $18.50. Reaction of natality indices to variation in natural and behavioral physiological aspects of human reproduction. Sophisticated use of statistics in unified approach to deriving existing models for conception risks, family building, distribution of interval between births. Some exercises, with answers. PJC

APPLICATIONS (ECONOMICS), P. *Lecture Notes in Economics and Mathematical Systems-85: Economics of Involuntary Transfers: A Unified Approach to Pollution and Congestion Externalities.* T. Page. Springer-Verlag, 1973, x + 159 pp, $8.10 (P). Review of economics literature on pollution and congestion problems, followed by development of a model and its application to air pollution and health data in London. Pollution and congestion are analyzed as involuntary transfers, or "uncompensated disservices." Evidence presented suggests health effects of air pollution decrease more than proportionately with decreases in pollution itself. PJC

APPLICATIONS (ENGINEERING), P. *Nonlinear Elasticity.* Ed: R.W. Dickey. Acad Pr, 1973, ix + 404 pp, $12.50. Proceedings of an April 1973 symposium at Madison, Wisconsin: 11 research papers, photocopied from typed manuscripts. LAS

APPLICATIONS (ENGINEERING), T(17: 2). *Mathematical Methods in Chemical Engineering, V. 3: Process Modeling, Estimation, and Identification.* John H. Seinfeld, Leon Lapidus. P-H, 1974, xiii + 545 pp, $19.95. Self-contained introduction to mathematical methods in chemical process modeling. Concerns primarily probability theory, stocastic processes, parameter estimation and (linear and nonlinear) system identification. Many examples, exercises and references to recent research papers. DFA

APPLICATIONS (ENGINEERING), S(15-16), P, L. *Units, Dimensional Analysis and Physical Similarity.* B.S. Massey. Van N-Rein, 1971, viii + 140 pp, $11.50. Details of "physical algebra" in which a symbol represents not just a number but a physical quantity--a number plus a unit of measurement. Basic principle is "dimensional homogeneity"; units of both sides of an algebraic equation must agree; this principle restricts the possible forms of the equation. Useful for checking results, designing experiments, solving problems for which a complete solution is not available. Includes careful definitions of all physical units of measurement, plus a table of dimensionless constants. PJC

APPLICATIONS (LINGUISTICS), P. *Prague Studies in Mathematical Linguistics, V. 4.* Ed: Ján Horecký, *et al.* U of Alabama Pr, 1972, 254 pp, $12. 16 papers (all in English) by Czechoslovak linguists on quantitative and algebraic linguistics. LAS

APPLICATIONS (LINGUISTICS), P, L. *Formal Logic and Linguistics.* Ernesto Zierer. Mouton, 1972, 92 pp, fl.14 (P). Third book in series *Mathematical Models in Linguistics.* Collection of 87 illustrations of the use of concepts and framework of mathematical logic in linguistics. Examples gathered by topic: sentential logic, predicate calculus, logic of relations, deduction in formal logic, sentential calculus. Would be of greater value with more exposition interleaved with the examples. PJC

APPLICATIONS (MANAGEMENT), L. *The Algorithm Writer's Guide.* S.M. Wheatley, A.W. Unwin. Longman, 1972, 130 pp, $5 (P); $9. An elementary introduction to management (*not* computer) algorithms, discussing intuitive and formal schemes for converting a set of verbal instructions into an efficient, well laid-out flow chart. LAS

APPLICATIONS (MODELLING), P, L. *Lecture Notes in Economics and Mathematical Systems-80: International Seminar on Trends in Mathematical Modelling.* Ed: Nigel Hawkes. Springer-Verlag, 1973, vi + 288 pp, $10.80 (P). Proceedings of a UNESCO-CNRS seminar in Venice, 1971, with a strong emphasis on futures research. Includes several contributions by the authors of *Limits to Growth.* A rich source of mathematical applications to contemporary problems. LAS

APPLICATIONS (OCEANOGRAPHY), P. *Waves on Beaches and Resulting Sediment Transport.* Ed: R.E. Meyer. Acad Pr, 1972, vii + 462 pp, $16. Proceedings of the October 1971 seminar sponsored by the Mathematics Research Center and the Coastal Engineering Research Center at Madison, Wisconsin. JAS

APPLICATIONS (PHYSICS), S, P, L. *Practical Quantum Mechanics.* Siegfried Flügge. Grund. math. Wissenschaften, B. 177, 178. Springer-Verlag, 1971. *V. I:* xiv + 341 pp, $22.20; *V. II,* xii + 287 pp, $19.10. A collection of solved problems exhibiting a wide variety of basic quantum mechanics applications. An excellent reference for the student attempting to apply quantum mechanics techniques to his own field. JC

APPLICATIONS (PHYSICS), P. *Spectral Theory and Problems in Diffraction.* Ed: M. Sh. Birman. Topics in Math. Physics, V. 2. Consultants, 1968, vii + 134 pp, $20 (P). Eight papers from the mathematics and physics departments of Leningrad State University translated

from the Russian. Semmingly a high price even given the (unnecessarily) nice type setting. JAS

APPLICATIONS (PHYSICS), P, *Spectral Theory*. Ed: M. Sh. Birman. Topics in Math. Physics, V. 3. Consultants, 1969, vi + 93 pp, $20 (P). Seven more translations from Leningrad State University. JAS

APPLICATIONS (PHYSICS), T(14-15), S, *Mechanics and Applied Calculus, 8 Vols*. Open University. Har-Row, 1972, 664 pp, $31.45 (P). Workbooks for 16 unit course, bound in 8 separate modules for prices from $2.95-$4.50@. Published as part of a multi-media course for the Open University in Great Britain, these modules are designed to be used for self-instruction in conjunction with a specific standard text (Smith and Smith, *Mechanics*, Wiley, 1971) and supplemented by a series of 16 films (at $125@) and three audio tapes (at $7@). The course covers basic mechanics, work and energy, equilibrium and stability, vibrations, Fourier analysis, variable mass and orbits. Although cleverly and artistically prepared, the modules by themselves would be relatively useless unless used as intended. Completed details of the modules and related films can be found in the Harper-Row catalogue. LAS

APPLICATIONS (PHYSICS), P, *Interdisciplinary Mathematics, V. IV-VI*. Robert Hermann (18 Gibbs St, Brookline, MA, 02146), 1973. *V. IV: Energy Momentum Tensors*, $8 (P); *V. V: Topics in General Relativity*, $9 (P); *V. VI: Topics in the Mathematics of Quantum Mechanics*, $13 (P). Advanced lecture notes on applications of differential geometry; a continuation, not of the more basic *Interdisciplinary Mathematics, V. I-III*, but of the author's many previous treatises on mathematical physics. Rough notes, privately printed. LAS

APPLICATIONS (PHYSICS), T(16-17: 1), S*, L*, *General Theory of Relativity*. C.W. Kilmister. Pergamon Pr, 1973, ix + 365 pp, $6.50 (P); $11.50. A valuable and unusual treatise combining a 100 pp. exposition of the theory with reprints (or English translations) of eleven classic papers by, e.g., Riemann, Einstein, Fock, Oppenheimer. LAS

APPLICATIONS (PHYSICS), T*(16-18: 2), P**, L***, *Gravitation*. Charles W. Misner, Kip S. Thorne, John Archibald Wheeler. Freeman, 1973, xxvi + 1279 pp, $19.95 (P); $39.50. A definitive, stunning introduction to geometrodynamics (general relativity) in the crisp modern symbolism of differential geometry. Massive (5.5 lbs, nearly 1300 pp), it is written on two tracks: the first is focused on key physical ideas, the second (over 80% of the whole) on mathematical enrichment. Dozens of distinct essays ("boxes") summarize, accent and extend key issues. Exercises, bibliography (nearly 2000 items) and extensive index. An unrivalled exposition of one of the supreme theories of mathematical physics. LAS

APPLICATIONS (PHYSICS), T(17: 2), *An Introduction to Fluid Dynamics*. G.K. Batchelor. Cambridge U Pr, 1970, xviii + 615 pp, $9.50 (P). A reprinting of the 1967 edition (TR, October 1969). Mathematics prerequisites: vector algebra and tensors. After three introductory chapters, treats uniform incompressible viscous fluids, effects of viscosity, irrotational flow and vorticity in inviscid fluids. TAV

APPLICATIONS (PHYSICS), S(19), P, *Geometry, Physics, and Systems*. Robert Hermann. Dekker, 1973, x + 304 pp, $17.50. An extensive development of the tools of modern differential geometry which then provides a background for a final chapter on thermodynamics. The author's intent seems to be to use this last chapter not as an end in itself but rather as an example of the potential of this sort of geometry for a wide variety of physics, engineering and even systems theory.JAS

APPLICATIONS (PHYSICS), P, *Lecture Notes in Physics-20: Statistical Mechanics and Mathematical Problems*. Ed: A. Lenard. Springer-Verlag, 1973, viii + 247 pp, $8.20 (P). A collection of lecture series from the Rencontres in Mathematics and Physics at the Battelle Institute, Seattle during the summer of 1971. JAS

APPLICATIONS (PHYSICS), P, L, *Foundations of Quantum Mechanics*. Ed: B. d'Espagnat. Acad Pr, 1971, xiv + 480 pp, $32.50. 23 discursive lecture series from the 1970 International Summer School of Physics at Lake Como, Italy: philosophical foundations, measurement problems, hidden variables, interpretations. Lecturers include E. Wigner and L. deBroglie. LAS

APPLICATIONS (PHYSICS), P, *Lecture Notes in Physics-18 & 19: Proceedings of the Third International Conference on Numerical Methods in Fluid Mechanics*. Ed: Henri Cabannes, Roger Temam. Springer-Verlag, 1973; *V. I: General Lectures, Fundamental Numerical Techniques*, vii + 186 pp, $6.40 (P); *V. II: Problems of Fluid Mechanics*, vii + 275 pp, $9.20 (P). Papers from Paris, July, 1972; the three survey lectures are in the first volume. LAS

APPLICATIONS (PHYSICS), S(16-18), L, *Theory of Relativity Based on Physical Reality*. L. Jánossy. Akademiai Kiado, 1971, 317 pp, $13.20. Book has a very personal style--typical of those about relativity. Author examines carefully the experimental bases for the theory. He arrives at the special and general theories by extending ideas of Lorentz. Detailed table of contents but no index. DG

APPLICATIONS (PHYSICS), P, *Lecture Notes in Physics-21: Optimization and Stability Problems in Continuum Mechanics*. Ed: P.K.C. Wang. Springer-Verlag, 1973, 94 pp, $6 (P). Papers by: Halkin on the method of Dubovitskii-Milyutin; Shield on optimum design of structures through variational principles; Wu, Chwang and Wang on optimization problems in hydrofoil propulsion; Infante on stability of general dynamical systems; Barston on stability of dissipative systems. From an August 1971 symposium at USC. DFA

APPLICATIONS (PHYSICS), P, *Lecture Notes in Physics-16: Phasenübergang 1. Art bei Gittergasmodellen*. Hans-Otto Georgii. Springer-Verlag, 1972, ix + 167 pp, $5.80 (P). An account, for specialists, of the theory of phase transitions of the first kind in models of a lattice gas with pairwise interactions. JD-B

Reviewers Whose Initials Appear Above

David Appleyard, Carleton; Paul J. Campbell, St. Olaf; James Cederberg, St. Olaf; John Dyer-Bennet, Carleton; Steven Galovich, Carleton; David Grimsrud, St. Olaf; Lorraine L. Keller, St. Olaf; Roger B. Kirchner, Carleton; Richard S. Kleber, St. Olaf; Loren C. Larson, St. Olaf; Pierre J. Malraison, Carleton; R.W. Nau, Carleton; J. Arthur Seebach, St. Olaf; Lynn A. Steen, St. Olaf; T.A. Vessey, St. Olaf.

NEWS AND NOTICES

EDITED BY RAOUL HAILPERN, SUNY AT BUFFALO

Readers are invited to contribute to the general interest of this Department by sending news items to Mathematical Association of America, 1225 Connecticut Avenue, NW, Washington, D. C. 20036. Items must be submitted at least two months before publication can take place.

PERSONAL ITEMS

Professor R. V. Hogg, University of Iowa, represented the Association at the inauguration of Dr. W. J. Bakrow as President of Saint Ambrose College on October 7, 1973.

Armstrong State College: Assistant Professor D. Z. Kilhefner, Ashland College, has been appointed Assistant Professor; Associate Professor R. M. Summerville, Chairman of the Department of Mathematics and Computer Science, has been promoted to Professor.

Bradley University: Assistant Professor H. W. McCurdy has been promoted to Associate Professor; Professor M. G. Moore, Chairman from 1958 to 1970, has retired after being at Bradley for $30\frac{1}{2}$ years.

Brooklyn College: Drs. Anthony D'Aristotle, State University at Geneseo, and Kishore Marathe, State University at Buffalo, have been appointed Assistant Professors.

Colorado State University: Associate Professor F. R. DeMeyer has been promoted to Professor; Assistant Professors W. E. Brumley, K. F. Klopfenstein, and Bennet Manvel have been promoted to Associate Professors.

Mankato State College: Assistant Professor R. B. Mericle has been promoted to Associate Professor; Professor V. D. Turner has been appointed Chairman of the Mathematics Department.

Marshall University: Dr. H. P. Greenough, Indiana University, has been appointed Assistant Professor; Assistant Professors W. C. Sisarcick and W. T. Whitley have been promoted to Associate Professors.

Northern Michigan University: Dr. William Mitchell, University of Wisconsin-Madison, has been appointed Assistant Professor; Assistant Professors William Mutch and Robert Myers have been promoted to Associate Professors.

Shippensburg State College: Associate Professor William Gould has been promoted to Professor; Assistant Professor W. R. Weller has been promoted to Associate Professor.

Southern Illinois University: Dr. L. E. Knop, University of Utah, has been appointed Lecturer; Assistant Professor R. C. Shock has been promoted to Associate Professor.

University of Tennessee at Chattanooga: Dr. Edward Rozema, Purdue University, has been appointed Assistant Professor; Associate Professor J. G. Ware, Chairman of the Mathematics Department, has been promoted to Professor; Guerry Professor Winston L. Massey retired on August 31, 1973, after 41 years of service, with the title of Guerry Professor Emeritus; Assistant Professor Dorothy C. Martin retired on August 31, 1973, with the title of Assistant Professor Emeritus.

Assistant Professor J. W. Brewer, University of Kansas, has been promoted to Associate Professor.

Assistant Professor D. S. Cochran, Lake Forest College, has been appointed Director of the Computer Center.

Assistant Professor James Dombek, State College of Arkansas, has been promoted to Associate Professor.

Dr. Robert Ducharme, Florida State University, has been appointed Assistant Professor at Baldwin Wallace College.

Dr. R. M. Gravina, Lowell State College, has been promoted from Assistant Professor to Associate Professor.

Associate Professor J. E. Hall, Chairman of the Mathematics Department of the University of Wisconsin — Extension, received the Kiekhofer Memorial Teaching Award, for Excellency in Teaching, in May 1973.

Assistant Professor Edward LeCuyer, Western New England College, has been appointed Chairman of the Department of Mathematics.

Assistant Professor John Lucas, University of Wisconsin, Oshkosh, has been promoted to Associate Professor.

Dr. J. T. McLean, Ohio State University, has been appointed Assistant Professor at Ohio Northern University.

Mrs. Alice G. Meissner, University of Delaware, has been appointed Instructor at Chatham College.

Professor Amin Muwafi, American University of Beirut, is on sabbatical leave at the University of California, Davis.

Professor Abba Newton, Vassar College, retired on July 1, 1973, with the title of Professor Emeritus.

Associate Professor J. C. Nichols, Thiel College, has been appointed Chairman of the Department of Mathematics.

Assistant Professor N. F. Page, University of North Carolina, Greensboro, has been appointed Associate Professor and Department Chairman at High Point College.

Assistant Professor H. H. Suber, Salisbury State College, has been promoted to Associate Professor.

Professor and Dean Emeritus Arthur E. Gault, Bradley University, died on August 31, 1973, at the age of 84. He was a member of the Association for fifty years.

Professor Lloyd L. Lassen, University of Texas at Arlington, died on May 14, 1973, at the age of 90. He was a member of the Association for seventeen years.

Dr. Herman Walton Smith, Professor Emeritus, Oklahoma State University, and Professor Emeritus, University of South Carolina, died on September 10, 1973, at the age of 81. He was a member of the Association for fifty years.

A COMPANION FOR THE MAA SABBATICAL EXCHANGE SERVICE

The MONTHLY has received information about an interesting companion to SEIS. It is called the FACULTY EXCHANGE CENTER (F. E. C.). Members who participated in SEIS may wish to explore the opportunity provided by F. E. C. for coverage beyond mathematics faculties. We reproduce below the text of an announcement provided to the MONTHLY by Professor John Joseph of F. E. C.

"The Faculty Exchange Center is entering its second year. Its initial year was successful. The first F. E. C. Catalog, to be released this fall, will show a membership representing over thirty disciplines from nearly a hundred colleges and universities, including a number of institutions from overseas.

Founded by professors who are of the opinion that teaching and travel are compatible, and that their colleagues as well as their ideas should be interchanged among campuses, the Faculty Exchange Center aims to make it possible for an interested faculty member to exchange position for a year or less with a colleague from another institution, either on this continent or abroad where the language of instruction is English. The Center will bring

to the attention of its members the names of others interested in an exchange through the publication of a catalog containing the names of the instructors and their institutions, their field of specialization and the region where they would like to teach. The catalog will be made available to registered members only.

Upon receipt of the catalog, the member will match himself with one or more colleagues and initiate correspondence with them to work out the details of exchange. When these arrangements have been completed, further efforts of the persons involved working through their Departments and Administration, can bring about an agreement regarding their mutual interests leading to a two-way exchange.

The F. E. C. is opening avenues for a new educational experience for faculty as well as for students and the institutions themselves. The Center will hopefully fulfill the widespread desire to teach in other institutions, a desire that reflects a general awareness of the benefits of teaching and study in different geographical and cultural settings. The reciprocal agreements that the Center encourages should also benefit the students by exposing them to ideas of other faculty members who can make available to them a wider range of knowledge and experience as well as courses and seminars.

The increased mobility of the academic manpower generated through the F. E. C. seems to be welcomed by college administrators especially during a period of limited institutional budget. Through the services of the Center it is possible in some cases to provide scholars needed to fill certain specialties in departmental curricula without the need to employ them on a regular basis."

Information may be obtained by writing to F. E. C., Box 1866, Lancaster, Pennsylvania 17604.

SABBATICAL LECTURESHIPS AT THE UNIVERSITY OF MASSACHUSETTS

The Department of Mathematics and Statistics of the University of Massachusetts, Amherst, Massachusetts, expects to have available a limited number of Sabbatical Lectureships for the academic year 1974–75. These lectureships will be open to faculty members of four-year colleges, or universities without a Ph. D. program in mathematics, who wish to spend their sabbatical leaves at the University of Massachusetts. Normally these individuals will be expected to have attained a Master's degree in mathematics, but not a Ph. D. Participants in the program will be half-time lecturers on the University faculty and will be required to teach one course per semester. Normally, they will also be expected to enroll in two courses and one seminar. Stipends up to $6000 for the academic year are available. In addition, tuition will be waived. Interested persons should write to: Professor Robert Blattner, Head, Department of Mathematics and Statistics, University of Massachusetts, Amherst, Massachusetts 01002.

MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

MARCH MEETING OF THE MISSOURI SECTION

The annual Spring meeting of the Missouri Section of the MAA was held at Southeast Missouri State University, Cape Girardeau, on March 23 and 24, 1973. There were 87 persons registered of which 11 were students.

The officers for the next year are: Chairman: Edward Andalafte, University of Missouri-St. Louis; Vice Chairman: Jerry Wilkerson, Missouri Western College; Secretary-Treasurer: T. L. Hicks, University of Missouri-Rolla.

The following presentations were made at the Friday session:

1. *What is a non-Archimedean field*?, by Leon Palmer, Southeast Missouri State University.
2. *The* 10-*adic integers*, by Lyle Pursell, University of Missouri-Rolla.
3. *An out-of-date computer program* by Dana Nau, Student, University of Missouri-Rolla.
4. *Newton: The man*, by Randy Makin, Student, Drury College.
5. *Problems and open questions in mathematics that originate in computer science*, by Paul Blackwell, University of Missouri-Columbia.
6. *Adaptive statistical inference*, by S. K. Katti, University of Missouri-Columbia.
7. *What is an Automaton*? Banquet Address by F. E. Hohn, University of Illinois.

The Saturday session consisted of the Business Meeting and the following one-hour invited talks:

1. *Consequences of continuity*, by R. P. Boas, Northwestern University, President of the MAA.
2. *The counter-revolution in mathematics education*, by Shirley Hill, University of Missouri-Kansas City.

T. L. Hicks, *Secretary-Treasurer*

OCTOBER MEETING OF THE NORTH CENTRAL SECTION

The fall meeting of the North Central Section of the MAA was held at the University of Minnesota, Duluth, on October 20, 1973.

The following papers were presented:

1. *The color problem for knots*, by J. F. Detlef, University of Minnesota, Morris.
2. *Generalized Fibonacci sequences and their period modulo m*, by G. E. Bergum, Sr., South Dakota State University.
3. *A recursive method for partial fractions*, by K. C. Schmidt, Moorhead State College.
4. *Spaces of solvability for operators on metric spaces*, by D. K. Cohoon, University of Minnesota, Minneapolis.
5. *A project in group theory*, by J. A. Gallian, University of Minnesota, Duluth.

A panel: "Applied Mathematics in the Undergraduate Curriculum" was moderated by Professor J. L. Nelson, University of Minnesota, Duluth, and presented by Professor Charles McLarnan, Macalester College; Professor P. M. Ryan, Gustavus Adolphus College; Professor Lynn Steen, St. Olaf College.

The invited speaker, Professor R. P. Boas, Northwestern University, president of the MAA, had as the topic for his address, "Consequences of Continuity."

H. M. Anderson, *Secretary-Treasurer*

1973 CONTRIBUTING MEMBERS AND SPECIAL GIFTS

The Association expresses its appreciation to 183 of its members who have elected to be Contributing Members, Sponsors, or Patrons for 1973. It is particularly gratifying to note that this represents an increase of almost 50 per cent from the number of such members in 1972. There were 174 Contributing Members in 1973. In addition, the following members were 1973 Sponsors and Patrons:

Patrons

Leonard Gillman
H. D. Mills
1 Anonymous Patron

Sponsors

W. E. Deskins
W. E. Hartnett
P. T. Mielke
I. J. Schoenberg
W. B. Temple
E. D. Weinstock

In addition, the Association acknowledges with deepest gratitude the following special gifts received during 1973:

Anonymous gifts of $5,000 and $1,000, the former from a member of the 1000 Club, in support of the Visiting Lecturers and Consultants Program;

A gift of $1,000 from H. M. Gehman, a member of the 1000 Club;

A gift of $1,000 from Leonard Gillman, making him the fourth member of the 1000 Club;

A gift of $1,000 from R. L. Wilder, making him the fifth member of the 1000 Club;

A gift of $375 from an anonymous donor;

A gift of $100 from R. D. Edwards;

A gift of $100 from W. L. Duren; and

A gift of $50 from W. S. Loud in support of the Visiting Lecturers and Consultants Program.

OFFICERS AND COMMITTEES AS OF FEBRUARY 1, 1974

General Offices: 1225 Connecticut Avenue, N.W., Washington, D. C. 20036
Executive Director: A. B. WILLCOX
Executive Director Emeritus: H. M. GEHMAN
Editorial Director: RAOUL HAILPERN

OFFICERS

President, R. P. BOAS, Northwestern University (1973–74)
President-Elect, H. O. POLLAK, Bell Telephone Laboratories (1974)
First Vice-President, IVAN NIVEN, University of Oregon (1974–75)
Second Vice-President, JUNE P. WOOD, South Texas College (1973–74)
Editor, ALEX ROSENBERG, Cornell University (1974–78)
Secretary, H. L. ALDER, University of California, Davis (1970–74)
Treasurer, LEONARD GILLMAN, University of Texas (1973–77)

ADDITIONAL MEMBERS OF THE BOARD OF GOVERNORS

Ex-Presidents

VICTOR KLEE, University of Washington (1973–78)
E. E. MOISE, Queens College (CUNY) (1969–74)
G. S. YOUNG, University of Rochester (1971–76)

Elected Members of the Finance Committee
R. D. ANDERSON, Louisiana State University (1974–77)
G. B. PRICE, University of Kansas (1972–75)

Governors-at-Large
DEBORAH T. HAIMO, University of Missouri at St. Louis (1974–76)
SHIRLEY A. HILL, University of Missouri, Kansas City (1972–74)
P. D. LAX, New York University (1973–75)
MARY E. RUDIN, University of Wisconsin, Madison (1973–75)
E. H. SPANIER University of California, Berkeley (1974–76)
S. K. STEIN, University of California, Davis (1972–74)

Editor of the MATHEMATICS MAGAZINE
G. N. WOLLAN, Purdue University (1971–75)

Sectional Governors (*July* 1, 1971–*June* 30, 1974)
Florida, HERMAN MEYER, University of Miami
Illinois, ARNOLD WENDT, Western Illinois University
Iowa, R. V. HOGG, University of Iowa
Louisiana-Mississippi, B. E. MITCHELL, Louisiana State University
Maryland-D. C.-Virginia, S. B. JACKSON, University of Maryland
Michigan, M. S. KLAMKIN, Ford Motor Company
North Central, C. M. BRADEN, Macalester College
Philadelphia, W. E. BAXTER, University of Delaware
Southern California, T. M. APOSTOL, California Institute of Technology
Texas, C. J. PIPES, Southern Methodist University

Sectional Governors (*July* 1, 1972–*June* 30, 1975)
Allegheny Mountain, R. G. AYOUB, Pennsylvania State University
Indiana, P. T. MIELKE, Wabash College
Kentucky, R. C. BUEKER, Western Kentucky University
Metropolitan New York, MEYER JORDAN, Brooklyn College (CUNY)
Nebraska, P. A. HAEDER, University of Nebraska, Omaha
Northern California, MARY V. SUNSERI, Stanford University
Oklahoma-Arkansas, T. W. CAIRNS, University of Tulsa
Rocky Mountain, D. C. BENSON, South Dakota School of Mines and Technology
Wisconsin, E. F. WILDE, Beloit College

Sectional Governors (*July* 1, 1973–*June* 30, 1976)
Kansas, H. L. THOMAS, Kansas State College
Missouri, R. W. FREESE, St. Louis University
New Jersey, S. L. GREITZER, Rutgers University, Newark
Northeastern, P. J. DAVIS, Brown University
Ohio, S. E. BOHN, Miami University
Pacific Northwest, H. E. REINHARDT, University of Montana
Seaway, M. W. POWNALL, Colgate University
Southeastern, B. F. BRYANT, Vanderbilt University
Southwestern, E. L. WALTER, Northern Arizona University

COMMITTEES OF THE ASSOCIATION

Terms of members expire, except where otherwise noted, at the Annual Meeting in January following the last year of service listed below. For temporary committees, no terms are listed since they are automatically discharged at the expiration of the President's term of office, which is the Annual Meeting in January 1975.

Executive Committee

R. P. Boas, *Chairman* (1973–74); H. L. Alder (1970–74), Leonard Gillman (1973–77), Ivan Niven (1974–76), H. O. Pollak (1974), Alex Rosenberg (1974–78), June P. Wood (1973–74), all *ex officio.*

Finance Committee

R. P. Boas, *Chairman* (1973–74), *ex officio*; H. L. Alder (1970–74), *ex officio*, R. D. Anderson (1974–77), Leonard Gillman (1973–77), *ex officio*, G. B. Price (1972–75).

Budget Review Subcommittee: G. B. Price, *Chairman* (1972–75); R. D. Anderson (1974–77), all *ex officio.*

Investment Committee: Leonard Gillman, *Chairman* (1973–77), *ex officio*; H. L. Alder (1973–74), *ex officio*, E. A. Cameron (1973–75).

Committee on Advisement and Personnel

J. C. Eaves, *Chairman* (1972–74); A. T. Basil (1973–75), F. A. Brooks, Jr. (1972–74), Craig Comstock (1973–75), Betty C. Detwiler (1973–75), G. E. Noether (1972–74), R. L. Wilson (1974–76).

Committee on Corporate Members

R. E. Gaskell, *Chairman* (1970–74); F. A. Brooks, Jr. (1970–74), R. C. Di Prima (1974–76), R. L. Graham (1974–76), T. R. Jenkins (1973–75), M. S. Klamkin (1970–74), T. J. Rivlin (1973–75), Leonard Tornheim (1973–75), A. B. Willcox (1974–76).

Committee on Earle Raymond Hedrick Lectures

R. D. Anderson, *Chairman* (1971–74); David Gale (1972–75), P. D. Lax (1973–76).

Committee on Educational Media

D. I. Schneider, *Chairman* (1974–76); A. A. Blank (1972–74), Alice B. Dickinson (1972–74), Robert Kalin (1972–74), J. D. E. Konhauser (1973–75), D. P. Minassian (1973–75), B. B. Peterson (1973–75), Seymour Schuster (1974–76), J. E. Skelton (1972–74).

Committee on High School Contests

R. A. Artino, *Chairman* (1973–76); H. M. Cox, *Director* (1970–74), F. A. Brooks, Jr. (1973–75), A. M. Gaglione (1973–76), S. L. Greitzer (1972–74), Sarah T. Herriot (1972–74), Alfred Kalfus (1973–75), M. S. Klamkin (1972–74), N. S. Mendelsohn (1972–74), P. O. Presley (1972–74), Niel Shilkret (1973–74), J. R. Smart (1971–74).

Subcommittee on the USA Mathematical Olympiad: S. L. GREITZER, *Chairman* (1974–76); R. A. ARTINO (1974), *ex officio*, ALFRED KALFUS (1972–75), M. S. KLAMKIN (1974–76), P. A. PAIGE (1973–74), C. C. ROUSSEAU (1973–75), NURA D. TURNER (1973–75).

COMMITTEE ON INSTITUTES

W. F. LUCAS, *Chairman* (1973–75); R. V. HOGG (1973–75), J. J. MALONE, JR. (1973–74), L. D. MEEKER (1973–75), BARBARA L. OSOFSKY (1972–74), M. W. POWNALL (1972–74), D. P. ROSELLE (1973–75).

COMMITEE ON NATIONAL AWARDS AND PUBLIC REPRESENTATION

R. L. WILDER, *Chairman* (1974–76); B. H. COLVIN (1973–75), MARK KAC (1972–74), MARY E. RUDIN (1974–76), R. P. BOAS (1973–74), *ex officio*.

COMMITTEE ON PERFORMANCE CONTRACTING

R. P. DILWORTH, *Chairman*; V. H. HAAG, E. D. NICHOLS, ISABELLA P. RUCKER, H. J. ZASSENHAUS.

COMMITTEE ON PUBLICATIONS

E. F. BECKENBACH, *Chairman* (1974–76); R. P. BOAS (1973–75), D. E. CHRISTIE (1972–74), D. T. FINKBEINER II (1972–74), IVAN NIVEN (1972–74), GEORGE PIRANIAN (1974–76), L. A. STEEN (1973–75), A. C. TUCKER (1973–75), G. L. WEISS (1972–74), LEONARD GILLMAN (1973–77), *ex-officio*, JOSEPH HASHISAKI (1970–74), *ex officio*, ALEX ROSENBERG (1974–78), *ex officio*, G. N. WOLLAN (1971–75), *ex officio*.

Editorial Committee on Selected Papers at the Precalculus Level: R. V. ANDREE, *Chairman*; JOSEPHINE P. ANDREE, D. G. CHAKERIAN, GERALDINE C. DARDEN, J. D. NEFF.

Editorial Committee on Selected Papers in Algebra: M. SUSAN MONTGOMERY, ELIZABETH W. RALSTON, *Co-Chairmen*; S. R. GORDON, G. J. JANUSZ, MARTHA K. SMITH.

Editorial Committee on Selected Papers in Geometry: TILLA K. MILNOR, *Chairman*; T. F. BANCHOFF, ANN K. STEHNEY.

Subcommittee on Carus Monographs: D. T. FINKBEINER II, *Chairman* (1972–74); R. P. BOAS (1973–75), D. E. CHRISTIE (1972–74).

Subcommittee on Lester R. Ford Awards: E. F. BECKENBACH, *Chairman* (1974–76), *ex officio*; EMIL GROSSWALD (1974–76), D. E. RICHMOND (1972–74).

Subcommittee on MAA Studies in Mathematics: IVAN NIVEN, *Chairman* (1974–76); ALEX ROSENBERG (1974–76), G. L. WEISS (1974–76).

Subcommittee on Miscellaneous Publications: E. F. BECKENBACH, *Chairman* (1974–76); H. L. ALDER (1970–74), LEONARD GILLMAN (1973–77), all *ex officio*.

Subcommittee on Translations: IZAAK WIRSZUP, *Chairman*; S. H. GOULD, P. J. HILTON, BETTY J. HINMAN, G. K. KALISCH.

COMMITTEE ON RELATIONS WITH OFFICE OF EDUCATION

B. H. COLVIN, *Chairman*; W. L. DUREN, JR., BETTY J. HINMAN, B. E. MESERVE, D. I. SCHNEIDER, A. B. WILLCOX.

COMMITTEE ON SECONDARY SCHOOL LECTURERS

MARY W. GRAY, *Chairman* (1972–74); MARY-JANE CROSS (1973–75), J. R. HANNA (1973–75), VIVIENNE M. MAYES (1974–76), S. M. NEWBERGER (1974–76), J. P. SELDIN (1974–76), D. B. SMALL (1973–75).

COMMITTEE ON SECTIONS

L. E. MEHLENBACHER, *Chairman* (1971–74); ALFRED AEPPLI (1971–74), S. H. DOUGLAS (1972–75), L. H. LANGE (1973–76), ARNOLD WENDT (1974–77), A. B. WILLCOX, *ex officio.*

COMMITTEE ON SPECIAL FUNDS OF THE ASSOCIATION

R. L. WILDER, *Chairman* (1972–74); C. B. ALLENDOERFER (1974–76), E. A. CAMERON (1973–75), LEONARD GILLMAN (1973–74), W. L. HART (1973–75), B. W. JONES (1973–75).

COMMITTEE ON THE AWARD FOR DISTINGUISHED SERVICE TO MATHEMATICS

C. B. ALLENDOERFER, *Chairman* (1972–76); R. H. BING (1974–76), R. L. WILDER (1973–75).

COMMITTEE ON THE CHAUVENET PRIZE

FRANÇOIS TRÈVES, *Chairman* (1972–74); P. D. LAX (1974–76), C. D. OLDS (1973–75).

COMMITTEE ON THE EXCHANGE OF INFORMATION ON MATHEMATICS

D. W. LICK, *Chairman* (1973–75); R. A. ALO (1973–74), L. W. COHEN (1972–74), A. L. HAMMOND (1973–75), C. V. NEWSOM (1974–76), F. J. WEYL (1974–76).

COMMITTEE ON THE MEMBERSHIP OF THE ASSOCIATION

R. D. ANDERSON, *Chairman*; E. A. CAMERON, C. V. NEWSOM, B. E. RHOADES, R. B. WARFIELD, JR., A. M. WEDEL, JUNE P. WOOD, H. L. ALDER, *ex officio*, R. P. BOAS, *ex officio.*

COMMITTEE ON THE PUTNAM PRIZE COMPETITION

D. J. NEWMAN, *Chairman* (1972–74); A. P. HILLMAN, *Director* (1973–77), G. D. CHAKERIAN (1974–76), J. I. RICHARDS (1973–75).

COMMITTEE ON THE UNDERGRADUATE PROGRAM IN MATHEMATICS

D. W. BUSHAW, *Chairman* (1973–75); DEBORAH T. HAIMO (1973–75), T. E. HULL (1973–75), R. D. LARSSON (1973–75), W. F. LUCAS (1974–76), B. J. MARTIN (1973–75), A. P. MATTUCK (1974–76), T. A. PORSCHING (1974–76), R. A. ROBERTS (1973–75), ALEX ROSENBERG (1969–75), M. D. THOMPSON (1974–76), J. H. WELLS (1972–74), JUNE P. WOOD (1974–76), R. P. BOAS (1973–74), *ex officio.*

COMMITTEE ON TWO-YEAR COLLEGES

W. G. CHINN, *Chairman* (1971–74); D. J. ALBERS (1974–76), NORMAN BARTON (1974–76),

H. E. CHRESTENSON (1973–75), R. A. DEAN (1971–74), JOSEPH HASHISAKI (1971–74), ERWIN JUST (1973–75), C. A. LATHAN (1973–75), G. B. PRICE (1974–76), K. E. SCHONDELMAIER (1974–76), HELEN B. SINER (1973–75), ARNOLD WENDT (1971–74), K. W. WILLIAMS (1974–76), JUNE P. WOOD (1971–74).

COMMITTEE ON VISITING LECTURERS AND CONSULTANTS

M. W. POWNALL, *Chairman* (1974–76); W. R. BALLARD (1973–74), E. M. BEESLEY (1972–74), T. S. CHIHARA (1972–74), BRINDALL HORELICK (1974–76), W. M. MYERS, JR. (1973–75), J. R. WESSON (1974–76), MARY B. WILLIAMS (1972–74).

COMMITTEE TO EXPLORE THE POSSIBILITY OF MAKING TELEVISION PROGRAMS IN MATHEMATICS

C. V. NEWSOM, *Chairman*; A. A. BLANK, D. W. LICK, D. I. SCHNEIDER.

FILM PROJECTS ADVISORY COMMITTEE

A. A. BLANK, *Chairman* (1973–75); N. L. MAX, *Director* (1972–74), R. C. BUCK (1974–76) P. J. HILTON (1973–75), H. B. KELLER (1974–76), VICTOR KLEE (1974–76), P. D. LAX (1974–76), R. G. LONG (1973–75), EILEEN L. POIANI (1973–75), H. O. POLLAK (1972–74), SEYMOUR SCHUSTER (1974–76).

JOINT COMMITTEE ON EMPLOYMENT OPPORTUNITIES

Terms of members of this committee expire on February 28 of the last year of service listed.

J. I. ROSENBLATT, *Chairman* (1970–74, AMS), R. D. ANDERSON (1972–76), R. A. HANDELSMAN (1971–73, SIAM), W. N. SMITH (1972–76, MAA).

JOINT COMMITTEE ON PLACES OF MEETINGS

G. L. WALKER, *Chairman*; H. L. ALDER, EVERETT PITCHER, A. B. WILLCOX, all *ex officio*.

NOMINATING COMMITTEE FOR 1974

P. A. HAEDER, *Chairman*; M. W. POWNALL, MARY V. SUNSERI.

EDITORIAL BOARDS OF THE ASSOCIATION

AMERICAN MATHEMATICAL MONTHLY (all terms expire December 31, 1978).

Editor: ALEX ROSENBERG

Associate Editors: JOSHUA BARLAZ, JANE W. DI PAOLA, R. K. GUY, RAOUL HAILPERN, SHIRLEY A. HILL, E. S. LANGFORD, W. F. LUCAS, P. T. MIELKE, C. B. MOLER, D. P. ROSELLE, SEYMOUR SCHUSTER, J. A. SEEBACH, JR., IVAR STAKGOLD, E. P. STARKE, L. A. STEEN, J. H. WELLS.

MATHEMATICS MAGAZINE (all terms expire December 31, 1975).

Editor: G. N. WOLLAN

Associate Editors: L. C. EGGAN, J. S. FRAME, RAOUL HAILPERN, R. E. HORTON, L. M. KELLY, ADA PELUSO, B. L. SCHWARTZ, R. A. STRUBLE, WILLIAM WOOTON, P. J. ZWIER.

REPRESENTATIVES OF THE ASSOCIATION

On the Advisory Committee to the Topology Films Project:
G. S. YOUNG (1973–75)
On the Council of the Conference Board of the Mathematical Sciences:
H. L. ALDER, *ex officio*, R. P. BOAS, *ex officio*
On the Cooperative Committee on the Teaching of Mathematics and Science:
M. W. POWNALL (1972–74)
On Sections of the American Association for the Advancement of Science:
Section A: L. J. PAIGE (1974)
Section Q: L. J. PAIGE (1974)
Section T: W. J. LEVECQUE (1974)
Section U: R. V. HOGG (1974)
Section X: A. B. WILLCOX (1974)
On the Editorial Board of the TYC MATHEMATICS JOURNAL:
JOSEPH HASHISAKI (1974), RALPH MANSFIELD (1974)
On the Governing Council of Mu Alpha Theta:
G. B. PRICE (1973–75)
On the National Research Council:
G. S. YOUNG (July 1, 1971–June 30, 1974)
On the U. S. Commission on Mathematical Instruction:
W. G. CHINN (July 1, 1973–June 30, 1977), P. J. HILTON (July 1, 1970–June 30, 1974).

OFFICERS OF THE SECTIONS

ALLEGHENY MOUNTAIN SECTION

Chairman — C. A. CABLE, Allegheny College
First Vice-Chairman — J. B. DERR, West Virginia University
Second Vice-Chairman — C. A. HALL, University of Pittsburgh
Secretary-Treasurer — M. R. WOODARD, Indiana University of Pennsylvania.

FLORIDA SECTION

Chairman — P. E. MCDOUGLE, University of Miami
Vice-Chairmen — J. F. GOLIGHTLY, Jacksonville University
W. R. RICE, St Petersburg Junior College
Chairman-Elect — R. D. MCWILLIAMS, Florida State University
Secretary-Treasurer — F. L. CLEAVER, University of South Florida.

ILLINOIS SECTION

Chairman — N. E. FOLAND, Southern Illinois University
First Vice-Chairman — DALE WILLIAMS, Illinois Central College
Second Vice-Chairman — P. A. MCGILL, Illinois Central College
Chairman-Elect — R. E. BRYAN, Knox College
Secretary-Treasurer — H. C. SAAR, 2601 Glasgow Street, Joliet, Illinois 60455.

INDIANA SECTION

Chairman — E. H. MCKINNEY, Ball State University
Vice-Chairman — M. D. THOMPSON, Indiana University
Secretary-Treasurer — R. T. HOOD, Franklin College.

IOWA SECTION

Chairman — D. F. BAILEY, Cornell College
Chairman-Elect — D. H. PILGRIM, Luther College
Secretary-Treasurer — B. E. GILLAM, Drake University.

KANSAS SECTION

Chairman — J. M. MARR, Kansas State University
Vice-Chairman — P. R. FLUSSER, Ottawa University
Associate Chairman for Junior Colleges — G. A. KRAUSE, Butler County Community College
Secretary-Treasurer — ELLEN C. VEED, Ft. Hays Kansas State College.

KENTUCKY SECTION

Chairman — J. E. MACK, University of Kentucky
Vice-Chairman — P. E. BLAND, Eastern Kentucky University
Secretary-Treasurer — WALTER FEIBES, Western Kentucky University.

LOUISIANA-MISSISSIPPI SECTION

Chairman — ELEANOR B. WALTERS, Delta State College
Vice-Chairman (Louisiana) — L. S. HAW, Nicholls State University
Vice-Chairman (Mississippi) — D. J. CAVENY, University of Southern Mississippi
Secretary-Treasurer — P. L. FORD, McNEESE State University.

MARYLAND-DC-VIRGINIA SECTION

Chairman — GERALDINE A. COON, Goucher College
Vice-Chairman (Membership) — O. M. THOMAS, U. S. Naval Academy
Vice-Chairman (Programs) — R. M. DAVIS, Northern Virginia Community College
Secretary — J. M. SMITH, George Mason University
Treasurer — HEWITT KENYON, George Washington University.

METROPOLITAN NEW YORK SECTION

Chairman — M. J. HELLMAN, Long Island University
Vice-Chairman (Two-Year Colleges) — HOWARD KLEIMAN, Queensborough Community College
Vice-Chairman (Colleges) — I. H. ROSE, Herbert H. Lehman College of CUNY
Secretary — RORA F. IACOBACCI, St. John's University
Treasurer — AARON SHAPIRO, Brooklyn College.

MICHIGAN SECTION

Chairman — G. F. FEEMAN, Oakland University
Vice-Chairman — A. V. MARTIN, Grand Valley State College
Secretary-Treasurer — YOUSEF ALAVI, Western Michigan University.

MISSOURI SECTION

Chairman — E. Z. ANDALAFTE, University of Missouri at St. Louis
Vice-Chairman — J. M. WILKERSON, Missouri Western College
Secretary-Treasurer — T. L. HICKS, University of Missouri, Rolla.

NEBRASKA SECTION

Chairman — W. W. GUTZMAN, University of South Dakota
Chairman-Elect — G. W. JOHNSON, University of Nebraska, Lincoln
Past Chairman — M. C. THORNTON, University of Nebraska, Lincoln
Vice-Chairman — P. A. HAEDER, University of Nebraska, Omaha
Secretary-Treasurer — H. M. COX, University of Nebraska, Lincoln.

NEW JERSEY SECTION

Chairman — F. J. ALMGREN, JR., Princeton University
Vice-Chairman — F. E. CLARK, Rutgers University, New Brunswick
Secretary-Treasurer — M. E. WHITE, Stevens Institute of Technology
Associate Secretary-Treasurer — EILEEN L. POIANI, St. Peter's College.

NORTH CENTRAL SECTION

Chairman — SISTER RITA JEAN TAUER, College of St. Catherine
Chairman-Elect — SYLVAN BURGSTAHLER, University of Minnesota, Duluth
Secretary-Treasurer — H. M. ANDERSON, Gustavus Adolphus College.

NORTHEASTERN SECTION

Chairman — L. AILEEN HOSTINSKY, Connecticut College
Vice-Chairman — ANNE F. O'NEILL, Wheaton College
Secretary — G. W. BEST, Phillips Academy.

NORTHERN CALIFORNIA SECTION

Chairman — D. J. ALBERS, Menlo College
Vice-Chairman — K. R. REBMAN, California State University, Hayward
Secretary-Treasurer — N. H. FISHER, California State University, San Francisco.

OHIO SECTION

Chairman — J. F. LEETCH, Bowling Green State University
Chairman-Elect — L. J. GREEN, Case-Western Reserve University
Secretary-Treasurer — R. H. ROLWING, University of Cincinnati

OKLAHOMA-ARKANSAS SECTION

Chairman — BILL SPICER, Seminole Junior College
First Vice-Chairman — J. R. HODGES, University of Arkansas at Little Rock
Second Vice-Chairman — R. D. REYNOLDS, Claremore Junior College
Secretary-Treasurer — E. K. MCLACHLAN, Oklahoma State University.

PACIFIC NORTHWEST SECTION

Chairman — J. R. REAY, Western Washington State College
First Vice-Chairman — J. H. JORDAN, Washington State University
Chairman-Elect — THEODORE WHITE, Everett Community College
Secretary-Treasurer — J. E. CALVERT, University of Idaho
Second Vice-Chairman — NORMAN BARTON, Vancouver City College.

PHILADELPHIA SECTION

Chairman — W. J. PERVIN, Drexel University
Vice-Chairman — J. W. P. MAYER, Lebanon Valley College
Secretary-Treasurer — A. E. FILANO, West Chester State University.

ROCKY MOUNTAIN SECTION

Chairman — F. N. FISCH, University of Northern Colorado
Meeting Chairman — R. R. Gutzman, Colorado School of Mines
Vice-Chairman — J. C. DAVIS, Mesa College
Secretary-Treasurer — D. J. STERLING, Colorado College.

SEAWAY SECTION

Chairman — W. C. STONE, Union College
First Vice-Chairman — D. O. MCKAY, University of Western Ontario
Second Vice-Chairman — C. A. LATHAN, Monroe Community College
Secretary-Treasurer — E. C. STOPHER, State University College at Oswego.

SOUTHEASTERN SECTION

Chairman — H. V. PARK, North Carolina State University
Vice-Chairman — D. L. HUNTER, Central Piedmont Community College
Chairman-Elect — J. H. WAHAB, University of South Carolina
Secretary-Treasurer — J. D. NEFF, Georgia Institute of Technology.

SOUTHERN CALIFORNIA SECTION
Chairman — D. L. OUTCALT, University of California, Santa Barbara
Vice-Chairman — D. H. POTTS, California State University at Northridge
Secretary-Treasurer — JOHN GREEVER, Harvey Mudd College.

SOUTHWESTERN SECTION

Chairman — E. D. GAUGHAN, New Mexico State University
Vice-Chairman — R. D. MEYER, Northern Arizona University
Secretary-Treasurer — ALVIN SWIMMER, Arizona State University.

TEXAS SECTION

Chairman — B. L. SANDERS, Texas Christian University
Vice-Chairman — D. E. EDMONDSON, University of Texas at Austin
Level I Director — JUNE P. WOOD, South Texas Junior College
Level II Director — GEORGE COPP, North Texas State University
Director-at-Large — VIVIENNE M. MAYES, Baylor University
Secretary-Treasurer — J. C. BRADFORD, Abilene Christian College
Past Chairman — R. A. NORTHCUTT, Southwest Texas University.

WISCONSIN SECTION

Chairman — P. R. BENDER, Marquette University
Vice-Chairman — C. E. JOHNSON, University of Wisconsin, Madison
Secretary-Treasurer — R. D. WAGNER, University of Wisconsin, Madison.

CALENDAR OF FUTURE MEETINGS

Summer Meeting 1974: There will be no joint summer meeting in 1974, in order that mathematicians may attend the International Congress of Mathematicians to be held in Vancouver, British Columbia, August 21–29, 1974.

Fifty-eighth Annual Meeting, Washington, D. C., January 25–27, 1975.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Editorial Director.

ALLEGHENY MOUNTAIN, Allegheny College, Meadville, Pennsylvania, May 3–4, 1974.

FLORIDA

ILLINOIS, Knox College, Galesburg, May 10–11, 1974.

INDIANA

IOWA

KANSAS, Ottawa University, Ottawa, Spring 1974.

KENTUCKY

LOUISIANA-MISSISSIPPI

MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA

METROPOLITAN NEW YORK, College of Mount St. Vincent, Riverdale, April 28, 1974.

MICHIGAN, Central Michigan University, Mount Pleasant, May 3–4, 1974.

MISSOURI

NEBRASKA

NEW JERSEY, Princeton University, Princeton, October 12, 1974.

NORTH CENTRAL

NORTHEASTERN, Lowell Technical Institute, Lowell, Massachusetts, November 30, 1974.

NORTHERN CALIFORNIA, Chabot College, Hayward, February 1975.

OHIO, Muskingum College, New Concord, May 3–4, 1974.

OKLAHOMA-ARKANSAS

PACIFIC NORTHWEST, University of British Columbia, Vancouver, August 21–24, 1974 (business meeting only — no general meeting).

PHILADELPHIA

ROCKY MOUNTAIN

SEAWAY, St. John Fisher College, Rochester, N. Y., November 1–2, 1974.

SOUTHEASTERN

SOUTHERN CALIFORNIA

SOUTHWESTERN

TEXAS

WISCONSIN, Marquette University, Milwaukee, May 3–4, 1974.

FUTURE MEETINGS OF OTHER ORGANIZATIONS

AMERICAN ASSOCIATION FOR THE ADVANCEMENT OF SCIENCE

AMERICAN MATHEMATICAL SOCIETY, Washington, D. C., January 23–26, 1975.

AMERICAN SOCIETY FOR ENGINEERING EDUCATION, Rensselaer Polytechnic Institute, Troy, New York, June 17–20, 1974.

ASSOCIATION FOR COMPUTING MACHINERY, San Diego, California, November 11–13, 1974.

ASSOCIATION FOR SYMBOLIC LOGIC

FIBONACCI ASSOCIATION

INSTITUTE OF MATHEMATICAL STATISTICS

MU ALPHA THETA, University of Arkansas, Fayetteville, August 4–7, 1974.

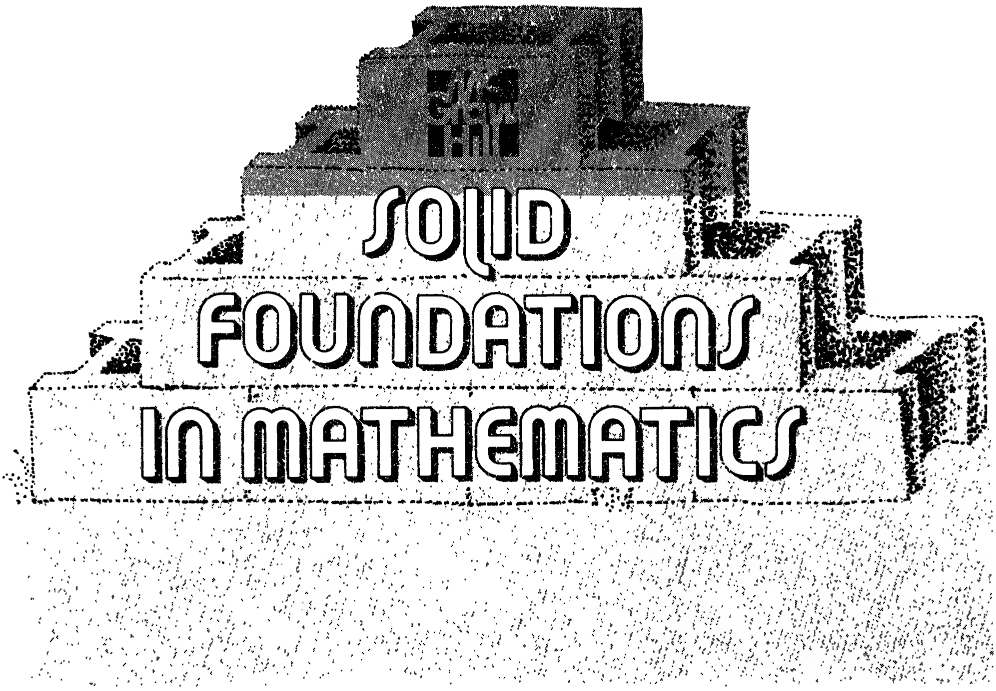
NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS, Washington, D. C., January 25–26, 1975. (Joint meeting with MAA.)

OPERATIONS RESEARCH SOCIETY OF AMERICA, San Juan, Puerto Rico, October 16–18, 1974.

PI MU EPSILON, Western Michigan University, Kalamazoo, August 19–20, 1975.

SCHOOL SCIENCE AND MATHEMATICS ASSOCIATION, Sheraton-Gibson Hotel, Cincinnati, Ohio, November 7–9, 1974.

SOCIETY FOR INDUSTRIAL AND APPLIED MATHEMATICS, Montana State University, Bozeman, June 24–26, 1974.



AN ELEMENTARY APPROACH TO FUNCTIONS

Henry R. Korn and **Albert W. Liberi**, both of Westchester Community College
1974, 480 pages, (035339-5), $9.95; Solutions Manual, (035341-7), $2.50

Following a classical, intuitive approach, this class-tested book prepares students for an introductory-level calculus course. The material is developed through problems and applications rather than through set theory. Graphing techniques are used extensively. Including over 300 examples with step-by-step solutions and over 1600 exercises, the book provides a strong background in algebraic techniques and a thorough treatment of analytic geometry.

FUNDAMENTAL MATHEMATICS, Fourth Edition

Thomas L. Wade, Florida State University, and **Howard E. Taylor**, West Georgia College
1974, 608 pages, (067652-6), $11.50; Instructor's Manual, (067657-7), $2.50

This revision of a popular freshman-level text begins with a combined study of the basic structures of arithmetic and algebra and leads into functions, graphs, elements of trigonometry, logarithms, compound interest and annuities, probability and elements of statistics. Maintaining the conversational style of the third edition, some of the revisions include: an expanded and new chapter on graphing, a completely rewritten chapter on probability and a brief introduction to linear programming with some updating to business applications. This text is suitable for a one- or two-term elementary algebra course for non-majors.

INTRODUCTION TO LINEAR ALGEBRA AND DIFFERENTIAL EQUATIONS

John W. Dettman, Oakland University
1974, 419 pages, (016588-2), $11.95; Instructor's Manual, (016603-x), $2.25

This text presents a balanced and unified treatment of linear algebra and differential equations using examples, problems and applications to stress their interaction and interrelationship. With carefully-selected worked-out examples in every section, the book proceeds gradually from elementary and familiar concepts to the more advanced and abstract ideas.

THE AMERICAN MATHEMATICAL MONTHLY

(Founded in 1894 by Benjamin F. Finkel)

THE OFFICIAL JOURNAL OF

THE MATHEMATICAL ASSOCIATION OF AMERICA

VOLUME 81 **NUMBER 5**

CODEN: AMMYAE

CONTENTS

(*Continued on inside cover*)

MAY **1974**

NOTICE TO AUTHORS

Research papers per se are unsuitable; see statement of policy (Vol. 81, p. 1).

Please follow the format in current issues of the Monthly. Manuscripts must be legibly typewritten or reproduced from typewritten copy, double spaced with wide margins. Three copies should be submitted to the appropriate editor and one kept by the author as protection against loss. The author's full address *must* appear at the end of the manuscript.

Backlog: Main Articles 12 months, Math. Notes 25 months, Research Problems 14 months, Classroom Notes 22 months, Math. Education 16 months.

EDITORIAL CORRESPONDENCE AND MAIN ARTICLES: to Alex Rosenberg, Department of Mathematics, Cornell University, Ithaca, N.Y. 14850; NOTES, etc.: to the corresponding Associate Editor; ADVERTISING CORRESPONDENCE: to Raoul Hailpern, Mathematical Association of America, SUNY at Buffalo, Buffalo, N. Y. 14214; CHANGE OF ADDRESS and SUBSCRIPTIONS: to A. B Willcox, Mathematical Association of America, 1225 Connecticut Ave., N.W., Washington, D.C. 20036.

Annual dues for members of the Association (including a subscription to the American Mathematical Monthly) are $14.00 for each of the first two years of membership and $18.00 thereafter. Student Membership is available with annual dues of $9.00. For nonmembers the subscription price is $18.00.

Published by the Association at Washington, D. C., and Menasha, Wisconsin, during the months of January, February, March, April, May, June–July, August–September, October, November, December.

Second-class postage paid at Washington, D. C., and additional mailing offices.

PRINTED IN THE UNITED STATES OF AMERICA

ALGEBRAIC TILING

S. K. STEIN

A variety of problems concerning number theory, tiling Euclidean space by cubes or by cross-shaped clusters of cubes, coding theory, and gambling, have led to questions in group theory, usually involving finite, or at least finitely generated, abelian groups. We shall discuss some of these problems, their history, and, when convenient, some details of their algebraic solutions. A reader who is familiar with vector spaces and with the quotient groups and homomorphisms of abelian groups will be able to follow the presentation without trouble. It is hoped, in particular, that the discussion will be accessible to upper-division students.

1. Preliminaries. The words "tiling" or "tesselation" usually call to mind congruent copies of a triangle or of a convex quadrilateral tiling the plane. Perhaps "tiling" may remind us of the herringbone pattern composed of translates of an L-shaped set in brick pavements. In any case, copies of some set by some collection of motions fill up another set without overlap (except perhaps along common borders).

Several types of tiling will concern us. In one case, we consider tilings of Euclidean n-space R^n by translates of a cube or by translates of a certain union of a finite number of cubes. The union of these translates will be R^n and the interiors of distinct translates will not intersect.

Secondly, we shall be involved with a group G and two subsets of G: A and B, such that each element of G is uniquely expressible in the form ab, $a \in A$, $b \in B$. We may think of G as being tiled by copies of B or, symmetrically, by copies of A. We shall write $G = (A, B)$, and speak of a **factoring** of G by subsets A and B.

Another important type of tiling is the following: Let G be an abelian group written additively, and $\{s_1, s_2, \cdots, s_k\}$ a set of integers. Each s_i determines a function $\bar{s}_i\colon G \to G$ by $\bar{s}_i(g) = s_i g = g + g + \cdots + g$, s_i times if s_i is positive, and $(-g) + (-g) + \cdots + (-g)$, $|s_i|$ times if s_i is negative. If each non-zero element of G is uniquely expressible in the form $\bar{s}_i b$ for b in some fixed set $B \subseteq G$ and some i, we write

$$G - \{0\} = \{s_1, s_2, \cdots, s_k\} : B$$

and say "$\{s_1, s_2, \cdots, s_k\}$ **splits** $G - \{0\}$". In this case, $G - \{0\}$ is tiled by k copies of B.

In special cases, the splitting of a group is equivalent to the factoring of a different group. To be specific, let p be a positive prime integer and let $C(p)$ be the additive group of the integers modulo p. Assume that $\{s_1, s_2, \cdots, s_k\}$ splits $C(p) - \{0\}$. Let Z_p be the field of integers modulo p and let G_p be its multiplicative group. Then we have the factoring

$$G_p = (\{s_1, s_2, \cdots, s_k\},\ B).$$

For instance, since $C(13) - \{0\} = \{\pm 1, \pm 2\}\colon \{1, 3, 4\}$, we have

$$G_{13} = (\{\pm 1, \pm 2\}, \{1,3,4\}).$$

Before going on to the tilings that we shall treat in detail (all of which concern commutative structures), let us illustrate the notion of a tiling by showing how a purely combinatorial problem can lead to a problem concerning the tiling of a nonabelian group.

Consider the set of $n!$ linear arrangements of the integers $1, 2, \cdots, n$. Call two such arrangements **adjacent** if one is obtainable from the other by a single transposition. The question is:

Is there a set B of arrangements of $1, 2, \cdots, n$ such that each of the $n!$ arrangements is in B or adjacent to precisely one member of B?

The case $n = 2$ is the only one for which the answer is known to be "yes". Since there are $n(n-1)/2$ arrangements adjacent to a given one, the set B, if it exists, would have

$$\frac{n!}{1 + n(n-1)/2}$$

elements. So, it is necessary that $1 + n(n-1)/2$ divide $n!$. Consequently, if $1 + n(n-1)/2$ is divisible by a prime that is larger than n, then the answer is "no". Rothaus and Thompson [**1**] obtained this stronger result:

THEOREM. *If $1 + n(n-1)/2$ is divisible by a prime that is larger than $2 + \sqrt{n}$, then there does not exist a set B of arrangements of $1, 2, \cdots, n$ such that each of the $n!$ arrangements of $1, 2, \cdots, n$ is either in B or adjacent to precisely one member of B.*

As stated, the theorem is purely combinatorial. To obtain their result, Rothaus and Thompson rephrased the problem in terms of factoring the symmetric group S_n:

Let T be the set of transpositions, together with the identity permutation, in the symmetric group S_n. Is there a subset B of S_n such that (T, B) is a factoring of S_n?

Rothaus and Thompson used the theory of group representations to obtain their result. The original question has still not been completely answered.

THE MINKOWSKI-HAJÓS PROBLEM

The next five sections follow the evolution of a problem of Minkowski, from its origins in number theory, through its resolution in abelian groups, and then describes the problems that grew out of the solution.

2. Minkowski's conjecture. The most dramatic work in factoring, that of Hajós [7] in 1942, solved a problem that Minkowski [**12**] raised in 1907. Minkowski first considered a question in number theory, quickly transformed it to one about vectors, and this to a problem about tiling space with congruent cubes. Hajós expressed this problem in terms of factoring a finite abelian group and solved it. Let us follow these transitions in detail, which in total are almost as startling as the metamorphosis of a caterpillar to a butterfly.

The original problem concerns the simultaneous approximation of several real numbers by rational numbers:

Let $a_1, a_2, \cdots, a_{n-1}$ *and* $t > 1$ *be real numbers. Do there exist integers* $x_1, x_2, \cdots, x_n$ *such that* $0 < x_n < t^{n-1}$ *and*

$$\left|a_1 - \frac{x_1}{x_n}\right| < \frac{1}{x_n t}, \left|a_2 - \frac{x_2}{x_n}\right| < \frac{1}{x_n t}, \cdots, \left|a_{n-1} - \frac{x_{n-1}}{x_n}\right| < \frac{1}{x_n t}\ ? \tag{2.1}$$

The case $n = 2$, for instance, concerns the approximation of a single real number a_1 by a rational number x_1/x_2 such that $|a_1 - x_1/x_2| < 1/x_2 t$ and $0 < x_2 < t$. (Note that these two inequalities imply that $|a_1 - x_1/x_2| < 1/x_2^2$.) If $t = 2$, and $a_1 = \frac{1}{2}$, then x_2 would have to be 1 and the inequality $|a_1 - x_1/x_2| < 1/x_2 t$ could not be satisfied. However, as will be shown later, if t is not an integer (2.1) can be satisfied.

We shall follow the evolution of the problem in terms of the specific case $n = 3$, for it illustrates the essentials for arbitrary n and is easier to describe.

Minkowski's question for $n = 3$ may be rephrased, after the clearing of denominators, as follows:

Let a_1, a_2, *and* $t > 1$ *be real numbers. Do there exist integers* x_1, x_2, x_3, *not all* 0, *such that*

$$\begin{aligned} &|tx_1 + 0x_2 - a_1 t x_3| < 1 \\ &|0x_1 + tx_2 - a_2 t x_3| < 1 \\ &|0x_1 + 0x_2 + (1/t^2)x_3| < 1? \end{aligned} \tag{2.2}$$

The determinant of the 3-by-3 matrix formed from (2.2) by removing x_1, x_2, x_3 has the value 1. So, Minkowski raised this more general question:

Let

$$\begin{bmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{bmatrix} \tag{2.3}$$

be a real matrix with determinant 1. *When do there exist integers* x_1, x_2, x_3, *not all* 0, *such that*

$$\begin{aligned} &|b_{11}x_1 + b_{12}x_2 + b_{13}x_3| < 1 \\ &|b_{21}x_1 + b_{22}x_2 + b_{23}x_3| < 1 \\ &|b_{31}x_1 + b_{32}x_2 + b_{33}x_3| < 1? \end{aligned} \tag{2.4}$$

Such integers x_1, x_2, x_3 do not always exist. For example, the only integral solution of

$$\begin{aligned} |1x_1 + b_{12}x_2 + b_{13}x_3| &< 1 \\ |0x_1 + \phantom{b_{12}}1x_2 + b_{23}x_3| &< 1 \\ |0x_1 + \phantom{b_{12}}0x_2 + \phantom{b_{13}}1x_3| &< 1 \end{aligned}$$

is $(0,0,0)$. Clearly, x_3 must be 0, then $x_2 = 0$, finally $x_1 = 0$. To find the extra condition on the matrix (2.3) that would guarantee a nontrivial solution for the inequalities (2.4), Minkowski transformed the question into a geometric one.

Observe that the three column vectors

$$v_1 = \begin{bmatrix} b_{11} \\ b_{21} \\ b_{31} \end{bmatrix} \quad v_2 = \begin{bmatrix} b_{12} \\ b_{22} \\ b_{32} \end{bmatrix} \quad v_3 = \begin{bmatrix} b_{13} \\ b_{23} \\ b_{33} \end{bmatrix}$$

span a parallelepiped whose volume is 1. For convenience we shall identify a vector with the point whose coordinates are the components of the vector. The question now reads:

Let v_1, v_2, and v_3 be three vectors in R^3 that span a parallelepiped of volume 1. *When are there integers x_1, x_2, x_3, not all* 0, *such that the vector*

$$x_1v_1 + x_2v_2 + x_3v_3 \tag{2.5}$$

lies in the **interior** *of the* 2 *by* 2 *by* 2 *cube C whose edges are parallel to the axes and whose center is the origin*?

If, instead of demanding that $x_1v_1 + x_2v_2 + x_3v_3$ lie in the interior of C, we ask only that it lie in C—perhaps on the surface of C—then Minkowski showed that the answer is "always". His argument goes like this: Let D be the 1 by 1 by 1 cube whose edges are parallel to the axes and whose center is the origin. Form the set of all translates of D by vectors of the form (2.5). (Note that x_1, x_2, x_3 are integers.) Such a set of translates, we shall call a lattice of translates.

Pair off each point of the form (2.5) with the parallelepiped obtained by translating by that vector the parallelepiped P spanned by v_1, v_2, v_3. Since the volume of P is 1, the number of points of the form (2.5) in a large region of volume V (say, similar to P) is approximately V. Thus there are approximately V translates of the unit cube C in that region. Since C has volume 1, these translates cannot be disjoint—they must overlap, perhaps only at their surface. Hence there are two vectors v and v^* of the form (2.5) such that

$$(v + C) \cap (v^* + C) \neq \varnothing,$$

hence also two points c and c^* in C such that $v + c = v^* + c^*$, or

$$v - v^* = c^* - c. \tag{2.6}$$

Now $c^* - c$ is in the cube D and $v - v^*$ is a non-zero vector of the form (2.5). Thus inequalities (2.4), with $<$ replaced by $\leqq$, do have a non-trivial solution.

As observed above, if

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad v_2 = \begin{bmatrix} b_{12} \\ 1 \\ 0 \end{bmatrix} \quad v_3 = \begin{bmatrix} b_{13} \\ b_{23} \\ 1 \end{bmatrix},$$

inequalities (2.4) have only the trivial solution. More generally, if the vectors

$$u_1 = \begin{bmatrix} c_{11} \\ c_{21} \\ c_{31} \end{bmatrix} \quad u_2 = \begin{bmatrix} c_{12} \\ c_{22} \\ c_{32} \end{bmatrix} \quad u_3 = \begin{bmatrix} c_{13} \\ c_{23} \\ c_{33} \end{bmatrix} \tag{2.7}$$

are simply another basis for the lattice spanned by

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad v_2 = \begin{bmatrix} b_{12} \\ 1 \\ 0 \end{bmatrix} \quad v_3 = \begin{bmatrix} b_{13} \\ b_{23} \\ 1 \end{bmatrix}, \tag{2.8}$$

then the inequalities

$$\begin{aligned} |c_{11}x_1 + c_{12}x_2 + c_{13}x_3| &< 1 \\ |c_{21}x_1 + c_{22}x_2 + c_{23}x_3| &< 1 \\ |c_{31}x_1 + c_{32}x_2 + c_{33}x_3| &< 1 \end{aligned} \tag{2.9}$$

have no non-zero integer solution. This leads to Minkowski's conjecture, which we state for dimension 3.

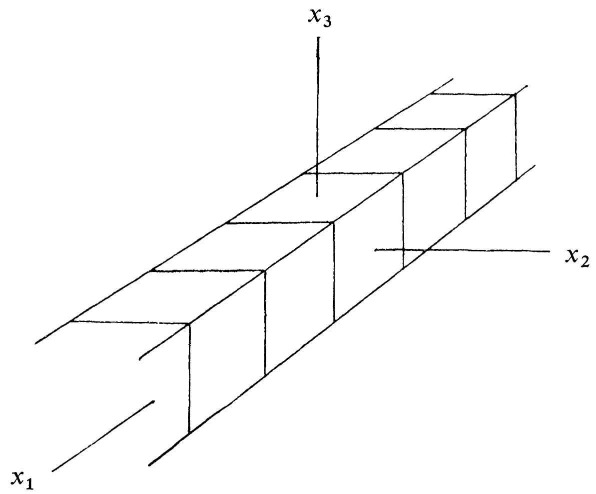
Minkowski's Conjecture. Let (c_{ij}) be a three-by-three matrix whose determinant is 1. If the only integral solution of inequalities (2.9) is $(0,0,0)$, then the lattice spanned by (2.7) can also be spanned by vectors of the form (2.8) (or vectors differing from (2.8) only by changing the role of the three axes).

The validity of this conjecture would imply that all entries in at least one row of the matrix (c_{ij}) would have to be integers. In particular, if t is not an integer, inequalities (2.2) would have a non-zero solution. In other words, Equations (2.4) have only the trivial solution $(0,0,0)$ if and only if the real matrix (2.3) of determinant 1 is unimodularly equivalent to a triangular matrix with ones on the diagonal. (For the case where t is an integer see [**11**] pp. 109–110.)

Minkowski also expressed his conjecture geometrically. The assertion that a lattice of translates of the unit cube C has a basis of, say, the form

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad v_2 = \begin{bmatrix} b_{12} \\ 1 \\ 0 \end{bmatrix} \quad v_3 = \begin{bmatrix} b_{13} \\ b_{23} \\ 1 \end{bmatrix},$$

implies, first of all, that the unit cube whose center is $(1,0,0)$ is present in the lattice of translates. This means that the two unit cubes whose centers are $(0,0,0)$ and $(1,0,0)$ *share a complete two-dimensional face.* Since a lattice is homogeneous—every cube in it playing the same role—the lattice must be composed of files of cubes, parallel to the x_1 axis: one such (endless) file is shown in this figure:



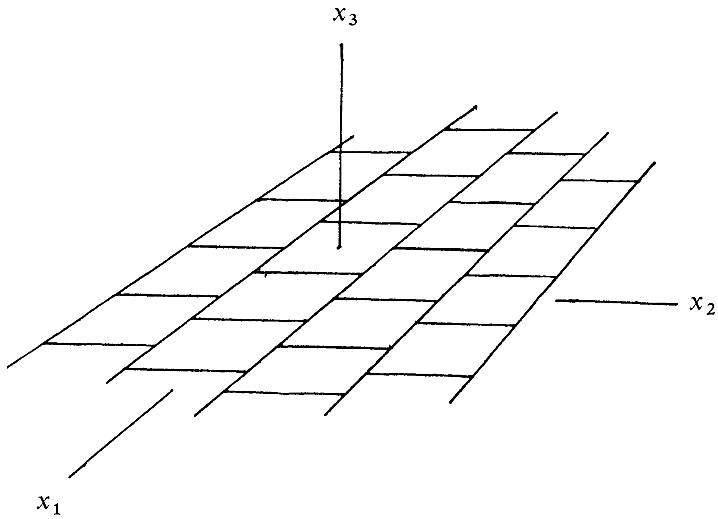
The presence of the vector

$$v_2 = \begin{bmatrix} b_{12} \\ 1 \\ 0 \end{bmatrix}$$

is then equivalent to the existence of another file of cubes at the same height, tangent to the first array, but perhaps slid along the x_1 direction. Hence the tiling of R^3 includes a tiling of the slab $|x_3| \leq \frac{1}{2}$, as shown in the figure on page 451.

Translates of this slab make up the original tiling of R^3. In other words, the tiling of R^3 can be built step by step, cubes forming files, then files forming slabs. This is how Minkowski's conjecture for general n reads in geometric terms:

MINKOWSKI'S CONJECTURE (**geometric form**): *If a lattice of unit cubes tiles R^n, then some pair of cubes share a complete $(n-1)$-dimensional face.*



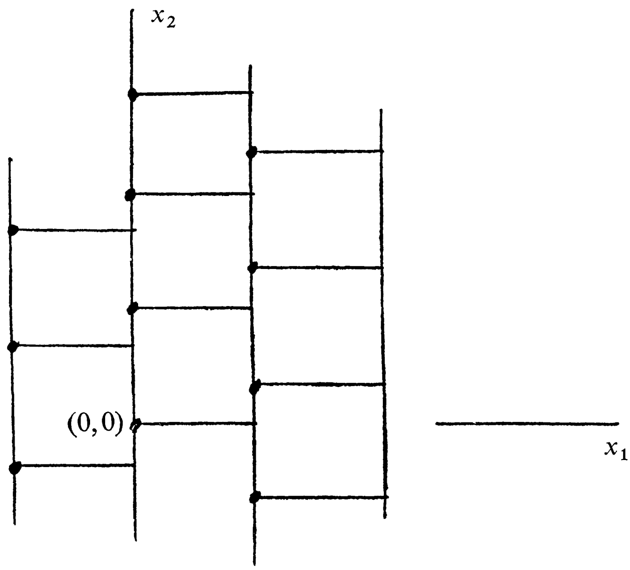
This formulation appears in [**12**, p. 74]. The problem, in arithmetic form, was raised eleven years earlier [**11**, §37].

3. Hajós' confirmation of Minkowski's conjecture. Minkowski easily settled his conjecture for $n = 2$ or 3. Jansen in 1909 [**9**] took care of $n = 4$, 5, and 6. In 1930, Keller [**10**] generalized the conjecture by removing the assumption that the cubes form a lattice. Using only the assumption that parallel unit cubes tile R^n, he proved that for any two of the cubes there is a coordinate axis such that along that axis the coordinates of their midpoints differ by an integer. Reviewing the work done up to 1940, Perron [**13**] remarked, "I must confess that in most of the papers I came to one or more places where I could not follow the line of reasoning. So I do not really know how far the assertions are in fact proved. Partly, but not completely, this may be because I lack the slightest intuitive picture of n-dimensional space, an insight with which the other authors seem to be endowed." In this paper, Perron verifies Keller's version of Minkowski's conjecture for $n \leqq 6$ and reduces the problem for any dimension n to a finite one concerning 2^n parallel unit cubes whose centers are located in a special way in the 2^n unit cubes that compose a cube of side 2. And there he left the problem, whose complexity grows rapidly with the dimension.

Soon after, in 1942, Hajós [7] settled Minkowski's conjecture, where it is assumed that the cubes form a lattice, after first casting it into the form of an equivalent conjecture about finite abelian groups. We shall describe this final formulation, but refer the reader to Fuchs [**6**, pp. 318–323] for the simplest exposition of Hajós' proof, which depends on the group ring of a finite abelian group with integer coefficients.

Let us consider only the case $n = 2$, the plane, for it illustrates the general idea and is easy to draw. Imagine, then, that the plane is tiled by a lattice of unit squares

parallel to the coordinate axes. Describe the location of each square by its lower left corner. These points form a group L^*.



First of all, Hajós reduced the general case to that in which the points in L^* have rational coordinates. A linear transformation of the form $(x, y) \rightarrow (m_1 x,\ m_2 y)$, where m_1 and m_2 are positive integers, distorts L^* to a subgroup L whose points have integer coordinates and simultaneously transforms the unit squares to m_1 by m_2 rectangles, still parallel to the axes.

This is now the situation. We have the group H of all points with integer coordinates and a subgroup L, which consists of the lower left corners of a family of m_1 by m_2 rectangles that tile the plane. Hajós then expressed Minkowski's conjecture in terms of the quotient group $G = H/L$.

Let $u_1 = (1,0)$ and $u_2 = (0,1)$ be the standard unit vectors in H. Let $a_1 = u_1 + L$ and $a_2 = u_2 + L$ be the corresponding elements in $G = H/L$. (We write the elements of G multiplicatively and those of H additively.)

The assumption that the rectangles tile the plane is composed of two conditions: that they cover the plane and that they do not overlap (except at their borders).

The assumption that the m_1 by m_2 rectangles cover the plane reads, in terms of the group H, as follows:

For any element $h \in H$, there are an element $l \in L$ and integers e_1 and e_2, $0 \leqq e_i < m_i$, $i = 1, 2$, such that

$$h = l + e_1 u_1 + e_2 u_2.$$

In terms of the quotient group, this reads: Every element $g \in G$ can be written in the form

$$g = a_1^{e_1} a_2^{e_2}, \qquad 0 \leqq e_i < m_i, \qquad i = 1,2.$$

The assumption that the m_1 by m_2 rectangles do not overlap reads, in terms of H: If $l_1 \in L$ and $l_2 \in L$, and $0 \leqq e_i,\ e_i' < m_i$, $i = 1,2$, and

$$l_1 + e_1 u_1 + e_2 u_2 = l_2 + e_1' u_1 + e_2' u_2,$$

then $e_1 = e_1'$ and $e_2 = e_2'$. In terms of G this condition reads:

$$\text{If } a_1^{e_1} a_2^{e_2} = a_1^{e_1'} a_2^{e_2'},\ 0 \leqq e_i,\ e_i' < m_i,\ i = 1,2, \text{ then } e_i = e_i',\ i = 1,2.$$

The assumptions of Minkowski's conjecture are now expressed completely in terms of the group G (which, incidentally, has $m_1 m_2$ elements). How does the conclusion read?

Imagine that two rectangles share a complete edge parallel, say, to the x_2-axis. There is no loss of generality in assuming that the common edge is the right edge of the rectangle whose lower left corner is at the origin. Thus

$$m_1 u_1 \in L.$$

In terms of G, this assertion is expressed in the equation

$$a_1^{m_1} = 1.$$

This suggests Hajós' translation of Minkowski's conjecture in n-space:

(3.1) *Let G be a finite abelian group and let $a_1, a_2, \cdots, a_n$ be n elements of G. Let the order of a_i be at least m_i, $i = 1, 2, \cdots, n$. Assume that each element of G is uniquely expressible in the form*

$$a_1^{e_1} a_2^{e_2} \cdots a_n^{e_n},$$

$0 \leqq e_i < m_i$, $i = 1, 2, \cdots, n$. *Then there is at least one integer i such that* $a_i^{m_i} = 1$.

This theorem concerns the factorization of the group G into n sets,

$$G = (\{1, a_1, \cdots, a_1^{m_1 - 1}\},\ \{1, a_2, \cdots, a_2^{m_2 - 1}\}, \cdots, \{1, a_n, \cdots, a_n^{m_n - 1}\}),$$

and asserts that at least one of the factors is a group. It was in this form that Minkowski's problem was finally solved.

4. Hajós generalized. With Hajós' solution of Minkowski's problem, interest in tiling by congruent cubes disappeared. No one seems to have pursued Keller's conjecture which removes the "lattice" assumption nor his problem concerning 2^n cubes in n-space.

But Hajós' theorem did inspire new questions and new work. Perhaps the most interesting is Rédei's generalization [**15**] in 1965 of Hajós' theorem, where the factors are no longer required to be "front ends" of cyclic groups.

Rédei's Theorem. *Let G be a finite abelian group and let $A_1, A_2, \cdots, A_m$ be subsets of G, each of which contains the identity element and each has prime order. Assume that G is factored*

$$G = (A_1, A_2, \cdots, A_m)$$

in the sense that each element of G is uniquely expressible as a product

$$a_1 a_2 \cdots a_m,$$

$a_i \in A_i$. *Then at least one A_i is a group.*

(It is not hard to reduce Hajós' theorem to the case where each m_i is prime. So Rédei's is indeed a generalization of Hajós'.)

Wittman [**22**] in 1969 simplified part of Rédei's proof, the important case where $G = C(p) \times C(p)$, the square of a cyclic group of prime order. Both Rédei and Wittman used characters of abelian groups, the factorization of polynomials, and, as did Hajós, the group ring.

Moving in another direction, Bernstein [**2**] generalized Hajós' theorem to finite nonabelian groups in which every cyclic subgroup of composite order is normal.

5. Good groups. Hajós' theorem also raised questions about the form of factorizations. Let us look at just one of them, which was answered over a period of fifteen years in a series of papers by Hajós [**8**], Rédei [**14**], Sands [**18, 19, 20, 21**], and de Bruijn [**4, 5**].

Hajós calls a subset A of a finite abelian group G **periodic** if there is an element g in G, other than the identity element, such that $gA = A$. (The conclusion of Hajós' theorem is equivalent to the assertion that one of the sets

$$\{1, a_i, a_i^2, \cdots, a_i^{m_i - 1}\}$$

is periodic.) Observe that a subset A is periodic if and only if there is a subgroup H of G, $|H| > 1$, such that A has a factorization, $A = (H, B)$.

In [**8**] Hajós proved that if G is a cyclic group of prime power order, $C(p^n)$, and $G = (A, B)$, then at least one of the subsets A and B is periodic. Rédei [**14**] obtained the same conclusion when G is of the form $C(p) \times C(p)$. These papers helped initiate the search for "good" groups. A finite abelian group G is **good** if in each factorization $G = (A, B)$, at least one of A and B is periodic. Sands [**20**], summarizing the efforts of several mathematicians, provided this complete list of the good groups:

$$(p^n, q),\ (p^2, q^2),\ (p^2, q, r),\ (p, q, r, s)$$

$$(p^3, 2, 2),\ (p^2, 2, 2, 2),\ (p, 2^2, 2),\ (p, 2, 2, 2, 2)$$

$$(p, q, 2, 2),\ (p, 3, 3),\ (3^2, 3),\ (2^n, 2)$$

$$(2^2, 2^2),\ (p, p),$$

and their subgroups. Here, p, q, r, s denote distinct primes; in each case p may equal 2. The notation $(a, b, \cdots, c)$ is short for the direct product $C(a) \times C(b) \times \cdots \times C(c)$.

Along the way, many side results were obtained. For instance, Sands [**18**] proved that if a finite cyclic group G is factored, $G = (A, B)$, and $|A|$ is a power of a prime, then A or B is periodic. In [**19**], Sands proved that if the finite abelian group G is factored, $G = (A, B)$, $|A|$ is a power of a prime, and if $(|A|, |B|) = 1$, then there is a subgroup H in G such that $G = (A, H)$. It is not known whether the assumption that the order of A is a prime power is necessary.

6. Another relation to geometric tiling problems. The preceding section was concerned with the relation of a factor of G to some subgroup H of G. The first such theorem is Lagrange's, which asserts that any subgroup H of G is a factor of G. Similar questions and results are to be found in the theory of convex bodies. For instance, for any convex body in the plane a densest packing by translates is provided by a *lattice* packing.

Zassenhaus in [**23**] remarked, concerning a different geometric problem, "It is highly interesting to observe that one of the densest X-admissible point sets turns out to be a 'lattice with a base', i.e., a point set which is the union of a finite number of translates of a geometric lattice...the fact that (vaguely formulated) optimal discrete distributions tend to be lattices with a base has been known to every scientist interested in solid state physics since Bragg's and von Laue's discoveries. ...Is it reasonable to assume that lattices with a base form a pattern of optimal packings?" Zassenhaus was presumably referring to the fact that when a liquid solidifies it tends to form a crystal which usually is the configuration that minimizes the total internal energy.

One algebraic analog of this expectation, for tilings rather than for packings in general, runs as follows. Let $G = Z \times Z \times \cdots \times Z$, the free abelian group with n generators (the analog of n-dimensional Euclidean space) with addition coordinatewise. Let A be a finite subset of G. Assume that there is a set B such that $G = (A, B)$. Is there then a subgroup H of G and a *finite* set S in G such that $G = (A, H, S)$? The set $H + S$ is the analog of "a lattice with a base". The answer is not known. In [**33**] is an example of a symmetric star body that tiles R^{10} as a lattice with a base but not as a lattice. Robinson [**17**] exhibited 32 finite subsets of $Z \times Z$ whose translates tile $Z \times Z$ but do not tile $Z \times Z$ in any way that resembles a lattice with a base. The general question for translates of a single finite subset of $Z \times Z$ remains open.

Tiling by Certain Star Bodies

The theory of convexity contains many results on tiling, packing, and covering by translates of a convex body. If the interiors of a family of convex bodies in Euclidean space are pairwise disjoint, the family is called a **packing**; if the union of the family is all of the space, the family is called a **covering**. Thus a tiling by convex

bodies is simultaneously a packing and a covering. We will also speak of lattice tilings. First of all, a family of vectors form a **lattice** if they are a group under addition and have no accumulation point. In a **lattice tiling**, the tiling family consists of the translates of a fixed convex body by the vectors of a lattice. The body is said to tile in a **lattice manner**. These definitions extend from convex bodies to star bodies. (A star body contains a point from which the entire body is "visible".) Two types of star bodies, called "crosses" and "semi-crosses", are of special interest, in part because their tiling problems, though fairly general, can be treated algebraically, and in part because of their appearance in combinatorial and coding theory. The next two sections concerns these star bodies.

7. Tiling Euclidean space by crosses or semi-crosses. Let k and n be positive integers. Any translate of the $kn+1$ unit n-dimensional cubes whose edges are parallel to the coordinate axes and whose centers are the $kn+1$ points specified by the n-tuples

$$(0,0,\cdots,0),\ (j,0,\cdots,0),\cdots,(0,0,\cdots,j),$$

$j = 1,2,\cdots,k$, is called a (k,n)-**semi-cross**. A (k,n)-semi-cross consists of n arms of length k attached at facets of a "corner" cube.

A (k,n)-**cross** is any translate of the $2kn+1$ unit cubes whose centers are at the $2kn+1$ points specified by the n-tuples

$$(0,0,\cdots,0),\ (\pm j,0,\cdots,0),\cdots,\ (0,0,\cdots,\ \pm j),$$

$j = 1,2,\cdots,k$. The (k,n)-cross is centrally symmetric.

We shall assume, unless otherwise stated, that in the tilings by crosses or semi-crosses, the centers of the cubes have integer coordinates. Essentially we are replacing R^n by Z^n. The next few theorems, which suggest the type of algebraic problems such tilings raise, are taken from Stein [**32**].

Theorems 7.1 and 7.3 show why lattice tilings by semi-crosses or crosses are specially amenable to algebraic treatment.

THEOREM 7.1. *The (k,n)-semi-cross tiles R^n in a lattice manner if and only if the set $\{1,2,\cdots,k\}$ splits an abelian group G of order $kn+1$, that is*

$$G-\{0\} = \{1,2,\cdots,k\}:\{g_1,g_2,\cdots,g_n\}, \tag{7.2}$$

where $\{g_1,g_2,\cdots,g_n\} \subseteq G$.

Proof. Let Z^n be the free abelian group with n generators. If the (k,n)-semi-cross tiles Z^n in a lattice manner, let H be the subgroup of Z^n occupied by the centers of the corners of the semi-crosses. Let G be the quotient group Z^n/H. Let $f\colon Z^n \to G$ be the natural homomorphism and let $g_j = f(E_j)$ where E_j is the basic unit vector in the jth direction, $E_j = (0,\cdots,0,1,0,\cdots,0)$, a 1 in the jth coordinate. Note that Z^n is the union of the $kn+1$ cosets of the form $c+H$, where c is a center of a cube in the semi-cross

whose corner is at the origin. Thus, as c runs through the $kn+1$ points iE_j, $1 \leqq i \leqq k$, $1 \leqq j \leqq n$, together with the origin, $f(c)$ runs through the elements of G. Thus we have the splitting,

$$G - \{0\} = \{1, 2, \cdots, k\}: \{g_1, g_2, \cdots, g_n\}.$$

Conversely, assume that G is an abelian group of order $kn+1$ and that each non-zero element is of the form ig_j, $1 \leqq i \leqq k$, $1 \leqq j \leqq n$. Define $f: Z^n \to G$ to be the unique homomorphism such that $f(E_j) = g_j$ and let H be the kernel of f. Then, as may be checked directly, the set of semi-crosses whose corner cubes have their center in H constitute a tiling of R^n.

In a similar manner, it can be shown that the (k, n)-cross tiles R^n in a lattice manner if and only if the set $\{\pm 1, \pm 2, \cdots, \pm k\}$ splits an abelian group of order $2kn+1$. In both these results, call G "the quotient group of the tiling". In particular, since $C(13) - \{0\} = \{\pm 1, \pm 2\}: \{1, 3, 4\}$, the $(2,3)$-cross tiles R^3.

So the question, "when does a (k, n)-semi-cross or cross tile R^n in a lattice manner," is reduced to a question concerning the splitting of finite abelian groups. Either question is far from being answered. The following two theorems represent only the first steps toward a complete solution.

THEOREM 7.3. ([**32**, Theorem 4.8]). *If p is a prime, then the $(p-1, n)$-semi-cross tiles R^n in a lattice manner for an infinitude of n such that $(p-1)n+1$ is prime.*

The next theorem shows that a semi-cross or cross can tile a given space in many geometrically distinct ways. Preprints are available from the authors.

THEOREM 7.4. (W. Hamaker-S.K. Stein). *Let p be an odd prime, b an integer greater than* 1, *and $n = (p^b - 1)/(p-1)$. Then the $(p-1, n)$-semi-cross tiles R^n and the $((p-1)/2, 2n)$-cross tiles R^{2n} in a lattice manner. Moreover, any abelian group of order p^b can be prescribed as the quotient group of either tiling.*

Theorem 7.4 shows, for instance, that the $(2,4)$-semi-cross tiles R^4 with quotient group $C(9)$ and also $C(3) \times C(3)$. There are in fact at least two geometrically inequivalent lattice tilings with quotient group $C(9)$. One has its corners at

$$\{(x_1, x_2, x_3, x_4) \,|\, x_1 + 4x_2 + 7x_3 + 3x_4 \equiv 0\,(9)\}$$

and the other at

$$\{(x_1, x_2, x_3, x_4) \,|\, x_1 + 4x_2 + 7x_3 - 3x_4 \equiv 0\,(9)\}.$$

With the aid of Theorem 7.1 and its companion for crosses, it is easy to show that the $(1, n)$-semi-cross tiles R^n with any abelian group of order $n+1$ as quotient group and that the $(1, n)$-cross tiles R^n with any abelian group of order $2n+1$ as quotient group.

Among the many questions suggested by these theorems, we state just two. In a

geometric tiling of R^n by crosses (parallel to the axes), one of which has its center at the origin, must the centers of all the crosses have integer coordinates? (For the $(1,3)$-semi-cross in R^3, the answer is "no".)

Secondly, if a (k,n)-semi-cross tiles R^n, $n \geqq 3$, is k bounded in terms of n? In [**32**] it was shown that in the case of a lattice tiling, where $kn+1$ is prime, then $k < 2n-2$. Hamaker [**26**] removed the assumption that $kn+1$ is prime. For crosses, a bound is known for any tiling, [**32**, Theorem 3.2].

8. Combinatorial and coding problems. Problems in packing, covering, or tiling by figures closely related to semi-crosses or crosses have appeared independently in such separate fields as combinatorics and coding theory.

The combinatorial case goes back to a gambling problem first investigated by Taussky and Todd [**34**] in 1948. This is their description of the problem: "A bettor tries to forecast the results of 13 games (win, loss, or tie). He 'knows' the results of 9, say. But the remaining 4 are uncertain. To make sure of getting the remaining 4 right, he would have to make 3^4 entries. But if he thinks 12 right will be the best submitted, he asks what is the smallest number of entries which will ensure that no matter what happens he will have at least 3 right out of 4."

This question suggested a general combinatorial problem:

Let X be a set with q elements and let n be a positive integer. Let S be the n-fold cartesian product, $X \times X \times \cdots \times X$. How small a subset $B \subseteq S$ can be found such that each element of S differs from some element of B in at most one coordinate?

(The gambler's problem is the case $q = 3$ and $n = 4$.) The minimal size of B is usually denoted $\sigma(n,q)$. Since a given point in S differs in exactly one coordinate from $n(q-1)$ points, clearly

$$\sigma(n,q) \geqq \frac{q^n}{n(q-1)+1}.$$

If S is pictured as an n-dimensional chess board of side q, then the set B can be interpreted as a minimal set of castles that attack or occupy every cube of the board.

At the outset, the combinatorial question was cast in algebraic terms. Let

$$G = C(q) \times C(q) \times \cdots \times C(q), \ n \text{ times}.$$

Let $A \subseteq G$ be the $n(q-1)+1$ elements

$$(8.1) \qquad (0,0,\cdots,0),\ (i,0,\cdots,0),\ (0,i,0,\cdots,0),\cdots,(0,\cdots,0,i)$$

$1 \leqq i \leqq q-1$. Then the combinatorial question now reads:

How small a subset B can be found in G such that $A + B = G$?

The symbol $A + B$ denotes the set of elements of the form $a + b$, a in A and b in B.

Since it is *not* assumed that each element of G is uniquely of the form $a + b$, this question concerns covering, not tiling. However, most of the early work was devoted

to the special case of finding a factoring; in this case $n(q-1)+1$ must divide q^n. The first general result in this direction is due independently to Zaremba [**35**] and Mauldon [**31**] in 1951:

THEOREM 8.2. *Let p be a prime number, let a be an integer larger than* 1, *and let $n = (p^a - 1)/(p-1)$. Then the group $G = [C(p)]^n$ has a tiling $G = (A, B)$, where A is prescribed in* (8.1).

Proof. Each non-zero element of $H = [C(p)]^a$ has order p. Select a non-zero element g_1. It generates a group G_1 of order p. Select g_2 not in G_1. It generates a group G_2 of order p that meets G_1 only in the element 0. Continuing in this way, select elements $g_1, g_2, \cdots, g_n$ such that

$$H - \{0\} = \{1, 2, \cdots, p-1\} : \{g_1, g_2, \cdots, g_n\}.$$

Next define a homomorphism

$$f: [C(p)]^n \to [C(p)]^a$$

by mapping $(0, \cdots, 1, \cdots, 0)$—where 1 is the ith place—onto g_i. Then $(A, f^{-1}(0))$ is a factoring of $G = [C(p)]^n$.

Theorem 8.2 includes the particular problem of the gambler when $p = 3$, $a = 2$, and $n = (p^2-1)/(p-1) = 4$, showing that $3^4/[4(3-1)+1] = 9$ forecasts suffice.

Zaremba [**36**] also treated the case where q is a power of a prime. For convenience, we include the short proof published in 1969 by Losey [**30**].

THEOREM 8.3. *Let q be a power of a prime p, let a be an integer larger than* 1, *and let $n = (q^a - 1)/(q-1)$. Then the group $G = [C(q)]^n$ has a tiling (A, B) where A is prescribed in* (8.1).

Proof. The vector space V^a of dimension a over the Galois field $\mathrm{GF}(q)$ is the union of $n = (q^a-1)/(q-1)$ lines through the origin. On each line, select a point g_i other than the origin. Let V^n be the vector space of dimension n over $\mathrm{GF}(q)$ and define a linear map

$$T: V^n \to V^a$$

by setting

$$T(0, \cdots, 1, \cdots, 0) = g_i,$$

(where a 1 is in the ith place and 0's elsewhere). Then it can be shown directly that

$$G = (A, T^{-1}(0)),$$

where G is considered as the additive group of V^n.

Note that G in the preceding proof is *not* $[C(q)]^n$. The additive structure of V^n is

the n-fold sum of the additive group of GF(q), hence the sum of copies of $C(p)$. The set $T^{-1}(0)$ is *not* a subgroup of $[C(q)]^n$ (unless q is itself a prime). This is a consequence of the following theorem, due to Zaremba [**36**].

THEOREM 8.4. *If q is composite, the group $G = [C(q)]^n$ has no factoring of the type (A, B), where A is prescribed in* (8.1) *and B is a group.*

Proof. Assume that in such a factorization B is a group. Let

$$f\colon G \to G/B$$

be the natural homomorphism. The ith axis of G,

$$\{(0, \cdots, x, \cdots, 0) \,|\, x \in C(q),\ x \text{ in the } i\text{th place}\},$$

is a subgroup of G. Let $H_i \subseteq G/B$ be the image of the ith axis under the homomorphism f. Then

$$G/B = H_1 \cup H_2 \cup \cdots \cup H_n$$

and any pair of the H_i's meet only at the element 0 in G/B. Each H_i is isomorphic to $C(q)$.

As Baer observes in [**24**, p. 337], if a finite abelian group H is expressed as the union of at least two subgroups, any two of which meet only at $\{0\}$, then every non-zero element of H has the same order.

[To show this, assume $H = A \cup B \cup C \cup \cdots$. Let $a \in A - \{0\}$ and $b \in B - \{0\}$. Assume $mb = 0$. We shall show $ma = 0$. Clearly, $a + b \notin A \cup B$, hence is in a subgroup C that meets A and B only at $\{0\}$. Hence $m(a + b) \in C$. But $m(a + b) = ma \in A$. Thus $ma = 0$. From this it follows that all non-zero elements of A and of B have the same order, which, being non-zero, must be a prime.]

The factorings of $[C(q)]^n$ in Theorems 8.2 and 8.3 provide tilings of R^n by $(q-1, n)$-semi-crosses and, if q is odd, by $((q-1)/2, 2n)$-crosses). If q is prime the tiling is by a lattice. If q is a power of a prime, but not prime, the tiling is by a lattice with a base.

The problem of tiling $[C(q)]^n$ arose, as we saw, from a combinatorial covering problem. The same problem is of importance in coding theory, where it grew out of a packing, rather than covering, problem.

In the coding case, X is a set of q "symbols" and X^n is the set of q^n "messages" of length n that can be written with those symbols. Assume that when a message of length n is transmitted, at most one of the n symbols is received erroneously. Therefore, when a message m is sent, any of $1 + n(q - 1)$ sequences can be received. Call this set of possibilities $A(m)$. Because of the possible ambiguity, the sender and the receiver must agree in advance on a list of possible messages, called "code words," $m_1, m_2, \cdots, m_k$ such that for $i \neq j$, $A(m_i)$ and $A(m_j)$ are disjoint. When each m_i is again interpreted as the position of a castle, no two of the k castles must attack the

same point. Instead of a (minimal) covering problem, we have come to a (maximal) packing problem. The two problems become identical if we insist that the covering have no overlap and the packing fill X^n. Codes that meet this stringent demand are called "perfect single-error correcting codes." For a survey of the construction of such codes, see van Lint [**28, 29**].

References

1. O. Rothaus and J. G. Thompson, A combinatorial problem in the symmetric group, Pacific J. Math., 18 (1966) 175–178.

Minkowski-Hajós

2. H. J. Bernstein, Extension of Hajós' factorization theorem to some non-abelian groups, Comm. Pure Appl. Math., 21 (1968) 289–311.

3. H. Davenport, Recent progress in the geometry of numbers, Proc. Int. Congr. Math., (1950) 166–174.

4. N. G. de Bruijn, On the factorization of finite abelian groups, Indag. Math. Kon. Ned. Akad. Wetensch. Amsterdam, 15 (1953) 258–264.

5. ———, On the factorization of cyclic groups, *ibid.*, 370–377.

6. L. Fuchs, Abelian Groups, Pergamon Press, 1967.

7. G. Hajós, Über einfache und mehrfache Bedeckung des n-dimensionalen Raumes mit einen Würfelgitter, Math. Zeit., 47 (1942) 427–467.

8. ———, Sur la factorisation des groupes abéliens, Časopis Pěst. Mat. Fys., 74 (1949) 157–162.

9. H. Jansen, Lückenlose Ausfüllung des R_n mit gitterförmig angeordneten n-dimensionalen Quadern, Dissertation Kiel, 1909.

10. O. H. Keller, Über die lückenlose Einfüllung des Raumes mit Würfeln, J. Reine Angew. Math., 163 (1930) 231–248.

11. H. Minkowski, Geometrie der Zahlen, Leipzig, 1896.

12. ———, Diophantische Approximationen, Leipzig, 1907.

13. O. Perron, Über lückenlose Ausfüllung des n-dimensionalen Raumes durch kongruente Würfel, Math. Zeit., 46 (1940) 1–26, 161–180.

14. L. Rédei, Zwei Lücken Sätze über Polynome in endlichen Primkörpern mit Anwendung auf die endlichen Abelschen Gruppen und die Gausseschen Summen, Acta Math., 79 (1947) 273–290.

15. ———, Die neue Theorie der endlichen abelschen Gruppen und Verallgemeinerung des Hauptsatzes von Hajós, Acta Math. Acad. Sci. Hung., 16 (1965) 329–373.

16. ———, Berichtigung zu meiner Arbeit, "Die neue Theorie der endlichen abelschen Gruppen und Verallgemeinerung des Hauptsatzes von Hajós", Acta Math. Acad. Sci. Hung., 17 (1966) 46.

17. R. M. Robinson, Undecidability and nonperiodicity for tilings of the plane, Invent. Math., 12 (1971) 177–209.

18. A. D. Sands, On the factorization of finite abelian groups, Acta Math., 8 (1957) 65–86.

19. ———, On a problem of L. Fuchs, J. Lond. Math. Soc., 37 (1962) 277–284.

20. ———, On the factorizations of finite abelian groups, II, Acta Math., 13 (1962).

21. ———, Factorization of cyclic groups, Proc. Colloq. Abelian Groups, Hungary, Sept. 1963, (1964).

22. E. Wittman, Einfacher Beweis des Hauptsatzes von Hajós-Rédei für elementare Gruppen von Primzahlquadratordnung, Acta Math. Sci Acad. Hung., 20 (1–2) (1969) 227–230.

23. H. Zassenhaus, Modern developments in the theory of numbers, Bull. Amer. Math. Soc., 67 (1961) 426–439.

Tiling by Certain Star Bodies

24. R. Baer, Partitionen endlichen Gruppen, Math. Zeitschr., 75 (1961) 333–372.

25. J. P. Conlan, Tiling space with the aid of the holomorph, J. Comb. Theory, 14 (1973) 167–172.

26. W. Hamaker, Factoring groups and tiling space, Aequationes Math., to appear.

27. M. Herzog and J. Schönheim, Group partition, factorization, and the vector covering problem, Canad. Math. Bull., 15 (1972) 207–214.

28. J. H. van Lint, Coding Theory, Lecture Notes in Mathematics, 201, Springer, 1971.

29. ———, A survey of perfect codes, Rocky Mountain Jour. Math., to appear.

30. G. Losey, Note on a theorem of Zaremba, J. Comb. Theory, 6 (1969) 208–209.

31. J. G. Mauldon, Covering theorems for groups, Quart. J. Math. Oxford (2), 1 (1950) 284–287.

32. S. K. Stein, Factoring by subsets, Pacific J. Math., 22 (1967) 523–541.

33. ———, A symmetric star body that tiles but not as a lattice, Proc. Amer. Math. Soc., 36 (1972) 543–548.

34. O. Taussky and J. Todd, Covering theorems for groups, Ann. Soc. Polonaise de Math., 21 (1948) 303–305.

35. S. K. Zaremba, A covering theorem for abelian groups, J. London Math. Soc., 26 (1951) 71–72.

36. ———, Covering problems concerning abelian groups, J. London Math. Soc., 27 (1952) 242–246.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA, DAVIS, CALIFORNIA 95616.

FOLIATIONS OF 3-MANIFOLDS

MAURICE COHEN

The theory of foliations has its roots in the study of differential equations in the nineteenth century and has recently been a very active area of topology. The modern theory started in 1944 and until 1969 the most striking examples and theorems were all concerning foliations of codimension one on the 3-sphere and other 3-manifolds. The statements of these results of G. Reeb, A. Haefliger, S. Novikov and J. Wood, all of which will be discussed, are very geometric and within reach of the imagination and our usual 3-dimensional intuition (together with a few drawings). The definition of foliation will be vague at first and made gradually precise. We shall begin with a few words about ordinary differential equations in the plane and the statement of the Poincaré-Bendixson theorem, so that it later becomes clear how foliations generalize differential equations and how questions about foliations arise naturally from the study of the qualitative behavior of solutions of ordinary differential equations.

For our purposes, a differential equation in the plane is given by a system

$$\begin{cases} \dfrac{dx}{dt} = f(x,y) \\[2ex] \dfrac{dy}{dt} = g(x,y), \end{cases}$$

where f and g have continuous partial derivatives and are not both zero at the same point. Through each point P in the plane we then have a line L_P with direction numbers

$$\frac{dx}{dt}, \frac{dy}{dt} \ .$$

A solution—or integral—curve of the equation is a curve C, such that for each P on C, L_P is tangent to C at P. The existence and uniqueness theorem for solutions of differential equations then says that maximal integral curves provide a partition of the plane by disjoint lines: a **foliation** of the plane by curves. Suppose now that D is a closed disk in the plane, and that C is an integral curve of the given system which crosses the boundary of D and then stays in the interior of D. The Poincaré-Bendixson theorem then asserts that C spirals towards a closed integral curve Γ in D (Figure 0). Note that the theorem states the existence of a closed integral curve and describes the behavior of nearby integral curves.

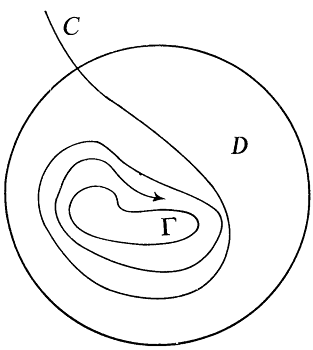


FIG. 0

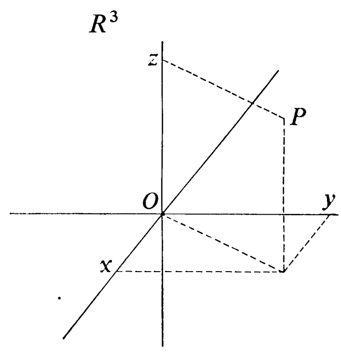


FIG. 1

Let R^3 denote 3-space. In a cartesian coordinate system, a point P in R^3 will have coordinates (x, y, z) and then $R^3 = \{(x,y,z) \mid x, y, z \text{ are real numbers}\}$ (Figure 1). The simplest foliation of R^3 is given by the collection of all planes of equation $z = c$, where the constant c is any real number (Figure 2). This decomposition of R^3 by disjoint surfaces is called the **trivial foliation** of R^3. For each c, the plane $z = c$ is called a **leaf** of the foliation. Notice that each point of R^3 is on a leaf and that the leaves are disjoint.

Consider in the yz-plane the collection of all lines $y = k$, for each $k \geqq 1$, and

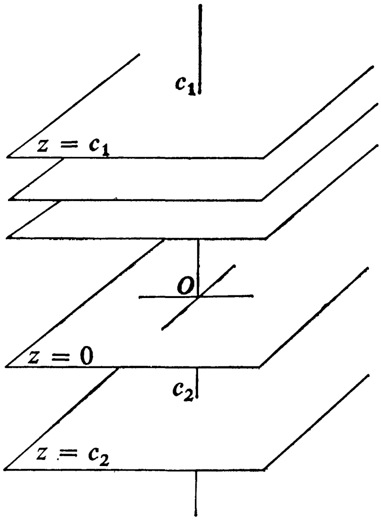


FIG. 2

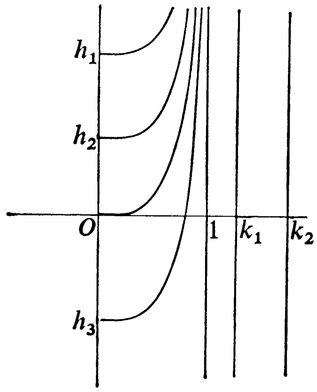


FIG. 3

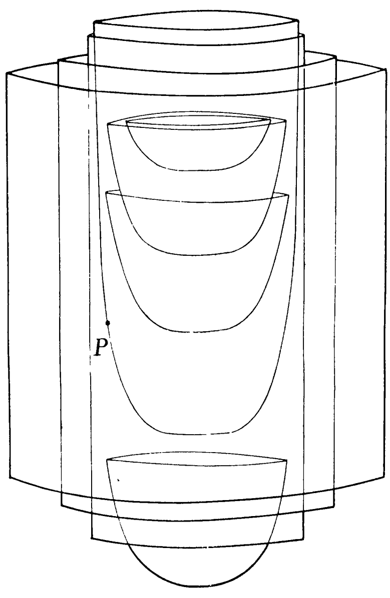


FIG. 4

the collection of all curves of equation

$$
(*) \qquad z = \begin{cases} e^{1/(1-y)} + h, & \text{for } \frac{1}{2} < y < 1 \\ h, & \text{for } 0 \leqq y < \frac{1}{4} \\ \text{the appropriate expression to make } z \text{ infinitely} \\ \text{differentiable as a function of } y, \text{ for } \frac{1}{4} \leqq y \leqq \frac{1}{2} \end{cases}
$$

for each constant h (Figure 3). Notice that the line $y = 1$ is an asymptote for each of the curves given by (*) and that each point in the half plane $y \geqq 0$ is on exactly one curve of either of the two collections defined above. Now rotate the yz-plane about the z-axis. Each curve of each collection generates a surface of revolution in R^3 (Figure 4). We obtain a foliation of R^3. Each leaf of this foliation is either a cylinder of equation $x^2 + y^2 = k^2$, with $k \geqq 1$, or a test-tube like surface inside the cylinder $x^2 + y^2 = 1$, asymptotic to that cylinder.

In the example of Figure 4 let us think of the leaves as being made of rubber. Then, given any point P in R^3, we can cut out a small enough solid ball centered at P, deform the pieces of leaves making up this ball and then stretch them indefinitely transforming the ball into R^3, but this time with the trivial foliation (Figure 5). This is always the case: every foliation locally looks like (in the above sense) the trivial foliation of R^3.

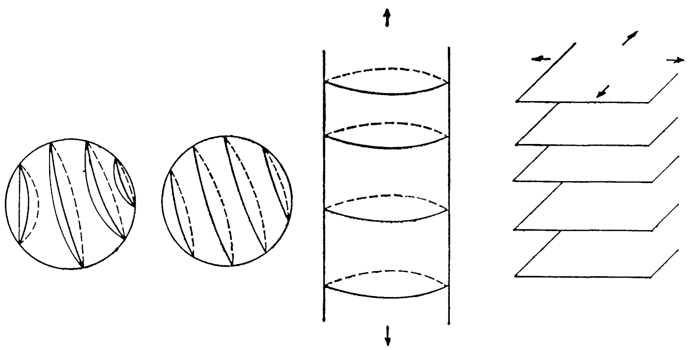


FIG. 5

Each leaf L of a foliation of R^3 is a surface which we require to have a tangent plane at each point. We can then choose at each point P on L a unit vector n_P normal to the leaf. Since each point of R^3 is on exactly one leaf, we have determined a vector field $n = \{n_P \mid P \text{ in } R^3\}$, which is said to be normal to the foliation. We ask that n_P can be chosen to vary smoothly (i.e., differentiably of all orders) with P (Figure 6).

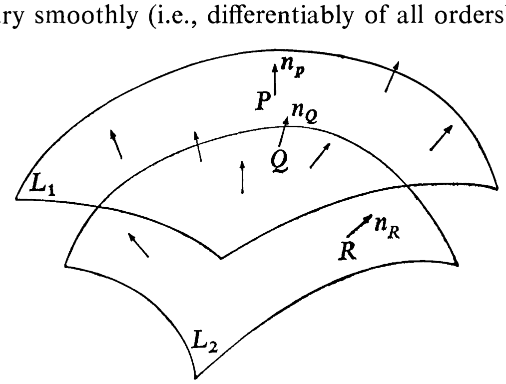
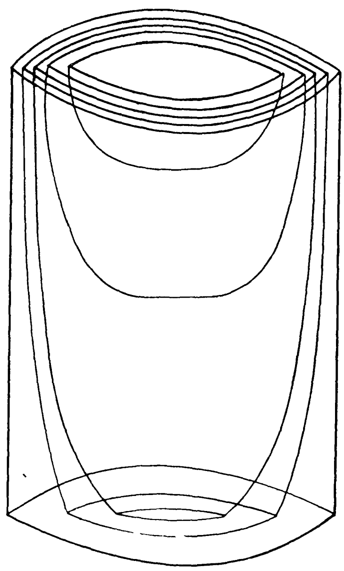
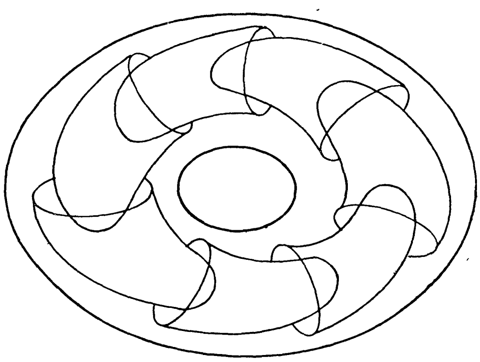


FIG. 6



(i)



(ii)

FIG. 7

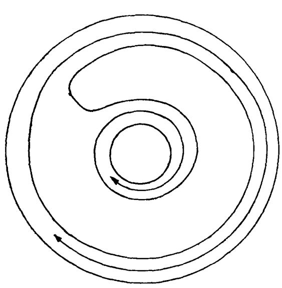


FIG. 8

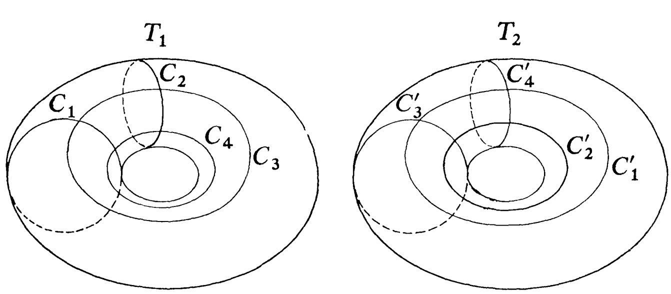


FIG. 9

In other words, if P has coordinates (x,y,z), then componentwise $n_P = (f_1(x,y,z), f_2(x,y,z), f_3(x,y,z))$ can be chosen such that f_1, f_2, f_3 have continuous partial derivatives of all orders. This is asking the foliation to be **smooth**.

The foliation of R^3 in Figure 4 is smooth. We could have obtained a very similar foliation by rotating about the z-axis the curves in the yz-plane given by

$$y = k,\ k \geqq 1$$

and those given by

$$z = \frac{y}{1-y} + h$$

for $0 \leqq y < 1$, and each constant h. The resulting foliation of R^3 (which looks so much like the one in Figure 4, that the reference is again Figure 4) fails to be smooth. The trouble occurs at each point on the cylinder $x^2 + y^2 = 1$: the surfaces inside the cylinder do not approach the cylinder steeply enough. We shall come back to the question of smoothness in connection with Haefliger's theorem.

Consider the foliation of R^3 in Figure 4, discard all the leaves outside the cylinder $x^2 + y^2 = 1$ (keeping the cylinder), cut the cylinder with the planes $z = 0$ and $z = 1$ and retain only the part of it for which $0 \leqq z \leqq 1$. We obtain a solid can, the interior of which is decomposed in disjoint surfaces, these surfaces intersecting the top and the bottom of the can in two sets of concentric circles (Figure 7(i)). Bend the can and glue its top to its bottom, matching concentric circle with concentric circle of same radius. It can be done in a way to make the wall of the can into a smooth empty torus while the surfaces inside the can become smooth surfaces inside the torus (Figure 7(ii)). The result is a foliation of the solid torus called the **Reeb component**. The empty torus is a leaf of this foliation. The other leaves look like snakes winding indefinitely around the torus, eating each other and themselves up and tending towards the empty torus (Figure 7(ii)). Figure 8 shows a section of the torus leaf and of one inside leaf of the Reeb component.

The 3-sphere S^3 is the set of points at distance one from the origin in R^4, 4-dimensional euclidean space; i.e.,

$$S^3 = \{(x,y,z,t) \mid x^2 + y^2 + z^2 + t^2 = 1\}.$$

One can visualize S^3 in many different ways. One possibility is to consider two solid tori T_1, T_2 and to attach T_2 on the outside of T_1 by glueing the empty torus boundary of T_2 to the empty torus boundary of T_1, matching the meridian circles of one to the parallel circles of the other and vice-versa. The resulting space is S^3. Figure 9 shows two solid tori with some meridian and parallel circles drawn on their boundaries; the glueing is done so that C_i is matched with C_i' for $i = 1,2,3,4$.

Another way to picture S^3 is to think of R^3 with a point adjoined at infinity. S^3 minus a point is R^3 in very much the same manner as S^2 minus a point is topo-

logically R^2: think of stereographic projection or stretching a punctured rubber balloon as in Figure 10. Figure 11 shows a combination of the two ways of looking

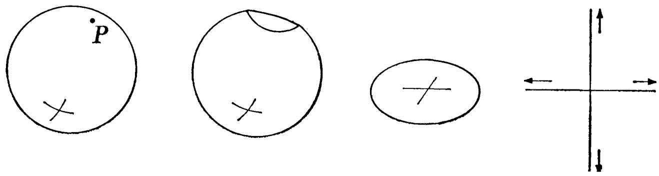


FIG. 10

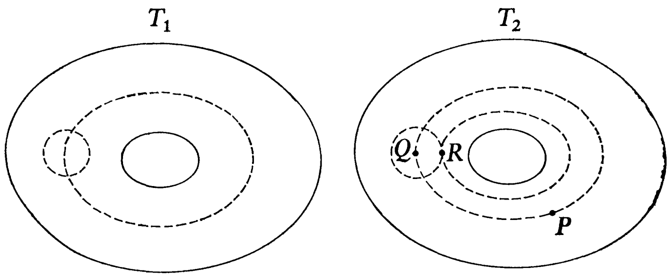


FIG. 11 (i)

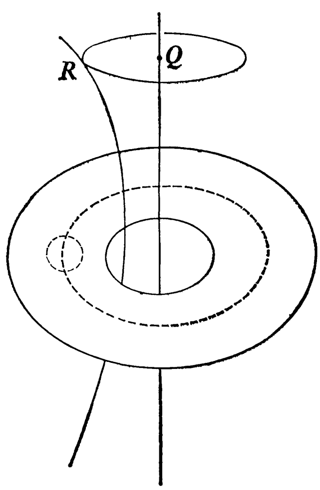


FIG. 11 (ii)

at S^3: in (i) we have T_1 and T_2 before glueing while in (ii) we have S^3 minus the point P of T_2 which has been thrown to infinity after the glueing.

If we foliate T_1 and T_2 each as the Reeb component and then attach T_2 to T_1 as above, we obtain a foliation of the resulting S^3 called the **Reeb foliation** of S^3. One leaf of this foliation is an empty torus. All other leaves look like snakes winding

one way on one side of the empty torus leaf, perpendicularly on the other side of it. Picturing S^3 as in Figure 11, Figure 12 shows the empty torus leaf and parts of two leaves, one on either side of it. Reeb described this foliation in the 1940's.

The Reeb foliation of S^3 is smooth. By that we mean that if we remove any one point of S^3, we obtain R^3 and a foliation of R^3 which is smooth, i.e., it is possible to choose a unit normal vector field to it with components $f_1(x, y, z)$, $f_2(x, y, z)$, $f_3(x, y, z)$ which have continuous partial derivatives of all orders. Can a foliation of S^3 be even smoother, in the sense that the functions f_1, f_2 and f_3 are analytic, i.e., are equal to their Taylor series? (A function with continuous partial derivatives may have a Taylor series which diverges or converges to a different function.) Such a foliation is said to be **analytic.** For example, the trivial foliation of R^3 is analytic and the Reeb foliation of S^3 is not analytic. In 1958 A. Haefliger proved in his thesis that no foliation of S^3 is analytic. Let us introduce the notion of holonomy of a leaf and indicate how it is involved in Haefliger's proof.

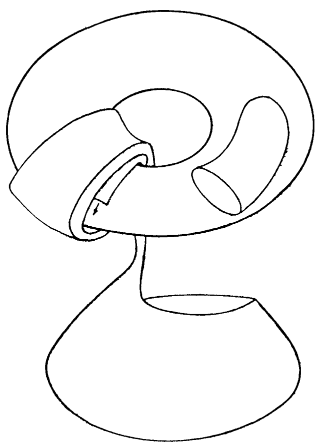


FIG. 12

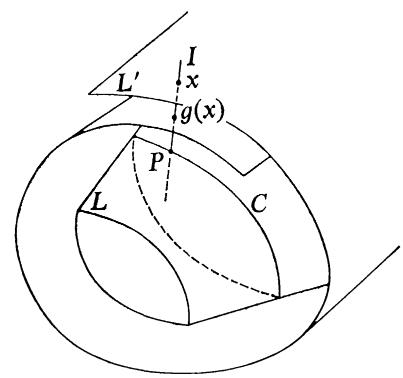


FIG. 13

Let L be a fixed leaf of a foliation and let C be a closed curve on L. Pick a point P on C and let I be a small segment of midpoint P, perpendicular to L at P. Imagine moving P around C returning it to its original position in one unit of time, while keeping I perpendicular to L. At time $t = 0$ a point x of I is on some leaf L' of the foliation and as P describes C, the point of intersection of L' with the moving segment I varies along the line carrying I. Let $g(x)$ be the position of this point at time $t = 1$ (Figure 13). Let I' be a small enough subsegment of I such that if x is on I', then $g(x)$ is on I (rather than on the line carrying I or at infinity on that line). Representing I' as the interval of real numbers $(-1, 1)$ and I as some larger subset of the set of real numbers R, this procedure defines a function $g: (-1, 1) \to R$. The collection of all these functions for a fixed leaf L (and various curves C), modulo an appropriate equivalence relation, is called the **holonomy** of the leaf L.

A little thought will convince the reader that the holonomy of L describes the behavior of leaves near L. It also turns out that the functions in the holonomy of any leaf of a foliation are just as smooth as the foliation. For example, if the foliation is analytic then each function g is analytic. In the case of the trivial foliation of R^3, it is geometrically clear (Figure 14) that the holonomy of any leaf consists of only the identity function $g(x) = x$, which is indeed analytic.

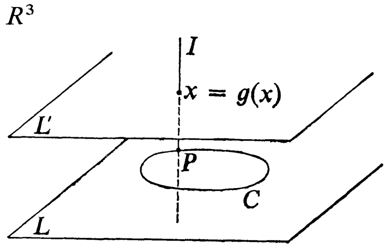


FIG. 14

Let us return to S^3 and suppose we have an arbitrary foliation of it. Haefliger proceeds by locating a leaf L of the foliation, a closed curve C on L and a point P on C such that the associated function $g: (-1,1) \to R$ is given by $g(x) = x$ for $x \leqq 0$ and such that $g(x) < x$ for $x > 0$. In other words, g is the identity function for $x < 0$ and is different from the identity function for $x > 0$. This behavior is impossible for an analytic function. Hence g is not analytic and the given foliation must not have been analytic. The procedure used to locate the leaf L relies heavily on the qualitative theory of differential equations in the plane and in particular the Poincaré-Bendixson theorem.

In 1964 S. Novikov proved that any foliation of S^3 contains a Reeb component. In other words, no matter how you foliate S^3 you must use a leaf which is an empty torus, which may be knotted, and inside this torus the leaves are self-eating snakes. The difficult proof uses ideas involved in the proof of Haefliger's theorem and the concept of a closed transversal to a foliation. A **closed transversal** to a foliation is a smooth closed curve which is never tangent to, nor lies in, a leaf of the foliation: it must intersect transversally each leaf it touches (but does not necessarily meet every leaf). For example, there is no closed transversal to the trivial foliation of R^3: Any smooth closed curve C in R^3 contains a point P whose height, say z_0, above the xy-plane is maximal. The curve C, being smooth, must be tangent to the plane $z = z_0$ which is a leaf of the foliation and hence C is not a closed transversal (Figure 15). On the other hand, the curve running down the middle of a solid torus is a closed transversal to the Reeb component (Figure 16). To prove his theorem, given any foliation of S^3, Novikov exhibits a leaf L such that no closed transversal of the foliation intersects L. He then proceeds to show that L has to be an empty torus which is the boundary leaf of a Reeb component.

Actually, Novikov proves the same result not just for S^3 but for a class of manifolds including any simply connected compact 3-manifold. A **3-manifold** is a subset

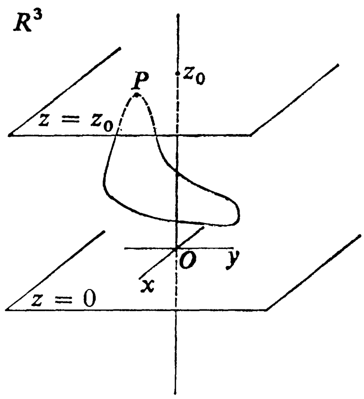


FIG. 15

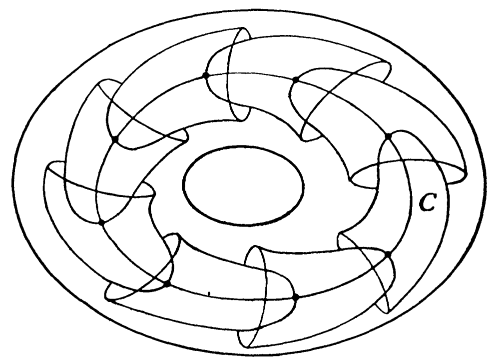


FIG. 16

of some euclidean space R^k, which around each of its points looks topologically and smoothly like R^3. A 3-manifold is **simply connected** if any closed curve contained in it can be continuously deformed to a point while remaining in the manifold. A 3-manifold is **compact** if it is a bounded subset of R^k (i.e., contained in some sphere centered at the origin of R^k) and if moreover, whenever a sequence of its points approaches a limit in R^k, that limit has to be a point of the manifold. For example, S^3 is a simply connected compact manifold, R^3 is a simply connected manifold which is not compact and the interior of a solid torus (i.e., the space occupied by the air in a tire) is a 3-manifold which fails to be simply-connected. Let us mention in this connection the famous unsolved Poincaré conjecture, which states that any compact simply connected 3-manifold is topologically equivalent, i.e., homeomorphic, to S^3.

Given a 3-manifold M, a **smooth foliation of** M is a decomposition of M by surfaces, called the **leaves** of the foliation, such that, in each neighborhood of each point of M which "looks like" R^3, the decomposition gives rise to a smooth foliation of R^3.

Lickorish, Novikov, Zeischang and Wood have proved that any compact 3-manifold M can be foliated. The theorem of Novikov then states that when M is simply connected, a Reeb component has to be used.

Let us return to S^3, considering it as the sphere of unit radius centered at the origin O in R^4. For each point P on S^3 let T_P be the three dimensional hyperplane in R^4 perpendicular to the radius OP at P. T_P is called the **tangent plane** to S^3 at P. In each T_P pick a unit vector v_P in such a way that v_P varies smoothly with P: that is, if P in S^3 has coordinates (x, y, z, t) with $x^2 + y^2 + z^2 + t^2 = 1$ and if componentwise

$$v_P = (f_1(x, y, z, t), f_2(x, y, z, t), f_3(x, y, z, t), f_4(x, y, z, t)),$$

then f_1, f_2, f_3, f_4 have continuous partial derivatives of all orders. The collection v

of all vectors v_P, $v = \{v_P\}$, is then called a **smooth unit tangent vector field** to S^3. Figure 17 depicts a two dimensional representation. It is well known that a two dimensional analogue, that is, a smooth unit vector field on S^2, (considered as the sphere of unit radius at the origin in R^3), does not exist: one cannot comb the hair on a billiard ball.

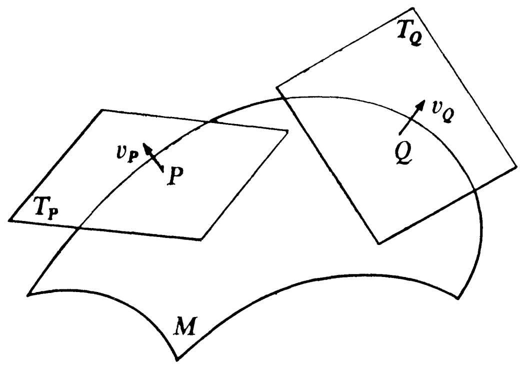


FIG. 17

Given a smooth foliation of S^3 and a smooth unit tangent vector field v on S^3, we shall say that v is tangent to the foliation at a point P of S^3 if v_P is tangent to the leaf of the foliation on which P is. If there is no point of S^3 at which v is tangent to the foliation, we say that v is transverse to the foliation. For example, if $n = \{n_P\}$ a vector field normal to a foliation then n is transverse to that same foliation.

If the vector field v is given, there does not necessarily exist a foliation to which v is transverse. One example is constructed as follows. Define a function H which assigns to each point P of coordinates (x, y, z, t) of S^3 (i.e., $x^2 + y^2 + z^2 + t^2 = 1$) the point $Q = H(P)$ of S^2 of coordinates $(2(xz + yt), 2(yz - xt), x^2 + y^2 - z^2 - t^2)$. It is easy to verify that the sum of the squares of the coordinates of Q equals 1 and therefore Q is on S^2, the sphere of radius one at the origin in R^3. This function is called the Hopf map and has the following property: the points of S^3 which are sent by H to the same point of S^2 all lie on a great circle of S^3 (exercise for the reader: draw a picture!). For each P on S^3 we let v_P be a unit tangent vector at P to the great circle through P on which H is constant. The function H being smooth, one can choose (at each P there are two possible choices) v_P to vary smoothly with P. The smooth unit tangent vector field v to S^3 thus obtained is not transverse to a foliation.

Even though an arbitrary smooth unit tangent vector field to S^3 need not be transverse to a foliation, it has been proved by J. Wood, in 1968, that it can be deformed into one which is transverse to a foliation. Let us define more precisely what we mean by deformation. Let $v = \{v_P\}$ be the vector field; v_P is a vector of origin P in the tangent plane T_P to S^3 at the point P of S^3. Think of v_P as the position at time $t = 0$ of a vector which rotates about P in T_P. Denote by

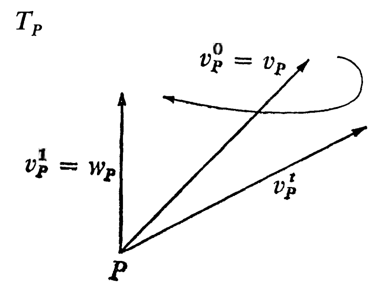


FIG. 18

v_P^t the position of the vector at time t. To say v_P^t rotates means that its length is constant in t. Put $w_P = v_P^1$. We have a vector field v^t at each time t with $v^0 = v$ and $v^1 = w$. If P has coordinates (x, y, z, u) with $x^2 + y^2 + z^2 + u^2 = 1$ and componentwise

$$v_P^t = (f_1^t(x, y, z, u), f_2^t(x, y, z, u), f_3^t(x, y, z, u), f_4^t(x, y, z, u)),$$

where f_1^t, f_2^t, f_3^t and f_4^t have continuous partial derivatives of all orders with respect to x, y, z, u and t, then we say that v^t is a **smooth deformation or smooth homotopy** of v into w. Wood's theorem states that any vector field tangent to S^3 is smoothly homotopic to one which is transverse to a foliation (Wood states and proves the theorem for arbitrary compact 3-manifolds).

Since 1968 the subject of foliations of 3-manifolds has flourished. The knowledge about foliations has increased to the point where one uses algebraic topology and its huge machinery to study questions of existence and classification of foliations. The field is still young enough though to have retained all its geometric appeal. These new developments are surveyed in the article of H. B. Lawson listed in the references.

References

1. A. Haefliger, Structures feuilletées et cohomologie à valeur dans un faisceau de groupoïdes, Comm. Math. Helv., 32 (1958) 248–329.

2. A. Haefliger, Variétés feuilletées, Ann. Scuola Norm. Sup. Pisa (3), 16 (1962) 367–397.

3. H. B. Lawson, Jr., Foliations, to appear in Bulletin of the American Math. Soc.

4. S. P. Novikov, The Topology of Foliations (in Russian), Trudy Moskov. Mat. Ob., 14 (1965) 248–278.

5. G. Reeb, Sur certaines propriétés topologiques des variétés feuilletées, Act. Sci. Ind., no. 1183, Hermann, Paris, 1952.

6. J. Wood, Foliations on 3-manifolds, Ann. of Math., 89 (1969) 336–358.

DEPARTMENT OF MATHEMATICS, SIR GEORGE WILLIAMS UNIVERSITY, MONTREAL 107, QUEBEC, CANADA.

OPTIMAL VELOCITY IN A RACE

JOSEPH B. KELLER

1. Formulation. We wish to determine how a runner should vary his speed $v(t)$ during a race of distance D in order to run it in the shortest time. Previously we formulated this problem in the following way [1]:

The time T to run the race is related to $v(t)$ and D by

$$D = \int_0^T v(t)\,dt. \tag{1.1}$$

The velocity v satisfies the equation of motion

$$\frac{dv}{dt} + \frac{v}{\tau} = f(t). \tag{1.2}$$

Here v/τ is a resistive force per unit mass and τ is a given constant, while $f(t)$ is the propulsive force per unit mass. Initially

$$v(0) = 0. \tag{1.3}$$

The force $f(t)$ is controlled by the runner, but it cannot exceed the maximum value F,

$$f(t) \leqq F. \tag{1.4}$$

The force also affects the quantity $E(t)$ of available oxygen in the muscles per unit mass, since oxygen is consumed in reactions which release the energy used in running. Energy is used at the rate fv, and we assume that oxygen is supplied by breathing and circulation at the rate σ in excess of that supplied in the non-running state. Then the oxygen balance equation is

$$\frac{dE}{dt} = \sigma - fv. \tag{1.5}$$

In (1.5) and elsewhere we measure oxygen in units of the amount of energy it would yield in a reaction. Initially

$$E(0) = E_0. \tag{1.6}$$

Since $E(t)$ can never be negative, we have

$$E(t) \geqq 0. \tag{1.7}$$

The problem now is this:

PROBLEM. *Find $v(t)$, $f(t)$, and $E(t)$ satisfying* (1.2)–(1.7) *so that T, defined by* (1.1) *is minimized. The four physiological constants τ, F, σ, and E_0 and the distance D are given.*

This is a problem in the calculus of variations with differential equations and inequalities as constraints. Since there are two differential equations and three unknown functions, it is called a problem of optimal control theory. The force $f(t)$ may be thought of as the control variable, and it is at the disposal of the runner.

We shall reformulate this problem in Section 2 and solve it in Section 3. Then we shall compare the predictions of the theory with the world records at distances D from 50 yards to 10,000 meters.

2. Reformulation. It is convenient to eliminate both $f(t)$ and $E(t)$ from the problem by expressing them in terms of $v(t)$. To do so we first note that (1.2) gives f directly in terms of v. When we use (1.2) to eliminate f from (1.4), we obtain

$$\frac{dv}{dt} + \frac{v}{\tau} \leqq F. \tag{2.1}$$

Next we use (1.2) to eliminate f from (1.5). Then we can integrate the resulting equation and utilize (1.6) to get

$$E(t) = E_0 + \sigma t - \frac{v^2(t)}{2} - \frac{1}{\tau} \int_0^t v^2(s)\, ds. \tag{2.2}$$

This equation gives E in terms of v, so we can use it to eliminate E from (1.7) with the result

$$E_0 + \sigma t - \frac{v^2(t)}{2} - \frac{1}{\tau} \int_0^t v^2(s)\, ds \geqq 0. \tag{2.3}$$

Now that f and E have been eliminated, the problem is to find $v(t)$ satisfying (1.3), (2.1), and (2.3) so that T, defined by (1.1), is minimized. Instead of minimizing T for given D, we shall consider the equivalent problem of maximizing D with T given. It is easy to see that these two problems yield the same relation between T and D.

3. Solution. Since $v(0) = 0$, the rate of doing work fv is zero initially no matter what value f has. Therefore initially f can be as large as possible without energy consumption, so we may set $f(0) = F$. We assume that $f(t)$ remains equal to F during some initial interval $0 \leqq t \leqq t_1$ where t_1 is to be determined. If it turns out that $t_1 = 0$ then this assumption is vacuous. Upon using this assumption in (2.1) we obtain

$$\frac{dv}{dt} + \frac{v}{\tau} = F, \qquad 0 \leqq t \leqq t_1. \tag{3.1}$$

The solution of (3.1) satisfying (1.3) is

$$v(t) = F\tau(1 - e^{-t/\tau}), \qquad 0 \leqq t \leqq t_1. \tag{3.2}$$

Upon using (3.2) in (2.3) we obtain

$$E_0 + \sigma t - F^2\tau^2 \left(\frac{t}{\tau} + e^{-t/\tau} - 1\right) \geqq 0, \qquad 0 \leqq t \leqq t_1. \tag{3.3}$$

If $\sigma \geqq F^2\tau$, (3.3) is satisfied for all $t \geqq 0$ and then (3.2) is the optimal velocity for all t. However, this is unrealistic, so we shall ignore it and consider only the case $\sigma < F^2\tau$. Then (3.3) holds only for $0 \leqq t \leqq T_c$ where T_c is the unique positive root of (3.3) with the equality holding.

Now if $T \leqq T_c$ we set $t_1 = T$ and (3.2) yields the optimal velocity throughout the race. We shall call such races "short sprints" or "dashes." For them (3.2) and (1.1) yield

$$D = F\tau^2\left(\frac{T}{\tau} + e^{-T/\tau} - 1\right), \qquad 0 \leqq T \leqq T_c. \tag{3.4}$$

The length of the longest dash we shall call D_c, which is given by (3.4) with $T = T_c$.

If $T > T_c$ then we must have $t_1 < T$, and it remains to find t_1 and the function $v(t)$ for $t_1 < t \leqq T$. It is clear that the solution must satisfy $E(T) = 0$, i.e., the oxygen supply must be used up at the end of the race. We shall assume that $E(t) = 0$ throughout an interval $t_2 \leqq t \leqq T$ at the end of the race, the length of which is to be determined. If it turns out that $t_2 = T$, this assumption is also vacuous. Thus we suppose that for some $t_2 \geqq t_1$ we have

$$E(t) = 0, \qquad t_2 \leqq t \leqq T. \tag{3.5}$$

We now use (2.2) for E in (3.5) and differentiate with respect to t to obtain

$$\sigma - \frac{1}{2}\frac{dv^2}{dt} - \frac{v^2}{\tau} = 0, \qquad t_2 \leqq t \leqq T. \tag{3.6}$$

This is an ordinary differential equation for v^2 which has the solution

$$v^2(t) = \sigma\tau + [v^2(t_2) - \sigma\tau]\, e^{2(t_2 - t)/\tau}, \qquad t_2 \leqq t \leqq T. \tag{3.7}$$

The value $v(t_2)$ has not yet been determined, nor has condition (3.5) been satisfied. Instead we have set $dE/dt = 0$, so (3.5) will hold if we make $E(t_2) = 0$.

We have now found $v(t)$ to be given by (3.2) in the interval $0 \leqq t \leqq t_1$ and by (3.7) in the interval $t_2 \leqq t \leqq T$. To find v in the remaining interval $t_1 \leqq t \leqq t_2$ we use these results in (1.1) to write D in the form

$$\begin{aligned} D = &\int_0^{t_1} F\tau(1 - e^{-t/\tau})\,dt + \int_{t_1}^{t_2} v(t)\,dt \\ &+ \int_{t_2}^{T} \{\sigma\tau + [v^2(t_2) - \sigma\tau]\, e^{2(t_2 - t)/\tau}\}^{\frac{1}{2}} dt. \end{aligned} \tag{3.8}$$

We must choose v to maximize D subject to the constraint $E(t_2) = 0$, which was mentioned above. Therefore we consider the functional $D + \lambda E(t_2)/2$ where λ is a Lagrange multiplier, $E(t)$ is given by (2.2) and D is given by (3.8). We equate to zero the first variation of this functional and obtain

$$(3.9)\quad \int_{t_1}^{t_2} \delta v(t)\,dt + v(t_2)\,\delta v(t_2) \int_{t_2}^{T} \{\sigma\tau + [v^2(t_2) - \sigma\tau] \cdot e^{2(t_2-t)/\tau}\}^{-\frac{1}{2}} e^{2(t_2-t)/\tau} dt$$
$$- \frac{\lambda}{2} v(t_2)\,\delta v(t_2) - \frac{\lambda}{2\tau} \int_{t_1}^{t_2} 2v(t)\,\delta v(t)\,dt = 0.$$

In order that (3.9) hold for arbitrary $\delta v(t)$, the coefficient of δv in the integrand must vanish and so must the coefficient of $\delta v(t_2)$. This yields the two equations

$$(3.10)\qquad v(t) = \tau/\lambda, \qquad t_1 \leqq t \leqq t_2,$$

$$(3.11)\qquad \lambda = 2 \int_{t_2}^{T} \{\sigma\tau + [v^2(t_2) - \sigma\tau] e^{2(t_2-t)/\tau}\}^{-\frac{1}{2}} e^{2(t_2-t)/\tau} dt.$$

From (3.10) we obtain the interesting result that $v(t)$ is constant from t_1 to t_2.

Now by equating $v(t_1)$ given by (3.2) to $v(t_1)$ given by (3.10) we obtain

$$(3.12)\qquad F(1 - e^{-t_1/\tau}) = 1/\lambda.$$

Next we rewrite (3.11), using (3.10) for $v(t_2)$ and evaluating the integral. This yields

$$(3.13)\qquad \lambda = 2\left(\sigma - \frac{\tau}{\lambda^2}\right)^{-1} \left[\left\{\sigma\tau + \left(\frac{\tau^2}{\lambda^2} - \sigma\tau\right) e^{2(t_2-T)/\tau}\right\}^{\frac{1}{2}} - \frac{\tau}{\lambda}\right].$$

Then we set $E(t_2) = 0$, using (2.2) for E and in it using (3.2) and (3.10) for $v(t)$. This gives

$$(3.14)\qquad E_0 + \sigma t_2 - \frac{\tau^2}{2\lambda^2} - F^2\tau\left(-\frac{3\tau}{2} + t_1 + 2\tau e^{-t_1/\tau} - \frac{\tau}{2} e^{-2t_1/\tau}\right) - \frac{\tau}{\lambda^2}(t_2 - t_1) = 0.$$

The three equations (3.12)–(3.14) determine t_1, t_2, and λ, provided that $t_1 \leqq t_2 \leqq T$. If the equations yield $t_1 > t_2$, then the maximum occurs at the endpoint $t_1 = t_2$ and (3.14) shows that $t_1 = t_2 = T_c$.

To find when the latter case occurs, we view (3.13) as a biquadratic for λ and solve it to obtain the four roots $\pm(\tau/\sigma)^{\frac{1}{2}}$ and $\pm(\tau/\sigma)^{\frac{1}{2}}[1 - 4\exp 2(t_2 - T)/\tau]^{\frac{1}{2}}$. The negative roots are not consistent with (3.12), and $(\tau/\sigma)^{\frac{1}{2}}$ violates (3.11), so the only admissible root is

$$(3.15)\qquad \lambda = (\tau/\sigma)^{\frac{1}{2}}[1 - 4e^{-2(T-t_2)/\tau}]^{\frac{1}{2}}.$$

We now use (3.15) for λ in (3.12), set $t_1 = t_2 = T_c$, and solve the resulting equation for T. The solution is T^* defined by

$$(3.16)\qquad T^* = T_c + \tau\{\log 2 - \tfrac{1}{2}\log[1 - \sigma F^{-2}\tau^{-1}(1 - e^{-T_c/\tau})^{-2}]\}.$$

For $T \geqq T^*$, the equations (3.12), (3.13) or (3.15), and (3.14) yield t_1, t_2 and λ. For $T_c \leqq T \leqq T^*$ we have instead $t_1 = t_2 = T_c$ and (3.12) yields λ.

Once t_1, t_2, and λ have been found, $v(t)$ is given by the three expressions (3.2), (3.7), and (3.10). When these expressions for $v(t)$ are used in (3.8), it yields the following result for D:

$$
\begin{aligned}
D = {} & F\tau^2\left(\frac{t_1}{\tau} + e^{-t_1/\tau} - 1\right) + \frac{\tau(t_2 - t_1)}{\lambda} \\
& + \tau(\sigma\tau)^{\frac{1}{2}}\Bigg[-\frac{1}{\lambda}\left(\frac{\tau}{\sigma}\right)^{\frac{1}{2}} - \tanh^{-1}\frac{1}{\lambda}\left(\frac{\tau}{\sigma}\right)^{\frac{1}{2}} \\
& - \left\{1 + \left(\frac{\tau}{\lambda^2\sigma} - 1\right)e^{-(2/\tau)(T-t_2)}\right\}^{\frac{1}{2}} \\
& + \tanh^{-1}\left\{1 + \left(\frac{\tau}{\lambda^2\sigma} - 1\right)e^{-(2/\tau)(T-t_2)}\right\}^{\frac{1}{2}}\Bigg], \qquad T \geqq T_c.
\end{aligned} \tag{3.17}
$$

For $T \leqq T_c$, D is given by (3.4).

In obtaining the optimal solution $v(t)$, we assumed that the inequality (1.4) is an equality in the initial interval $0 \leqq t \leqq t_1$. We also assumed that the inequality (1.7) is an equality in the final interval $t_2 \leqq t \leqq T$. These two assumptions were based on physical reasoning. They could have been avoided by using the following fact from the general theory of optimal control, which is nearly obvious: An optimal solution consists of a finite number of arcs each satisfying one or more of the equalities in the constraints, together with a finite number of arcs satisfying none of the equalities. To use this fact we would have had to consider an arbitrary number of intervals, instead of just three. Then we would have had to show that the optimal solution is the one with just three intervals located as we assumed them to be. Instead of that we could try to show directly that our solution yields the minimum value of T. Since we shall do neither of these two things, we have shown only that our solution yields a stationary value of T rather than a minimum.

4. Determination of the constants from the world records. The four constants τ, F, σ, and E_0 can be found by comparing the predictions of the theory with the world records. To find them we consider first the results for 50 yds., 50 m., 60 yds., 60m., 100 yds., 100m., 200 yds., and 200 m., which we assume to be dashes. Then T is related to D by (3.4), which involves only τ and F. We choose, τ and F to minimize the sum of the squares for these eight dashes of the relative errors,

$$\Sigma(T_{\text{record}} - T_{\text{calculated}})^2 / T^2_{\text{record}}.$$

This minimization is carried out on a computer by varying τ and F and searching for the minimum. Once τ and F are found in this way, we determine σ and E_0 to minimize the sum of the squares of the relative errors for 14 records at distances from 400 m. to 10,000 m. In doing so we calculate T from (3.17) after solving (3.12)–(3.14) for t_1, t_2, and λ. Finally we calculate T_c from (3.3) and use it in (3.4) to get $D_c = 291$ m.

Since D_c is greater than 200 m. and less than 400 m., our assumption that the eight shortest distance races are dashes is confirmed and the calculation of the constants is consistent. The values of the constants and the calculated values of T, t_1, and $T-t_2$ are given in [**1**].

In Figure 1, a curve of the calculated value of the average velocity D/T is shown for $D \leqq 2000$ m., together with points based on the world records.

It is to be noted that the theory yields no final "kick," which runners often use. Instead it yields a slowing down in the last one or two seconds—a negative kick—which can be understood by thinking about driving a car a fixed distance in the shortest time with a limited amount of fuel. Primarily the theory confirms the accepted view that for distances greater than one half mile, a constant speed is best, and it refines that view by specifying the best initial and final velocity variations.

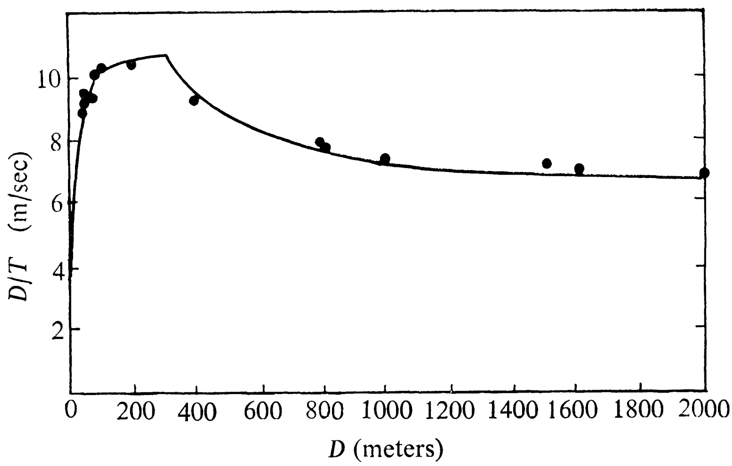


FIG. 1. The average speed D/T for a race of distance D. Points represent the world records and the curve is based upon the theory. For $D <$ 291 meters, D and T are related by (3.4) while for $D >$ 291 meters they are related by (3.17). The constants are those determined by least squares.

A simple energy balance theory to explain the world records was proposed by A. V. Hill in 1928, (see Lloyd [**2**]). His theory is simply that $v(t)$ is constant, and its value is determined by setting $E(T) = 0$. We shall show now that for $D \gg D_c$, our theory yields the same result for D/T as his does. In this case we can replace t_2 by T and t_1 by 0 with an error that tends to zero as D tends to infinity, and we can neglect the kinetic energy term $F^2\tau^2(t_1/\tau + e^{-t_1/\tau} - 1)$ compared to the other terms in (3.14). Then (3.14) becomes $E_0 + \sigma T = v^2(t_1)T/t$, which is just Hill's equation for the case of our resistive force $-v/\tau$. Solving this equation for $v(t_1)$ and noting that $v(t_1) = D/T$ when the end intervals are neglected, we obtain

$$\frac{D}{T} \sim \left(\frac{\tau E_0}{T} + \sigma\tau\right)^{\frac{1}{2}}, \qquad D \gg D_c. \tag{4.1}$$

This is just Hill's result when the power usage is v^2/τ, which it is in our case, (see Lloyd [2], eq. (5)). From (4.1) we see that the square of the average speed is a linear function of $1/T$. On the other hand, for the short sprints, which are not covered by Hill's theory, T/τ is large so we can omit $e^{-T/\tau}$ from (3.4). Then (3.4) yields D as a linear function of T, which is in fair agreement with the data.

This work was supported in part by NSF Grant No. NSF–GP–32996 $x2$.

References

1. J. B. Keller, A theory of competitive running, Physics Today, 26 (1973) 42–47.
2. B. B. Lloyd, The energetics of running: an analysis of world records, Advancement of Science, 22 (1966) 515–530.

COURANT INSTITUTE OF MATHEMATICAL SCIENCES, NEW YORK UNIVERSITY, 251 MERCER ST. NEW YORK, N. Y. 10012.

AN ELEMENTARY PROOF OF KOLMOGOROV'S THEOREM

A. S. CAVARETTA, JR.

Recently in this MONTHLY I. J. Schoenberg [2] has written a detailed and elementary account of the Landau problem concerning sharp inequalities between the supremum norms of derivatives. Professor Schoenberg there suggests a method of proof which offers many new insights; in particular, he presents an elegant discussion of the uniqueness of the extremizing functions and an extension of the problem to complex valued functions. Consistent with his purpose of presenting material accessible to, say, a calculus student, Professor Schoenberg proves only the theorems involving low order derivatives; the method he has developed, though completely general, requires appropriate differentiation formulae and, when high order derivatives are involved, these formulae become quite complicated.

The present paper is written in the same spirit as Professor Schoenberg's and indeed can be considered as an appendix to his paper. Our concern is the Landau problem on $(-\infty, \infty)$, which was first completely solved by Kolmogorov. We prove the theorem for all values of n in an elementary way, using only Rolle's theorem and the Leibniz formula for differentiation of a product. The approach is first to prove the result for functions having an integral period; the general result follows from the special case by an elementary approximation argument.

1. The Euler spline functions and statement of Kolmogorov's theorem. In this section we review briefly the background of our problem as it is developed in Professor Schoenberg's paper. Throughout we use the notations set forth in his paper, and for details concerning this section the reader should consult the earlier paper.

Let $n \geqq 2$. We consider the class of real valued functions $f(x)$ defined on $\mathbb{R}$ which are bounded and have a bounded nth derivative $f^{(n)}(x)$. More precisely, we assume that

$$f(x) \in C^{(n-1)}(\mathbb{R}), \tag{1.1}$$

$$f^{(n-1)} \text{ is piecewise continuously differentiable for all real } x, \tag{1.2}$$

$$\text{both } f(x) \text{ and } f^{(n)}(x) \text{ are bounded.} \tag{1.3}$$

We define the norm of this class by $\|f\| = \sup_{x \in R} |f(x)|$.

Among such functions there is one which is of particular importance and has appeared in connection with several different problems. It is the Euler spline $\mathscr{E}_n(x)$. Schoenberg's paper contains an elementary construction of $\mathscr{E}_n(x)$ for each n, and so we only recall those properties relevant to our problem. In particular, $\mathscr{E}_n(x)$ is characterized by the following properties:

$$\mathscr{E}_n(\nu) = (-1)^\nu \quad \text{for all integers } \nu, \tag{1.4}$$

$$\|\mathscr{E}_n\| = 1, \tag{1.5}$$

$$|\mathscr{E}_n^{(n)}(x)| \text{ is constant,} \tag{1.6}$$

and

$$\begin{cases} \text{for } n \text{ even, } \mathscr{E}_n^{(n)}(x) \text{ changes sign precisely at the half integer points } \nu + \frac{1}{2}; \\ \text{for } n \text{ odd, } \mathscr{E}_n^{(n)}(x) \text{ changes sign precisely at the integers } \nu. \end{cases} \tag{1.7}$$

In addition, we observe that

$$\mathscr{E}_n(x) \text{ is periodic with period 2} \tag{1.8}$$

and

$$\text{on each interval } \mathscr{E}_n(x) \text{ is strictly monotone.} \tag{1.9}$$

For the most part, these properties have simple geometric interpretations; for example (1.4), (1.5), and (1.9) say, that $\mathscr{E}_n(x)$ equioscillates between the values 1 and -1 at the integers. Following Professor Schoenberg, we set

$$\gamma_{n,\nu} = \|\mathscr{E}_n^{(\nu)}\|, \ (\nu = 0, 1, \cdots, n), \ \gamma_{n,0} = 1. \tag{1.10}$$

We can now state the

THEOREM OF KOLMOGOROV. *If $f(x)$ is such that*

$$\|f\| \leqq 1, \ \|f^{(n)}\| \leqq \gamma_{n,n} \tag{1.11}$$

then

$$\|f^{(\nu)}\| \leqq \gamma_{n,\nu} \quad \textit{for } \nu = 1, 2, \cdots, n-1. \tag{1.12}$$

The constants $\gamma_{n,\nu}$ are *best* constants because the Euler spline $\mathscr{E}_n(x)$ satisfies (1.11) and furnishes the equality sign in (1.12) simultaneously for all values of ν. For a more general statement of Kolmogorov's theorem let $F(x)$ be chosen in our class and set

$$M_\nu = \|F^{(\nu)}\| \qquad \nu = 0, \cdots, n. \tag{1.13}$$

Then Kolmogorov's theorem can be stated as

$$\begin{aligned} M_\nu &= C_{n,\nu} \cdot M_0^{1-(\nu/n)} M_n^{\nu/n} \text{ where} \\ C_{n,\nu} &= \gamma_{n,\nu} \cdot \gamma_{n,n}^{-\nu/n}, \qquad (0 < \nu < n). \end{aligned} \tag{1.14}$$

That (1.14) follows from (1.12) is immediately seen by setting $f(x) = aF(bx)$ and determining a and b so that $f(x)$ satisfies the conditions

$$\|f\| = 1, \qquad \|f^{(n)}\| = \gamma_{n,n}. \tag{1.15}$$

2. Proof of the theorem for functions having an integral period. Let ν be fixed, $1 \leqq \nu \leqq n-1$. We assume f is given, satisfying (1.11) and having integral period k. Without loss of generality, we assume that k is even. Thus

$$\begin{aligned} f(x+k) &= f(x) \\ \mathscr{E}_n(x+k) &= \mathscr{E}_n(x). \end{aligned} \tag{2.1}$$

So we can think of these functions as defined for x taken modulo k; thus for example, the zeros of these functions are to be counted per period k, and $\mathscr{E}_n(x)$ exhibits precisely k points of equioscillation per period k.

Now define $\alpha > 0$ by the equality

$$\|f^{(\nu)}\| = \alpha\gamma_{n,\nu}. \tag{2.2}$$

We assume that $\alpha > 1$ and derive a contradiction. Choose x_0 and x_1 so that

$$|f^{(\nu)}(x_0)| = \|f^{(\nu)}\| \tag{2.3}$$

and

$$\alpha\mathscr{E}_n^{(\nu)}(x_0 - x_1) = f^{(\nu)}(x_0). \tag{2.4}$$

Finally set

$$h(x) = \mathscr{E}_n(x - x_1) - \frac{1}{\alpha} f(x). \tag{2.5}$$

From the hypothesis (1.11) and $1/\alpha < 1$, $h(x)$ surely has k distinct zeros, due to the

equioscillation of $\mathscr{E}_n(x - x_1)$. Hence by repeated use of Rolle's theorem $h^{(v)}(x)$ also has at least k distinct zeros.

We first consider the case $v < n - 1$. Then by (2.2), (2.3) and (2.4) x_0 is a local extreme point for both $f^{(v)}(x)$ and $\mathscr{E}_n^{(v)}(x - x_1)$, and so

$$f^{(v+1)}(x_0) = \mathscr{E}_n^{(v+1)}(x_0 - x_1) = 0; \tag{2.6}$$

hence $h^{(v)}(x_0) = h^{(v+1)}(x_0) = 0$. So on taking to account the k zeros of $h^{(v)}(x)$ and the double zero of $h^{(v)}(x)$ at x_0, we conclude that $h^{(v+1)}$ has at least $k+1$ distinct zeros. It follows by Rolle's theorem that

$$h^{(n)}(x) = \mathscr{E}_n^{(n)}(x - x_1) - (1/\alpha)f^{(n)}(x) \tag{2.7}$$

must have at least $k + 1$ sign changes. But by (1.11) and $1/\alpha < 1$,

$$\left|\frac{1}{\alpha}f^{(n)}(x)\right| < \left|\mathscr{E}_n^{(n)}(x - x_1)\right| \tag{2.8}$$

and so $h^{(n)}(x)$ has exactly k sign changes. Hence a contradiction and so $\alpha \leqq 1$, which proves the theorem by (2.2).

For the case $v = n - 1$, we observe that by (1.11) and $1/\alpha < 1$, $h(x)$ exhibits k zeros where the function actually changes sign; hence by Rolle's theorem $h^{(n-1)}(x)$ also has k zeros where $h^{(n-1)}(x)$ changes sign. Moreover, from (2.4) $h^{(n-1)}(x_0) = 0$. But x_0 is also a local extreme point of $h^{(n-1)}(x)$ since by (2.2), (2.3), and (2.4) we see that $\mathscr{E}_n^{(n)}(x - x_1)$ must change sign at x_0. Therefore $h^{(n-1)}(x)$ has at least $k + 1$ distinct zeros. Hence $h^{(n)}(x)$ has at least $k + 1$ changes of sign, and this is a contradiction, as before.

3. Proof of the general result. In the previous section we have proved the normalized Kolmogorov theorem under the restriction that f have integral period. We now prove the general result using this special case. Let f be an arbitrary function in our class and as before set

$$M_v = M_v(f) = \|f^{(v)}\|, \qquad v = 0, \cdots, n. \tag{3.1}$$

We assume conditions (1.11) and prove (1.12).

It is easy to see that M_v, $v = 1, \cdots, n-1$, must be finite. While there are many proofs of this, we prefer the following which is in the spirit of our previous arguments. Consider for example M_1. We define the oscillating polynomial

$$p(x) = \alpha(x-1)(x-2)\cdots(x-n) \tag{3.2}$$

where α is chosen so large that

$$\alpha n! > M_n \tag{3.3}$$

and

$$\alpha(\tfrac{1}{2})^n > M_0. \tag{3.4}$$

Define $a < 1$ by

$$p(a) = \begin{cases} + M_0 & n \text{ even} \\ - M_0 & n \text{ odd} \end{cases} \tag{3.5}$$

and $b > n$ by

$$p(b) = M_0. \tag{3.6}$$

Finally set

$$m_1 = \max_{x \in [a,b]} |p'(x)|. \tag{3.7}$$

We claim that $M_1 \leqq m_1$. For if not, there is a point x_0 such that

$$|f'(x_0)| > m_1. \tag{3.8}$$

Now by (3.2) and (3.4), $p(x)$ is oscillating between $+ M_0$ and $- M_0$ and so we can surely find $x_1 \in (a, b)$ such that

$$p(x_1) = f(x_0). \tag{3.9}$$

Translating f by $t = x_1 - x_0$, we have

$$f(x_1 - t) = p(x_1) \tag{3.10}$$

and

$$|f'(x_1 - t)| > |p'(x_1)|. \tag{3.11}$$

Geometric considerations together with (3.10) and (3.11) show that

$$h(x) = p(x) - f(x - t) \tag{3.12}$$

has at least $n + 1$ zeros in $[a, b]$. So by Rolle's theorem $h^{(n)}(x)$ has at least 1 zero which is impossible by (3.3). So our claim is established for M_1, and thus for M_ν, $\nu = 1, \cdots, n - 1$.

Our next goal is to associate with f a periodic function F in such a way that $M_\nu(F)$ will be close to $M_\nu(f)$. To do this, we need the following auxiliary function:

$$g(x) = \begin{cases} 1 & -1 \leqq x \leqq 1 \\ (-1)^n(x-2)^n \sum_{k=0}^{n-1} \binom{n+k-1}{k} (x-1)^k & 1 < x < 2 \\ (-1)^n(x+2)^n \sum_{k=0}^{n-1} \binom{n+k-1}{k} (x+1)^k & -2 < x < -1 \\ 0 & |x| \geqq 2. \end{cases}$$

By construction g is in our class and has bounded derivatives up to order n. Now let

k be a positive integer and define $F_k(x)$ by the relations

$$F_k(x) = f(x)g\left(\frac{x}{k}\right) \qquad -2k \leqq x \leqq 2k \tag{3.14}$$

and

$$F_k(x+4k) = F_k(x) \qquad \text{for all } x. \tag{3.15}$$

Thus $F_k(x)$ has period $4k$ and $F_k(x)$ is in our class since $g(x/k)$ has zeros of multiplicity n at $\pm 2k$.

To compute $M_\nu(F_k)$ we apply the Leibniz formula to the product $f(x)g(x/k)$. According to our first argument in this section and the construction of g, we can choose a constant A such that

$$\begin{aligned} M_\nu(f) < A \qquad \nu = 0, \cdots, n, \\ M_\nu(g) < A \qquad \nu = 0, \cdots, n. \end{aligned} \tag{3.16}$$

Then for any $\nu = 0, \cdots, n$, and $-2k \leqq x < 2k$,

$$\begin{aligned} (F_k(x))^{(\nu)} &= (f(x)g(x/k))^{(\nu)} \\ &= f^{(\nu)}(x)g\left(\frac{x}{k}\right) + \sum_{i=1}^{\nu} \binom{\nu}{i} f^{(\nu-i)}(x)\frac{d^i}{dx^i} g\left(\frac{x}{k}\right). \end{aligned} \tag{3.17}$$

We observe that for each x

$$g(x/k) \to 1 \text{ as } k \to \infty, \tag{3.18}$$

and so

$$f^{(\nu)}(x)g(x/k) \to f^{(\nu)}(x) \text{ as } k \to \infty. \tag{3.19}$$

As for the finite sum appearing in (3.17), we have that

$$\frac{d^i}{dx^i} g\left(\frac{x}{k}\right) = \frac{1}{k^i} g^{(i)}\left(\frac{x}{k}\right), \qquad (i = 1, \cdots, \nu), \tag{3.20}$$

and so

$$\left| \sum_{i=1}^{\nu} \binom{\nu}{i} f^{(\nu-i)}(x) \frac{d^i}{dx^i} g\left(\frac{x}{k}\right) \right| \leqq \frac{1}{k} \cdot 2^\nu \cdot A^2. \tag{3.21}$$

Hence from (3.17), (3.19), and (3.21) we conclude that

$$\lim_{k \to \infty} M_\nu(F_k) = M_\nu(f), \qquad \nu = 0, \cdots, n. \tag{3.22}$$

Now the desired result is immediate. Let $\alpha < 1$ and consider αf. Then

$$\| \alpha f \| \leqq \alpha < 1 \tag{3.23}$$

and

$$\| \alpha f^{(n)} \| \leqq \alpha \gamma_{n,n} < \gamma_{n,n}. \tag{3.24}$$

Now

$$\lim_{k \to \infty} M_\nu(\alpha F_k) = M_\nu(\alpha f), \qquad \nu = 0, 1, \cdots, n \tag{3.25}$$

and hence, for sufficiently large k, we have

$$M_0(\alpha F_k) \leqq 1 \tag{3.26}$$

and

$$M_n(\alpha F_k) \leqq \gamma_{n,n}. \tag{3.27}$$

So by section 2

$$M_\nu(\alpha F_k) \leqq \gamma_{n,\nu} \qquad (\nu = 1, \cdots, n-1),$$

whence it follows by (3.25) that

$$M_\nu(\alpha f) \leqq \gamma_{n,\nu} \qquad (\nu = 1, \cdots, n-1). \tag{3.28}$$

Letting α tend to 1, we conclude that

$$M_\nu(f) \leqq \gamma_{n,\nu} \qquad (\nu = 1, \cdots, n-1). \tag{3.29}$$

4. One final remark. The geometric arguments used in Section 2 will also give a solution of the Landau problem for the half line [**1**]. Nevertheless, the construction of the extremal function for the half line case [**3**] requires various compactness arguments and fixed point arguments, and so the Landau problem for the half line remains far from elementary.

References

1. A. S. Cavaretta, Jr., Remarks on Landau's problem for the half line, (to appear).

2. I. J. Schoenberg, The elementary cases of Landau's problem of inequalities between derivatives, this MONTHLY, 80 (1973) 121–158.

3. I. J. Schoenberg and A. S. Cavaretta, Solution of Landau's problem concerning higher derivatives on the half line, MRC Technical Summary Report 1050, Madison, Wisconsin, 1970. Also in Proc. of the Intern. Conf. on Constructive Function Theory, Golden Sands (Varna) May 19–25, 1970, Publ. House Bulgarian Acad. Sci., Sofia, (1972) 297–308.

DEPARTMENT OF MATHEMATICS, CALIFORNIA INSTITUTE OF TECHNOLOGY, PASADENA, CA 91109
PRESENT ADDRESS: DEPT. OF MATH., KENT STATE UNIVERSITY, KENT, OH44242.

A NEW LOOK AT THE THREE CIRCLES THEOREM

H. S. BEAR

1. Introduction. The setting for the Hadamard three circles theorem (see, e.g., [4], p. 120) is a plane annulus, $R_0 \leqq |z| \leqq R_1$. Let f be analytic on the annulus and let M_0, M_1, M_r be the maximum values of $|f|$ on the three circles of radii R_0, R_1, and r, where $R_0 \leqq r \leqq R_1$. The theorem is then the assertion that if $\log r = (1-\theta)\log R_0 + \theta \log R_1$, for $0 \leqq \theta \leqq 1$, then

$$\log M_r \leqq (1-\theta)\log M_0 + \theta \log M_1 . \tag{1}$$

In words, $\log M_r$ is a convex function of $\log r$.

The Three Circles Theorem is remarkable for its lack of obvious intuitive content. However, the statement can be clarified substantially by noticing that it is not really a theorem about analytic functions, but a theorem about subharmonic functions. (Definitions follow in section 2; the critical fact is that $\log|f|$ is subharmonic.) If v is a subharmonic real valued function, and M_0, M_1, M_r denote its maximum values on the circles of radius R_0, R_1, r, then

$$M_r \leqq (1-\theta)M_0 + \theta M_1 ,$$

where, as before, $\log r = (1-\theta)\log R_0 + \theta \log R_1$.

Our aim in this note is to reexamine the Three Circles Theorem and to reformulate it so that among other things it has rememberable content. This reformulation also has the virtue that it admits straightforward generalizations which are of interest in themselves.

2. Definitions and relevant facts. We assemble in this section the facts we shall need. The reader will find very clear expositions in the text by Ahlfors [1]. The information about non-continuous subharmonic functions can be found, for example, in [5].

A real valued function $u(z) = u(x,y)$ is **harmonic** in a domain D, provided it satisfies the Laplace Equation identically in D: $u_{xx} + u_{yy} \equiv 0$. A linear combination of harmonic functions is again harmonic.

A function v is **subharmonic** in D if $-\infty \leqq v(z) < \infty$, v is upper semicontinuous (given $A > v(x_0, y_0)$, there is $\delta > 0$ such that $v(x,y) < A$ if $(x-x_0)^2 + (y-y_0)^2 < \delta^2$), and v satisfies the condition that whenever v is dominated by some harmonic function u on the boundary of a subdomain $D' \subset D$, then $v \leqq u$ on D'. A harmonic function is clearly also subharmonic.

Any upper semicontinuous function (and hence any subharmonic function) attains a maximum value on any compact set.

Every harmonic function is uniquely determined by its boundary values, and a nonconstant harmonic function takes its maximum and minimum values only on the boundary of a domain.

If f is analytic, then $\operatorname{Re} f$ is harmonic and $|f|$ is subharmonic; $\log|f|$ is harmonic if $f \neq 0$, and subharmonic in any case.

The Dirichlet problem is solvable for any plane domain D whose complement has no components which are single points. This means that for any given continuous real function g on ∂D (the boundary of D) there is a continuous function u on $\bar{D}$ (the closure of D) so that u agrees with g on ∂D.

3. A proof of the theorem. We shall consider instead of the annulus a doubly connected domain bounded by two curves C_0 and C_1, with C_0 inside C_1. Thus $\bar{D} = D \cup C_0 \cup C_1$. This setting provides a mild generalization of the standard theorem, but more important, frees us from some irrelevant information we have in the special case.

Let u be the function which is continuous on $\bar{D}$, harmonic on D, equal to 0 on C_0 and 1 on C_1. Such a function exists because the Dirichlet problem is solvable on D; it is unique because a harmonic function is determined by its boundary values. Since a nonconstant harmonic function assumes its maximum and minimum values only on the boundary, $0 < u(z) < 1$ for $z \in D$.

Clearly, u assumes every value between 0 and 1 by the intermediate value theorem. Let $C_\theta = \{z : u(z) = \theta\}$, for $0 \leqq \theta \leqq 1$. Then the sets C_θ are disjoint closed sets whose union is $\bar{D}$. (Actually, the sets C_θ are disjoint curves.)

Now consider any function w which is continuous on $\bar{D}$, harmonic on D, and constant on each of the curves C_0 and C_1. To be specific, suppose $w = a$ on C_0, and $w = b$ on C_1. Then we claim that w is a linear function of θ; that is, for $z \in C_\theta$,

$$w(z) = (1 - \theta)a + \theta b. \tag{2}$$

Since $u = 0$ on C_0 and $u = 1$ on C_1, on $C_0 \cup C_1$ we have the identity

$$w(z) = (1 - u(z))a + u(z)b. \tag{3}$$

Since (3) holds on the boundary of D and both sides are harmonic functions, (3) is an identity on $\bar{D}$. On the curve C_θ, $u = \theta$, which gives (2).

The function u furnishes a single "coordinate," θ, for the domain such that any harmonic function constant on each of the curves C_0 and C_1 is a linear function of θ.

Now let u be as before (0 on C_0, 1 on C_1, and harmonic), and let v be any harmonic or subharmonic function on $\bar{D}$, without the assumption that v is constant on C_0 and on C_1. If M_0 and M_1 are the maximum values of v on C_0 and C_1, then on $C_0 \cup C_1$ we have

$$v(z) \leqq (1 - u(z))M_0 + u(z)M_1. \tag{4}$$

Since (4) holds on the boundary of D, it holds on all of D, because v is subharmonic and the right side of (4) is harmonic. If $z \in C_\theta$, $u(z) = \theta$, and (4) becomes

$$v(z) \leqq (1 - \theta)M_0 + \theta M_1 \quad (z \in C_\theta). \tag{5}$$

If M_θ denotes the maximum value of v on C_θ, then clearly from (5) we also have

$$M_\theta \leqq (1 - \theta)M_0 + \theta M_1. \tag{6}$$

We shall regard (6) *as our reformulation of the Three Circles Theorem*; ***that is***, *the maximum value of a subharmonic function is a convex function of* θ.

For the annulus $R_0 \leqq |z| \leqq R_1$, the function $\log|z|$ is harmonic and constant on circles. The function

$$u(z) = \frac{\log|z| - \log R_0}{\log R_1 - \log R_0}$$

is the harmonic function 0 on R_0 and 1 on R_1, which accounts for the messy but explicit classical formulation of the theorem.

4. Generalization to multiply connected domains. The Three Circles Theorem as formulated above generalizes naturally to domains of higher connectivity. We illustrate with a domain D bounded by three curves C_1, C_2, C_3.

Let u_1 be the continuous function on $\bar{D} = D \cup C_1 \cup C_2 \cup C_3$ which is 1 on C_1 and 0 on $C_2 \cup C_3$, and harmonic in D. Let u_2 and u_3 be defined similarly, so that u_i is 1 on C_i and zero on the remainder of the boundary. For each i, $0 < u_i < 1$ on D, and since $u_1 + u_2 + u_3 = 1$ on $C_1 \cup C_2 \cup C_3$, $u_1 + u_2 + u_3 \equiv 1$ on D.

For any triple $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ with $0 \leqq \alpha_i$ and $\alpha_1 + \alpha_2 + \alpha_3 = 1$, let $D_\alpha = \{z \in D: u_i(z) = \alpha_i\}$. The sets D_α are closed disjoint sets whose union is $\bar{D}$; in particular, $D_{(1,0,0)} = C_1$, $D_{(0,1,0)} = C_2$, and $D_{(0,0,1)} = C_3$. Some sets D_α are necessarily empty. For example, $D_{(\frac{1}{2},\frac{1}{2},0)} = \varnothing$, since if $u_3(z) = 0$, then $z \in C_1 \cup C_2$, and hence, $u_1(z) = 1$ or $u_2(z) = 1$. We think of the triple α as a kind of convexity coordinate which expresses the "distance" of a point of D from each boundary curve.

Let w be harmonic on $\bar{D}$ and constant on each C_i: $u = a$ on C_1, $u = b$ on C_2, $u = c$ on C_3. Then $w = au_1 + bu_2 + cu_3$ on the boundary of D (because of the definition of the u_i), and hence, on all of D. If $z \in D_\alpha$, $\alpha = (\alpha_1, \alpha_2, \alpha_3)$, then

$$w(z) = a\alpha_1 + b\alpha_2 + c\alpha_3.$$

That is, w is a linear function of α on D if w is harmonic and constant on each C_i.

If v is subharmonic on $\bar{D}$, and M_1, M_2, M_3 are the maximum values of v on C_1, C_2, C_3, then for $z \in C_1 \cup C_2 \cup C_3$,

$$v(z) \leqq u_1 M_1 + u_2 M_2 + u_3 M_3. \tag{7}$$

Since v is subharmonic and the right side of (7) is harmonic, the inequality (7) holds for all $z \in D$. Hence, if $z \in D_\alpha$, $u_i(z) = \alpha_i$ and

$$v(z) \leqq \alpha_1 M_1 + \alpha_2 M_2 + \alpha_3 M_3.$$

If M_α denotes the maximum of the subharmonic function v on the compact set D_α, then

$$M_\alpha \leqq \alpha_1 M_1 + \alpha_2 M_2 + \alpha_3 M_3, \tag{8}$$

and M_α is a convex function of α on D.

For a function f analytic on D and continuous on $\bar{D}$, let us write $\|f\|_i$ for the maximum of $|f|$ on C_i, and $\|f\|_\alpha$ for the maximum of $|f|$ on D_α. Then since $|f|$ and $\log|f|$ are subharmonic, we have

$$\|f\|_\alpha \leqq \alpha_1 \|f\| + \alpha_2 \|f\|_2 + \alpha_3 \|f\|_3$$

$$\log \|f\|_\alpha \leqq \alpha_1 \log \|f\|_1 + \alpha_2 \log \|f\|_2 + \alpha_3 \log \|f\|_3 .$$

Exponentiating both sides of the last inequality gives

$$\|f\|_\alpha \leqq \|f\|_1^{\alpha_1} \|f\|_2^{\alpha_2} \|f\|_3^{\alpha_3}. \tag{9}$$

Inequalities like (9) have been used by Bishop [2] and Creese [3] to introduce new norms into function algebras and to study their maximal ideal spaces.

As a final comment, we remark that these ideas have immediate and obvious extensions. One can consider, for example, boundaries more complicated than curves, or plane domains of higher connectivity. The development above also applies to arbitrary linear spaces of continuous functions on a compact space (in place of the harmonic functions), and thereby also to complex function algebras.

References

1. Lars V. Ahlfors, Complex Analysis, McGraw-Hill, New York, 1966.
2. Errett Bishop, Holomorphic completions, analytic continuations, and the interpolation of semi-norms, Ann. Math., 78 (1963) 468–500.
3. Thomas M. Creese, Norm interpolation in function algebras and the piecemeal recovery of maximal ideal spaces, *Function Algebras*, ed. by Frank T. Birtel, Scott-Foresman, Chicago, 1966, 70–83.
4. Zeev Nehari, Conformal Mapping, McGraw-Hill, New York, 1952.
5. Walter Rudin, Real and Complex Analysis, McGraw-Hill, New York, 1966.

Department of Mathematics, University of Hawaii, Honolulu, Hawaii 96822.

THE DERIVATIVE SONG

Words by Tom Lehrer — Tune: "There'll be Some Changes Made"

You take a function of x and you call it y,
Take any x-nought that you care to try,
You make a little change and call it delta x,
The corresponding change in y is what you find nex',
And then you take the quotient and now carefully
Send delta x to zero, and I think you'll see
That what the limit gives us, if our work all checks,
Is what we call dy/dx,
It's just dy/dx.

QUERIES

Edited by A. C. Zitronenbaum

This Department welcomes queries from readers about mathematics at the collegiate level, such as sources for exposition of a particular topic from a special point of view, references to vaguely remembered articles, descriptions of special kinds of courses or teaching methods, and methods for constructing illustrative examples for exercises of particular kinds (questions on research topics should, in general, be addressed to the "Queries Department" of the Notices of the American Mathematical Society). Replies will be forwarded to the questioner and may also be edited into a composite answer for publication in this Department. Consequently all items submitted for consideration for possible publication should include the name and complete mailing address of the person who is to receive the reply. Queries and answers should be sent to A. C. Zitronenbaum, Department of Mathematics, Cornell University, Ithaca, NY 14850.

9. D. B. Small. I would appreciate receiving ideas for alternatives to the "standard" class-lecture format for teaching mathematics.

10. H. W. Vayo and C. Payne. We are interested in sources of information on finite matrices whose elements are operators; particularly the relationships between the operator entries and the matrix itself regarded as another operator. We have been told that this area is mostly folklore, is this so?

MATHEMATICAL NOTES

Edited by David Roselle

Material for this Department should be sent to David Roselle, Department of Mathematics, Louisiana State University, Baton Rouge, LA 70803.

RELATIVE COLORINGS AND THE FOUR-COLOR CONJECTURE

Roy B. Levow

The four-color conjecture (FCC) claims that the vertices of any planar graph may be colored with four colors in such a way that no two adjacent vertices are colored with the same color, or dually that any planar map can be colored with four colors in such a way that no two adjacent regions are colored with the same color.

Kainen and Saaty [2] have considered the relation between relative coloring and FCC. A section subgraph G' of a graph G is the subgraph of G containing all edges of G which connect vertices of G'. If G is a planar graph and G' is a section subgraph of G with a given fixed coloring, then a coloring of G whose restriction to G' is the same as the given coloring of G' is a relative coloring of (G, G'). The relative chromatic number $\chi(G, G')$ of (G, G') is the maximum number of new colors needed in any relative coloring of (G, G'), where the maximum is taken over all

colorings of G'. Kainen and Saaty have shown that FCC is equivalent to the conjecture: For any pair (G, G') with G planar and G' a (possibly empty) section subgraph of G, we have $\chi(G, G') \leqq 4$. They also posed the question of whether FCC is equivalent to the following statement, noting that it clearly implies FCC.

CONJECTURE. *For any pair (G, G') with G planar and G' a non-null connected section subgraph of G, we have $\chi(G, G') \leqq 3$.*

Before proving that the above conjecture is equivalent to FCC, we first establish the following equivalence:

LEMMA 1. *Let G be a planar graph. FCC is equivalent to the statement: For each planar drawing of G there is a four-coloration in which the boundary of each region (face) contains vertices of at most three colors.*

Proof. Given a drawing of G, augment the drawing by adding a new vertex in each region and connecting it with new edges to each of the vertices on the boundary of the region in which it is located. The resulting graph is planar, and thus four-colorable by hypothesis. The restriction of the resulting coloration to the original graph is as required.

The converse is trivial.

LEMMA 1$'$. *Let G be a plane map. FCC is equivalent to the statement: There is a four-coloration of G in which the regions adjacent to each vertex are colored in at most three colors.*

Proof. This is the dual of Lemma 1.

We are now prepared to prove

THEOREM 1. *Let G be a planar graph. FCC is equivalent to the statement: For any pair (G, G') with G' a non-null connected section subgraph of G, $\chi(G, G') \leqq 3$.*

Proof. In any drawing of G the vertices of $G - G'$ adjacent to vertices of G' must lie on the boundary of a single region of that drawing restricted to $G - G'$ because G' is connected. Thus by Lemma 1, there is a four-coloring of $G - G'$ in which the vertices of $G - G'$ adjacent to vertices of G are colored with at most 3 colors. A relative coloring of (G, G') may, therefore, be produced by using new colors only for these vertices. At most three such new colors, one for each old color, are required, so $\chi(G, G') \leqq 3$.

References

1. F. Harary, Graph Theory, Addison-Wesley, Reading, Mass., 1969.

2. P. C. Kainen, Relative colorings of graphs, J. Combinatorial Theory, 14 (1973) 259–262.

3. T. L. Saaty, Thirteen colorful variations on Guthrie's four-color conjecture, this MONTHLY, 79 (1972) 2–43.

DEPARTMENT OF MATHEMATICS, FLORIDA ATLANTIC UNIVERSITY, BOCA RATON, FLORIDA 33432.

DIFFERENTIABILITY IN BANACH ALGEBRAS

J. J. Buoni

In the literature there are many ways of defining a derivative in a Banach space; all of which are generalizations of the ordinary quotient

$$\lim_{t\to 0} \frac{F(x+t)-F(x)}{t} = \frac{dF(x)}{dt}.$$

In this note a definition from the finite dimensional setting [see **3**] is generalized to the infinite dimensional setting and a class of functions is exhibited which have this property. Although these results appear in [**2**], the use of the right and left multiplication operator and the theory of several commuting operators is avoided and only elementary techniques are employed.

Let W be a Banach algebra with an identity 1, and let F be defined on a domain $\mathscr{D}$, an open connected set, of W, with range in W. Fundamental in the theory of differentiation on W is the Frechet derivative. The function F is said to be Frechet differentiable at a point $x \in \mathscr{D}$ if there exists a continuous linear function of t, say $dF(x, \cdot)$, such that

$$\| F(x+t) - F(x) - dF(x,t) \| \leqq o(\| t \|). \tag{1}$$

The process of extending analytic functions of a complex variable to Banach algebras is well known [**1**, p. 118]. This motivates the first definition.

Definition A. The function F is said to be differentiable at a point $x \in \mathscr{D}$ if for all y in some neighborhood V of x the function can be represented in the form

$$F(y) = (2\pi i)^{-1} \int_C f(\lambda)(\lambda 1 - y)^{-1} d\lambda \tag{2}$$

for a suitable analytic function f, where C consists of a finite number of rectifiable Jordan curves which are contained in the domain of analyticity of f, and encloses $\sigma(y)$, for $\forall y \in V$ where $\sigma(y)$ denotes the spectrum of y.

The next definition can be found in [**1**, p. 114] where W is a commutative algebra or in [**3**] where W is a finite dimensional algebra.

Definition B. The function F is said to be differentiable at $x \in \mathscr{D}$ if it is Frechet differentiable at x and if $dF(x, \cdot)$ is an element of the enveloping algebra of W, the smallest (norm) closed subalgebra of $B(W)$ containing the left and right multiplication operators, where $B(W)$ is the algebra of bounded linear operators on W equipped with the usual norm.

We shall show that Definition A implies Definition B; however, we need the following

Lemma. *Let U be an open set containing $\sigma(x)$ and Γ a compact subset of $\mathbb{C}$,*

the complex plane, whose intersection with U is empty. Then there exists a sphere about x of radius $\delta > 0$, $S_\delta(x)$, and a constant depending on x, $M(x)$, such that for all $\lambda \in \Gamma$, and for all $y \in S_\delta(x)$, $\|(\lambda - y)^{-1}\| \leqq M(x)$.

Proof. Since $\|(\lambda - x)^{-1}\|$ is a continuous mapping from Γ, a compact subset of the resolvent of x, into the positive reals, there exist positive constants N and M such that for all $\lambda \in \Gamma$, $M \leqq \|(\lambda - x)^{-1}\| \leqq N$.

Let U be any open set containing $\sigma(x)$. By the upper semi-continuity of the spectrum there exists $\delta > 0$ such that $\delta \leqq (N+1)^{-1}$ and that for all $y \in S_\delta(x)$, $\sigma(y) \subset U$. Now for all $\lambda \in \Gamma$, λ is in the resolvent of both x and y, where $y \in S_\delta(x)$, and by the second resolvent equation, we have

$$(\lambda - y)^{-1} - (\lambda - x)^{-1} = (\lambda - x)^{-1}(x-y)(\lambda-y)^{-1} \quad [\mathbf{1}, \text{ p. } 115].$$

This leads to the inequality

$$\begin{aligned}\|(\lambda-y)^{-1}\| &\leqq \|(\lambda-x)^{-1}\| \; \|1 + (x-y)(\lambda-y)^{-1}\| \\ &\leqq \|(\lambda-x)^{-1}\|(1 + \delta\|(\lambda-y)^{-1}\|).\end{aligned}$$

Since $1 - \delta\|(\lambda-x)^{-1}\| \geqq 1 - \delta N > 0$, the above inequality yields that

$$\begin{aligned}\|(\lambda-y)^{-1}\| &\leqq \{1 - \delta\|(\lambda-x)^{-1}\|\}^{-1}\|(\lambda-x)^{-1}\| \\ &= \{(\|(\lambda-x)^{-1}\|)^{-1} - \delta\}^{-1} \leqq \{(1/M) - \delta\}^{-1}.\end{aligned}$$

This yields the result with $M(x)$ set equal to $\{(1/M) - \delta\}^{-1}$.

THEOREM. *Definition A implies Definition B.*

Proof. If $F(x)$ satisfies Definition A then $F(x) = (2\pi i)^{-1}\int_C f(\lambda)(\lambda 1 - x)^{-1}d\lambda$. With Cauchy's formula in mind, we note that a reasonable candidate for $dF(x,t)$ is $(2\pi i)^{-1}\int_C (\lambda 1 - x)^{-1}t(\lambda 1 - x)^{-1}f(\lambda)d\lambda$. Let V be an open set containing $\sigma(x)$ such that C encloses V. Then by the upper semi-continuity of the spectrum there exists $\delta > 0$ such that $\|t\| < \delta$ implies $\sigma(x+t) \subset V$. Now

$$\begin{aligned}&F(x+t) - F(x) - dF(x,t) \\ &= (2\pi i)^{-1}\int_C \{(\lambda 1 - x - t)^{-1} - (\lambda 1 - x)^{-1} - (\lambda 1 - x)^{-1}t(\lambda 1 - x)^{-1}\}f(\lambda)d\lambda \\ &= (2\pi i)^{-1}\int_C \{(\lambda 1 - x - t)^{-1}(\lambda 1 - x)(\lambda 1 - x)^{-1} - (\lambda 1 - x - t)^{-1}(\lambda 1 - x - t) \\ &\qquad (\lambda 1 - x)^{-1} - (\lambda 1 - x)^{-1}t(\lambda 1 - x)^{-1}\}f(\lambda)d\lambda \\ &= (2\pi i)^{-1}\int_C \{(\lambda 1 - x - t)^{-1}[(\lambda 1 - x) - (\lambda 1 - x - t)](\lambda 1 - x)^{-1} \\ &\qquad - (\lambda 1 - x)^{-1}t(\lambda 1 - x)^{-1}\}f(\lambda)d\lambda\end{aligned}$$

$$= (2\pi i)^{-1}\int_C [(\lambda 1-x-t)^{-1}t(\lambda 1-x)^{-1} - (\lambda 1-x)^{-1}t(\lambda 1-x)^{-1}]f(\lambda)d\lambda$$

$$= (2\pi i)^{-1}\int_C \{(\lambda 1-x-t)^{-1} - (\lambda 1-x)^{-1}\}t(\lambda 1-x)^{-1}f(\lambda)d\lambda$$

$$= (2\pi i)^{-1}\int_C (\lambda 1-x-t)^{-1}t(\lambda 1-x)^{-1}t(\lambda 1-x)^{-1}f(\lambda)d\lambda,$$

where the last equation follows by the same preceding steps. So,

$$(3) \qquad \| F(x+t) - F(x) - dF(x,t) \| \leqq (2\pi)^{-1}\|(\lambda 1-x-t)^{-1}\| \; \|t\|^2 \|(\lambda 1-x)^{-1}\|^2 \|f\| \, l(C),$$

where $\|f\|$ indicates the sup norm of $f(\lambda)$ and $l(C)$, the length of C.

With $y = x + t$, and C a compact set, find δ and $M(x)$ as in the Lemma. Set $K = (2\pi)^{-1}(M(x))^3 \|f\| l(C)$. Then for any $\varepsilon > 0$, set $\delta_1 = \min(\varepsilon, \delta, 2\varepsilon/K)$. Thus for any $\varepsilon > 0$, if we select t such that $\|t\| < \delta_1$, we find that

$$(3) \leqq K\|t\|^2 \leqq o(\|t\|).$$

It remains to show that $t \to dF(x,t)$ belongs to the closed enveloping algebra of W. This follows immediately, because the integral $1/2\pi i \int_C(\lambda 1-x)^{-1}t(\lambda 1-x)^{-1} \cdot f(\lambda)d\lambda$, is the limit, in the norm of $B(W)$, of Riemann sums

$$\sum_{k=0}^{n} (\lambda_k 1-x)^{-1}t(\lambda_k 1-x)^{-1}f(\lambda_k)(\lambda_k - \lambda_{k-1}).$$

This completes the proof.

Set $W = \mathbb{C} \oplus \mathbb{C}$ with the usual norm and operations. Then W is a Banach algebra with an identity. Define $F: W \to W$ by $F(\alpha,\beta) = (\beta,\alpha)$ for all $(\alpha,\beta) \in W$, then F is Frechet differentiable but does not satisfy Definition B [**1**, 115]. On the other hand, if we define $F: W \to W$ by $F(\alpha,\beta) = (0,1)$ for all $(\alpha,\beta) \in W$, then F satisfies Definition B but not A.

NOTE: I would like to thank the referee for some helpful comments concerning this manuscript.

References

1. E. Hille, Functional analysis and semi-groups, Amer. Math. Soc. Colloq. Publ., 31 (1948).

2. J. Schwartz, Two perturbation formulas, Comm. Pure Appl. Math., 8 (1955) 371–376.

3. A. Trampus, Differentiability and analyticity in linear algebras, Duke Math. J., 27 (1960) 431–441.

DEPARTMENT OF MATHEMATICS, YOUNGSTOWN STATE UNIVERSITY, YOUNGSTOWN, OHIO 44503.

POINT SEPARATING ALGEBRAS OF POLYNOMIALS

D. J. NEWMAN

For a given collection of polynomials one would like to know whether all other polynomials are obtainable from them by polynomial operations. For this to hold it is clearly necessary that the collection separate points (including "infinitely close" points). Very often in approximation problems this separation condition is also sufficient, see e.g., [**1**], and it is natural to ask whether it is in our case.

There is quite a different answer to this question depending on whether we construe "polynomial" to mean a polynomial in 1-variable or in several variables. In several variables polynomials have a complicated behavior at ∞ (unlike the situation for 1-variable) and one can give counter-examples based on this behavior. This will be shown below where we shall also obtain the affirmative result in the case of 1-variable. Our proof of this is fairly elementary, but we must mention that a much shorter proof is possible, based on local rings and a theorem of Nakayama.

Before turning to the precise statements and proofs, however, allow us to make one curious observation. If our collection consisted of two polynomials, $p(x)$ and $q(x)$ say, then the point separation condition would amount to the fact that the system

$$\frac{p(x)-p(y)}{x-y}=0, \qquad \frac{q(x)-q(y)}{x-y}=0,$$

has no solutions. Similarly, if our collection consisted of three polynomials $p(x)$, $q(x)$, $v(x)$, then we would require that

$$\frac{p(x)-p(y)}{x-y}=0, \qquad \frac{q(x)-q(y)}{x-y}=0, \qquad \frac{v(x)-v(y)}{x-y}=0$$

have no solutions. "In general" two equations in two unknowns have a solution, but three equations in two unknowns do not. We conclude that "in general," two polynomials do not generate everything but that three polynomials do!

In what follows the underlying field will be understood to be the complex numbers, $\mathbb{C}$, although the results extend to any algebraically closed field. We say that a sub-algebra A of $\mathbb{C}[x_1, x_2, \cdots, x_n]$ is "point separating' if

(a) $1 \in A$,

(b) For $\alpha, \beta \in \mathbb{C}^n$, $\alpha \neq \beta$, $\exists P \in A$ with $P(\alpha) \neq P(\beta)$,

(c) For $\alpha \in \mathbb{C}^n$, $\delta \in \mathbb{C}^n$, $\delta \neq 0$, $\exists P \in A$ with $\delta \cdot \nabla P(\alpha) \neq 0$.

(Here $\delta \cdot \nabla$ is a shorthand for the differential operator

$$\delta_1 \frac{\partial}{\partial x_1} + \delta_2 \frac{\partial}{\partial x_2} + \cdots \delta_n \frac{\partial}{\partial x_n}, \text{ where } \delta = (\delta_1, \delta_2, \cdots, \delta_n).)$$

We can now produce our several variable counter example. Consider, namely, the algebra A in $\mathbb{C}[x,y]$ of all polynomials of the form $p(x)+M(x,y)(xy-1)$. This algebra is certainly proper since it only contains polynomials which remain bounded as $y\to\infty$ on the hyperbola $xy=1$. On the other hand, A is point separating as the following shows:

Clearly $1\in A$. Now let $(\alpha_1,\alpha_2)\neq(\beta_1,\beta_2)$ if $\alpha_1\neq\beta_1$ then the choice $P=x$ works for (b), if $\alpha_1=\beta_1\neq 0$ then $P=xy-1$ works, and if $\alpha_1=\beta_1=0$, then $P=y(xy-1)$ does the job. Finally let $(\delta_1,\delta_2)\neq 0$. If $\delta_1\neq 0$ then the choice $P=x$ satisfies (c). If $\delta_1=0$, $\alpha_1\neq 0$ then the choice $P=xy-1$ works, and if $\delta_1=0$, $\alpha_1=0$ then $P=y\,(xy-1)$ satisfies (c) and the verification is complete.

We can now turn to the 1-variable case, and we first prove a simple consequence of the property of point separation. This is the fact that *any* interpolation problem is solvable.

LEMMA 1. *Let A be point separating in $\mathbb{C}[x]$ and let $p(x)$, $q(x)$ be any polynomials with $q(x)$ not identically 0. There is a polynomial $M(x)$ such that $M(x)\,q(x)+p(x)\in A$.*

Proof. We first observe that for any λ there is always an $M_1(x)$ with $M_1(\lambda)\neq 0$, and $M_1\,q\in A$. It suffices to consider q which are linear since the general result will then follow by multiplication. If $q(x)=x-\alpha$, where $\alpha\neq\lambda$, we use (a) and (b) to produce a polynomial in A which vanishes at α but not at λ. If $q(x)=x-\lambda$, on the other hand, we use (a) and (c) to produce a polynomial in A which vanishes simply at λ. In either case the desired $M_1(x)$ is produced.

We now proceed by induction on $\deg q(x)$. The result is trivial for $\deg q(x)=0$ and we show that its truth for $q(x)$ implies its truth for $(x-\lambda)\,q(x)$. So determine $M(x)$ such that $p(x)+M(x)\;q(x)\in A$. By our previous remark we may choose $M_1(x)\;q(x)\in A$ with $M_1(\lambda)=M(\lambda)$. This tells us, however, that

$$p(x)+\frac{M(x)-M_1(x)}{x-\lambda}(x-\lambda)\,q\,(x)\in A$$

and the induction is complete.

It is necessary, at this point, to shift our attention to *subfields* of the field of rational functions, $\mathbb{C}(x)$ rather than subalgebras of $\mathbb{C}[x]$. We cite namely the theorem of Lüroth [**2**].

LEMMA 2. *The only possible subfields of $\mathbb{C}(x)$ which contain $\mathbb{C}$ are the fields $\mathbb{C}(r(x))$, where $r(x)$ is a rational function. If in addition the subfield contains a non-constant polynomial then $r(x)$ can be chosen as a polynomial.*

As for the second part, let $p(x)\in\mathbb{C}(r(x))$ where $p(x)$ is a non-constant polynomial. This means that, for some rational function R, $R(r(x))=p(x)$. If ∞ is a pole of R then $r(x)$ can not equal ∞ for any finite x and $r(x)$ is itself a polynomial. If, on the

other hand, $\alpha \in \mathbb{C}$ is a pole of R then the range of $r(x)$ misses the value α so that $1/r(x) - \alpha$ is a polynomial and of course

$$\mathbb{C}(r(x)) = \mathbb{C}\left(\frac{1}{r(x) - \alpha}\right).$$

Using this result we next obtain

LEMMA 3. *Let A be any subalgebra of $\mathbb{C}[x]$. Either $A \subset \mathbb{C}[p(x)]$ for some polynomial $p(x)$ of degree > 1 or else A contains a non-trivial ideal in $\mathbb{C}[x]$.*

Proof: We may assume that $A \neq \mathbb{C}$. If we form the quotient field of A we may apply Lemma 2, and conclude that this quotient field is $\mathbb{C}(p(x))$, $p(x)$ a non-constant polynomial. If $\deg p(x) > 1$ then we are done, so let us assume that $\deg p(x) = 1$. This means of course that the quotient field of A consists of all rational functions. In particular, x itself lies in the quotient field and this means that there exists a $P(x) \in A$ with $x\ P(x) \in A$. Suppose $\deg P(x) = k > 0$, we claim then that A contains the principal ideal generated by $P^{k-1}(x)$.

Namely, it follows easily that every polynomial may be expressed as a sum of terms $x^i P^j(x)$ where $0 \leqq i \leqq k-1$, $0 \leqq j$. Thus every multiple of $P^{k-1}(x)$ is expressible as a sum of terms $x^i P^l(x)$ where $0 \leqq i \leqq k-1 \leqq l$. But $x^i P^l(x) = (xP)^i P^{l-i}$ and since $xP, P \in A$ it follows that these terms all lie in A. Thus it follows that every multiple of $P^{k-1}(x)$ does indeed lie in A and the proof is complete.

It is now easy to prove our

THEOREM. *If A is a point separating algebra in $\mathbb{C}[x]$ then $A = \mathbb{C}[x]$.*

Proof: Observe, first of all, that if $\deg p(x) > 1$ then the algebra $\mathbb{C}[p(x)]$ is not point separating. Indeed there is a point α where $p'(\alpha) = 0$ and at this point all the $P'(\alpha) = 0$ for $P \in \mathbb{C}[p]$. Since A is point separating, then it follows by Lemma 3 that A contains a non-trivial ideal, say $(q(x))$. If $p(x)$ is any polynomial, however, $p(x) + M(x)\,q(x) \in A$ by Lemma 1. Combining these two facts tells us that $p(x) \in A$ and we conclude that $A = \mathbb{C}[x]$, as required.

References

1. L. Nachbin, Sur les Algèbres Denses..., C. R. Acad. Sci. Paris, 1949, p. 228.
2. B. L. van der Waerden, Modern Algebra, I. Ungar, New York, 1949, Vol. I, page 198.

DEPARTMENT OF MATHEMATICS, YESHIVA UNIVERSITY — BELFER, NEW YORK, N. Y. 10033.

EDITED BY RICHARD GUY

In this Department the Monthly presents easily stated research problems dealing with notions ordinarily encountered in undergraduate mathematics. Each problem should be accompanied by relevant references (if any are known to the author) and by a brief description of known partial results. Manuscripts should be sent to Richard Guy, Department of Mathematics, Statistics, and Computing Science, The University of Calgary, Calgary 44, Alberta, Canada, T2N 1N4.

THE LARGEST GRACEFUL SUBGRAPH OF THE COMPLETE GRAPH

SOLOMON W. GOLOMB

1. Introduction. A **graceful numbering** for a graph with n nodes and e edges has been defined [**1**] as an assignment of a subset of the numbers $\{0, 1, \cdots, e\}$ to the n nodes in such a way that the edges receive all the numbers from 1 to e, where the **number** of an edge is computed to be the absolute value of the difference between the node numbers at its endpoints. (This has also been termed a **β-valuation** for a graph in [**2**] and [**3**].) Any graph which admits a graceful numbering is called a **graceful graph**. In [**1**] it is proved that the complete graph K_n is never graceful for $n \geqq 5$, and it is natural to inquire as to the size (measured by the number of edges) of the largest graceful subgraph of K_n. Since it is known (see [**1**] or [**3**]) that the complete bipartite graph $K_{a,b}$ is always graceful, we have the example of $K_{[n/2],[(n+1)/2]}$, which is a graceful subgraph of K_n having $[n^2/4]$ edges. In this paper, we exhibit graceful subgraphs of K_n, for all n, having $[n^2/4]+n-2$ edges. For $n \leqq 7$, this size is known by exhaustive search to be best possible. For $n \geqq 8$, no larger examples are known, so that it becomes a reasonable conjecture, to be either proved or disproved, that this is truly the maximum size.

2. The Construction. Given the n nodes, we partition them into two sets, A and B, of $[(n+1)/2]$ and $[n/2]$ nodes respectively. We form the complete bipartite graph connecting all nodes of A to all nodes of B, which has $[n^2/4]$ edges. Moreover, we connect one of the nodes of A to all the other nodes of A, and one of the nodes of B to all of the other nodes of B. The total number of edges is then

$$\left[\frac{n^2}{4}\right] + \left(\left[\frac{n}{2}\right] - 1\right) + \left(\left[\frac{n+1}{2}\right] - 1\right) = \left[\frac{n^2}{4}\right] + (n-2),$$

for all $n \geqq 2$.

The node numbers are assigned as follows: In the set A, the special node which is joined to all the others is numbered "0", and the others are numbered $1, 2, 3, \cdots, [(n-1)/2]$. In the set B, the special node which is joined to all the others is numbered $e = [n^2/4] + (n-2)$, and the others are numbered $e + 1 - j[(n+3)/2]$ for $j = 1, 2, \cdots, [(n-2)/2]$. These numbers may also be represented as $h[(n+3)/2] - 2$ for $h = 2, 3, \cdots, [n/2]$.

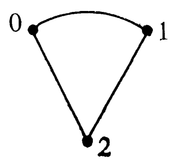
From this numbering, it is readily seen that the edges in set A have the numbers $1, 2, \cdots, [(n+1)/2]$ while the edges in set B have the numbers $j[(n+3)/2]-1$ for $j = 1, 2, \cdots, [(n-2)/2]$. Moreover, the edges connecting set A to set B have all the numbers on the range from $h[(n+3)/2]-2$ down to

$$h\left[\frac{n+3}{2}\right]-2-\left[\frac{n-1}{2}\right]=(h-1)\left[\frac{n+3}{2}\right],$$

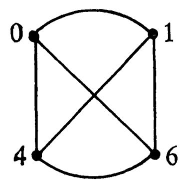
for every $h = 2, 3, \cdots, [n/2]$. This gives every edge number from 1 to e once and only once in the entire graph.

3. Examples. The following illustrations portray the constructions just described for $n = 2, 3, \cdots, 10$.

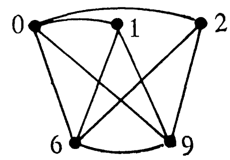
$n = 2$



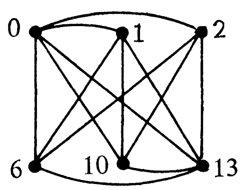
$n = 3$



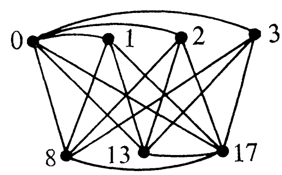
$n = 4$



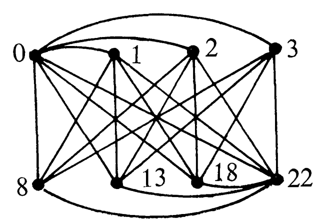
$n = 5$



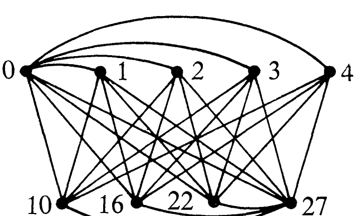
$n = 6$



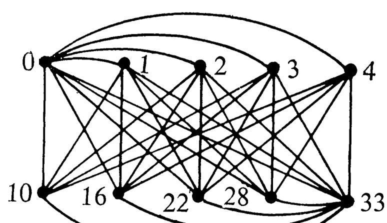
$n = 7$



$n = 8$



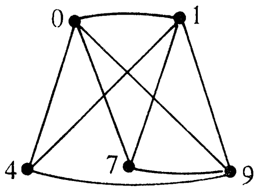
$n = 9$

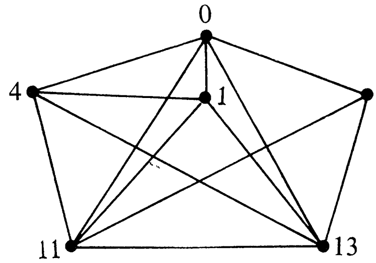


$n = 10$

While no larger graceful graphs are known for any of these values of n, the examples presented here are not necessarily unique. There is, first of all, the trivial renumbering which replaces each node number m by $e - m$, but leaves all edge

numbers unchanged. In the following illustration, we see an essential renumbering, for $n = 5$, of what is abstractly the same graph, using an obvious modification of the original construction. We also show another example for $n = 6$ which achieves the maximum number of edges by an entirely different method. (The two examples for $n = 6$ involve non-isomorphic graphs, since the first uses K_6 with two disjoint edges deleted, while the second uses K_6 with two adjacent edges deleted.)





This research was supported in part by the U.S. Army Research Office under Grant No. DA–ARO–D–31–124–173–G167.

References

1. S. W. Golomb, How to Number a Graph, in "Graph Theory and Computing," R. C. Read, editor, Academic Press, New York, 1972, 23–37.

2. R. A. Duke, Can the complete graph with $2n+1$ vertices be packed with copies of an arbitrary tree having n edges? this MONTHLY, 76 (1969) 1128–1130.

3. A. Rosa, On certain valuations of the vertices of a graph, Proc. Internat. Sympos. Rome 1966, Dunod, Paris & Gordon and Breach, New York, 1967, 349–355.

POWELL HALL, UNIVERSITY OF SOUTHERN CALIFORNIA, LOS ANGELES, CALIFORNIA 90007.

CLASSROOM NOTES

EDITED BY DAVID ROSELLE

Material for this Department should be sent to David Roselle, Department of Mathematics, Louisiana State University, Baton Rouge, LA 70803.

A NOTE ON THE WELL ORDERING OF CARDINALS

C. METELLI AND L. SALCE

In a naive approach to cardinals, not requiring a previous introduction of ordinals, it may be interesting to give a direct proof of the fact that cardinals are well ordered by their natural order.

We recall that, if Γ is a non-void set of cardinals, and for each $\gamma \in \Gamma$, X_γ is a set

of cardinality γ, an order relation $\leqq$ is given in Γ by letting $\gamma \leqq \delta$ $(\gamma, \delta \in \Gamma)$ if there exists an injection $\phi_{\gamma\delta}: X_\gamma \to X_\delta$. (The antisymmetric property is provided by the Schroeder-Bernstein theorem.) The fact that Γ is well ordered (in particular, totally ordered) by $\leqq$ is an immediate consequence of the following, purely set-theoretical proposition:

THEOREM. *Let A be a non-empty set, $\{X_\alpha\}_{\alpha \in A}$ a family of sets. Then for a convenient $\bar{\alpha} \in A$ there exists an injection $\phi_{\bar{\alpha}\alpha}: X_{\bar{\alpha}} \to X_\alpha$ for every $\alpha \in A$.*

The proof is based on the Axiom of Choice, in its following (equivalent) forms: THE PRODUCT AXIOM: The direct product of a non-empty family of non-empty sets is non-empty, and ZORN'S LEMMA: Any partially ordered set which is inductive, (i.e., in which every totally ordered subset has an upper bound), has at least one maximal element.

Proof. The statement is trivial if $X_{\bar{\alpha}} = \varnothing$ for some $\bar{\alpha} \in A$. Assume then $X_\alpha \neq \varnothing$ for each α: let $X = \prod_{\alpha \in A} X_\alpha$ be the cartesian product of the X_α, and $\pi_\alpha: X \to X_\alpha$ the α-projection of X. Consider the set $\mathscr{B} = \{B \subseteq X \mid \pi_\alpha | B$ is injective for each $\alpha \in A\}$. Then X is non-empty by the Product Axiom, and if $x \in X$, then $\{x\} \in \mathscr{B}$. Thus $\mathscr{B}$ is non-empty. $\mathscr{B}$ is partially ordered by inclusion, and is inductive, because, if $\mathscr{C}$ is a totally ordered subset of $\mathscr{B}$, $\bigcup_{B \in \mathscr{C}} B$ is an upper bound of $\mathscr{C}$; thus Zorn's Lemma applies, and $\mathscr{B}$ has a maximal element M. Then for a suitable $\bar{\alpha} \in A$, $\pi_{\bar{\alpha}} | M$ is surjective: for otherwise, again by the Product Axiom, $Y = \prod_{\alpha \in A} (X_\alpha - \pi_\alpha(M))$ would be non-empty, and $y \in Y$ implies $M \cup \{y\} \in \mathscr{B}$; but $y \notin M$, thus contradicting the maximality of M. Hence, for some $\bar{\alpha} \in A$, $\pi_{\bar{\alpha}} | M$ is a bijection, so that, for each $\alpha \in A$, the map $\phi_{\bar{\alpha}\alpha} = \pi_\alpha \circ (\pi_{\bar{\alpha}} | M)^{-1}: X_{\bar{\alpha}} \to X_\alpha$ is the required injection.

The interested reader may now wonder whether the preceding statement is in fact equivalent to the Axiom of Choice. (We are grateful to the referee for having brought this to our attention.) The answer is affirmative: Assuming our statement as a starting point, the Product Axiom follows. Suppose our theorem is true, and let $\{X_\alpha\}_{\alpha \in A}$ be a non-empty family of non-empty sets. To prove that the set $X = \prod_{\alpha \in A} X_\alpha$ is non-empty, we shall point out an element of X, i.e., a map $\phi: A \to \bigcup_{\alpha \in A} X_\alpha$, such that, for every $\alpha \in A$, $\phi(\alpha) \in X_\alpha$ (Dugundji [**1**, p. 22]). We know that, for a convenient $\bar{\alpha} \in A$, there exists a map $\phi_{\bar{\alpha}\alpha}: X_{\bar{\alpha}} \to X_\alpha$ for every $\alpha \in A$. Fix an $x \in X_{\bar{\alpha}}$, and let $\phi: A \to \bigcup_{\alpha \in A} X_\alpha$ be the map such that $\phi(\alpha) = \phi_{\bar{\alpha}\alpha}(x)$. Then $\phi \in X$.

Lavoro eseguito nell'ambito dell'attivitá dei Gruppi di Ricerca Matematici del Consiglio Nazionale delle Ricerche.

Reference

1. James Dugundji, Topology, Allyn and Bacon, Boston, 1966.

UNIVERSITÁ DI PADOVA, SEMINARIO MATEMATICO, VIA BELZONI 3, 35100 PADOVA, ITALY.

References

1. I. Kaplansky, Fields and Rings, The University of Chicago Press, Chicago, 1969.
2. E. Schenkman, Group Theory, Van Nostrand, Princeton, N. J. 1965.

DEPARTMENT OF MATHEMATICS, BOWDOIN COLLEGE, BRUNSWICK, MAINE 04011.

KRASNOSELSKI'S THEOREM ON THE REAL LINE

D. F. BAILEY

In [2] Krasnoselski established the following result. (See page 30 of [1] for a discussion in English.)

THEOREM. *If K is a convex, closed, bounded subset of a uniformly convex Banach space and if f is a mapping of K into a compact subset of K such that $\| f(x) - f(y) \| \leqq \| x - y \|$, then the sequence obtained by choosing x_1 in K and defining $x_{n+1} = \frac{1}{2}[x_n + f(x_n)]$ converges to some z in K and $f(z) = z$.*

Below we give a proof for the special case in which K is a closed interval of the real line, which is sufficiently simple for presentation to students enrolled in a class in advanced calculus. We restate this special case of Krasnoselski's theorem as follows.

THEOREM. *If f takes $[a,b]$ into itself and $|f(x) - f(y)| \leqq |x - y|$, then the sequence obtained by choosing x_1 in $[a,b]$ and defining $x_{n+1} = \frac{1}{2}[x_n + f(x_n)]$ converges to some z in $[a,b]$ and $f(z) = z$.*

Proof. We assume that the sequence $\{x_n\}$ does not converge to a fixed point of f and arrive at a contradiction. There follow several initial observations:

(1) The sequence $\{x_n\}$ does not converge. For if $\{x_n\}$ converges to z, then so does $\{x_{n+1}\}$. But $\{x_{n+1}\}$ converges to $\frac{1}{2}[z + f(z)]$. Thus $z = \frac{1}{2}[z + f(z)]$ which in turn implies $f(z) = z$.

(2) There is no subsequential limit of $\{x_n\}$ which is fixed under f. For if z is a subsequential limit which is fixed under f, it follows that

$$|z - x_{n+1}| = \left|\tfrac{1}{2}[z + f(z)] - \tfrac{1}{2}[x_n + f(x_n)]\right| \leqq \tfrac{1}{2}|z - x_n| + \tfrac{1}{2}|f(z) - f(x_n)| \leqq |z - x_n|.$$

This inequality implies there is only one subsequential limit contrary to (1).

(3) Since $[a,b]$ is compact it follows that $\{x_n\}$ has at least one subsequential limit. Furthermore, there exists some subsequential limit w such that $f(w) > w$. For if $f(p) \leqq p$ for all subsequential limits p, let z denote the greatest lower bound of the set of subsequential limits. Then z is itself a subsequential limit and if $f(z) < z$, it follows that $\frac{1}{2}[z + f(z)]$ is a subsequential limit smaller than z. Hence $f(z) = z$ contrary to (2).

(4) There exists $\varepsilon > 0$ such that $|f(z) - z| \geqq \varepsilon$ for all subsequential limits z. For if not designate by w_n that subsequential limit such that $|w_n - f(w_n)| < 1/n$. Now note that any subsequential limit of $\{w_n\}$ is fixed under f and is also a subsequential limit of $\{x_n\}$. This contradicts (2).

Finally, to complete the proof observe that there exists (by (3) and (4)) a largest subsequential limit satisfying $f(y) > y$; call this subsequential limit W. Let $Q = \frac{1}{2}[W + f(W)]$ and note that $f(W) > Q > W$ and $f(Q) < Q$. By (4) there exists a smallest subsequential limit greater than W satisfying $f(x) < x$; call this limit R. It follows that $W < R < f(W)$. Next note that $f(R) < W$; for, if not, $A = \frac{1}{2}[R + f(R)]$ satisfies $W < A < R$ and by definition of W and R, $f(A) = A$, contrary to (2). Thus $f(R) < W < R < f(W)$. It then follows that $|f(R) - f(W)| > |R - W|$. This contradicts the hypothesis of the theorem and establishes the result.

REMARK: To see that the iteration scheme described in the theorem does not apply to arbitrary continuous mappings of a closed interval into itself, consider the following example. Define f, taking the closed unit interval into itself by

$$f(x) = \begin{cases} 3/4 \text{ if } 0 \leqq x \leqq 1/4 \\ -3x + (3/2) \text{ if } 1/4 < x < 1/2. \\ 0 \text{ if } 1/2 \leqq x \leqq 1. \end{cases}$$

Note that $x_1 = 1/4$ yields $x_2 = 1/2$, $x_3 = 1/4$, etc.

References

1. F. F. Bonsall, Lectures on some Fixed Point Theorems of Functional Analysis, Tata Institute of Fundamental Research, Bombay, 1962.

2. M. A. Krasnoselski, Two remarks on the method of successive approximations, Uspehi, Math. Nauk (N. S.), No. 1, 10 (1955) 123–127.

DEPARTMENT OF MATHEMATICS, CORNELL COLLEGE, MOUNT VERNON, IOWA 52314.

A QUICK PROOF FOR A ONE-DIMENSIONAL VERSION OF LIAPOUNOFF'S THEOREM

R. E. JAMISON

A remarkable and important result of Liapounoff asserts that if $\mu_1, \mu_2, \cdots, \mu_n$ are finite non-atomic measures on a measure space X, then, as A ranges over all measurable subsets of X, the points $(\mu_1(A), \mu_2(A), \cdots, \mu_n(A))$ determine a closed convex set in R^n [2]. The proof of this is rather difficult, although short (but not elementary) arguments exist [3]. In simpler, one-dimensional versions, however this result often makes an appearance in beginning graduate courses in analysis.

This note presents a quick proof of one such version by using a Zorn's lemma argument — a technique too often left unexploited in measure theory courses. The notation and terminology established in [1] will be used.

THEOREM. *Suppose that μ is a regular measure defined on the Borel subsets of a compact Hausdorff space X. If the measure of any singleton subset of X is 0, then for each β in $[0, \mu(X))$ there is a compact subset D of X with $\mu(D) = \beta$.*

Proof. We claim first that if $(D_i)_{i \in I}$ is a decreasing family (not necessarily countable) of compact sets in X, then

$$\mu(\bigcap_{i \in I} D_i) = \inf_{i \in I} \mu(D_i).$$

Since the measure is monotone, the measure of the intersection is clearly less than or equal to the infimum. Suppose that the infimum is λ and that λ is strictly larger than the measure of the intersection. Then we can choose, by regularity, an open set U such that $\bigcap_{i \in I} D_i \subset U$ and $\mu(U) < \lambda$. The sets $D_i \cap U'$ form a decreasing family of compact sets whose intersection is empty; thus for some i, $D_i \cap U'$ is empty. Whence $D_i \subset U$, so $\mu(D_i) \leqq \mu(U) < \lambda$ for that i, contradicting the infimal definition of λ.

Now let β be a positive number less than $\mu(X)$. We assert that by Zorn's lemma there is a minimal compact subset D of X with $\mu(D) \geqq \beta$. For if $(D_i)_{i \in I}$ is any decreasing chain of such sets, we have

$$\beta \leqq \inf \mu(D_i) = \mu(\bigcap D_i).$$

That is, $\bigcap D_i$ is a lower bound for the chain. Suppose that $\mu(D) > \beta$. Then D must be nonvoid, so let x be any point of D. Then $\mu(x) = 0$, by hypothesis. By regularity, there is then an open set U with $x \in U$ and $\mu(U) < \mu(D) - \beta$. Thus

$$\mu(D) \leqq \mu(D \cap U') + \mu(U) < \mu(D \cap U') + (\mu(D) - \beta).$$

So $\beta < \mu(D \cap U')$. But $D \cap U'$ is compact and properly contained in D since $x \notin D \cap U'$; this is contrary to the minimality of D. Therefore we must have had $\mu(D) = \beta$.

References

1. E. Hewitt and K. Stromberg, Real and Abstract Analysis, Springer-Verlag, New York, 1969.

2. A. A. Liapounoff, On completely additive vector-functions, Izv. Akad. Nauk SSSR, 4 (1940), 465–478.

3. Joram Lindenstrauss, A short proof of Liapounoff's convexity theorem, J. Math. Mech., 15 (1969) 971–972.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF WASHINGTON, SEATTLE, WASHINGTON 98195.

MATHEMATICAL EDUCATION

EDITED BY SHIRLEY HILL AND PAUL MIELKE

Material for this Department should be sent to Shirley Hill, Department of Mathematics, University of Missouri, Kansas City, MO 64110, or to Paul Mielke, Department of Mathematics, Wabash College, Crawfordsville, IN 47933.

A SEMESTER OF MATHEMATICS EACH MONTH

W. C. RAMALEY

The past three years Colorado College has offered practically all courses in blocks of time $3\frac{1}{2}$ weeks in length. The college calendar consists of 9 such blocks, four comprising an autumn semester and the other five a spring semester. Blocks within a semester are separated by $4\frac{1}{2}$ day weekends. With this calendar, called The Colorado College Plan or more simply the Plan, a student normally takes one course during each of the nine blocks. Block-long courses are intended to cover as much material as normal 3-semester-hour courses. When we speak of a "course" in the following discussion, such a course is meant.

A very few courses are offered at a half-time rate over more than one block and sometimes a student can pair two such courses over a period of several blocks. However, no student can take more than two academic courses concurrently. The Department of Mathematics attempted some of these "extended half-courses" in the first year of the Plan, but offered none the next two years.

At the start of the Plan, there was essentially no change in our course offerings. For example, the calculus sequence, which had been four semesters long (with the last being differential equations), became four blocks long. No longer would students take mathematics all the time, but by the end of the sophomore year the typical math or science major would have taken at least these four courses and perhaps the course in linear algebra.

Experience with the Plan has caused us to change some course content and method of presentation. The close working relationship possible with students under the Plan lends itself to direct dialogue. Thus we may lecture less. Furthermore, we offer more courses now in an independent-directed study mode than before. Also, there is more flexibility in the course content. For example, in 1972–73 we did not offer a course in advanced calculus, but rather, offered a special topics course in complex variables which could, in part, cover some of the traditional material of advanced calculus.

This should not be taken to mean that we can teach courses successfully in a manner lacking rather strict internal structure. Indeed, if we allow a student to proceed gradually into a problem or area, then he is likely to waste the entire block. There is

almost no time left to accomplish anything if a week has been spent deciding what is to be done. Those students unwilling or unable to submerge themselves in intensive study for a month do not profit from their course to a very great extent. In fact, some faculty argue that the haste they find made necessary by the Plan completely precludes mathematics as a civilized activity suited to undergraduates or to their faculty.

Whereas in advanced courses you can sometimes feel your way along with your class, at the elementary level pressure can develop at a frightening rate. With 20 to 25 students, all of whom have different backgrounds, the first course in calculus is apt to be the worst of all. In part, courses further along in the sequence assume that the first course teaches manipulative skills and problem-solving abilities that enable the student to successfully continue the sequence. Furthermore, other departments have their own expectation of a student who has taken a calculus course. These problems are not unique to beginning calculus, but they seem greatest there.

Think about what you might like to get done in your own first calculus course. Then realize that if you spend two days reviewing at the start of the course, there will be only 15 class days left, including the day for the final. Of course, technically the students are "yours" all day long, but whether there exists a one-to-one correlation between time spent and material mastered is at least questionable.

By the second year of the Plan it was apparent that the content in the calculus courses would have to be changed. All work on series was postponed until the course on differential equations. Thus, that course could cover its own material for only 8 or 9 days. Now we plan to have the calculus be 4 blocks long and introduce a fifth course for differential equations. In all courses, content is constantly being examined closely to see whether it can be omitted.

It rapidly became obvious that none of our courses, elementary or advanced, could humanely cover as much material as could be covered in a 3-hour semester course. The fraction may be 4/5, 2/3, or some other number, but no one in the Mathematics Department seriously argues for a fraction close to one. This may not be bad in itself, but it should be taken into consideration when the role of mathematics as a service-department is discussed. It *must* be considered in school transfers and graduate study.

Most classes meet twice a day and most faculty hold both meetings in either the morning or the afternoon. Usually one session is for problems and one for new material, although of course some mixing occurs—as it should. Almost any combination of times and meetings is possible, but the students usually feel that they are unduly worked if class meets formally for much more than two hours during the day.

A common teaching pattern is to have many quizzes, possibly some hour exams, and a final examination. Frequent tests insure that students quickly know what they don't understand. To get behind by more than a day or two is a very serious thing. To fall behind a whole week is almost certain disaster. This is true for faculty as well since no one can teach classes other than his own except under really extraordinary circumstances. There just is not time for meeting classes and students other than your

own. Thus, sickness or any other serious interruption of the course is more hazardous than under other calendars.

An advantage that the Plan does offer to a student whose education is disrupted for more than a few days is that generally only one course need be lost. Should someone break a leg or take a month off, he loses only one course.

Having courses start and stop abruptly and with little continuity from block to block is a serious problem. It is difficult for a student suddenly to quit one full-time topic and start on a new, usually unrelated, subject. The same is somewhat true for the faculty, but at least we are usually in the same area each block.

Sequences of courses that build upon each other are harder to take and harder to teach and there is a tendency for more "one-shot" or "side-track" courses that delve into a non-standard, but worthwhile, subject on an *ad hoc* basis. Sequence courses are harder to take, in part, because there is time needed for "review" and the "review" often seems completely new to the students. Such courses are harder to teach because, as in first year calculus, there exist such definite expectations of the students.

There is the danger that "one-shot" courses, no matter how successful, may reduce the student's mastery of a well-defined body of mathematical knowledge that is thought of as comprising a major. Some of our faculty feel this has already happened.

The Plan here is operated by a college for undergraduates and where the entire faculty teaches all the time. There is essentially no time whatsoever for any other work. Members of the Natural Science Division report they spend an average of 36 hours/week directly on class work [**1**, p. 44]. The students feel they own their instructor and his time under the Plan since they do not have to share him with another class concurrently taught. It is hard to get time to do anything other than help students on a daily basis. Time for personal research and development is a major problem for 3/4 of the faculty [**1**, p. 48]. Even at a teaching college time is needed to prepare courses and to keep up personal skills and interests. Our administration realizes this and in an attempt to correct the difficulty is considering a plan to give each faculty member one block off each year.

A departmental colleague wrote [**2**, p. 59]: "For me, the greatest failing (of the Plan) has been the inability to teach courses without giving up tremendous amounts of my time. ... I envisioned students doing reading outside of class and coming to class with questions, complaints, and comments which would generate a good discussion on the subject matter. I envisioned doing away with lectures, with the students having responsibility for obtaining some kind of overall view of the subject, as we proceeded through it. This overall view would give them a meaningful framework to work in, and provide a basis for formulating their questions and stimulating discussion. Only *one* course in my years here has worked that way. Generally, most students, even in upper division courses, simply can't read 10 pages of mathematics without getting stumped on page 1 and getting almost nothing out of the last 9 pages. They need the material broken up into smaller bits; they need to be led through

numerous examples; they need to see solutions to problems; they need to have their own work criticized carefully, etc."

Hindsight more often finds missed opportunities and drawbacks than it finds fortuitous choices and advantages. Conversely, advantages are more clearly seen than disadvantages before a change occurs. Certainly at the Plan's inception the advantages seemed obvious to a majority of the general faculty (and an overwhelming majority of them still supports the Plan).

Many of the advantages of the Plan are difficult to test, but nonetheless seem very real. Foremost among these advantages is the intensive submergence in the course. The class, its size limited to 25, often develops a cohesive nature where the instructor is working on the material with students he comes to know personally and who know and help each other. Students appear to like the Plan and work diligently to make it work. A class going well for them can be worked on full-time and one going poorly will soon be over. Class attendance is essentially 100%. Last, the flexibility of class scheduling and the class's informality are advantages to instructor and student alike. Thus, the entire academic atmosphere is more pleasant.

It was hoped that the breadth and depth of academic material would be increased with this intensive study. We lack an objective measure of this. We rather doubt it for most of our students in most mathematics courses. In fact, as I have indicated, the students probably know less material. However, perhaps the intensive nature of the Plan causes them to grasp more firmly the material that is covered and better prepares them for the independent work envisaged by my hopeful colleague.

There are no simple answers to the problems that the Plan creates. Calendar changes, like textbook changes, are not a panacea. Anyone considering the Plan, or a modification, must decide whether the problems are worse than the ones faced under his present calendar.

References

1. *Statistical Results of the All-Student and All-Faculty Evaluation Surveys of the Colorado College Plan*, May 1972 (prepared by the Office of the Dean of the College).

2. *Anthology of Faculty Comments and Opinions from the All-Faculty Evaluation of the Colorado College Plan*, May 1972 (prepared by the Office of the Dean of the College).

DEPARTMENT OF MATHEMATICS, FORT LEWIS COLLEGE, DURANGO, COLORADO 81301.

A STRUCTURE FOR UNSTRUCTURED LECTURES

D. FERGUSON

The single element of the classroom situation which is most opposed to *reality* is its "structure." Nowhere else in his life does a student so often find himself (1) enclosed by walls that meet in right angles, (2) seated in a chair nailed to the floor, (3) facing a rectangular green mask, (4) looking to the only individual without a seat, whose task it is to teach him something. As a result of all of this, a student is consciously, and often painfully, aware of the full duration of the class period.

During the past quarter, the author was involved in the teaching of an elementary algebra course to students with deficiencies in basic mathematics. (The course was offered, as a part of the program of the Council on Educational Development, to college students at UCLA.) The class was composed largely of minorities with Chicanos, Blacks, and American Indians being the major representatives.

During the first week of class, it became quite apparent that the usual lecture style would be completely inadequate, or a total disaster.

Suggestions for improving student-teacher relationships were solicited from the students in open discussions. Several suggestions were made; including having students present problems from the homework, having students discuss topics that interest them, and allowing for any special projects that any creative student might suggest.

In an effort to remove some of the structured nature of the classroom situation, the author proposed a type of intellectual game that was met with favorable responses by the students. In considering what type of game should be used, the author immediately eliminated all previous "well structured" mathematical games. It was his opinion that the game should possess an element of chance, while yet being intellectually stimulating. The game should be flexible enough to allow the most simple problems along with the "almost" impossible. The game items should be related to the material under discussion, and should have a self-contained review that would eliminate the long boring conventional pre-test review sessions.

After some consideration, the author arrived at the following:

Rules for Contest

1. Each team will be determined at random by selecting slips from an envelope.
2. The quiz items will consist of algebra problems. Some problems will be taken from the text, some problems will be taken from other algebra books, others will be constructed by the teacher.
3. The captains of the teams will rotate, so that each player will at one time serve as captain.
4. There will be two envelopes. One envelope will contain the regular quiz items; worth 1 point, 2 points, 3 points, lose your turn, lose 1 point, 1 point free or take another turn. The other envelope will contain the bonus items. The bonus questions will be considerably more difficult than the others. Each bonus item is worth 5 points.
5. Teams will alternate in selecting items from the envelope of their choice. The point value of each question and the amount of time allotted will be stated on each slip.
6. If the team that selects the question cannot answer in the allotted time, then the other team may respond, provided the question is not of a true-false or multiple choice nature.
7. A winner is determined each day. During the following class meeting new teams may be formed.

In selecting their own items, the students are *actively* involved. When they miss a problem they tend to remember why.

This method may be used to expand on various topics, as well as to present a limited amount of new material. The author usually discusses new material during the first part of the hour, and serves as quiz master the rest of the hour.

The author was quite pleased with the results. Some unusually difficult problems for this class were perceived and solved. Students immediately learned to find counter-examples to false statements, and plausible reasons (if not rigorous proofs) for those statements that they believed to be true.

The game served three major functions: (1) It retained the interest of students for long periods. (2) It helped to imprint the basic facts in the students' minds. (3) It provided an avenue for stimulating discussions on mathemetical topics.

It is the belief of the author that exercises similar to the one in this brief article would be extremely useful to any instructor teaching a similar course. The flexibility is immense, since the instructor possesses the sole power to determine the difficulty and scope of the items. The game is no more limited than the limitations on the instructor's own creativity!

691 Lenering Ave. No. 15, Los Angeles, CA 90024.

LECTURER PROGRAM AS PUBLIC RELATIONS

Mary Gray

Part of the explanation of the current job crisis lies in the public's anti-mathematical bias. Many students do not take math courses because they feel that they are not relevant. The mathematical community is partly responsible, for in the days of abundance some of us assumed an elitist, unresponsive attitude towards the public, towards students, towards other disciplines and towards society's problems. Now we need to work on changing our image.

There exists at least one project which can be used for this purpose: the MAA Secondary School Lecture Program. In many Sections there is a lively exchange between high schools and colleges with the lecturers serving as ambassadors for their discipline and their schools. A good lecturer can turn on many of the students turned-off mathematics and stimulate interest in studying mathematics — with a career in mind, for general cultural enrichment or for the uses to which mathematics can be put. The lectures are becoming increasingly application-oriented as participants have been formulating responses to the frequently asked question: What can I do with mathematics?

Not only can lecturers interest the students in general in the study of mathematics, but another purpose which can be served by the lectures is to draw into the various fields of mathematics the groups which have traditionally been excluded: women and minority groups. They have more than the usual share of anti-mathematical bias, due to conditioning and due to their treatment in the profession. Attracting them to the study of mathematics can be accomplished by the selection of lecturers — to present working mathematicians from these groups as models, topics — to highlight some of

the contributions made by women and minorities in mathematics and to study the phenomenon of their small numbers in the field, and audiences — to reach women and minority students, teachers and parents, both for lectures on special topics and for the talks on pure and applied mathematics.

Lecturers are also taking on the role of consultants on curriculum and related matters and in many cases speaking to groups of parents, a particularly beneficial activity as far as public relations are concerned. Anything that can be done to bring to a wider audience more understanding and appreciation for mathematics is a useful exercise.

The program needs wider support as it is now operating in only a few sections of the country, leaving high schools in many areas without these valuable contacts. The existing programs are funded in many cases with small amounts from MAA contest funds or from contributions from local businesses; in a few states the state board of education or analogous agency has provided support. Efforts are being made to secure some national funding.

Meanwhile, each Section might view the program as part of the desperately needed public relations campaign that the mathematics community must conduct and accordingly bolster its program or initiate one if there is none in existence. Brochures describing existing programs and offering suggestions to MAA Sections are available from the MAA, 1225 Connecticut Ave., N. W., Washington, D. C. 20036.

Department of Mathematics and Statistics, The American University, Washington, D. C. 20016.

PROBLEMS AND SOLUTIONS

Edited by Emory P. Starke

All problems (both elementary and advanced) proposed for inclusion in this Department should be sent to E. P. Starke, 1000 Kensington Ave., Plainfield, NJ 07060. Proposers of problems are urged to enclose any solutions or information that will assist the editors. Ordinarily, problems in well-known textbooks and results in generally accessible sources are not appropriate for this Department. No solutions (except those accompanying proposals) should be sent to Professor Starke.

ELEMENTARY PROBLEMS

Solutions of Elementary Problems should be sent to Problems Group, Mathematics Department, University of Maine, Orono, ME 04473. To facilitate their consideration, solutions of Elementary Problems in this issue should be typed (with double spacing) and should be mailed before August 31, 1974.

An asterisk (*) means neither the proposer nor the editors supplied a solution.

E 2474. *Proposed by Paul Abad and David Friedman, San Francisco Unified School District*

Let $X_1, \cdots, X_n$ be independent random variables such that each X_j is uniformly distributed over the interval $[0, a_j]$ with $0 < a_1 < \cdots < a_n$. Let $Y = \mathrm{Max}\,[X_1, \cdots, X_n]$. Find a formula for the probability that $Y = X_i$ in terms of i and the a_j. (Cf. Problem 71–14, S.I.A.M. Review 13 (1971), p. 378.)

E 2475. *Proposed by W. J. Berger, District of Columbia Teachers College*

Under what conditions can the four tritangent circles of a triangle be rearranged so as to be mutually tangent?

E 2476*. *Proposed by W. J. Berger, D.C. Teachers College*

Under what conditions can the five tetratangent spheres of a tetrahedron be rearranged so as to be mutually tangent?

E 2477. *Proposed by A. W. Walker, Toronto, Canada*

A straight line L meets the sides BC, CA, AB of a triangle ABC with orthocenter H at X, Y, Z; DE is a diameter of the circle ABC. Through X, Y, Z lines $B'C'$, $C'A'$, $A'B'$ are drawn parallel to AE, BE, CE to form a triangle $A'B'C'$ oppositely similar to ABC. If D', E', H' are the images of D, E, H for this similarity, prove that in general

(a) The lines AA', BB', CC', DD', HH' concur, so that $ABCD$ and $A'B'C'D'$ ($ABCH$ and $A'B'C'H'$) are oppositely similar perspective cyclic (orthocentric) quadrangles;

(b) The lines DH' and HD' meet at the invariant point of the similarity, and $DHD'H'$ is a cyclic quadrangle;

(c) The axis L is perpendicular to DD' and bisects EE'.

E 2478. *Proposed by Charles Wells, Case Western Reserve University*

All functions map the reals to the reals. Call f and g *similar* if $f = h^{-1}gh$ for some bijective h.

(a) Are sin and cos similar?

(b) Characterize those a, b such that $f(x) = x^2$ and $g(x) = x^2 + ax + b$ are similar.

E 2479. *Proposed by J. H. Blau, Antioch College*

Let n be a fixed natural number. It is well known that the functional equation

$$f(x + y^n) = f(x) + (f(y))^n$$

has many discontinuous solutions if $n = 1$. Discuss the situation for other values of n. (The function f is a real-valued function of a real variable.)

SOLUTIONS OF ELEMENTARY PROBLEMS

Reflecting Property of an Oblate Spheroid

E 2413 [1973, 435]. *Proposed by C. B. Grosch, Control Data Corporation, Minneapolis*

Consider an oblate spheroid and the circle in its equatorial plane which is the locus of the foci of meridian ellipses. Show that any ray that originates on the circle will be reflected to the circle after a reflection from the interior of the spheroid. (On the spheroid, the angle of incidence equals the angle of reflection.)

I *Composite solution edited from those submitted by L. Kuipers, Southern Illinois University, and E. Trost, Technikum Winterthur, Switzerland.* Let S be a point on the spheroid at which we are to reflect a ray. Assume that S is not a pole and does not lie in the equatorial plane, since the theorem is obvious if either of these conditions is true. Since the spheroid is a surface of revolution, the normal to the surface at S passes through the axis of the spheroid, so these two lines determine a plane Γ. Since Γ intersects the spheroid in an ellipse, the theorem is true for a ray lying in Γ, and the normal to the spheroid at S bisects the angle between the ray and its reflection.

Consider the oblique circular cone with vertex at S and having as base Σ the circle of foci for the spheroid. The stated problem then follows from the proof of the following theorem:

THEOREM. *Let S be the vertex and Σ the base of an oblique circular cone. Let T denote a point in the interior of Σ and coplanar with S and the normal to Σ at its center. Consider the pencil of planes passing through ST. If ST is a bisector in two of the resulting plane sections of the cone, then all sections have this property.*

Proof. Let M and r denote the center and radius of circle Σ, and let Δ be a plane of the pencil cutting Σ at A and B. Let $\phi < \pi/2$ be the angle between MT and AB. With $AT = u$, $BT = v$, and $MT = t$, we easily find that

$$u, v = \sqrt{r^2 - t^2 \sin^2\phi} \pm t\cos\phi. \tag{1}$$

Letting $\measuredangle\ MTS = \alpha$, $\measuredangle\ ATS = \theta$, and $ST = m$, we have, since plane SMT is normal to plane Σ,

$$\cos\theta = \cos\alpha\cos\phi, \tag{2}$$

$$SA^2 = m^2 + u^2 - 2um\cos\theta, \quad SB^2 = m^2 + v^2 + 2vm\cos\theta. \tag{3}$$

The assumption that $\sphericalangle\ AST = \sphericalangle\ BST$ is equivalent to

$$SA^2/SB^2 = u^2/v^2. \tag{4}$$

Combining (3) and (4) we get

$$2uv\cos\theta = m(v-u). \tag{5}$$

Using (1) and (2), (5) becomes

$$[mt + (r^2 - t^2)\cos\alpha]\cos\phi = 0. \tag{6}$$

By assumption, (6) holds for at least one angle $\phi \neq \pi/2$ (for the plane Γ in the spheroid, $\phi = 0$), whence (6) holds for every ϕ.

We have actually proved that if the property of the hypothesis of this theorem holds for just one plane section where $\phi \neq \pi/2$, then it holds for all plane sections.

II. *Solution by the proposer.* Let a denote the equatorial radius of the spheroid, b the polar radius, and c the radius of the circle of foci; then $a^2 = b^2 + c^2$. Choose a coordinate system such that the coordinates of the point of reflection on the spheroid are $(0, a\cos\theta, b\sin\theta)$ and let the ray originate at $(c\cos\phi, c\sin\phi, 0)$. The unit vector in the direction of the incident ray is then

$$\boldsymbol{x} = p^{-1}(-c\cos\phi\boldsymbol{i} + (a\cos\theta - c\sin\phi)\boldsymbol{j} + b\sin\theta\,\boldsymbol{k}), \tag{1}$$

where $p = \sqrt{a^2 + c^2\cos^2\theta - 2ac\cos\theta\sin\phi}$. The unit normal at the point of reflection on the spheroid is

$$\boldsymbol{n} = q^{-1}(b\cos\theta\boldsymbol{j} + a\sin\theta\,\boldsymbol{k}),$$

where $q = \sqrt{a^2 - c^2\cos^2\theta}$. The unit vector in the direction of the reflected ray is now

$$\boldsymbol{r} = \boldsymbol{x} - 2(\boldsymbol{x}\cdot\boldsymbol{n})\boldsymbol{n}.$$

Let $\boldsymbol{z}$ denote the vector from the center of the spheroid to the point at which the reflected ray pierces the equatorial plane. Then

$$\boldsymbol{z} = a\cos\theta\boldsymbol{j} + b\sin\theta\,\boldsymbol{k} + s\boldsymbol{r},$$

for an appropriate scalar multiple s of $\boldsymbol{r}$; s is determined by the fact that the $\boldsymbol{k}$-component of $\boldsymbol{z}$ must be 0. Substituting (1), (2), and (3) in this equation and simplifying shows that $s = q^2/p$ and thus

$$\boldsymbol{z} = -cp^{-2}(q^2\cos\phi\boldsymbol{i} + (p^2\sin\phi - 2ac\cos\theta\cos\phi)\boldsymbol{j}).$$

Since $\boldsymbol{z}\cdot\boldsymbol{z} = c^2$, it follows that the reflected ray passes through the circle of foci.

Also solved by Michael Goldberg. Only the proposer's solution was complete and correct.

Editor's comment. This result extends to three dimensions the well-known property of the ellipse that all rays emanating from one focus are reflected through the other focus. This property generalizes immediately to a prolate spheroid, and present problem shows how it applies to an oblate spheroid. It is not clear what can be said about an arbitrary ellipsoid.

Chess Champ's Chances

E 2414 [1973, 559]. *Proposed by J. G. Wendel, University of Michigan*

In one form of chess match $2n$ games are played, wins count 1 point each, draws $\frac{1}{2}$, losses are worth 0. In order to win the match, the defender needs only score at least n, while the challenger must achieve at least $n+\frac{1}{2}$. Suppose that the two players are of equal strength, and that the probability of a draw is a constant δ. Prove or disprove: the defender's chance of keeping his title is an increasing function of δ.

I. *Solution by K. Alam and K. Seo, Clemson University.* Since the players are of equal strength, and the probability of a draw is δ, the probability of a win by either player is $\frac{1}{2}(1-\delta)$. Also, the probability that the defender's score exceeds n equals the probability that the challenger's score exceeds n. If $A_n = A_n(\delta)$ is the probability that both players score n points, the probability that a player scores more than n points is $\frac{1}{2}(1-A_n)$. Therefore, the probability of the defender's keeping his title is

$$P_n(\delta) = A_n + \tfrac{1}{2}(1-A_n) = \tfrac{1}{2}(1+A_n).$$

Thus $P_n(\delta)$ is an increasing function of δ if and only if $A_n(\delta)$ is an increasing function of δ.

The probability that there will be k wins by the champion, k wins by the challenger, and $2n-2k$ draws is a multinomial probability

$$\frac{(2n)!}{(k!)^2(2n-2k)!}\delta^{2n-2k}\left(\frac{1-\delta}{2}\right)^{2k},$$

so the probability A_n can be written as

$$A_n(\delta) = \sum_{k=0}^{n} \frac{(2n)!}{(k!)^2(2n-2k)!}\delta^{2n-2k}\left(\frac{1-\delta}{2}\right)^{2k}. \tag{1}$$

Differentiating A_n with respect to δ, we see that the (continuous) derivative $A_n'(\delta)$ satisfies

$$A_n'(0) = -n2^{-2n+1}\binom{2n}{n} < 0,$$

so that $A_n(\delta)$ and hence $P_n(\delta)$ are decreasing functions of δ for δ near 0. Therefore the assertion that $P_n(\delta)$ is increasing on $[0,1]$ is false.

II. *Solution by E. M. Klimko and M. F. Neuts, SUNY at Binghamton.* The proposed statement is false. Consider the case $n = 1$. The defender can lose the title only if he either loses both games or draws one game and loses the other. The probability of the champion's keeping the title is therefore given by

$$P_1(\delta) = 1 - \left(\frac{1-\delta}{2}\right)^2 - 2\delta\left(\frac{1-\delta}{2}\right) = \tfrac{1}{4}(3\delta^2 - 2\delta + 3),$$

which is concave upwards in δ and achieves its minimum for $\delta = \frac{1}{3}$.

The following theorem holds: *The probability that the defender keeps his title is, for general n, a strictly concave upwards function of δ in $[0,1]$ which reaches its unique minimum inside the interval $(0,\frac{1}{2})$.*

Also solved by D. M. Bloom, L. J. Dickson (Australia), D. Ž. Djoković, Jordi Dou (Spain), E. S. Eby, R. J. Evans, Ann Goodsell & N. J. Kuenzi & J. Oman, S. H. Greene, F. A. Hacker, Ellen Hertz, G. A. Heuer and K. Heuer (Germany), N. L. Johnson, E. F. Knapp, J. R. Kuttler, Harry Lass, Joel Levy, Carolyn MacDonald, W. D. Markel, J. Noailles & J. Choné (France), Larry Olson, G. S. Rogers, Steven Russ, Bruce Schatz, Paul Smith, W. N. Smith, Wolfe Snow, P. K. Stockmeyer, Walt Stromquist, E. T. H. Wang, P. H. Young, and the proposer.

Editor's comment. Alam and Seo note that A_n is the solution to the differential equation

$$(2\delta - 1)(1 - \delta) A_n'' - [4n - 2 - (4n - 3)\delta] A_n' + 2nA_n = 0$$

with the boundary conditions $A_n(0) = 2^{-2n}\binom{2n}{n}$ and $A_n(1) = 1$.

Let δ_n be the value of δ that minimizes $P_n(\delta)$ in the interval [0,1]; as Solution II shows, $\delta_1 = 1/3$. Numerical calculations suggest that δ_n decreases monotonically as $n \to \infty$ and apparently approaches 0, although no analytic proof of this has been found.

Notice that the model of the match is probabilistically equivalent to the way such a match is really played, viz., the match goes to a *maximum* of $2n$ games, but is terminated as soon as the champion amasses at least n points or the challenger amasses at least $n + \frac{1}{2}$ points. (The reader may remember that in the recent Fischer-Spassky match, $n = 12$.) A reasonable estimate for δ for two strong, equally matched players might be 2/3, a figure much greater than any δ_n. It appears that in real life anyway, the champion should play conservatively in order to keep his title.

A Curious Property of 10

E 2415 [1973, 559]. *Proposed by C. D. H. Cooper, Macquarie University, Australia*

Find all positive integers n having the property that each positive divisor (> 1) of n has the form $a^r + 1$ where a, r are integers and $r > 1$.

Solution by the proposer. Let S be the set of all positive integers all of whose divisors have the above form. Clearly every positive divisor of an element of S is itself in S.

Let $n = a^r + 1 \in S$. We can suppose that a is taken to be as small as possible and hence cannot be expressed as b^s for $s > 1$. If r is odd, then $a + 1$ is a positive divisor of n whence $a + 1 = b^s + 1$ for some b, s with $s > 1$, a contradiction. Hence r is even and so every element of S can be expressed in the form $a^2 + 1$.

An odd number $m \in S$ must clearly have the form $4a^2+1$. Let p,q be odd prime numbers in S where $p \leqq q$, $p = 4a^2+1$, $q = 4b^2+1$ and suppose that $pq = 4c^2+1 \in S$. Clearly $a \leqq b < c$. Then $q(p-1) = 4(c^2-b^2)$ whence $q \mid (c+b)$ or $q \mid (c-b)$. In either case, $p \leqq q < 2c$ and so $pq < 4c^2$, a contradiction. Hence every composite element of S is even.

Let $p = a^2+1$ be a prime element of S and suppose that $2p = b^2+1 \in S$. Then $p = b^2-a^2 = (b-a)(b+a)$ so that $b-a=1$, $b+a=p$ and thus $p-1 = 2a$. But $p-1 = a^2$. Hence $a=2$ and $p=5$.

The only composite element of S which is a product of two primes is thus 10. Hence there can be no product of three primes in S and so no other composite elements in S. The only positive integers with the stated property are therefore 1, 10, and all primes of the form n^2+1.

Also solved by M. G. Greening (Australia), L. E. Mattics, David Spear, Guy Torchinelli, Phil Tracy, R. H. Warren, and Charles Wexler.

Editor's comment. Spear notes that the question of whether there are infinitely many numbers of the stated type, is in essence one of the famous unsolved problems of number theory.

Zabek, Kummer, and Divisibility of $\binom{n}{r}$

E2416 [1973, 559]. *Proposed by F. T. Howard, Wake Forest University*

Let $p_1,\cdots,p_k$ be distinct primes, $e_1,\cdots,e_k$ arbitrary nonnegative integers, and r a fixed positive integer. Prove that there are infinitely many positive integers n with the property that $p_i^{e_i}$ $(i = 1,2,\cdots,k)$ is the highest power of p_i which divides the binomial coefficient $\binom{n}{r}$.

Solution by David Singmaster, Istituto di Matematica, Pisa, Italy, and Polytechnic of the South Bank, London, England. This problem is an extension of the note of Simmons [**4**]. It is most easily attacked by use of the following theorem and its corollary of Zabek [**7**]; see also [**6**], [**2**], and [**5**], Theorem 38.

THEOREM. *Let p be a prime. For $r>0$, suppose $p^m \leqq r < p^{m+1}$. Then the sequence of residues $\binom{n}{r} \pmod{p^e}$, for $n = r, r+1,\cdots$, is periodic with minimal period p^{m+e}.*

COROLLARY. *Let $d = \prod p_i^{e_i}$. For $r>0$ and each i, suppose*

$$p_i^{m_i} \leqq r < p_i^{m_i+1} .$$

Then the sequence of residues $\binom{n}{r} \pmod{d}$, for $n = r, r+1,\cdots$, is periodic with minimal period $\prod p_i^{e_i+m_i}$.

Write $p^k \| m$ to mean $p^k \mid m$ but $p^{k+1} \nmid m$. To show that there are infinitely many n such that $p_i^{e_i} \| \binom{n}{r}$, we consider the sequence $\binom{n}{r} \pmod{\prod p_i^{e_i+1}}$. Then we need only find one such n, for everything in its residue class $\pmod{\prod p_i^{m_i+e_i+1}}$ will have the same property. (Indeed, this will show that the set of such n has positive density.)

Further, since the periods for the prime power factors $p_i^{e_i+1}$ are pairwise relatively prime, it suffices to show this result for prime powers; i.e., to show for $r > 0$ and for any prime power p^e, there is an n such that $p^e \parallel \binom{n}{r}$.

To show this result for prime powers, we use the following theorem due to Kummer [**3**]; see also [**1**] and [**5**], Theorem 6.

THEOREM. *For any prime p, we have $p^e \parallel \binom{n}{r}$ if and only if e is the number of carries in the addition $r + (n-r)$ when done in p-ary arithmetic, or equivalently, the number of borrows in the p-ary subtraction $n - r$.*

Again suppose $p^m \leqq r < p^{m+1}$ and let $n = p^{m+e} + r - p^m$. One easily sees that the p-ary subtraction $n - r$ has exactly e borrows, so we have found an n such that $p^e \parallel \binom{n}{r}$ and we are done.

References

1. L. E. Dickson, History of the Theory of Numbers, Vol. 1, Chelsea, New York, 1952, p. 270, item 71.

2. D. Drazin, Complements and comments, this MONTHLY, 77 (1970) 1078–1079.

3. E. E. Kummer, Über die Ergänzungssätze zu den allgemeinen Reciprocitätsgesetzen, J. Reine Angew. Math., 44 (1852) 93–146.

4. G. J. Simmons, Some results concerning the occurrence of specified prime factors in $\binom{n}{r}$, this MONTHLY, 77 (1970) 510–511, (MR 42, # 4476).

5. D. Singmaster, Divisibility of binomial and multinomial coefficients by primes and prime powers, to appear.

6. W. F. Trench, On periodicities of certain sequences of residues, this MONTHLY, 67 (1960) 652–656, (MR 23A, # 2365).

7. S. Zabek, Sur la périodicité modulo m des suites des nombres $\binom{n}{k}$, Ann. Univ. Mariae Curie-Sklodowska Sect. A., 10 (1956), 37–47, (MR 20, # 1653).

Also solved by Jeanne Agnew, Harry Lass, W. F. Trench, Charles Wexler, and the proposer.

Covering $3 \times 2m$ Checkerboards

E 2417 [1973, 559]. *Proposed by Ioan Tomescu, University of Bucharest, Rumania*

The number of ways of filling a $2 \times n$ rectangle with dominoes (i.e., with 1×2 rectangles) is well known (see Problem E 1470 [1962, 61]). On page 139 of his book *Polyominoes*, S. W. Golomb asks for the corresponding result for $3 \times n$ rectangles.

Let $U(n)$ be the number of ways of covering a $3 \times n$ rectangle with dominoes. Obviously $U(n) = 0$ if n is odd. Show that

$$U(2m) = \frac{1}{2\sqrt{3}}[(\sqrt{3}+1)(2+\sqrt{3})^m + (\sqrt{3}-1)(2-\sqrt{3})^m].$$

Solution by D. Ž. Djoković, University of Waterloo, Canada. We look at the rectangle of size $3 \times 2m$ and consider the sides of length 3 as vertical. Let a_m be the

number of coverings of this rectangle such that the domino in the lower right corner is vertical and let b_m be the number of coverings such that three dominoes touch the right side of length 3. Then we have the recurrence relations

$$a_{m+1} = 3a_m + b_m,\ b_{m+1} = 2a_m + b_m\ (m \geqq 1). \tag{1}$$

Since $a_1 = b_1 = 1$, we find $a_2 = 4$, $b_2 = 3$. Using $U(2m) = 2a_m + b_m$, we get $U(2) = 3$, $U(4) = 11$. Replacing m by $m+1$ in (1) and eliminating a's or b's, we get the fact that the a_m's as well as the b_m's satisfy

$$x_{m+2} - 4x_{m+1} + x_m = 0 \quad (m \geqq 1). \tag{2}$$

It follows that $U(2m)$ also satisfies (2) and consequently we have

$$U(2m) = \lambda(2+\sqrt{3})^m + \mu(2-\sqrt{3})^m$$

for suitable λ and μ. Using the known values of $U(2)$ and $U(4)$ we find

$$\lambda = \frac{1}{2\sqrt{3}}(\sqrt{3}+1), \quad \mu = \frac{1}{2\sqrt{3}}(\sqrt{3}-1).$$

Also solved by M. T. Bird, Dan Bump, J. Choné (France), George Dahir, L. J. Dickson (Australia), R. A. Gibbs, S. H. Greene, M. G. Greening (Australia), M. S. Klamkin, M. J. Knight, Harry Lass, O. P. Lossers (Netherlands), Jacob Strum & Michael Steiner (Israel), Phil Tracy, J. B. van Rongen (Netherlands), Aleksandras Zujus, and the proposer.

Subsemigroups of Composite Natural Numbers

E2418 [1973, 560]. *Proposed by C. A. Nicol, University of South Carolina*

Characterize those subsets S of the natural numbers with the property that every sum of elements taken from S (repetitions allowed) is composite.

I. *Solution by John Christopher, California State University at Sacramento, and M. J. Knight, California Institute of Technology.* (*Editor's Composite*). Let S be a set of natural numbers and let S^* be the set of all sums of elements taken from S. (I.e., S^* is the subsemigroup generated by S.) Then every element of S^* is composite if and only if the elements of S have a common prime factor p which does not lie in S.

The condition is clearly sufficient. To see that it is also necessary, suppose the greatest common divisor (GCD) of S is 1. Then S has a finite subset $T = \{c_1, c_2, \cdots, c_n\}$ with $\mathrm{GCD}(T) = 1$. Hence there exist integers $x_1, x_2, \cdots, x_n$ such that $c_1x_1 + c_2x_2 + \cdots, + c_nx_n = 1$. Collecting terms of like sign, we have an equation of the form $a - b = 1$, where $a, b \in S^*$. Since $(a,b) = 1$ it follows by Dirichlet's Theorem that the sequence $\{a + kb\colon k = 1, 2, \cdots\}$ contains (infinitely many) primes. But $a + kb \in S^*$ for every natural number k, implying that S^* contains primes, a contradiction. Thus $\mathrm{GCD}(S) > 1$ so that there exists a prime p such that $p \mid x$ for all $x \in S$. Obviously, however, $p \notin S$ since $S \subseteq S^*$ (one element sums are allowed).

II. *Solution by Leon Mattics, University of South Alabama.* Suppose $\mathrm{GCD}(S) = 1$. Let $a \in S$ have prime factorization $a = p_1^{e_1} p_2^{e_2} \cdots p_n^{e_n}$. Choose $c_i \in S$ such that $(p_i, c_i) = 1$, $i = 1, 2, \cdots, n$. Let

$$b = \sum_{i=1}^{n} \left(\prod_{j \neq i} p_j \right) c_i .$$

Then $b \in S^*$ and $(a, b) = 1$. Hence (as above) S^* contains primes.

Also solved by W. E. Bolton, S. D. Bronn, Jordi Dou (Spain), R. J. Evans, G. F. Feissner, P. K. Garlick, M. G. Greening (Australia), David Grinstein, D. Z. Kilhefner, Joel Levy, T. E. Moore, H. Niederreiter, Lothar Redlin, Kenneth Rosen, Steven Russ, Paul Smith, Joel Spencer, Walter Stromquist, Phil Tracy, J. B. van Rongen (Netherlands), and the proposer.

Editor's note. If single element "sums" are disallowed, then the common prime factor may be an element of S. About half of the solvers so interpreted the problem.

The key to the result is that if GCD $(S) = 1$, then S^* contains primes. It is known, however, that if GCD $(S) = 1$, then the subsemigroup S^* generated by S must contain all but a finite number of natural numbers. Levy refers to Theorem 1.4.1 of Kemeny and Snell, *Finite Markov Chains*, Van Nostrand, Princeton, 1959, and Moore refers to John C. Higgins, *Subsemigroups of the additive natural numbers*, Fibonacci Quart. 10 (1972), 225–230.

ADVANCED PROBLEMS

All solutions of Advanced Problems should be sent to J. Barlaz, Rutgers — The State University, New Brunswick, N. J. 08903. Solutions of Advanced Problems in this issue should be typed (with double spacing) on separate, signed sheets and should be mailed before August 31, 1974.

An asterisk (*) means neither the proposer nor the editors supplied a solution.

5970. *Proposed by Albert Wilansky, Lehigh University*

Let A be an algebra with identity and let I, J be two distinct ideals of co-dimension 1. Show that $IJ \cup JI$ spans $I \cap J$. (That is, every member of $I \cap J$ is a linear combination with scalar coefficients of members of $IJ \cup JI$.) Show also that IJ need not span $I \cap J$.

5971. *Proposed by William Sánchez and David Spear, New York City*

Can the positive reals $\mathbb{R}_+$ be partitioned into two sets both of which are closed under addition? Can these sets be Lebesgue-measurable?

5972. *Proposed by Gregory Wene, University of Iowa*

Let R be a ring and let * be the statement: There exists a positive integer $n \geqq 2$ such that $a^n = a$ for each $a \in R$.

For each ring R satisfying *, define the μ-value of R to be $\mu = \min\{m \geqq 2 \mid m$ a positive integer and $a^m = a$ for each $a \in R\}$. It is easily shown that 18 is not the μ-value for any ring. Find all positive integers k such that k is the μ-value for some ring and show that there are infinitely many positive integers which are not μ-values for any ring.

5973. *Proposed by G. Tsintsifas, Thessalonika, Greece*

Let $G = \{A_1, A_2, \cdots, A_n\}$ be a bounded set of points in the plane. If any three of these points can be covered by a strip of breadth d, show that G can be covered by a strip of breadth $2d$.

Find also the minimum real number k so that any point set G with the given property can be covered by a strip of breadth kd.

5974. *Proposed by M. S. Klamkin, Ford Motor Company*

Prove that aside from a polynomial of integration of degree $2n-1$,

$$x^{2n-1}\int\frac{dx}{x^2}\int\frac{dx}{x^2}\cdots\int\frac{dx}{x^2}\int x^{2n-1}F(x)dx = \iint\cdots\int F(x)(dx)^{2n}$$

where there are $2n$ integrals on each side.

5975. *Proposed by Richard Rado, University of Reading, England, and Albert Wilansky, Lehigh University*

An ordinal type A is called *thick* if there exists a family $(S_\alpha\colon \alpha\in A)$ of sets with $S_\alpha \subset S_\beta$ properly for $\alpha<\beta$ and card $\cup\{S_\alpha\colon \alpha\in A\} <$ card A. Show that R (the reals) is thick and that any well ordered type (i.e., ordinal number) is thin (= not thick).

* Determine other thick and thin ordinal types.

SOLUTIONS OF ADVANCED PROBLEMS

Free Abelian Groups

5900 [1973, 324]. *Proposed by E. R. Gentile, University of Buenos Aires, Argentina*

Let A and B be abelian groups (or modules over a principal ideal domain) such that $A\otimes B$ is a nonzero free abelian group (module). Prove that A and B are free.

Solution by D. F. Anderson, University of Chicago. First we assume that A and B are torsion-free and hence flat (J. Rotman, *Notes on Homological Algebra*, Theorem 4.23). Since B is torsion-free we have the exact sequence $0 \to Z \to B$; and thus $0 \to A \to A\otimes B$ is exact since A is flat. Hence A is a subgroup of $A\otimes B$ and thus free. Similarly B is free.

Now the general case. By P. A. Griffith, *Infinite Abelian Group Theory*, Theorem 39, we have

$$(A\otimes B)/t(A\otimes B) \approx (A/tA)\otimes(B/tB).$$

$A\otimes B$ is free, hence $t(A\otimes B)=0$. It follows that A/tA and B/tB are free; thus the exact sequence $0\to tA\to A\to A/tA\to 0$ splits, so that $A = tA\oplus A^*$, where $A^*\approx A/tA \neq 0$, is free. Similarly $B = tB\oplus B^*$, where $B^*\neq 0$, is free.

$$A\otimes B \approx (tA\otimes tB)\oplus(tA\otimes B^*)\oplus(tB\otimes A^*)\oplus(A^*\otimes B^*).$$

Now $A \otimes B \neq 0$ is free and we must also have $tA \otimes B^* = 0$. But B^* is free, hence $tA = 0$ and A is free. Similarly B is free.

Also solved by S. D. Brown & J. K. Cooper, D. L. Costa, Sam Cox (Puerto Rico), Tom Head, A. A. Jagers (Netherlands), R. E. Johnson, F. A. Oliva II, D. B. Shapiro, and the proposer.

Cox observes that a solution appears in Mark Ramas, *Free Exterior Powers*, J. of Algebra 19 (1971), 110–115. Head notes that the problem may be found in Laslo Fuchs, *Infinite Abelian Groups*, Vol. I (1972), Ex. 5, p. 265. Head also offers a proof of a broader version of the problem for modules over an arbitrary commutative ring with identity.

The Ring Operation $ax - xa$

5901 [1973, 324]. *Proposed by E. D. Dixon, Tennessee Technological University*

If a and x are elements of a ring R we denote $[a,x] = [a,x]_1 = ax - xa$ and, in general, $[a,[a,x]_h] = [a,x]_{h+1}$ for all positive integers h. Show that if P is a polynomial with coefficients which are integers or coefficients which are in the center of R, then

(1) $[a,[P(a),x]_h] = [P(a),[a,x]]_h$ and

(2) if $[a,x]_h = 0$ then $[P(a),x]_h = 0$.

Solution by D. L. Costa, University of Kansas. We prove (1) by induction on h. Since the coefficients of P are central we get the following chain of equalities in the case $h = 1$.

$$\begin{aligned} [a,[P(a),x]] &= a[P(a),x] - [P(a),x]a \\ &= a(P(a)x - xP(a)) - (P(a)x - xP(a))a \\ &= (P(a)ax - P(a)xa) - (axP(a) - xaP(a)) \\ &= [P(a),[a,x]]. \end{aligned}$$

Next, if (1) holds for a particular integer h we have

$$\begin{aligned} [a,[P(a),x]_{h+1}] &= [a,[P(a),[P(a),x]_h]] = [P(a),[a,[P(a),x]_h]] \\ &= [P(a),[P(a),[a,x]]_h] = [P(a),[a,x]]_{h+1}, \end{aligned}$$

where we have used our knowledge of the case $h = 1$ to obtain the second equality.

We will now use (1) and induction to obtain (2). $[a,x] = 0$ if and only if x commutes with a, in which case x commutes with every power of a. But then x commutes with $P(a)$, i.e., $[P(a),x] = 0$. So (2) holds when $h = 1$.

Now suppose (2) holds for a particular integer h and that $[a,x]_{h+1} = 0$. Then $[a,[a,x]]_h = [a,[a,x]_h] = [a,x]_{h+1} = 0$, using (1) with $P(a) = a$. This implies that $0 = [P(a),[a,x]]_h = [a,[P(a),x]_h]$, using (1), which by the case $h = 1$ implies $0 = [P(a),[P(a),x]_h] = [P(a),x]_{h+1}$. Therefore (2) holds for $h + 1$ and the proof is complete.

Also solved by S. D. Bronn, E. W. Ewing, E. A. Herman, A. A. Jagers (Netherlands), Fumio Kubo (Japan), H. S. Lieberman, Gary McDonald & Merry McDonald, Paul Milnes, Wanda J. Mourant, L. A. Oldroyd, David Spear, Tae-il Suh, E. J. Taft, E. S. Tsai, E. T. Wong, and the proposer.

Bronn, Spear and Wong show that (1) holds if $P(a)$ is replaced by an element c which commutes with a. Bronn also establishes the formula $[a, [c, x]_n]_m = [c, [a, x]_m]_n$.

"Balancing" a Polynomial in $\mathbb{R}^m$;

5902 [1973, 324]. *Proposed by John H. Hubbard*

Integration is with respect to Lebesgue measure. Let $X \subset \mathbb{R}^m$ be a measurable set of finite measure. For any function $f\colon \mathbb{R}^m \to \mathbb{R}$, call $X_f^+ = \{x \in X : f(x) \geqq 0\}$, $X_f^- = \{x \in X : f(x) \leqq 0\}$. Prove that for any whole number $n \geqq 1$, there exists a polynomial function $p\colon \mathbb{R}^m \to \mathbb{R}$ of degree at most n such that for any polynomial function $q\colon \mathbb{R}^m \to \mathbb{R}$ of degree at most n satisfying $q(0) = 0$,

$$\int_{X_p^+} q = \int_{X_p^-} q.$$

Solution by the proposer. Call P_n the vector space of polynomial functions on $\mathbb{R}^m$ of degree at most n, and give P_n the norm $\| q \| = \int_X |q|$. This norm is of class C^1 on $P_n - \{0\}$, in fact

$$\frac{d}{dt}\| q_1 + tq_2 \|(0) = \int_X \frac{q_1}{|q_1|} q_2 .$$

Let $H \subset P_n$ be the hyperplane of functions vanishing at 0, and choose any $f \in P_n$ such that $f \notin H$. Since P_n is finite-dimensional, the norm function restricted to $H + f$ reaches its minimum at some function p. For any $q \in H$, the function $\| p + tq \|$ of t has a minimum at $t = 0$, and hence

$$0 = \frac{d}{dt}\| p + tq \|(0) = \int_X \frac{p}{|p|} q = \int_{X_p^+} q - \int_{X_p^-} q .$$

The following further questions may be of interest:

(1) Give necessary and sufficient conditions on X and n for p to be unique up to multiplication by a constant.

(2) For fixed X, one gets as n increases a sequence of polynomials of increasing degree. Do the coefficients tend to stabilize (either become fixed, or converge to some limit)?

(3) Is the problem still true in the space of real-analytic functions on X? (For this problem to be reasonable, one must assume X open and $0 \in X$.)

The Associative Ring $\mathbb{R}^2$

5903 [1973, 325]. *Proposed by G. A. Heuer, Concordia College, and Albert Wilansky, Lehigh University*

If B is a two-dimensional noncommutative algebra over $\mathbb{R}$ (the real numbers) it is known that the multiplication in B is given by $ab = f(a)b$ or by $ab = f(b)a$ for some linear functional f on B. (Cf. Wilansky, *Functional Analysis*, problem 40, p. 258.) Is there a noncommutative multiplication in the set $\mathbb{R}^2$ which, together with the usual vector addition makes $\mathbb{R}^2$ into an associative ring which is not an algebra?

Solution by Jürg Rätz, University of Bern, Switzerland. The answer is affirmative as can be seen from the following theorem.

THEOREM. *Let M be a real vector space of dimension* $\geqq 2$. *Then there exists a noncommutative multiplication* $*: M \times M \to M$ *such that* (a) $\langle M, +, * \rangle$ *is an associative ring, and* (b) *for any subfield S of $\mathbb{R}$ different from Q (the rational numbers), M is not an S-algebra; specifically, M is not an $\mathbb{R}$-algebra.*

Proof. Let $H = \{y_i : i \in I\}$ be a Hamel basis of $\mathbb{R}$ over Q satisfying $1 \in H$. If we define the additive function $f: \mathbb{R} \to \mathbb{R}$ by $f(y_i) = 1$ for all i in I, then $f(\mathbb{R}) = Q$ and $\{\lambda \in \mathbb{R} : f(\lambda x) = \lambda f(x)$ for every $x \in \mathbb{R}\} = Q$. Let B be a Hamel basis of M over $\mathbb{R}$, say $d, e \in B$, $d \neq e$. Define the linear functional $g : M \to \mathbb{R}$ by $g(d) = 1$ and $g(c) = 0$ for every $c \in B \backslash \{d\}$. Now the multiplication $*$ is constructed by $a * b = f[g(a)] \cdot b$ for all a, b in M. $g(d) = 1$, $f(1) = 1$, $g(e) = 0$, and $f(0) = 0$ imply that $d * e = e$, $e * d = 0$, i.e., the noncommutativity of $*$. From the vector space structure of M and the Q-homogeneity of f and $f(\mathbb{R}) = Q$ we deduce the associativity of $*$. It is easily seen that the expression $f[g(a)] \cdot b$ is biadditive with respect to a and b, so $*$ is distributive over $+$, i.e., our assertion (a) holds.

Let $\lambda \in \mathbb{R} \backslash Q$ be arbitrary. Then there exists $z \in \mathbb{R}$ with the property $f(\lambda z) \neq \lambda f(z)$. It follows from the vector space rules in M that $[\lambda(z \cdot d)] * d = f(\lambda z) \cdot d$ and $\lambda \cdot [(z \cdot d) * d] = [\lambda f(z)] \cdot d$, i.e., $[\lambda(z \cdot d)] * d \neq \lambda \cdot [(z \cdot d) * d]$. Therefore M is not an S-algebra as soon as S contains elements $R \backslash Q$, which proves assertion (b).

REMARK. (b) cannot be improved in the following sense: If M is a real vector space with ring structure, then M is a Q-algebra.

Also solved by A. A. Jagers (Netherlands) and by the proposers.

Semigroup with $aba = a$

5904 [1973, 325]. *Proposed by S. Y. Chen and S. C. Hsieh, National Tsing Hua University, Taiwan*

Consider a semigroup S in which each pair of elements a, b satisfy $aba = a$.

For such a in S let T_a be the set of elements b such that $ab = a$. Show that if, for any a, T_a contains more than half the elements of S then $T_b = S$ for all b in S.

Solution by J. R. Gilbert, University of New Mexico. First note that for all p in S, $ppp = p$, and so $p^4 = p^2$. Also $p^4 = p(pp)p = p$, and so $p^2 = p$.

Now let a be an element of S such that T_a contains more than half the elements of S. If $x \in T_a$ and $y \in T_a$ then $a = ax$, and so $xa = xax = x$. Similarly $ya = y$. Then $xy = (xa)(ya) = x(aya) = xa = x$. Thus $y \in T_x$ for any x and y in T_a. Therefore we need only show that $T_a = S$.

Assume, for the sake of contradiction, that there exists $f \in S$ with $f \notin T_a$. If $x \in T_a$, consider xf. If $xf \in T_a$, then by definition of T_a, $a(xf) = a$. But $a(xf) = (ax)f = af$, and so we have $af = a$ and $f \in T_a$. Since this cannot be, we must have $xf \notin T_a$ for every x in T_a. Since T_a contains more than half the elements of S there must be distinct x and y in T_a with $yf = xf$. But then $yfx = xfx = x$, so that $yx = y(yfx) = yfx = x$ since $p^2 = p$ for all p in S. But also $yx = y$ since x and y are in T_a. Thus $y = x$. This contradiction denies the existence of f and completes the proof.

We note that although we have assumed S to be finite, the above proof works perfectly well for infinite S if by "T_a contains more than half the elements of S" we mean T_a has greater cardinality than $S \backslash T_a$, the complement of T_a with respect to S.

Also solved by Shaggi Aggi, J. T. Arnold, E. T. Beasley, Jr., S. D. Bronn, J. H. Carruth, D. L. Costa, S. C. Currier, Jr., Hugo D'Alarcao, Ronald Davis, E. W. Ewing, M. G. Greening (Australia), Kit Hanes, E. A. Herman, Craig Huneke, A. A. Jagers (Netherlands), H. F. Kennison, D. J. Kleitman, Charlotte Krauthamer (Austria), T. G. Kucera, Eric Lofgren, Gary McDonald & Merry McDonald, H. L. Miller, Wanda J. Mourant, G. M. Reekie, Eugene Sadowski, David Spear, D. P. Sumner, R. J. Weber, E. T. Wong, John Woods, and the proposers.

UNSOLVED PROBLEMS

This list supplements lists printed on p. 711 of the June-July, 1969 issue of this MONTHLY and on p. 1033 of the November 1971 issue.

Any comments or solutions will be welcomed by the editors.

5715 [1970, 197] Free R-module
5723 [1970, 313] Maximum expectation
5749 [1970, 775] Inequality in a set of integers
5773 [1971, 84] Complete linear space
5777 [1971, 202] Divisors of $\binom{m}{n}$
5790 [1971, 410] Maps preserving linearity
5794 [1971, 411] Bessel function
5808 [1971, 798] Arithmetic progression
5819 [1971, 912] $h(f(x), g(x)) = x$

5860 [1972, 667] Mean value property
5861 [1972, 667] Rolle's theorem
E2289 [1971, 405] Problem in logic

In response to several requests, we include the following list of problems for which notes or partial solutions have been printed, but for which complete solutions are still solicited.

5020 [1963, 574] Diophantine equation
5027 [1964, 441; 1966, 206] Maximal nonsingular subspaces
5052 [1964, 328] Multiplicative seminorm
5125 [1964, 806] Addition chains of vectors
5217 [1965, 681] Uniform polynomial approximations
5364 [1967, 214] Arrays of lattice points
5415 [1969, 948] Differential equation.
5479 [1968, 1018] Dissection
5485 [1968, 419] Covering with squares
5499 [1968, 1019] Rational triangles
5529 [1970, 657] Definite integral
5609, [1969, 571] Cardinality
5634 [1969, 952] Diophantine equation
5648 [1969, 1156] Set of squares in a group
5650 [1970, 314] Rational distances on a conic
5827 [1973, 326] Automorphism of p-groups
5849 [1973, 701] Approximations for π
5867 [1973, 1148] Copositive quadratic forms
E1782 [1966, 670] Coloring a chessboard
E1910 [1968, 80] Special sequence of integers
E 1978 [1968, 781] Minimum number of intersections
E 2056 [1969, 195] A three circle configuration
E 2112 [1969, 699] Arranging odd squares in even groups
E 2303 [1972, 664] Wilsonian numbers
E 2308 [1972, 774] Weird numbers
E 2384 [1974, 170] Binomial coefficient summation

Editorial Note. The editors regret to announce that several contributions regarding problem E2344 (below) seem to have been misplaced. We request all solvers to resubmit their solutions.

E 2344 [1972, 303]. *Proposed by Jordi Dou, Barcelona, Spain*

Consider a square array of red dots and blue dots with 50 rows and 50 columns. Whenever two dots of the same color are adjacent in the same row or column, connect them with a segment of that color; if they are adjacent of different color, connect them with a black segment. There are 1269 red dots, among them 99 on the border, none of them at the corners. There are 1035 black segments. Find the number of red segments and the number of blue segments.

REVIEWS

EDITED BY J. ARTHUR SEEBACH, JR. AND LYNN A. STEEN

with the assistance of the mathematics departments of St. Olaf and Carleton Colleges

COLLABORATING EDITOR FOR FILMS: SEYMOUR SCHUSTER, CARLETON COLLEGE

We invite readers to submit reviews of significant recent college-level mathematics books. We especially encourage reviews based on classroom use, or comparative reviews of several related books. Reviews should ordinarily not exceed two pages (per book) typed double spaced. Manuscripts of reviews as well as books submitted for review should be sent to: Book Review Editor, American Mathematical Monthly, St. Olaf College, Northfield, MN 55057.

Why Johnny Can't Add: The Failure of the New Math. By Morris Kline. St. Martin's Press, New York, 1973. ii + 173 pp. $6.95. (Telegraphic Review, May 1973.)

The other day I was explaining the new math to my 81-year-old mother. Results that used to be handed down as rules to be memorized, I told her, are now presented as theorems, derived from axioms: for example, the result that zero times any number is zero. She snorted. "What's the problem? If you add up nothing any number of times, you still have nothing." "Gosh, Mom, you'll never remember the rule that way." "Let's see, I learned it in 1901. Today is Wednesday, the..." "Yes, yes. But if you just knew the distributive property and some simple facts about addition, and could remember a few artificial steps that graduate students usually forget,..." "But how do I know the distributive property always works?" "Well, you see, we adopt it as an axiom."

Morris Kline has for many years been the acknowledged champ of the critics of the new math. In *Why Johnny Can't Add* he collects his criticisms under one cover. His writing is clear and simple, though somewhat repetitious. It can be good-humored, and it is frequently biting. His personal opinions are expressed in strong terms, he is unafraid of going out on a limb, and he quotes thinkers such as Poincaré and Felix Klein to back him up.

The major innovation of the new math is a rigorous, deductive approach, and the author advances many reasons why this is contrary to pedagogy — for example, utility determines logic, rather than the other way around: we make multiplication commutative for complex numbers but not for matrices. Sometimes, though, what he criticizes as underlying philosophy is really just lack of judgement on the part of individual writers — for example, the insistence that a student justify each tedious step. (My mother hasn't regrouped since 1902.)

Incidentally, could Johnny add before? According to the author himself, no tests have yet shown a win for either side.

Professor Kline deplores the development of mathematics for its own sake, structure as a guiding theme, and pedantic precision of language. Here he gives the new math too much credit: surely, much of its writing is sloppier than the old math, what with multiplying by a numeral or a minus sign, total confusion about quotation marks, and unbelievable bloopers with sets — e.g., the set of all animals on this farm is the set {chickens, cows, dogs} and therefore has exactly three members.

As for the new subjects taken up in the new math, Professor Kline considers them either too specialized to be worth while, or pointless novelties, or necessarily insignificant. "Oh, see, Johnny has a set of marbles. Look, look, Billy has a set of marbles. See Billy's set. Here comes Mary. Mary gets all the marbles. Mary gets the union of Johnny's set and Billy's set. See Mary's union." (Attributed to an anonymous critic.) But surely some topics, such as inequalities, can be taught early with profit: just teach them sensibly.

In a stinging chapter on the deeper reasons for the new math, the author excoriates the pure mathematicians for their mathematical narrowness, ignorance of science, and lack of interest in the psychology of learning. This chapter will stimulate thought and raise hackles.

The author concludes with his own recommendations for reform. Briefly, mathematical education should be broad rather than deep, and the basic approach at all levels should be intuitive, and motivated by applicatons. The author would make a stronger case if he came up with some suggestions in detail, such as an actual course outline or some sample pages of text, including applications that the student can understand.

As far as I can tell, neither Professor Kline nor those he is criticizing ever consider the psychology of *mathematical* thinking. For instance, perhaps the associative and commutative laws should be withheld from children in favor of a combined "grouping" law, according to which the sum (or product) of a collection of numbers can be found by grouping and adding (or multiplying) in any order. Intelligent research into such questions would seem highly worth while.

LEONARD GILLMAN, The University of Texas at Austin

Polynomials, Power Series, and Calculus. By Howard Levi. Van Nostrand, Princeton, New Jersey, 1968. viii + 158 pp. $6.95.

In this very interesting book, the author proposes to replace the usual beginning calculus course by one geared more directly to the students' needs. Minutiae about continuous functions, discontinuities, and the Riemann integral are abandoned. Most students will need to be able to handle only the relatively simple class of piecewise analytic functions. And the problems they encounter involving integrals can be cast in the first place in terms of antiderivatives: "to take up the definite integral ⋯ would amount to preparing the way for introducing certain complications and then taking additional steps to avoid them."

The book begins with a study of polynomial functions and approximations by them, leading to a discussion of derivatives and their properties. Then comes a chapter on applications of derivatives and antiderivatives, after which the author moves on to a careful and detailed treatment of power series and analytic functions.

The most interesting part of the book is the chapter on applications of derivatives and antiderivatives, based on the author's article, *Integration, Anti-Differentiation and a Converse to the Mean Value Theorem*, this MONTHLY, 74 (1967), 585–586. The converse in question is the following seemingly innocuous observation about a function f continuous on an interval I: suppose there is a function F on I such that $F(v) - F(u)$ lies between $(v - u)\min_{[u,v]} f$ and $(v - u)\max_{[u,v]} f$ whenever $u < v$; then F is differentiable and $F' = f$. To decide that linear velocity should be defined as the derivative of distance, for example, the author does not refer to average velocities over ever-smaller distances and then appeal to the student's blurred feeling for limits. Instead he appeals to our resolute conviction that a greater velocity yields a greater distance, puts this down as an *axiom*, and then *derives* the result from the above lemma. Similar reasoning would lead to the definition of density as the derivative of mass, etc.

The same lemma is applied to obtain formulas for area, volume, arc length, and gravitational attraction. In these formulas, the symbol $\int_a^b f$ stands for $F(b) - F(a)$, where $F' = f$. In essence, the lemma allows one to set up the integral and prove the Fundamental Theorem at the same time. Thus, integrals are set up without reference to infinitesimals, least upper bounds, or other limiting processes. Textbooks from Granville on have argued particular applications this way, but to the reviewer's knowledge, this is the first text where the argument is systematic. (For the information of the reader, the Riemann integral is defined in an appendix.)

The elementary transcendental functions $f = \exp, \sin, \cos$ are introduced via the functional equations for $f(x + y)$, which lead to formulas for the derivatives and thence to the power series.

The author is eloquent when it comes to talking *about* his subject, but his presentation *of* the subject tends to be brief. There are not nearly enough exercises. All in all, the style seems more like that of an article for the MONTHLY, suggesting what should or could be done and how, than of a textbook for actual classroom use (although the author reports having used the book successfully). In any case, the book certainly belongs on the reference shelf of every thoughtful teacher of calculus.

LEONARD GILLMAN, University of Texas at Austin

FILMS

Can You Hear the Shape of a Drum? A lecture by Mark Kac. Produced by the Individual Lecturers Project of the Committee on Educational Media, supported by a grant from N. S. F. to the Mathematical Association of America; A. N. Feldzamen, Executive Producer. Distributor: Modern Learning Aids. 16 mm, color and sound, 67 minutes.

A clear and well-presented lecture is an enjoyable experience. It is what one hopes for but too rarely finds at colloquium talks and meetings. It is becoming rarer as travel funds become scarcer. So it may well be worthwhile from time to time to forego the obvious advantages of a live lecture in return for a first-rate presentation by a first-rate mathematician and expositor in the form of a moving picture.

The material presented by Professor Kac may well be too technical for most undergraduates to absorb. But it is important for both students and faculty members to learn that quite technical material can be presented with wit, with clarity, and above all, with enthusiasm.

The following mathematical problem is considered: if all the eigenvalues $\lambda_1 \leqq \lambda_2 \leqq \cdots$ of the membrane problem

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \lambda u = 0 \quad \text{in } \Omega$$

$$u = 0 \quad \text{on the boundary,}$$

are known, what can one say about the region Ω? While the original question is whether Ω is completely determined by its eigenvalues, Professor Kac confines himself to showing that certain of its properties are determined.

The most important concept conveyed by this lecture is what applied mathematics is (or at least ought to be) all about. Professor Kac begins with an interesting question concerning a physical system. By using a known mathematical model of the physical system, he converts the question into a mathematical question of mathematical interest. Professor Kac's knowledge of related mathematical models for quite different physical systems then permits him to bring his physical intuition to bear in formulating a plausible conjecture and in constructing first a heuristic proof and then a mathematical proof of this conjecture. The process evolves before one's eyes.

Seeing this movie ought to be a part of the education of every student, and certainly of every graduate student of mathematics.

The technical quality of the movie is excellent. It is clearly made by professionals to professional standards. The writing on the whiteboard is almost always clearly visible when it is needed.

The only change I could suggest would be to abridge a somewhat lengthy introduction of Professsor Kac at the beginning.

H. F. WEINBERGER, University of Minnesota

Editor's Note. Another version of the film "Can You Hear the Shape of a Drum?" deletes some of the highly technical details of the so-called "complete version" that Professor Weinberger reviewed. The shorter version, which is regarded as the primary one for advanced undergraduates, is 49 minutes in length; it, also, is available to buy or rent from Modern Learning Aids.

Calculus in Motion. Eight computer-generated films. Produced by Bruce and Katherine Cornwell in collaboration with Duane Bailey. Animated, silent, in color, with each film three to four-and-a-half minutes. Available in super 8mm Technicolor Magicartridges or Kodak Cassettes. Distributor: Houghton Mifflin Company.

This series of film loops is for use in the beginning calculus sequence. Its emphasis is on developing geometric intuition rather than presenting rigorous definitions and proofs. With the appropriate projection equipment, the films can be shown without darkening the classroom. Absence of a sound track coupled with stopmotion capabilities enable the instructor to state definitions, fill in proofs, ask questions and present other details to whatever extent is desired. Each film will now be discussed separately.

1. *Functions.* A function is presented as a series of arrows from one copy of the real numbers to another. No formulas are given. After several examples, the range line is rotated 90 degrees to exhibit the usual graph of a function. I would have preferred that the film state the formulas for the examples, but the class and instructor may learn by trying to determine what functions are shown.

2. *Limits.* The two methods used in the first film for presenting a function are now employed to describe the limit concept. There is a definite and explicit emphasis on the fact that $\lim_{x\to a} f(x)$ does not depend on $f(a)$. Unfortunately, no examples of undefined limits are given.

3. *Derivatives.* This film contains an excellent presentation of the derivative using a mobile differential triangle to graph the derivative. There is strong emphasis on the derivative as a derived *function*. The discussion is in terms of rectangular coordinates; the derivative is given as slope of the tangent with no explicit definition. While no examples of non-differentiable functions are treated explicitly, one such function appears in the later film on Rolle's Theorem and the Mean Value Theorem. The second derivative is also treated.

4. *Concavity and Points of Inflection.* The film states that a function is concave

up if it lies above the tangent and concave down if it lies below the tangent. (These definitions should be "a function is concave *up at a* if it lies above the tangent *to the graph at a*.") Concavity is also defined in terms of the direction of rotation of the tangent as a point moves along the graph. Then the second derivative is computed to indicate the relation between zeros of the second derivative and points of inflection.

5. *Rolle's Theorem and the Mean Value Theorem.* This film, which I judge to be very good, begins with an arbitrary function (discontinuous, with unequal values at endpoints) and changes it into one satisfying the hypothesis of Rolle's theorem. The knowledge of Rolle's Theorem is then applied to obtain the Mean Value Theorem. The film concludes with a function which fails to have a derivative at some point (with different left and right hand derivatives) and asks why the Mean Value Theorem fails to apply.

6. *The Definite Integral.* Inner and outer rectangular approximations of area are discussed. Areas are converted to a 1 by A rectangle.

7. *The Fundamental Theorem.* The graph of $f(x)$ and the derivative of the integral are compared for several examples. Use is made of the same differential triangle method as was introduced in the film on *Derivatives.* This film and the preceding one are both straightforward and standard, except for the method of graphing the derivative.

8. *Taylor Polynomials.* Graphs of Taylor polynomials up to degree 16 are exhibited for $\sin x$, $\exp(x)$, and a linear combination of sine and cosine. The graphical descriptions are very effective, with the intervals of convergence clearly illustrated.

The series is of uniform technical quality, but if budgetary considerations limited my choice to three films, I would pick *Functions, Derivatives,* and *Taylor Polynomials* as the three which deal most effectively with topics that seem to give some students difficulty.

PIERRE J. MALRAISON, JR., Carleton College

TELEGRAPHIC REVIEWS

Telegraphic reviews are designed to give prompt notice of new books with sufficient information to assist our readers in deciding whether to order an examination copy or to suggest library purchase. Possible uses are indicated as follows:

T = textbook P = professional reading
S = supplementary reading L = undergraduate library purchase
13 to 18 = freshman to second year graduate level usage
1 to 4 = appropriate time in semesters to cover text

Asterisks (*) or question marks (?) denote special positive or negative emphasis, respectively. Publishers are denoted by standard abbreviations; complete addresses may be found in *Books in Print*.

GENERAL, S, P, L**, *A Bibliography of Recreational Mathematics, V. 3*. William L. Schaaf. NCTM, 1973, x + 175 pp, $5 (P). Continues with works published since mid-1969, plus a few repeats. Overall organization remains largely the same. New features include chapter on classroom games and recreational activities, chronological synopsis of Martin Gardner's "Mathematical Games" column in *Scientific American*, and a 500-entry glossary of terms in recreational math. A valuable bibliography resource for student projects. PJC

GENERAL, T(13: 1), *Mathematics as A Second Language*. Joseph Newmark, Frances Lake. A-W, 1974, xiv + 482 pp, $10.95. A book which lives up to the expectations of the title. Written for the liberal-arts student with little or no mathematical preparation, it has put such topics as game theory, linear programming, probability and statistics at a level easily handled by a large cross-section of students. There are many examples directly related to most students' own experience. LLK

GENERAL, T(13: 1), *Fundamentals of Finite Mathematics*. Benjamin W. Volker, Andrew S. Wargo. Intext, 1972, xi + 368 pp, $9.95. Aimed at students with virtually no mathematical background or maturity. First 100 pages on logic and sets. Then probability, statistics, linear equations. Finally, a very elementary reference to linear programming. A reasonable number of problems with answers to some.LH

GENERAL, P, *Education in Applied Mathematics*. SIAM, 122 pp, $5 (P). Essays (with discussion) by C.C. Lin, P.D. Lax, E.W. Montroll, G.F. Carrier, J.B. Rosser, and R.E. Gomory from a 1966 S.I.A.M. conference. Reprinted from *Siam Review* 9 (1967) 289-415. Still a timely and useful document, although nearly a decade old. LAS

GENERAL, S, L, *Test Your Logic: 50 Puzzles in Deductive Reasoning*. George J. Summers. Dover, 1972, vi + 100 pp, $1.50 (P). The clues are given, either explicitly or implicitly, in story form for a master detective. Junior high algebra is helpful in ten of them; otherwise clear, constructive reasoning suffices. Hints at the bottom of the page offer fresh insight when needed. Very complete solutions. LCL

GENERAL, S, *Puzzles for Pleasure*. E.R. Emmet. Emerson, 1972, x + 310 pp, $6.95. 102 logical brain teasers, more than half of which "have not previously appeared in print." Complete solutions comprise half the volume. LAS

GENERAL, P. *Matematikken i Samfundet*. Else og Jens Høyrup. Gyldendals, 1973, 184 pp, KR32,50 (P). An analysis of the social (as distinct from the technical) role of mathematics in society. Examples and exhortation concerning such subtle factors as the effect of memorizing abstract concepts on the student's susceptibility to ideological manipulation, or the effect of school textbook illustrations on the development of a student's self-image. A pioneering effort; interesting, but not yet very profound. LAS

GENERAL, S(13). *Topics in Modern Mathematics, V. I.* T.D.H. Baber. Pitman, 1972, ix + 178 pp, £1.40 (P). Elementary but brief introduction to the real number system, set theory, inequalities, relations and functions, non-decimal arithmetic, digital computers, linear programming, matrices and vectors. Exercises. JNC

BASIC, T(13: 1). *Basic Mathematics*. Richard L. Steinhoff. McGraw, 1973, x + 308 pp, $5.95 (P). Prealgebra workbook concerning arithmetic and its applications. Substantial portion deals with formulas. LCL

BASIC, T(13: 1). *Pre-Algebra Mathematics*. Gerald S. Lieblich, Charles Leake. Merrill, 1973, xiv + 318 pp, $8.95. A response to an open admissions program at Bronx Community College. The text together with the student manual (exercise book) provide a behaviorist-oriented review of mathematics through arithmetic. Clearly done but very basic. JAS

BASIC, T(13: 1). *Applied Mathematics for Technical Programs: Algebra*. Robert G. Moon. Merrill, 1973, ix + 337 pp, $7.95 (P). Blocked out text (almost a programmed text) designed for use with taped explanations of examples illustrating each idea. Algebra through quadratics, fractions and graphing. JAS

BASIC, S(13). *Modern Mathematics for Managers*. K. Williams. Longman, 1972, viii + 112 pp, $3.50 (P). Directed to business managers with little mathematical skills. Hence, topics include sets, logic, Boolean algebra, group theory. Does discuss Bayes theorem, matrices. Can be read quickly--good exmaples, no problems, clear, lively, conversational style. LH

BASIC, T?(13: 1, 2). *Mathematics for Industrial Technicians*. Chester Pachucki. P-H, 1974, xvi + 457 pp, $9.95. Arithmetic, algebra and "advanced topics"--logarithms, geometry, trigonometry and complex numbers. A cookbook approach. For example, least common divisors are found by looking at prime factorizations with no mention of Euclidean algorithm. If one wishes to be competent in basic mathematical techniques this is an acceptable book. Examples and exercises are taken from real situations. The question is, do industrial technicians (or biologists, etc.) need their own special texts in mathematics? PJM

PRECALCULUS, T*(13: 2). *A Second Course in Algebra*. David L. Outcalt, J.P. Wood. Merrill, 1974, xi + 708 pp, $12.95. Very large, but a good selection of topics. Exceptional because it is well-written and readable. Also contains many examples, good problems, and emphasis on important statements with shading and blocking. LLK

PRECALCULUS, T(13: 1, 2). *Precalculus Mathematics: An Elementary Functions Approach.* David A. Sprecher. Har-Row, 1974, x + 363 pp, $10.95. A good pre-calculus book. Mathematically sound, but not too rigorous, the only flaw in my mind being too much introductory material on functions and graphing before the student gets to work with polynomials. Irrational exponents are done via Cantor's postulate and nested intervals, very neatly. Chapters on real numbers, beginning coordinate geometry, sets and functions, polynomial and rational functions, exponential, logarithmic and trigonometric functions, complex numbers and more coordinate geometry (graphing quadratics). PJM

PRECALCULUS, T(13: 1). *Essentials of Algebra and Elementary Functions.* H.S. Bear. Freel, 1973, xi + 493 pp, $8.50 (P). An exact copy of *Algebra and Elementary Functions* (TR, March 1973) with the deletion of chapters 14 and 15 (3-dimensional geometry). LLK

PRECALCULUS, T(13: 2). *Precalculus Mathematics: An Introduction.* Basil M. Wall, Charles R. Wall. Merrill, 1974, xii + 428 pp, $10.95. A large number of precalculus topics presented in "plane English" including matrices and determinants, analytic geometry, conic sections, and complex numbers. LLK

PRECALCULUS, T(13: 1). *Precalculus Mathematics: A Functional Approach.* Earl Swokowski. Prindle, 1973, viii + 514 pp, $12.95. A straightforward presentation of precalculus topics from algebra, trigonometry and analytic geometry. Good collections of word problems for this level. Also review exercises, both oral and written, at the end of each chapter. LLK

PRECALCULUS, T(13 : 1). *A Precalculus Course in Algebra and Trigonometry.* Earl W. Swokowski, *et al.* Prindle, 1973, 715 pp, $14.95 (P). The first ten chapters of his book *Precalculus Mathematics: A Functional Approach* each followed by sections entitled Homework Problems; Things to Study; Sample Test; and Programmed Supplement. LLK

EDUCATION, T(13-16: 1, 2). *Introduction to Modern Mathematics, Second Edition.* Dora McFarland, Eunice M. Lewis. Heath, 1973, xi + 478 pp, $9.50. Aimed at future elementary teachers and parents of grade school children. Topics: sets, numbers, numeration, natural numbers, whole numbers, arithmetic of whole numbers, integers, rational numbers, arithmetic of nonnegative rational numbers, real numbers, metric geometry, relations and fucntions, mathematical systems, probability. Scores of examples and exercises; some interesting historical notes. Designed for a two semester course. SG

EDUCATION, T(13-15: 1). *Mathematics for the Elementary School Teacher.* Eugene D. Nichols, Robert L. Swain. HR&W, 1971, xii + 518 pp, $9.95. Topics chosen to help an elementary school teacher understand arithmetic processes beginning with logic. Includes geometry and probability. LLK

EDUCATION, P, L. *Mathematics Education: The Sixty-ninth Yearbook of the National Society for the Study of Education, Part I.* Ed: Edward G. Begle. U of Chicago Pr, 1970, xi + 474 pp, $7. 14 reflections on the status and future of school mathematics, prepared following the curricular "shock wave of radical change" of the sixties. Contributors include R.L. Wilder, J.L. Kelley, H.O. Pollak. LAS

EDUCATION, S*(16), P**, *A Guide to Teaching About Computers in Secondary Schools.* Donald D. Spencer. Abacus, 1973, xii + 138 pp, $12.95. Comprehensive handbook for inexperienced teachers showing how computers may be used in secondary school education. Excellent reference source for textbooks, films and other teaching aids. Can also be used as a supplementary text in a methods course. RSK

EDUCATION, P, L, *The Use of Computers in Secondary School Mathematics.* Ed: Dudley L. Post. Entelek, 1970, xii + 252 pp, (P). This is a book for the administrator and the teacher planning to use the computer in the classroom. Time-sharing is exclusively stressed and BASIC is used in the examples. Includes extensive discussion of Computer Aided Instruction (CAI) of elementary mathematics in the classroom. The last chapter gives a good review of some recent results. RB

HISTORY, P, L, *Functional Analysis in Historical Perspective.* A.F. Monna. Wiley, 1973, viii + 167 pp, $16.50. A pastiche of quotations (untranslated, except for some of Peano's Italian) from the creators of functional analysis, held together with brief, informative commentary by the author. Requires real fluency in French and German (e.g., to read several *verbatim* pages of Grassmann) and costs 10¢ a (small) page. Otherwise would be a valuable resource for all mathematics majors. LAS

HISTORY, P, *Demotic Mathematical Papyri.* Richard A. Parker. Brown U Pr, 1972, xiv + 86 pp, and 25 plates, $25. Analysis of papyri containing 72 mathematical problems in arithmetic of fractions, plane and solid geometry, trigonometry, and arithmetic progressions. Surprising occurrence of use of fractions with numerators other than 1--earliest known use in Egypt (3rd century B.C.). Papyri are reproduced almost full-size: 11 fragments from Cairo Museum, 3 from British Museum, 1 from U. of Copenhagen, all from 3rd century B.C. to 2nd century A.D. *Demotic* (also called *enchorial*) was the popular and simplified form of later Egyptian writing, the formal form being *hieratic*. Both were cursive, distinguishing them from the earlier *hieroglyphics*. PJC

FOUNDATIONS, S(14-16), P, L*, *Philosophy and Mathematics: From Plato to the Present.* Robert J. Baum. Freeman, Cooper, 1973, x + 320 pp, $9.95; $6.95 (P). Brief introduction to and extensive excerpts from the writings of 12 classical and 3 contemporary philosophers that bear on the nature of mathematical truth and on the existence of mathematical objects. Designed to fill a gap in existing literature; deliberately avoids overlap with the existing anthologies concerned with foundations of mathematics and mathematical logic. A handy volume that collects in one place many pertinent passages. LAS

FOUNDATIONS, P*, *Lecture Notes in Mathematics-328; Decidable Theories II. The Monadic Second Order Theory of All Countable Ordinals.* J. Richard Büchi, Dirk Siefkes. Springer-Verlag, 1973, vi + 217 pp, $9 (P). Let MT[$\alpha$] and MT[co] denote the monadic second order theories of α and of all countable orders respectively. Büchi shows that MT[co] and MT[α] for $\alpha \leq \omega_1$ are decidable. The proofs are quite general and instructive. Siefkes and Büchi then characterize MT[co] and MT[α] for $\alpha \leq \omega_1$ by axiom systems. LCL

FOUNDATIONS, P. *Lecture Notes in Mathematics-344; Metamathematical Investigation of Intuitionistic Arithmetic and Analysis.* Ed: A.S. Troelstra. Springer-Verlag, 1973, xvii + 485 pp, $13.10 (P). Unpolished notes (largely by Troelstra with contributions by C. Smorynski, J.I. Zucker and W.A. Howard) based on courses at U. Utrecht and U. Amsterdam. Purpose is to give a "coherent presentation" of methods (not necessarily of results) in the metamathematical study of intuitionistic formal systems. LAS

COMBINATORICS, S(15-18), P, L. *Topics in Combinatorial Mathematics.* C.L. Liu. MAA, 1972, ii + 265 pp, $2.35 (P). Lecture notes (photo-offset from typescript) from 1972 MAA Cooperative Summer Seminar at Williams College. Some topics of purely mathematical interest, others of an applied nature; all involve recent results. Ramsey's theorem, matching theory, colorability, Pólya's theory of counting, sorting and scheduling algorithms, and more. PJC

NUMBER THEORY, P, L*. *Riemann's Zeta Function.* H.M. Edwards. Acad Pr, 1974, xiii + 315 pp, $21.50. A valuable addition to the literature. A study of Riemann's famous paper on the distribution of primes (a translation of that paper is included). The book is concerned with the Riemann hypothesis, the prime number theorem, the number of zeroes on $Re(s) = 1/2$ and other problems raised by Riemann in that paper. Also covered thoroughly is the Riemann-Siegel formula. Well-written. Should become the standard reference. SG

NUMBER THEORY, S*(15-17), P, L*. *Two Papers on Number Theory.* L.J. Mordell. VEB Deutscher Verlag, 1972, 75 pp, (P). Reprints of two works originally published by Cambridge U Pr: "Three lectures on Fermat's last theorem" (1921), and "A Chapter in the theory of numbers" (1947), inaugural lecture regarding rational and integer solutions of the Diophantine equation $y^2 = x^3 + k$. LCL

NUMBER THEORY, T(14-15). *Elementary Theory of Numbers.* Harriet Griffin. McGraw, 1954, ix + 203 pp, $2.45 (P). A paperback version of a text first published in 1954. Contains the standard material covered in an elementary number theory course. The writing is quite clear; some historical remarks are worked into the text; almost all of the problems are routine. SG

LINEAR ALGEBRA, T(14: 1). *Introduction to Linear Algebra and Differential Equations.* John W. Dettman. McGraw, 1974, xi + 404 pp, $11.95. After a first chapter on complex numbers, the book is evenly divided between linear algebra and differential equations. Good format--an introduction to each chapter states the contents of the chapter, then the author procedes to carefully follow this outline. Each chapter ends with a starred section which deals with more advanced topics.LLK

LINEAR ALGEBRA, T(14: 1). *Elements of Linear Algebra, Second Edition.* Lowell J. Paige, J. Dean Swift, Thomas A. Slobko. Xerox, 1974, ix + 287 pp, $10.95. Intended as a text for a one-semester course for students with a year of calculus. Topics: vectors and vector spaces; linear transformations and matrices, determinants, bilinear mappings and quadratic forms, complex numbers and polynomial rings, characteristic values and vectors, canonical forms, applications (simplex method, least-squares). Exercises have a wide range of difficulty. Illustrative examples used throughout. Clearly written. SG

LINEAR ALGEBRA, T(13-14: 1). *Linear Algebra; An Introduction*. Paul J. Knopp. Wiley, 1974, xvi + 435 pp, $10.95. Written to be used at either freshman or sophomore level. Not intended to teach methods of proof writing. Many examples and exercises. LLK

LINEAR ALGEBRA, T(16-18: 1, 2), L. *Finite Dimensional Multilinear Algebra, Part I*. Marvin Marcus. Dekker, 1973, x + 292 pp, $14.50. This first volume presents tensors and their transformations followed by tensor algebras, the latter emphasizing Grassmann and symmetric algebras and derivations. A no-nonsense definition-theorem-proof presentation in typescript. The unusual point of this book is its mass of problems with each section; the last exercise set is 31 pages long! A rather small subject index is supplemented by a very necessary and extensive notation index. JAS

ALGEBRA, T(17-18), S, P, L. *Lectures on Numerical Algebra*. A.S. Householder. MAA, 1972, iii + 257 pp, $3 (P). Notes based on lectures delivered at Williams College, summer 1972. The aim is to develop algorithms for solving systems of linear and non-linear equations, for inverting matrices, for finding eigenvalues and eigenvectors, and to derive error estimates. Familiarity with elementary linear algebra and complex variables is assumed. Can provide the basis for an undergraduate seminar. No exercises. SG

ALGEBRA, P. *Report of Algebra Group*. Queen's Papers in Pure and Appl. Math., No. 36. Queen's U, 1973, ii + 352 pp, $6.50 (P). Abstracts, preprints and occasional papers from the informal 1972-73 algebra group at Queen's U., Kingston, Ontario. LAS

ALGEBRA, T(15-17: 1), L. *Elementary Rings and Modules*. Iain T. Adamson. B&N, 1972, 136 pp, $6 (P). Straightforward presentation of modules--content parallels the elementary sections of the author's *Rings, Modules and Algebra*. The final third is concerned with commutative rings, primarily UFD's and Dedekind domains. LCL

ALGEBRA, T(17-18), P. *Rings with Polynomial Identities*. Claudio Procesi. Dekker, 1973, viii + 190 pp, $15.50. Systematic presentation of PI-rings: categorical properties, structure theorems, application to irreducible representations of rings, special topics. Emphasis on recent results; includes several open problems. LCL

ALGEBRA, P. *Lecture Notes in Mathematics-351: Quasi-Frobenius Rings and Generalizations QF-3 and QF-1 Rings*. Hiroyuki Tachikawa. Springer-Verlag, 1973, xi + 172 pp, $7.40 (P). An account of recent (last 25 years) results in the theory of quasi-Frobenius rings. The author proves the structure theorem for QF-3 rings; discusses dominent dimension, Nakayama's lemma, rings of finite representation type, and the double centralizer condition. SG

ALGEBRA, T(15-17), S. *Interdisciplinary Mathematics, V. I-III*. Robert Hermann (18 Gibbs St, Brookline, MA 02146), 1973. *V.I: General Algebraic Ideas*, $7.50 (P); *V. II: Linear and Tensor Algebra*, $6.50 (P); *V. III: Algebraic Topics in Systems Theory*, $10 (P). Informal, privately printed lecture notes which comprise an introduction to algebra strongly flavored with applications to physics and optimal control theory. The first two volumes are more or less standard algebra; the third is essentially applications of systems of linear differential equations. LAS

ALGEBRA, P, *Lecture Notes in Mathematics-353: Proceedings of the Conference on Orders, Group Rings and Related Topics.* Ed: Thomas G. Ralley. Springer-Verlag, 1973, x + 224 pp, $8.20 (P). From Ohio State U., May 1972, in honor of Hans Zassenhaus. LAS

ALGEBRA, T(15-16), L, *Elementary Modern Algebra.* Robert C. Thompson. Scott F, 1974, 472 pp, $10.95. An interesting approach to abstract algebra. The definitions of groups, rings and fields are given in the first fifty pages. The remainder of the book is devoted to examples: the complex field, the integers, polynomial rings, fields of fractions, mappings of rings. A final chapter on special topics includes valuations, p-adic representations, and continued fractions. The use of optional sections and subsections (e.g., proof of irrationality of e, roots of the derivative of a complex polynomial) makes the book suitable for students with a wide range of abilities. Problems are also divided into three types: computational, easy-theoretical, and hard-theoretical. This is convenient for the instructor, but sometimes upsets students ("I can't do that, it's a type III"). Index and supplementary readings. PJM

ALGEBRA, P, *The Arithmetics of Quadratic Jordan Algebras.* Michel L. Racine. Memoirs No. 136. AMS, 1973, vi + 125 pp, $3.20 (P). Study of the arithmetic of a quadratic Jordan algebra over the quotient field of a Dedekind domain. The author considers maximal orders of special and exceptional Jordan algebras, and determines the number of isomorphism classes of maximal orders over a local field. SG

ALGEBRA, P, *Ring Theory: Proceedings of the Oklahoma Conference.* Ed: Bernard R. McDonald, Andy R. Magid, Kirby C. Smith. Lect. Notes in Pure and Appl. Math., V. 7. Dekker, 1974, xvi + 295 pp, $7.75 (P). Fifteen survey papers "designed to inform mathematicians about current trends in ring theory." From a March 1973 conference at U. Oklahoma. LAS

ALGEBRA, P, *Lecture Notes in Mathematics-341, 342, 343: Algebraic K-Theory I-III.* Ed: H. Bass. Springer-Verlag, 1973. *V. I: Higher K-Theories*, xv + 335 pp, $10.70 (P); *V. II: "Classical" Algebraic K-Theory, and Connections with Arithmetic*, xv + 527 pp, $14.80 (P); *V. III: Hermitian K-Theory and Geometric Applications*, xv + 572 pp, $15.60 (P). Proceedings of the conference held at the Battelle Institute in Seattle, August 28-September 8, 1972. A comprehensive, coherent presentation of current research. LAS

ALGEBRA, T(18: 1), P, *The Structure of Linear Groups.* John D. Dixon. Van-N-Rein, 1971, iv + 183 pp, $6.95 (P). Finite and infinite groups having faithful linear representations. Chapters on primitive irreducible, finite non-modular, solvable, nilpotent, p-solvable, algebraic and periodic groups and one on the method of finite approximation follow a substantial general introduction. Many exercises and references. Well-suited either for a seminar or for self-study. DFA

CALCULUS, T(14: 1), *Calculus with Analytic Geometry.* Ralph Crouch, Albert Herr, Dorothy B. Sasin. Prindle, 1972, 234 pp, $11.95 (P). Topics in series and multivariable calculus to be used with *Calculus with Analytic Geometry* (TR, February 1972). Inexplicably, these two volumes have identical titles. LLK

CALCULUS, T(13-14: 1-3), *Introductory Mathematical Analysis: For Students of Business and Economics*. Ernest F. Haeussler, Jr., Richard S. Paul. Reston, 1973, xxxix + 600 pp, $13.95. Elementary functions, calculus, and matrix algebra with emphasis on applications to economics. Many examples. At a well-chosen level of rigor. FLW

CALCULUS, T(13-14: 4), L. *Mathematical Methods for Science Students, Second Edition*. G. Stephenson. Longman, 1973, ix + 528 pp, $4.50 (P). First half covers the standard topics of elementary and advanced calculus; the second half is concerned with basic linear algebra and applications to differential and integral equations. JJ

CALCULUS, T(13-14: 1-3), S. *An Introduction to Calculus and Algebra, 3 Volumes*. Open University. Har-Row. *V. 1: Background to Calculus*, 1971, x + 245 pp, $4.95 (P); *V. 2: Calculus Applied*, 1971, x + 242 pp, $4.95 (P); *V. 3: Algebra*, 1972, xii + 480 pp, $6.95 (P). A textbook covering a typical calculus-linear algebra-differential equations syllabus, extracted from the Open University's more comprehensive *Foundations Course* (TR, April 1974). This abridged version is designed for students more interested in science and technology than in mathematics *per se*. Some exercises are scattered throughout the text with more or less complete solutions given at the chapter ends. Additional exercises and supplementary material are provided by the coordinated series of workbook-modules *Elementary Mathematics for Science and Technology* (TR below). Although few proofs are given (reasonable, in view of the intended audience) these volumes retain the rather high level of definitional rigor present in the *Foundations Course* from which they are derived (e.g., functions nearly always denoted by $f:x \to 2x+1$; strong discussion of morphisms in the algebra sections). LAS

CALCULUS, T(13-14: 1-3), S. *Elementary Mathematics for Science and Technology, 9 Volumes*. Open University. Har-Row, 1972, 574 pp, $28.65 (P). Modular workbooks for a multi-media self-instructional course designed by the Open University of Great Britain. This course is a stripped down version of the comprehensive *Foundations Course* (TR, April 1974) designed for students of science and technology. It is based on the three volume text *An Introduction to Calculus and Algebra* (TR above) which is itself a modified version of the *Foundations Course* material. Each workbook is available separately at prices of $2.25-$4.25, as are 17 coordinated films ($125@) and 4 audio tapes ($7.50@). Syllabus covers sets, calculus, optimization, Taylor series, vectors, matrices, differential equations. LAS

COMPLEX ANALYSIS, P. *Lecture Notes in Mathematics-332: Séminaire Pierre Lelong (Analyse) Année 1971-1972*. Ed: Pierre Lelong. Springer-Verlag, 1973, 131 pp, $7.30 (P). From Institut Henri Poincaré. JAS

DIFFERENTIAL EQUATIONS, P. *Théorie des Perturbations et Méthodes Asymptotiques*. V.P. Maslov. Dunod, 1972, xvi + 384 pp. Contents: perturbation theory; regularization problem, equations with operator coefficients, strong convergence of operational equations, one-parameter semi-groups; asymptotic methods; equations of quantum mechanics, partial differential equations, local characteristic representation for certain equations, local and global asymptotic properties of solutions to various equations, quasi-classical formulas for solutions to quantum mechanical equations. Appendices by Arnol'd and Bouslaev. SG

DIFFERENTIAL EQUATIONS, T(14-16; 1), L. *A Short Course in Differential Equations, Fifth Edition.* Earl D. Rainville, Phillip E. Bedient. Macmillan, 1974, xi + 320 pp, $10.95. Only one major change from the 4th edition (TR, June 1969): the inclusion of a matrix approach to systems of linear equations (the elimination method is still used). Otherwise, the text remains clearly written with plenty of exercises, plenty of applications. SG

DIFFERENTIAL EQUATIONS, T(14-16). *Differential Equations, Second Edition.* Shepley L. Ross. Xerox, 1974, xi + 712 pp, $12.50. A well-written text. The first nine chapters are intended for a one semester introductory course: first and second order equations; series methods; linear systems; approximations; Laplace transform. The last five chapters present existence and uniqueness theory; Sturm-Liouville problems; nonlinear and partial differential equations. The exposition is very complete; definitions, theorems, and methods are always illustrated with examples. The problems are plentiful, although most of them are routine. SG

NUMERICAL ANALYSIS, P. *Diophantine Approximation and Its Applications.* Ed: Charles F. Osgood. Acad Pr, 1973, ix + 356 pp, $12.50. Fourteen papers presented at the conference held at the Naval Research Laboratory, Washington, D.C., June 1972. JAS

FUNCTIONAL ANALYSIS, P. *Minimum Principle and Maximality.* Gunter Ritter. Lect. Notes Series, No. 37. Aarhus U, 1973, i + 53 pp, $4 (P). Fairly technical lectures on constructions of a harmonic space from a system with the minimum principle and maximality. A harmonic space is (roughly) a sheaf of harmonic functions on a subset of R^n. PJM

FUNCTIONAL ANALYSIS, P. *Lecture Notes in Mathematics-346: Spectral Analysis of Nonlinear Operators.* Svatopluk Fučik, *et al.* Springer-Verlag, 1973, 287 pp, $10.10 (P). Let S, T be nonlinear operators between Banach spaces. λ is an eigenvalue if $(\lambda T - S)(x) = 0$ has a nontrivial solution. Main questions are the solvability of $(\lambda T - S)(u) = f$ and the structure of the set of all eigenvalues. Chapters: Preliminaries, Fredholm alternative for nonlinear operators, Ljusternik-Schnirelmann theory, Morse-Sard theorem, the converse of L-S theory. Appendices on applications to integral and differential equations. 107 references. RBK

FUNCTIONAL ANALYSIS, P. *Topological Riesz Spaces and Measure Theory.* D.H. Fremlin. Cambridge U Pr, 1974, xiv + 266 pp, $17.50. Measure theory has provided examples and representation results for linear functionals in functional analysis, but the author feels that measure theoretic techniques are foreign to the spirit of functional analysis. He proposes an alternative development of these results by means of Riesz spaces, or vector lattices. RBK

FUNCTIONAL ANALYSIS, P. *Lecture Notes in Mathematics-345: Proceedings of a Conference on Operator Theory.* Ed: P.A. Fillmore. Springer-Verlag, 1973, 228 pp, $8.20 (P). Seven papers from an informal "mini-conference" at Dalhousie U., Halifax, N.S., April 1973. LAS

OPTIMIZATION, S(17-18), P. *Lecture Notes in Economics and Mathematical Systems-88: Güterströme in Netzen.* H. Steckhan. Springer-Verlag, 1973, vii + 134 pp, $6.60 (P). Develops four new algorithms for optimizing flows in networks. Closely related to the work of Ford and Fulkerson. JD-B

OPTIMIZATION, T(17-18: 1, 2), S, P, L. *Nonserial Dynamic Programming.* Umberto Bertelè, Francesco Brioschi. Acad Pr, 1972, xii + 235 pp, $13.95. Considers discrete deterministic optimization problems, suggests standard formulations for them, defines three applicable classes of dynamic programming procedures, provides criteria for measuring the computational efforts involved, and constructs appropriate algorithms. No problem sets. FLW

OPTIMIZATION, T*(15-17: 2), S*, *Variational Methods in Optimization.* Donald R. Smith. P-H, 1974, xv + 378 pp, $14.95. Exceptionally clear introduction. Builds directly on min-max of calculus to abstraction of Euler-Lagrange theorem for functionals. Applied emphasis. Concepts introduced through application to non-trivial problems from mechanics, control theory, production and investment planning. Clear movement from problem to theory and back. Very readable style for student with 2 years of calculus. LH

OPTIMIZATION, S(15-16), *Optimization Techniques with Fortran.* James L. Kuester, Joe H. Mize. McGraw, 1973, 500 pp, $5.95 (P). A list of optimization algorithms. The computer implementations of these methods are presented in great detail, including FORTRAN source listings, input/output specifications, and sample solutions. However, the supporting analytic bases for the algorithms are not included. The primary roles of the book should be those of a supplementary text and a user's reference. JJ

OPTIMIZATION, P, *Lecture Notes in Economics and Mathematical Systems-86: Symposium on the Theory of Scheduling and Its Applications.* Ed: S.E. Elmaghraby. Springer-Verlag, 1973, viii + 437 pp, $13.20 (P). 26 papers on applications, theory and models from a May, 1972 ONR sponsored conference at North Carolina State U. at Raleigh. LAS

ANALYSIS, P, *Lecture Notes in Mathematics-322: Nonlinear Problems in The Physical Sciences and Biology.* Ed: Ivar Stakgold, Daniel D. Joseph, David H. Sattinger. Springer-Verlag, 1973, viii + 357 pp, $9.70 (P). The lecture series and individual talks from the July 1972 summer institute at Battelle, Seattle. JAS

ANALYSIS, P, *Variational Analysis: Critical Extremals and Sturmian Extensions.* Marston Morse. Wiley, 1973, xi + 260 pp, $17.50. An exposition of the calculus of variations, as developed by Morse and others, in sufficient generality to support the known applications to differential topology, optimal control theory, and mathematical physics. The first of two volumes; the proposed second volume is *Variational Topology.* JAS

ANALYSIS, P, *Vector and Operator Valued Measures and Applications.* Ed: Don H. Tucker, Hugh B. Maynard. Acad Pr, 1973, xvi + 458 pp, $18.50. Papers presented at a symposium held at Alta, Utah, August 1972. Includes an 835 item bibliography developed by participants at the symposium. LAS

ANALYSIS, P, *Approximation Theory.* Ed: G.G. Lorentz. Acad Pr, 1973, xv + 525 pp, $16. Proceedings of an international symposium held at U. Texas, January 1973. Includes seven long invited survey papers, and 50 short research notes. LAS

ANALYSIS, P, *American Mathematical Society Translations, Series 2, V. 102: Ten Papers in Analysis.* AMS, 1973, iv + 252 pp, $19.10.

APPLICATIONS, T*(15-17: 2), S*, P, L*, *Mathematical Models and Applications with Emphasis on the Social, Life, and Management Sciences.* Daniel P. Maki, Maynard Thompson. P-H, 1973, xv + 492 pp, $14.95. A wide choice of stimulating and interesting real life situations in which mathematical ideas can be introduced to provide precision of understanding. Exercises and open-ended projects having many "solutions" will keep the student actively involved in model building. Computer work is not integrated into the text, but opportunities for its use abound. The text can be used for a variety of courses--survey, in-depth, teacher preparation. LCL

APPLICATIONS (AGRICULTURE), T(16-17: 1), S, L, *Linear Programming Applications to Agriculture.* Raymond R. Beneke, Ronald Winterboer. Iowa State U Pr, 1973, vii + 244 pp, $9.95. A large number and variety of linear programming models requiring a thorough knowledge of agriculture, in addition to modest background in algebra, economics and computer science (programs for data processing are written in MPS for the IBM 360). The authors are very careful to draw attention to special features of each model. LCL

APPLICATIONS (BIOLOGY), T(16-18), P, L**, *Foundations of Mathematical Biology.* Ed: Robert Rosen. Acad Pr. *V. I: Subcellular Systems,* 1972, xxviii + 287 pp, $15; *V. II: Cellular Systems,* 1972, xxviii + 330 pp, $16; *V. III: Supercellular Systems,* 1973, xvii + 412 pp, $26. A massive text whose lengthy chapters are written by individuals with different expertise (e.g., Michael Arbib, N. Rashevsky, Robert Rosen). Intended to complement (not duplicate) existing texts, it is nevertheless a definitive, comprehensive, accessible survey of contemporary theoretical biology. A good place to begin for mathematicians untrained in this area. LAS

APPLICATIONS (DEMOGRAPHY), T*(16-17: 1), P, L, *Mathematical Models for the Growth of Human Populations.* J.H. Pollard. Cambridge U Pr, 1973, xii + 186 pp, $17.50. Concise exposition of the major deterministic and stochastic models of population growth with final chapters on special problems: two-sex models, extinction of surnames, analysis of organizations. Omits consideration of numerical data analysis. Presumes thorough knowledge of probability and linear algebra. Exercises (with solutions in back), good bibliography and author and subject indices. LAS

APPLICATIONS (ECONOMICS), T(15-17: 1), S, L, *Elementary Mathematical Macroeconomics.* David A. Bowers, Robert N. Baird. P-H, 1971, xv + 304 pp, $10.75. Puts aggregate supply and demand into mathematical language (i.e., derivatives, partial derivatives, difference equations). Develops several models to compute equilibrium levels of output, prices, wages, unemployment and effects of monetary and fiscal policy on these levels. Models of economic growth. Very clear exposition but could use more exercises. LH

APPLICATIONS (ENGINEERING), T(13-16: 1), S, P, *Logic Design and Algorithms.* D. Zissos. Oxford U Pr, 1972, x + 458 pp, $29. A collection of explicit algorithms for circuit design. Requires no previous design work, very little math or electronics background. Techniques are based on Boolean algebra of first chapter. Technical, practical rather than theoretical. Many fully solved model problems, many illustrations. LH

APPLICATIONS (PHYSICS), T(17-18: 1), L. *A Pedestrian Approach to Quantum Field Theory*. Edward G. Harris. Wiley, 1972, xii + 167 pp, $11.95. Lean notes intended for non-theoretically minded physics graduate students. Begins with a concise treatment of the Hilbert space formalism and concludes with a "facing up" to the "distressing" fact that throughout quantum electrodynamics "some quantities that are presumed to be small...turn out in fact to be infinite." Contains some problems with solutions. LAS

APPLICATIONS (PHYSICS), T(14: 1), *An Introduction to the Theory of Mechanics, Eighth Edition*. K.E. Bullen. Cambridge U Pr, 1971, xvi + 365 pp, $14.50. For the first undergraduate course. Assumes concurrent study of calculus, provides vector analysis. A multitude of exercises. Little different from the seventh edition, which made the change from cgs to S.I. units. DFA

APPLICATIONS (PHYSICS), S, L, B, *Collection of Problems in Classical Mechanics*. G.L. Kotkin, V.G. Serbo. Transl: D. ter Haar. Pergamon Pr, 1971, viii + 278 pp, $8.50. 289 problems corresponding to material in standard textbooks on mechanics. Complete solutions consume three-fourths of the volume, and many study limiting cases and suggest variations and generalizations. Pleasing style. Valuable to the physics student. DFA

APPLICATIONS (PHYSICS), P, *Applications of the Theory of Distributions*. Romulus Cristescu, Gheorghe Marinescu. Transl: Silviu Teleman. Wiley, 1973, 227 pp, $11.95. An exposition of the theory of distributions with concrete examples of applications to ordinary and partial differential equations, physics, probability and linear dynamical systems. Translation of Romanian edition. References, no index. RBK

APPLICATIONS (PHYSICS), P, *Graphical Methods of Spin Algebras in Atomic, Nuclear, and Particle Physics*. E. El Baz, B. Castel. Dekker, 1972, x + 428 pp, $19.50. Visual mnemonics for quantum mechanics: bra and ket vectors, 3jm and 3nj symbols, tensors, spherical harmonics. Second half applies these schemata to various topics in nuclear and atomic physics. LAS

APPLICATIONS (SOCIAL SCIENCE), L, *The Study of Games*. Elliott M. Avedon, Brian Sutton-Smith. Wiley, 1971, xiv + 530 pp, $11.50. Selected readings and topical bibliographies on all aspects of games--in history, anthropology, folklore, social science; for recreative, military, business, educational, therapeutic purposes. A valuable reference to supplement standard mathematical sources. LAS

Reviewers Whose Initials Appear Above

David F. Appleyard, Carleton; Ralph Bjork, St. Olaf; Paul J. Campbell, St. Olaf; Judith N. Cederberg, St. Olaf; John Dyer-Bennet, Carleton; Steven Galovich, Carleton; Loren Haskins, Carleton; James Johnson, St. Olaf; Lorraine L. Keller, St. Olaf; Roger B. Kirchner, Carleton; Richard S. Kleber, St. Olaf; Loren C. Larson, St. Olaf; Pierre J. Malraison, Carleton; J. Arthur Seebach, Jr., St. Olaf; Lynn A. Steen, St. Olaf; Frank L. Wolf, Carleton.

NEWS AND NOTICES

EDITED BY RAOUL HAILPERN, SUNY at Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to Mathematical Association of America, 1225 Connecticut Avenue, NW, Washington, D. C. 20036. Items must be submitted at least two months before publication can take place.

PERSONAL ITEMS

University of Alaska: Assistant Professors Barbara M. Lando and C. A. Lando have been promoted to Associate Professors.

University of Connecticut: Assistant Professor Manuel Lerman, Yale University, has been appointed Associate Professor; Mr. Robert Weber, Cornell University, has been appointed Instructor; Associate Professor J. H. Neuwirth has been promoted to Professor.

Dickinson College: Associate Professors L. W. Baric and P. E. Martin have been promoted to Professors.

East Texas State University: Dr. Marian Joan Kelterborn, Rutgers University, has been appointed Assistant Professor; Associate Professor Howard Lambert has been promoted to Professor; Associate Professor W. W. Taylor retired on June 1, 1973, with the title of Professor Emeritus.

Emory University: Dr. R. L. Kruse, Sandia Corporation, has been appointed Associate Professor; Dr. J. C. Wiener, Washington University, has been appointed Assistant Professor; Mr. D. L. Greenwell, Vanderbilt University, has been appointed Instructor.

Illinois State University: Assistant Professor George Andria, University of Pittsburgh, has been appointed Associate Professor; Associate Professors Lawrence Eggan and Phares O'Daffer have been promoted to Professors; Assistant Professor Stephen Friedberg has been promoted to Associate Professor.

Kansas State College of Pittsburg: Assistant Professor G. L. McGrath has been promoted to Associate Professor; Professor R. G. Smith retired on May 31, 1973, with the title of Professor Emeritus.

Miami University: Mr. D. D. Berkey, University of Cincinnati, has been appointed Instructor; Assistant Professor D. O. Koehler has been promoted to Associate Professor; Professor S. E. Bohn has been appointed Chairman of the Mathematics Department.

Michigan Technological University: Assistant Professor K. J. Heuvers has been promoted to Associate Professor; Associate Professor O. G. Ruehr has been promoted to Professor.

Northern State College: Dr. John Camden, University of Texas, has been appointed Assistant Professor; Professor Elton Fors was appointed Chairman of the Department of Mathematics on July 1, 1973.

Rockhurst College: Assistant Professor Merrill Goldberg, University of Colorado, has been appointed Assistant Professor; Mr. Steve Hansen, University of Missouri, Kansas City, has been appointed Computer Center Director; Assistant Professor Elizabeth Berman has been appointed Acting Chairman of the Mathematics Department.

Seattle Pacific College: Dr. Ed Beardslee, Pennsylvania State University, has been appointed Assistant Professor; Dr. L. J. Montzingo, Chairman of the Mathematics Department, has been appointed Director of the School of Natural and Mathematical Sciences.

University of Texas, Austin: Professor R. H. Bing, University of Wisconsin, has been appointed Professor; Dr. Charles Friedman, MIT, has been appointed Assistant Professor; Associate Professor Milo Weaver retired on August 31, 1973, with the title of Associate Professor Emeritus.

Valdosta State College: Dr. P. J. Buckhiester, Clemson University, has been appointed Assistant Professor; Associate Professor R. C. Moore has been promoted to Professor.

Washington University: Drs. R. A. Levaro, University of Illinois, and E. N. Wilson, Brandeis University, have been appointed Assistant Professors.

Weber State College: Assistant Professor Richard Miller has been promoted to Associate Professor; Assistant Professor Patricia Fernandez has been promoted to Associate Professor and appointed Chairman of the Department of Mathematics.

University of West Florida: Assistant Professor S. E. Shamma has been promoted to Associate Professor; Assistant Professor C. M. Bundrick has been promoted to Associate Professor; Assistant Professor F. N. Moore received one of three awards for outstanding teaching for 1972–1973; Associate Professor D. R. Byrkit was appointed Chairman of the Mathematics Department on December 15, 1972.

Assistant Professor Kenneth Batker, Pacific Lutheran University, has been promoted to Associate Professor.

Dr. William Burns, Mathematician-Analyst at the National Security Agency, has been appointed Assistant Professor at Florida Institute of Technology.

Dr. L. A. Dysart, Jr., Merrimack College, has been promoted to Assistant Professor.

Assistant Professor I. Kleiner, York University, has been promoted to Associate Professor.

Assistant Professor Robert Lewand, Windham College, has been elected Chairman of the Science Division.

Assistant Professor G. D. Richardson, East Carolina University, has been promoted to Associate Professor.

Dr. J. R. Shipman, Instructor of Mathematics at Alabama A and M University, has been promoted to Associate Professor and appointed Chairman of the Department of Mathematics and Physics.

Associate Professor R. F. Shortt, Keuka College, has been promoted to Professor.

Professor Bernard F. Dostal, Milwaukee, died on June 20, 1973, at the age of 85. He was a member of the Association for fifty-two years.

Brother Frank R. Gutting, Associate Professor at St. Mary's University, died on October 17, 1973. He was a member of the Association for sixteen years.

Professor Holbrook M. MacNeille, Case Western Reserve University, died on September 30, 1973, at the age of 66. He was a member of the Association for thirty-seven years.

Professor William J. Walbesser, SUNY at Buffalo, died on October 22, 1973, at the age of 45. He was a member of the Association for nineteen years.

GRADUATE EDUCATION CHANGES ARE URGED; RESEARCH QUESTIONED

Major changes in American graduate education are urged in a report released this weekend by a national panel of leading educators. The Panel on Alternate Approaches to Graduate Education said it questions the emphasis on research as the single criterion for evaluating all graduate schools and their faculties, and recommends that all doctoral students spend time working outside university walls in areas related to their major fields.

The panel, which examined graduate education over an 18-month period, was supported by the Graduate Record Examinations Board, whose policies affect the entrance requirements of most graduate institutions, and the Council of Graduate Schools in the United States, whose 307 members include universities awarding 98 percent of the nation's doctoral degrees.

J. Boyd Page, president of CGS, served as chairman of the 15-member panel. Educa-

tional Testing Service provided administrative support under the direction of I. Bruce Hamilton.

Among the panel's 26 recommendations are that:

• Graduate school faculty be encouraged to take a wider view of their professional roles, and the decisions "for tenure, promotion, and salary increments no longer (should be) based on the single criterion of research and publication."

• More experts who may not possess the usual academic credentials be added to graduate school faculties. "Successful achievers in business and government possess gifts and experience that could be of immense influence in redirecting academic energies toward the servicing of social needs."

• More intensified efforts be made to recruit able minority-group representatives and women to the faculties. "Statistics can be cited confirming that the politics of graduate education reflect the influence of a... discriminatory society."

• Often rigid institutional requirements, such as residency and fellowship policies, become more flexible to meet the needs of new groups of students; for example, part-time women students. "Graduate administrators and faculties must arrive at a new perception of the worth and dignity of 'recurrent' or 'intermittent' learners."

• Every graduate student should be required to undertake discipline-related work outside the university if he has not previously done so to insure that no advanced-degree candidate graduates without exposure to real working situations.

• Certain institutional policies be altered to allow faculty members more time to play a larger role in the solution of major societal problems.

"It is a matter of recreating the graduate faculty as leaders in the search for a new understanding of the possibilities of human society and of recreating the graduate institution as one that is capable of counseling political and cultural leaders on ways of assuring meaning to the structural changes of society now in progress," the panel concludes.

The 20,000-word report cites a "cultural lag" resulting from an enormous increase in the past 50 years in the proportion of persons obtaining graduate degrees. While this increase, according to the report, has dramatically altered "the relation between the university and society as a whole," there has been little change in the self-conceptions of graduate departments in the same 50-year period.

Following its specific recommendations, and suggestions for implementation, the report goes on to make projections about the future of graduate schools. Student populations, the panel forecasts, will be fairly evenly divided between the sexes; at least 20 percent of its numbers will be drawn from minority groups.

Because of recurrent education, the ages of students will correspond more closely with those of the general population. It will be standard practice for students and teachers alike to examine the social implications of projected research. Standards for measuring faculty performance will be applied to a great variety of professional activities. For instance, community activity could be part of the assessment process.

The panel also predicts that the graduate professor will become more of a "mentor and preprofessional counselor" through expanded use of new educational technology. In addition, significant lines of communication will connect graduate programs and schools of different functions with each other and with other institutions, such as two-year colleges and state education departments.

In addition to Page, the panel included:

Daniel Alpert, director, Center for Advanced Study, University of Illinois; Warren G. Bennis, president, University of Cincinnati; Albert H. Berrian, Associate Commissioner for Higher Education, New York State Education Department; Edward E. Booher, president, Books and Education Services Group, McGraw-Hill, Inc.; Jean W. Campbell, director,

Center for Continuing Education of Women, University of Michigan; Benjamin DeMott, professor of English, Amherst College; May N. Diaz, professor of anthropology, University of California at Berkeley; Patricia Albjerg Graham, professor of history and education, Barnard College and Teachers College, Columbia University; Cyril O. Houle, professor of education, University of Chicago; Robert F. Kruh, dean of the graduate school, Kansas State University; W. Edward Lear, dean of the school of engineering, University of Alabama; Lincoln E. Moses, dean of the graduate division, Stanford University; Rochus E. Vogt,, professor of physics, California Institute of Technology; Albert N. Whiting, president, North Carolina Central University.

Single copies of *Scholarship for Society* are $2; 10 or more, $1.50 each; 100 or more, $1 each. Orders should be addressed to: Panel Report, GRE Board, Educational Testing Service, Princeton, N. J. 08540. (Prepayment is requested with orders for fewer than 10 copies.)

Reprinted from HIGHER EDUCATION AND NATIONAL AFFAIRS, XXII (43) November 30, 1973

THE DOCTOR OF ARTS IN MATHEMATICS — WASHINGTON STATE UNIVERSITY

A program at Washington State University leading to the Doctor of Arts in Mathematics was authorized in October, 1973. In conformity with guidelines prepared by various national and regional groups, it is designed to prepare exceptionally well-qualified teachers of undergraduate mathematics. Although the program will be as demanding in its way as that leading to the traditional Ph. D. (which has been offered at Washington State University since 1959), it has the following distinctive features:

(1) Greater emphasis on breadth of course work, including requirements in Computer Science and other mathematics-related areas.

(2) A component concerned specifically with the teaching process, including courses in higher education generally and a seminar in problems of undergraduate mathematics instruction.

(3) A period of internship: at a suitable stage of his program, the student spends a quarter or more as a full-time teacher at a two- or four-year college under close supervision by an experienced teacher at the college.

(4) An original thesis which does not necessarily make a direct addition to mathematical knowledge as such, but may be concerned with educational, historical, or other aspects of mathematics.

Generally speaking, the D. A. differs from the Ph. D. by offsetting a reduced emphasis on mathematical research by greater breadth of study and more explicit attention to the process of communicating mathematics to students.

The program is so planned that a student entering it immediately after finishing a suitable undergraduate program and proceeding at a normal rate while holding a teaching or other assistantship should finish in four years.

Applicants will be selected for admission to this program and for financial support not only on their academic records but also on evidence of exceptional interest in and aptitude for college teaching.

Inquiries should be addressed to: Doctor of Arts Program, Department of Pure and Applied Mathematics, Washington State University, Pullman, WA 99163.

MATHEMATICAL SPECTRUM

Mathematical Spectrum (MS) is a magazine designed to provide useful background reading for student mathematicians; it should appeal to readers of all ages interested in

mathematics. MS concentrates on the ideas involved in mathematical work, rather than on technicalities, and its contributors range from world-famous mathematicians to students.

MS is published by the Applied Probability Trust, and all orders and correspondence concerning it should be sent to the Editor — *Mathematical Spectrum*, Hicks Building, The University, Sheffield S3 7RH, England.

Orders should be placed for complete volumes published in the academic year. Volume 6 (1973/74) consists of two issues; No. 1 was published in September 1973 and is still available, while No. 2 will be ready in March 1974. The price of the complete volume, including postage by surface mail, is US \$2.50 (£1.00; \$A.2.00) for overseas subscribers, and £0.50 for British and European subscribers. If a bulk order is placed for five or more copies of the same volume, a discount of 10% is allowed.

With the exception of Volume 1, all earlier volumes are still available at the back issue price of US \$2.50 (£1.00; \$A. 2.00) for all subscribers.

CENTER FOR PERSONALIZED INSTRUCTION

The Center for Personalized Instruction, a service to higher education in individualized instruction, announces a program of events of interest to college teachers and administrators who want to introduce new instructional techniques into the college classroom involving the ideas of mastery learning, self-paced study, the Keller Plan, the Personalized System of Instruction (PSI), modular courses, mini-courses, and similar methods:

Workshop, January 3–12, 1974, Washington, DC
Regional conference, March 16, 1974, Chicago
National conference, April 5–6, 1974, Washington, DC
Workshop, May 25–June 2, 1974, Chicago
Workshop, July, 1974, San Francisco.

The workshops are for teachers ready to prepare materials for their own courses. The conferences are more for information and orientation in this field. For details, write to the Center at 29 Loyola Hall, Georgetown University, Washington, DC 20007.

MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

THE FIFTY-SEVENTH ANNUAL MEETING OF THE ASSOCIATION

The Fifty-seventh Annual Meeting of the Mathematical Association of America was held at the San Francisco Hilton Hotel, San Francisco, from Thursday to Saturday, January 17 to 19, 1974, in conjunction with a meeting of the American Mathematical Society. The first panel discussion on Thursday morning was a joint meeting with the American Mathematical Society. There were 3410 persons registered including 1903 members of the Association.

Sessions were held on Thursday morning and Friday morning and Saturday morning and afternoon in the Continental Ballroom of the San Francisco Hilton Hotel. Presiding officers were Professor H. L. Alder at the first panel discussion on Thursday morning, Dean L. H. Lange at the second, Professor R. M. Robinson at the first lecture on Friday morning,

President R. P. Boas at the Retiring Presidential Address, Professor Mary V. Sunseri on Saturday morning, Professor W. G. Chinn at the first lecture on Saturday afternoon, Professor G. B. Pedrick at the second, and Professor Craig Comstock at the last.

The Program Committee consisted of G. L. Alexanderson, Chairman; W. G. Chinn, N. H. Fisher, G. B. Pedrick, R. M. Robinson, and Mary V. Sunseri.

FIRST SESSION OF THE ASSOCIATION

Joint Session with the American Mathematical Society

Panel Discussion: The Problem of Learning to Teach

A panel discussion with Professors P. R. Halmos, Indiana University, E. E. Moise, Queens College of the City University of New York, and George Piranian, University of Michigan, moderated by Professor H. L. Alder, University of California, Davis.

Professor Moise explained a scheme for turning the initial teaching experience into a group activity. See section 2 of the speaker's article entitled Jobs, Training, and Education for Mathematicians, *Notices of the AMS*, August, 1973, pp. 217–221.

Professor Piranian spoke on "Promotion of Participation". The first requirement for successful teaching is mastery of the subject. Next in importance is lively participation by the students, and this in turn usually requires deep involvement of the teacher. As teachers, we can promote our own participation by deviating from good textbooks and by deserting occasionally our excellent but yellow lecture notes. To generate participation by the entire class, we need communication between the students. We can foster such communication by challenging the students with problems not treated in standard texts, and by encouraging classroom discussion of these problems in an atmosphere of friendly competition. With proper care, the program of special problems serves to sharpen the focus of the course.

This was followed by a general discussion by the panel and the audience.

Panel Discussion: On Problem Solving

A panel discussion with Professor A. P. Hillman, University of New Mexico, Mr. M. S. Klamkin, Ford Scientific Laboratories, Professor George Polya, Stanford University, and Professor J. E. Wetzel, University of Illinois, Urbana.

Professor Polya presented the thesis: *The backbone of mathematics teaching is problem solving*, especially in the high school and first years of college. The student cannot learn just by listening. It is by doing problems that he digests the lecture, understands the subject, learns to apply and appreciate it. And how is the student's progress judged? By his problem-solving. Moreover, mathematics should contribute to general, liberal education. How? By the student's work, his problem-solving. Get it across to the student: *Use your head to learn the subject. Use the subject to learn how to use your head.*

Professor Hillman suggested that when the ratio of numbers of students to numbers of teachers increases from Plato's ideal of 1-to-1, it may be best to direct most of the class time toward the average students. This is especially true if the institution provides special tutoring for slower students and out-of-class challenge for those who learn more quickly. An important ingredient of the program for superior students is the availability of supplementary sets of challenging problems. Contests provide stimulation for both routine and special techniques of education as well as the recognition of achievement.

Mr. Klamkin reported that Professor Synge once picturesquely described the use of applied mathematics in its relation to solving a physical problem. It consists of three stages:

I. A dive from the world of reality into the world of mathematics.

II. A swim in the world of mathematics.

III. A climb back from the world of mathematics to the world of reality, carrying a prediction in our teeth.

Other mathematicians have expanded these three stages to five or more stages. In terms of one word descriptors of H. Pollak, we have

I. Recognition
II. Formulation
III. Solution — Feedback
IV. Computation
V. Communication

These stages were illustrated by means of specific physical examples.

Professor Wetzel first gave a summary of what could be called "classical" problem-solving, i.e., problem-solving in the sense of G. Polya. He then sketched a problem-solving course for teachers having its roots in the suggestions that appear in Polya's books. He made a number of specific suggestions for such a course. He concluded with an exhortation: Give the students the chance to experience the very real thrill of mathematical discovery by setting up "research situations" with well-chosen, non-routine problems. This is, after all, what mathematics is all about.

This was followed by a general discussion by the panel and the audience.

SECOND SESSION OF THE ASSOCIATION

The Logic of Equality, by Professor L. A. Henkin, University of California, Berkeley.

Can every equational identity (involving + and 0) that holds for the natural numbers, be derived from the commutative, associative, and zero laws? To answer this, it is necessary to use precise definitions stating when an equational identity is true of a given structure, and when it is derivable from other identities. These definitions permit us to explore the notion of logically complete deductive systems, as well as some simple elements of model theory, within the context of equational grammars. The original question is then answered affirmatively, although a similar question involving x, exp, and 1, as well as + and 0, is open. Finally, rules of derivability were considered for systems such as high school students implicitly use in solving equations. These turn out to be surprisingly more complicated than the rules needed for deriving equational identities from others.

Annual Business Meeting of the Association; Announcement of a Special Gift to the Association; the Association's Thirteenth Award for Distinguished Service to Mathematics; Award of the 1974 Chauvenet Prize.

Retiring Presidential Address: Interval Graphs and Allied Systems, with Relations to Molecular Genetics, Psychophysics, Archeology, Ecology, and the Inversion of Sparse Matrices, by Professor Victor Klee, University of Washington.

The lecture presented some simple but appealing notions that can be worked into courses in combinatorics, geometry, and linear algebra in order to illustrate some of the relationships of those subjects to molecular genetics, psychophysics, ecology, and archeology.

THIRD SESSION OF THE ASSOCIATION

Applications of Mathematics to the Biological Sciences

Mathematics in Pulmonary Physiology, by Professor J. W. Evans, University of California, San Diego.

Mathematics is employed widely in the study of pulmonary physiology. In this talk two examples were given which may be used as exercises in undergraduate courses. Some properties of convex functions were used to elucidate the behavior of gas transfer when the blood and air supplies in the lung are mismatched, and the mixing of gases in the lungs was studied using elementary techniques to obtain the eigenvalues and eigenvectors of a particular linear operator.

Graphs and Molecules, by Professor Joshua Lederberg, Stanford University.

For many years, organic chemists have manipulated abstract structures, called molecules, without identifying them as undirected graphs (with colored nodes, corresponding to atom types). A new effort at systematization has been motivated recently by research on the emulation of chemists' problem-solving behavior by computer programs. Successful applications of this approach in the field of analytical mass spectrometry was illustrated. In addition, the systematic enumeration of molecules allows the prediction of all possible molecular structures, and has implications for college-level education in chemistry, for the bibliographic ordering of data in chemical science, for patent law, for the discovery of new principles of chemical theory from empirical data, and for highly automated scanning of natural products and body fluids for new compounds of biomedical significance. The combinatorial theory of these graphs also requires further development and application if efficient computer algorithms dealing with organic molecular structures are to be achieved.

Manifolds, Machines, Models, and Computability, by Professor Hans Bremermann, University of California, Berkeley.

The dynamical systems of biology generally are of great complexity. Explicit orbits are frequently unobservable and transcomputational. The biological observables are the attractors (forms). A classification of forms and understanding the phenomena of simplicity and complexity of forms is a major challenge. Several complexity theories exist for finite state automata and for Turing machines. The state space description of automata is formally similar to dynamical systems theory. A unification of both theories and an extension of complexity theory from automata to dynamical systems is a challenging possibility.

FOURTH SESSION OF THE ASSOCIATION

Some Modern Work on Determinants, by Professor Olga Taussky, California Institute of Technology.

There has been a revival of interest in determinants recently. Some of the developments were discussed. They range from categorical aspects to combinatorial and numerical work. Also extensions of old determinantal theorems to new matrix theorems were noted. The main issues concern commutators, the map of $GL(n, F)$ into $SL(n, F)$, $SL(n, z)$.

What is Infinite Dimensional Topology?, by Professor R. D. Anderson, Louisiana State University.

The primary building blocks in this relatively new subject are Hilbert space, l_2; the Hilbert cube, Q; and the countable infinite product of lines, s. The study includes the internal topology of these spaces, homeomorphic classification of various linear infinite-dimensional spaces and representation and characterization theorems of manifolds modeled on such spaces. Exploitation of infinite product structures has recently led to surprisingly strong results and a developing cohesive theory which has many essential contacts with AR and ANR theory, with homotopy and simple homotopy theory, and with various aspects of hyperspace theory.

Computer Science and its Relation to Mathematics, by Professor D. E. Knuth, Stanford University.

A new discipline called Computer Science has recently arrived on the scene at most of the world's universities. The lecturer gave a personal view of how this subject interacts with mathematics, by discussing the similarities and differences between the two fields, and by examining some of the ways in which they help each other. A typical nontrivial problem was worked out in order to illustrate these interactions.

SPECIAL SESSIONS OF THE ASSOCIATION

Film showings were held in the Continental Ballroom of the San Francisco Hilton Hotel on Friday evening. The following films were shown:

7:30–8:45 P.M.	*An Application of Computer Graphics to Teaching Mathematics: Exhibition of Six Computer-Animated Super 8 Movies and Several Slide Sequences on Calculus, Statistics, Pure and Applied Analysis,* by Professors R. B. Kirchner and R. W. Nau, both of Carleton College.
9:00–9:22 P.M.	*An Allendoerfer film:* CYCLOIDAL CURVES OR TALES FROM THE WANKLENBURG WOODS (in color).
9:23 P.M.	*Films of the Topology Films Project* (in color and with sound narration)
9:23–9:53 P.M.	SPACE FILLING CURVES
9:55–10:09 P.M.	REGULAR HOMOTOPIES IN THE PLANE, PART I
10:10–10:29 P.M.	REGULAR HOMOTOPIES IN THE PLANE, PART II

MEETING OF THE BOARD OF GOVERNORS

The Board of Governors met on Wednesday at 9:00 A.M. in the California Room of the San Francisco Hilton Hotel, with 40 members present.

The Board approved the appointment by President Boas of the following Nominating Committee for 1974: P. A. Haeder, Chairman; M. W. Pownall, and Mary V. Sunseri.

Professor R. D. Anderson of Louisiana State University was elected a member of the Finance Committee for the term 1974–77 to succeed Professor G. S. Young.

Professors Joseph Hashisaki of Western Washington State College and C. A. Lathan of Monroe Community College, Rochester, New York, were elected Co-Editors of the TWO-YEAR COLLEGE MATHEMATICS JOURNAL, effective as of the date of the Association's assumption of responsibility for this journal. Their terms extend through 1978.

The Board elected Professor F. J. Almgren, Jr., of Princeton University as Hedrick Lecturer for 1975.

The Subcommittee on the USA Mathematical Olympiad was authorized to enter a team from the USA in the International Mathematical Olympiad to be held July 8 and 9, 1974, in Erfurt, D. D. R., if an invitation for such participation is received.

The Board approved two recommendations of the Committee on Two-Year Colleges, in a slightly modified form, namely:

Sections are urged to provide in their meetings for programs arranged by and for the two-year college faculty. Furthermore, each program for national meetings should include sessions of special interest to two-year college mathematicians. Program Committees should contain appropriate representation from two-year colleges and these members should be primarily responsible for that part of the program. In order to accommodate the programs in two-year college fields, it may be necessary to arrange parallel sections during some of the sessions.

On the recommendation of the ad hoc Committee on Women in Mathematics, the Board voted that the MAA participate in a joint committee with the AMS in an investigation of the status of women in mathematics.

The Finance Committee reported the receipt of a gift of $2,000 from the family of Marian Barr Gehman. The Board then approved the following resolution:

"The Board of Governors of the Mathematical Association of America gratefully acknowledges the generous gift of $2,000 from the family of Marian Barr Gehman. This gift, in memory of a person who gave so much of herself to the Association in her dual role as loyal assistant to and loving and supportive wife of Harry Gehman, will greatly assist the Association in maintaining its program of publications. The Board pledges to use the gift to support a publication which we believe would have been dear to Marian's heart."

The Finance Committee reported with great appreciation receipt of the following additional gifts: a gift of $1,000 from a foundation which has requested to remain anonymous, a gift of $1,000 each from two individuals, and gifts of $375, $100, and $50 from three individuals.

The Board voted to receive with gratitude a grant of $62,300 from the National Science Foundation for preparing jointly with NCTM a Source Book of Applications of Mathematics for Secondary School Teachers of Mathematics.

The Executive Director reported the membership of the Association as of December 31, 1973, as 18,634 members, very close to the maximum membership of the Association.

The Board voted to request the Secretary to submit an amendment to the By-Laws for a vote of the membership at the business meeting on January 26, 1975, in Washington, D. C., providing for an Editor of the TWO-YEAR COLLEGE MATHEMATICS JOURNAL to be a member of the Board of Governors, and an amendment for replacing all references in the By-Laws to "ordinary" members by "individual" members.

The Board approved the following resolution:

"The Mathematical Association of America respectfully urges the National Science Foundation and the Office of Education to institute sizeable programs of post-doctoral fellowships. Federally supported programs for both academic year and summer activities are required, in order to meet the following developing needs over the next generation:

(1) to assist and encourage departments to keep alive and vital professionally;

(2) to enable faculty members to become involved in new and evolving applications of mathematics to societal problems;

(3) to enable young and gifted research mathematicians, who are employed by departments which are not research-oriented, to maintain their research activity and growth.

As the professional organization primarily concerned with undergraduate education in mathematics, the Association is vitally concerned with the stultifying effect of the impending non-growth and decreased mobility phenomena likely to be experienced over the next 20-year period. Prospective long-term decrease of the college-age population and the youthful age distribution of the present mathematical faculty will make continuing additions of vital new faculty most difficult. No longer will departments be able to rely primarily on new faculty to keep the department up to date on new developments in our science. Thus, it is of vital importance to the continuing improvement of our undergraduate programs and to the intellectual life of our nation that opportunities be available for post-doctoral training of faculty members. Such training would be expected to meet the above-mentioned needs in the following ways:

"(1) With a relatively stable national faculty, fellowships similar to the earlier Science Faculty Fellowships would enable advanced subject matter training and would permit extended visits to centers of innovative curricular and instructional development. Such a program would help provide healthy and modern undergraduate programs for the years ahead.

(2) Applied mathematics is regaining its rightful place as an equal partner with pure mathematics. Many faculty members are anxious to receive broader training in the

newer and exciting areas of applications. Post-doctoral fellowships afford an ideal vehicle for such training. Moreover, with the continually evolving areas of applications, an organized post-doctoral program will be essential in order that meaningful new applications become part of a substantial number of undergraduate programs.

(3) It seems certain that a large number of gifted and able young mathematicians will spend most of their careers in departments which are not oriented toward research and even lack research libraries. In order that the talents of such persons not be lost to the nation and that full use be made of their abilities in both teaching and research, it is of highest importance to provide opportunities compensating for research isolation and involving time spent at active research centers both following completion of graduate school and after teaching for a period of time."

The Board voted not to hold a meeting this summer, since there will be no meeting of the Association because of the International Congress of Mathematicians in Vancouver, Canada. The next Board meeting was scheduled for January 24, 1975, in Washington, D. C. The Board also voted that the meeting of Section Officers, which normally would have been held in conjunction with the 1974 Summer Meeting, be postponed until the time of the Annual Meeting in January 1975.

ANNUAL BUSINESS MEETING OF THE ASSOCIATION

The Annual Business Meeting was held on Friday, January 18, 1974, in the Continental Ballroom of the San Francisco Hilton Hotel, with President Boas presiding. He opened the meeting with an announcement of a gift from Professor Mary P. Dolciani, which will enable the Association to initiate a new series of publications, to be called DOLCIANI MATHEMATICAL EXPOSITIONS. For details, see the "Announcement of Dolciani Mathematical Expositions" in this issue of the MONTHLY.

The Association's Thirteenth Award for Distinguished Service to Mathematics was made to Professor R. H. Bing of the University of Texas, Austin. The citation (which appears on pages 117–119 of the February issue of this MONTHLY) was prepared and read by Professor R. D. Anderson of Louisiana State University. Professor Bing, in accepting the Award, recalled that the last four decades had been very good to him, as a student, teacher, and researcher, and that it had been fun pushing back the frontiers. It had been fun working with organizations in mathematics and with the Association. He concluded his remarks as follows: "Our goal has exceeded our reach, and the road looks rocky. It looked rocky when we started down that road. Let us enjoy the challenge."

A specially-bound copy of the citation was presented to Mrs. Bing as a token of appreciation for her continuous support of her husband's multitude of activities.

The Chauvenet Prize for 1974 was awarded to Professor P. D. Lax of the Courant Institute of Mathematical Sciences for his paper "The Formation and Decay of Shock Waves," which appeared in this MONTHLY 79 (1972), 227–241. Further details concerning this Prize and its recipient appear on pages 119–120 of the February issue of this MONTHLY. Mrs. Lax was presented a copy of the certificate.

The Secretary reported the results of the balloting for officers, in which 2008 votes were cast. Dr. H. O. Pollak of Bell Telephone Laboratories was elected as President-Elect for 1974, Professor Ivan Niven of the University of Oregon as First Vice-President for the two-year term 1974–75, and Professor Deborah T. Haimo of the University of Missouri at St. Louis and Professor E. H. Spanier of the University of California, Berkeley, as Governors for the three-year term 1974–76.

The Secretary reported on some of the actions taken by the Board of Governors. He announced that the first annual list of members interested in a sabbatical exchange through

the MAA Sabbatical Exchange Information Service was now available free of charge from the Washington office of the MAA.

The Secretary paid tribute to the efforts of the local Committee on Arrangements for this meeting, who did so much to assure that all of its aspects would function to the complete satisfaction of everyone. He acknowledged a special debt of gratitude to the Chairman, Professor N. H. Fisher, who not only participated in all phases of arranging the meeting, but also served as Publicity Director.

As an item of new business, Dr. P. W. Healy presented the following resolution:

"Resolved, that the Editor of the MONTHLY reconsider the decision not to publish the comments of Professor J. M. Thomas on the article by David Gale in the 1969 volume of the MONTHLY."

After it was pointed out to Dr. Healy that the term of the previous Editor of the MONTHLY had expired since that decision was made, Dr. Healy felt that this should make it easier for the new Editor to consider the decision made by the previous Editor, and changed the resolution, replacing the word "reconsider" by "consider."

The Secretary then reviewed the facts of the case, in particular noting that the MONTHLY has a policy of not publishing any letters to the Editor and that none had been published.

Professor P. D. Lax then spoke in opposition to the resolution, recalling that when he first received Professor Thomas' letter, he was sympathetic because he remembered from the time he was a student the paper of L. L. Dines, and felt it was a marvelous piece of work. After correspondence with President Boas and Secretary Alder, he felt that the claim made by Professor Thomas in his letter was beside the point, since Professor Gale had not written about the theory of inequalities but of algorithms, which are quite different subjects. As a result he felt Professor Thomas' claims were not justified.

Dr. Healy urged that the new Editor consider the matter, pointing out that Professor Thomas is a distinguished member of the mathematical community and a former editor of a mathematical publication, who should be given more consideration than he had been given up to this point.

Dr. W. A. Beyer of the Los Alamos Scientific Laboratories also felt that the Editor should consider this matter. He felt that the matter had gotten too much involved in personalities.

The motion was then put to a vote and defeated by a large margin.

MEETINGS OF OTHER ORGANIZATIONS

The American Mathematical Society held sessions from Tuesday, January 15, to Friday, January 18. The forty-ninth Josiah Willard Gibbs Lecture was delivered by Professor P. A. Samuelson of the Massachusetts Institute of Technology on "Economics and Mathematical Analysis" on Tuesday at 8:30 P.M. in the Continental Ballroom. The Retiring Presidential Address was given by Professor Nathan Jacobson of Yale University on "Some Groups and Lie Algebras Defined by Jordan Algebras" on Wednesday at 2:45 P.M. in the Continental Ballroom.

There were two sets of Colloquium Lectures: Professor Louis Nirenberg of the Courant Institute of Mathematical Sciences, New York University, gave one set, entitled "Selected Topics in Partial Differential Equations" on Tuesday at 11:00 A.M., Wednesday at 11:00 A.M., and Thursday and Friday at 1:00 P.M. Professor J. G. Thompson of the University of Cambridge gave the other set on "Finite Simple Groups" on Tuesday at 1:30 P.M., Wednesday at 1:00 P.M. and on Thursday and Friday at 2:15 P.M.

The Committee on Employment and Educational Policy presented two panel discussions. The first, on "Non-academic Employment of Ph.D.'s," was held on Wednesday at 8:30 P.M. in the Continental Ballroom. It was moderated by Dr. H. O. Pollak of Bell Telephone Laboratories. The panelists were Dr. E. E. David, Jr., Executive Vice-President of Gould, Inc., and former Presidential Science Advisor; Dr. C. V. Newsom, former President, New York University; and John McQuown, Vice-President and Director of Management Sciences, Wells Fargo Bank, San Francisco. The second panel discussion, on "The Role of the Dissertation in the Ph.D. Program," was held on Thursday at 8:30 P.M. in the Continental Ballroom. It was moderated by Professor P. E. Thomas, University of California, Berkeley. The panelists were Professors William Browder, Princeton University, Karel de Leeuw, Stanford University; I. N. Herstein, University of Chicago; and Saunders MacLane, University of Chicago.

The AMS presented a two-day Preceptorial Introduction to Computer Science for Mathematicians on Sunday and Monday, January 13 and 14, in the California Room. The program was under the direction of Professor J. T. Schwartz, Courant Institute of Mathematical Sciences, New York University, and consisted of six lectures on various aspects of computer science intended to provide a concentrated introduction to the field, thus making it possible for the participants to judge if computer science is a subject which they would be interested in pursuing further. The speakers and their topics were Professor J. T. Schwartz, Courant Institute of Mathematical Sciences, New York University, "Pragmatic and Theoretical Considerations Concerning Programming," Professor R. M. Karp, University of California, Berkeley, "Lower and Upper Bounds on the Computational Complexity of Combinatorial Problems"; and Professor A. R. Meyer, Massachusetts Institute of Technology, "Discrete Computation: Theory and Open Problems."

Invited addresses were given as follows, all in the Continental Ballroom:

Intrinsic Distances, Measures, and Geometric Function Theory, Professor Shoshichi Kobayashi, University of California, Berkeley, Tuesday, 8:30 A.M.

Lagrangian Submanifolds, Professor A. D. Weinstein, University of California, Berkeley, Tuesday, 9:45 A.M.

Some Recent Developments in Combinatorics, Professor D. K. Ray-Chaudhuri, Ohio State University, Tuesday, 2:45 P.M.

Combinatorial Inequalities and Smoothness of Functions, Professor A. M. Garsia, University of California, San Diego, Tuesday, 4:00 P.M.

Some Problems of Mathematics and Science, Professor R. J. Duffin, Carnegie-Mellon University, Wednesday, 8:30 A.M.

Combinatorial Game Theory, Professor E. R. Berlekamp, University of California, Berkeley, Wednesday, 9:45 A.M.

Representations of Finite Chevalley Groups, Professor Louis Solomon, University of Wisconsin, Madison, Thursday, 3:30 P.M.

Some Applications of Module Theory to Functor Categories, Professor B. M. Mitchell, Rutgers University, New Brunswick, Friday, 3:30 P.M.

The Bôcher Memorial Prize was awarded on Wednesday at 2:15 P.M. to Professor D. S. Ornstein of Stanford University. The Prize was awarded to Professor Ornstein for his many contributions to ergodic theory and, in particular, for his paper entitled "Bernoulli Shifts with the Same Entropy are Isomorphic," *Advances in Mathematics* 4 (1970), 337–352.

The Conference Board of the Mathematical Sciences sponsored a panel discussion on "Mathematics and the Problems of Society" on Thursday at 2:30 P.M. in the Continental Ballroom, Parlors 1 through 4. The session was planned by Dr. D. L. Thomsen, Jr. of the SIAM Institute for Mathematics and Society. The members of the panel and titles of their presentations were:

Biological, Social and Cultural Evolution, Professor L. L. Cavalli-Sforza, Stanford University School of Medicine.

Mathematical Approach to Ecosystem Problems, Professor S. A. Levin, Cornell University.

Discrete Mathematics Applied to Environmental Problems, Professor F. S. Roberts, Rutgers University, New Brunswick.

Deployment of Emergency Vehicles, Dr. J. M. Chaiken, The RAND Corporation.

ARRANGEMENTS, ENTERTAINMENT AND RECREATION

The Committee on Arrangements consisted of N. H. Fisher, Chairman; H. L. Alder, W. G. Bade, W. G. Chinn, Daniel Gallin, Estelle M. Goldberg, R. S. Lehman, K. A. Ross, G. L. Walker.

Registration headquarters were located in the Tower Lobby of the Hilton Hotel. The, Mathematical Sciences Employment Register was maintained from 9:00 A.M. to 4:00 P.M. on Wednesday, January 16, and from 9:00 A.M. to 5:40 P.M. on Thursday through Saturday in the Imperial Ballroom. Book and educational media exhibits were displayed in the Hilton Plaza of the Hilton Hotel from noon to 5:00 P.M. on Tuesday, from 9:00 A.M. to 5:00 P.M. on Wednesday and Thursday, and from 9:00 A.M. to noon on Friday.

A No-Host Get-Together was held on Thursday from 4:30 to 6:00 P.M. in the California Room of the Hilton Hotel. A tour through the Lawrence Hall of Science of the University of California, Berkeley, was conducted on Friday afternoon at 3:00 P.M.

HENRY L. ALDER, *Secretary*

ANNOUNCEMENT OF DOLCIANI MATHEMATICAL EXPOSITIONS

At the Annual Business Meeting of the Association on January 18, 1974, in San Francisco, President Boas announced the establishment of a new series of publications by the Association which the Board of Governors has designated DOLCIANI MATHEMATICAL EXPOSITIONS. This series has been mads possible by a gift of $20,000 from Professor Mary P. Dolciani. The first volume in this series is MATHEMATICAL GEMS, by Professor Ross A. Honsberger of the University of Waterloo. It is now available from the Washington office of the Association. It is hoped that this series will become as famous as the Carus Monographs.

President Boas then read the following citation:

Mary Patricia Dolciani, Associate Provost for Academic Services at Hunter College of the City University of New York, is a native of New York City. She received her Bachelor of Arts degree at Hunter College in 1944 and has been back at Hunter since 1955.

In the intervening years, while holding various fellowships from 1944 to 1948, she received her Master of Arts and Doctor of Philosophy degrees in algebra at Cornell University and was a visiting member of the Institute for Advanced Study in Princeton From 1948 to 1955 she was instructor and then assistant professor at Vassar College.

At Hunter College since 1955, Dr. Dolciani has risen to the rank of Professor of Mathematics and has served as Chairman of the Department of Mathematics, as Associate Dean, and currently as Associate Provost for Academic Services.

For many years she has been deeply concerned with pedagogical problems and with innovative programs of instruction at all levels. These interests have taken her, for example, on a faculty fellowship to University College, London, and have led to her being a director and teacher at numerous New York State Education Department institutes for school mathematics teachers. She was an early member and vigorous writer of the School Mathematics Study Group. She is noted for the high idea content of her talks at

general sessions of the National Council of Teachers of Mathematics, where her dynamic presentations are enthusiastically acclaimed by huge audiences.

Less well known, perhaps, is the fact that she gives unstintingly of herself in less widely publicized appearances. We know, for example, of an instance in which she accepted an invitation to talk to the high school teachers of Monterrey, Mexico. Despite the sudden change to a much warmer climate, on each of two successive days with the aid of an interpreter she graciously gave two hour-and-a-half lectures with unabated vigor and devotion to her subject.

Her many services to the Association include, among others, terms as Visiting Secondary School Lecturer, as member of the Committee on Publications, and as Member-at-Large of the Board of Governors.

President Boas announced that in recognition of her many services and contributions to the mathematical community and the MAA in particular, the Board of Governors had voted to bestow an honorary Life Membership upon Professor Dolciani.

In the absence of Professor Dolciani, the certificate for honorary Life Membership was accepted in her behalf by Professor E. F. Beckenbach, who remarked that nothing but extreme illness in her family could have kept Professor Dolciani from this meeting. She had asked that her appreciation be expressed to the officers, Board of Governors, and the Committee on Publications for giving her the privilege and opportunity of serving the Association in this way. Professor Beckenbach noted that it is through the Committee on Special Funds of the Association, under the chairmanship of Professor R. L. Wilder, that such opportunities happen.

Professor Beckenbach conveyed Professor Dolciani's gratitude for the honorary Life Membership and the sentiment behind the presentation.

ACADEMIC MEMBERS ELECTED INTO THE ASSOCIATION

In accordance with the amendment adopted at the business meeting of the Association at Stillwater on August 30, 1961, the Board of Governors at its meeting in San Francisco, California, on January 16, 1974, elected to membership the twenty-fourth set of applicants for academic membership (for election of the other twenty-three sets, see the March and December issues of 1969, the April and November issues of 1970 and 1971, the May and December issues of 1972, and the May issue of 1973). Approval for election was given to the following fifteen applicants for academic membership:

Bowie State College, Bowie, Maryland
California State University, Northridge, California
California State University, San Diego, California
California State University, San Francisco, California
Clarkson College of Technology, Potsdam, New York
Grambling College, Grambling, Louisiana
Hendrix College, Conway, Arkansas
Kean College, Union, New Jersey
Montclair State College, Upper Montclair, New Jersey
Palm Beach Atlantic College, West Palm Beach, Florida
St. Johns University, Jamaica, New York
Tarleton State University, Stephenville, Texas
Three Rivers Community College, Poplar Bluff, Missouri
University of Hawaii, Honolulu, Hawaii.

HENRY L. ALDER, *Secretary*

BY-LAWS OF THE MATHEMATICAL ASSOCIATION OF AMERICA (INC.)

ARTICLE I — NAME, PURPOSE AND CORPORATE SEAL

1. This organization shall be known as

THE MATHEMATICAL ASSOCIATION OF AMERICA (INCORPORATED)

2. Its object shall be to assist in promoting the interests of the mathematical sciences in America, especially in the collegiate field, by holding meetings in any part of the United States or Canada for the presentation and discussion of mathematical papers, by the publication of mathematical papers, journals, books, monographs, and reports, by conducting investigations for the purpose of improving the teaching of mathematics, by accumulating a mathematical library and by cooperating with other organizations whenever this may be desirable for attaining these or similar objects.

3. The Corporate Seal of the Association shall have inscribed thereon the name of the Association and the words "Corporate Seal — Illinois".

ARTICLE II — MEMBERSHIP

1. There shall be two classes of members, ordinary and institutional.

2. Any person interested in the field of collegiate mathematics shall be eligible for election to ordinary membership in the Association.

3. Any institution, academic or corporate, interested in the support of collegiate mathematics shall be eligible for election to institutional membership in the Association.

4. Election to membership shall be by vote of the Board upon written application from the individual or institution seeking admission. In the case of individuals qualifying for student dues, the application shall be endorsed by two ordinary members of the Association.

ARTICLE III — BOARD OF GOVERNORS AND OFFICERS

1. The Officers of the Association shall be a President, a President-Elect (only during a year immediately prior to the expiration of a President's term), a Past-President (only during a year immediately following the expiration of a President's term), a First Vice-President, a Second Vice-President, an Editor of its publication entitled "THE AMERICAN MATHEMATICAL MONTHLY", a Secretary, and a Treasurer.

2. There shall be a Board of Governors (herein called"the Board") to consist of the officers, the ex-presidents for terms of six years after the expiration of their respective presidential terms, the Editor of its publication entitled MATHEMATICS MAGAZINE, the members of the Finance Committee, and additional elected members (herein called "Governors"). It shall be the function of the Board to supervise all scholarly and scientific activities of the Association, to administer and control these activities, and to authorize expenditures of funds of the Association.

3. There shall be an Executive Committee of the Board consisting of the President, the President-Elect (only during a year immediately preceding the expiration of a President's term), the Past-President (only during a year immediately following the expiration of a President's term), the two Vice-Presidents, the Editor of the AMERICAN MATHEMATICAL MONTHLY, the Secretary, and the Treasurer. It shall be the function of this Committee to review continually the policies and activities of the Association, to plan and organize new activities, to formulate in broad outline the programs of meetings and of publications, and in general to consider all matters of importance or interest to the Association. This Committee shall prepare the agenda for meetings of the Board and shall analyze the implications and aspects of all matters which are to come before the Board for decision. It shall present to the Board the viewpoints suggested by such analyses, as well as all such facts as may seem pertinent or as may in any way facilitate the Board's work.

4. At all meetings of the Board of Governors a quorum shall consist of not less than 25 per cent of the membership of the Board, and no business may be validly transacted at a meeting at which less than a quorum is present.

5. There shall be a Finance Committee responsible to the Board; at the direction of the Board it shall receive and administer the funds of the Association, control its properties and investments, make its contracts, and exercise such powers as may be delegated to it by the Board. This Committee shall consist of five members including the President, the Secretary, and the Treasurer.

6. The Board shall hold a meeting each year immediately preceding the annual business meeting of the Association. Other meetings of the Board may be held from time to time at the call of the President or of any six (6) members of the Board.

7. Notice of all meetings of the Board shall be given by the Secretary to each member of the Board at least fifteen (15) days prior to the date set therefor.

8. A member of the Board may waive notice with the same effect as if due notice had been given him.

9. The Board may refer a matter to a referendum mail vote of the entire membership and shall make such reference if a referendum is requested, prior to final action by the Board, by three hundred or more members. The taking of a referendum shall act as a stay upon Board action until the votes have been canvassed, and thereafter no action may be taken by the Board except in accordance with a plurality of the votes cast in the referendum.

ARTICLE IV — ELECTIONS, APPOINTMENTS, AND TERMS OF OFFICERS AND MEMBERS OF THE BOARD

1. (a) The membership at large shall elect biennially a President-Elect for a term of one year and a First Vice-President for a term of two years and shall elect annually two Governors for terms of three years. The President-Elect shall become President for a two-year term at the expiration of his one-year term as President-Elect and shall become Past-President for a one-year term at the expiration of his term as President.

(b) The membership in each Section shall elect triennially a Governor for a term of three years beginning July 1. For these elections at least two nominations shall be submitted to the members by a committee appointed for that purpose by the Chairman of the Section. A Governor who has moved his place of employment from the Section by which he was elected shall be considered to have ended his term of office on the Board.

(c) The Board shall elect at appropriate times by ballot and for terms stated: a Second Vice-President for two years, an Editor of the AMERICAN MATHEMATICAL MONTHLY, an Editor of MATHEMATICS MAGAZINE, a Secretary, and a Treasurer, each for five years, and members of the Finance Committee (other than the President, the Secretary, and the Treasurer) for four years.

(d) The beginning and end of the term of every officer and member of the Board (except as provided in Section (b) of this Article) shall occur at the adjournment of the annual business meeting. All officers and members of the Board shall hold over until their respective successors have been duly elected or appointed and qualified.

(e) The President shall be ineligible for reelection as President-Elect or as President. The Vice-Presidents, the Editors, and the Governors shall be eligible for reelection only after an interim equal to their respective terms of office except that Governors having served less than a year and a half shall be eligible for reelection for a term of three years.

(f) The Board shall have authority to fill vacancies *ad interim* in any office, including vacancies in the Board, and to make any other appointments necessary for the transaction of business of the Association.

(g) Elections by the Board shall be made from nominations by the Executive Committee. At least two nominations shall be made for each office to be filled in the case of the Second Vice-President and members of the Finance Committee. The Board may make additional nominations.

2. For general elections by the membership of the Association there shall be a Nominating Committee appointed annually by the President with the approval of the Board. The general election shall be conducted in two stages, a primary mail voting concluding approximately five months before the date of the annual meeting and a final voting concluding at the time of the annual meeting. For the primary voting the Nominating Committee shall prepare printed ballots with five or more nominees for President-Elect and three or more for each other office to be filled by the members. Blank spaces on the ballot shall be provided for write-in votes. From the results of the primary voting the Nominating Committee shall prepare a printed ballot for the final voting. This ballot shall be mailed to the membership approximately one month before the annual meeting and the voting shall close on the day of the annual business meeting. The final ballot shall present one nominee for President-Elect, to be selected by the Nominating Committee out of the three persons who received the most votes in the primary voting. For each other office the final ballot shall present two names, one being that of the person who received the highest vote in the primary voting.

3. The President shall be the Executive Officer of the Association, shall preside at all meetings of the Board of Governors and at the annual business meeting of the Association. He shall be Chairman of the Executive Committee and of the Finance Committee. He shall have the usual duties pertaining to his office and such other duties as may from time to time be assigned him by the Board of Governors.

4. In the absence of the President, the First Vice-President (or in his absence the Second Vice-President) shall have and exercise the powers of the President. The Board of Governors may assign to the Vice-Presidents such duties as may from time to time be determined.

5. The Secretary shall have the usual duties pertaining to his office, including the custody of the records of the Association and of its Corporate Seal, the keeping of minutes of the meetings of the Board of Governors and of the annual business meeting and special meetings, and the giving of due notice of all regular and special meetings of the Association and of the Board of Governors. The Secretary shall also have the duty of seeing that whenever Governors are elected, including the election of Governors to fill vacancies, a Certificate, under the Seal of the Association, giving the names of those elected and the terms of their office, shall be recorded in the Office of the Recorder of Deeds for Cook County, Illinois. Such Certificates shall be signed by the Secretary and verified by oath of the President.

6. The Treasurer shall have the usual duties pertaining to his office including the collection of dues and the supervision and safekeeping of the funds of the Association.

7. (a) There shall be an Executive Director who shall be a paid employee of the Association. He shall have charge of the central office of the Association and shall carry out such other duties as may be assigned to him by the Board. He shall be responsible to the Board and shall attend meetings of the Board, the Executive Committee, and the Finance Committee, except when they meet in executive session, but he shall not be *ex officio* a member of these bodies. He shall be especially responsible for implementing and coordinating Section activities.

(b) The Executive Director shall be elected by the Board under terms and conditions of employment fixed by the Finance Committee.

Article V — Business Meetings of the Association

1. A business meeting of the Association shall be held annually, at such time and place as the Board may direct. Other business meetings of the Association may be called from time to time by the Board or by the President of the Association to be held at such time and place as may appear from the call.

2. Notice of any business meeting of the Association shall be given by the Secretary to each member of the Association at least thirty (30) days prior to the date set for each meeting.

3. Any member of the Association may waive notice with the same effect as if due notice had been given him.

4. At all business meetings of the Association a quorum shall consist of not less than twenty-five (25) members and no business may be validly transacted at a meeting at which less than a quorum is present.

Article VI — Sections

1. In the interest of more effective promotion of the objectives of the Association on a local level, the United States, Canada and their possessions shall be subdivided by the Board of Governors into non-overlapping geographical areas, and a Section of the Association shall be established in each of these areas. The subdivision into non-overlapping areas may be changed by the Board, upon recommendation by the Committee on Sections (see paragraph 7).

2. Each member of the Association residing in the United States, Canada or their possessions shall belong to one and only one Section. He will belong to the Section in whose geographic area he resides, except that a member who resides in one area and is employed in a different area may elect the Section to which he prefers to belong. Any member may petition the Committee on Sections for reassignment of his membership to another Section.

3. Each Section shall adopt a set of By-Laws which, along with any subsequent changes, must be approved by the Board. The geographic area covered by a Section shall be described in the By-Laws for the Section.

4. If there are members of the Association residing in a geographic area in which no Section has been organized, any ten or more members of this Association residing or employed in this area may petition the Board for authority to organize a Section covering that area.

5. A group of not less than twenty-five members of an existing Section may petition the Board to partition the area and the Section into two or more Sections. The Board shall have authority to approve or deny this petition. The Board may specify conditions under which such action may be accomplished. It may conduct a poll of some or all members of the Association in the Section to determine the advisability of allowing the proposed partition of the Section. If separate Sections are approved then each new Section must prepare its own set of By-Laws to be approved by the Board.

6. A group of not less than twenty-five members residing or employed in that part of the area of an existing Section which they desire to become part of another existing Section may petition the Board to redefine the geographic boundaries of the Sections affected. The Board shall have authority to approve or deny this petition. It may conduct a poll of some of all members of the Sections involved to determine the advisability of permitting such action.

7. There shall be a standing Committee on Sections through which the Board shall maintain general supervision over the activities of all Sections. This Committee, in particular, shall study all matters involving creation of Sections or modification of boundaries of Sections and make appropriate recommendations to the Board.

8. The Association shall not be obligated to pay from its treasury any of the expenses of a Section except as the Board may provide.

Article VII — Official Publications

1. The Association shall publish at least one official journal, of which one shall be sent free to all members of the Association in accordance with Article VIII.

2. The Board shall have full control of the publication and sale of each official journal and of all other official publications.

3. There shall be appointed by the Board a body of Associate Editors for each official journal.

4. The Board shall from time to time, as the need arises, make special provision for the management of any other publications.

5. The Board shall fix the price of each official journal and of any other publications of the Association, but in no case shall an official journal be sold to nonmembers for less than the annual dues of ordinary members.

Article VIII — Dues

1. The Board shall establish the annual dues and privileges of membership for ordinary and institutional members. The dues of ordinary members shall include a subscription to one of the official journals.

2. All dues shall be payable on the first of January of each year. Should the annual dues of any member remain unpaid beyond a reasonable time, that member shall be dropped from the list after due notice.

3. New members entering the Association after April 1 of any year may have their dues prorated for the balance of the year, except when they desire to receive the full current volume of an official journal.

4. Any ordinary member who because of age is no longer in active service, who is in good standing at the time of his retirement, and who has been a member of the Association for twenty years, may, upon notifying the central office of said retirement, be exempt from the payment of dues, with the privilege of obtaining an official journal at an annual cost of half of the dues of an ordinary member.

Article IX — Amendments to the Articles of Association and By-Laws

1. Changes in the Articles of Association or amendments to the By-Laws may be made at any annual business meeting of the Association, or at any adjourned session thereof, or at any special meeting of the Association called for such purpose, by a two-thirds (2/3) vote of those present and entitled to vote; *provided* that due notice concerning such amendment shall have been printed in each official journal, or mailed to each member, at least one (1) month before the date of such meeting. The Secretary shall give such due notice when so instructed by a vote of the Board of Governors or when so petitioned by at least forty members of the Association.

2. No changes in the Articles of Association or amendments to these By-Laws shall have legal effect until a certificate thereof, verified by oath of the President and under Seal of the Association, attested by the Secretary, shall be recorded in the office of the Recorder of Deeds for Cook County, Illinois.

SUPPLEMENT TO 1973–74 COMBINED MEMBERSHIP LIST

A special supplement to the 1973–74 Combined Membership List has been published listing 1495 members of the Association whose names were omitted from the list prepared by the American Mathematical Society in September 1973. Copies of this supplement are available free of charge from the Washington Office to MAA members who received the 1973–74 CML.

NOVEMBER MEETING OF THE OHIO SECTION

The Ohio Section of the MAA held its annual Autumn Meeting at Lorain County Community College, Elyria, November 2 and 3, 1973. Two hundred thirty-nine people attended the meeting. Chairman Frederick Leetch presided; James Murtha was Program Chairman.

Unlike recent fall meetings, the program did not center entirely on one theme. The following invited addresses were presented on Friday:

Some possible effects of the computing art on the teaching of mathematics, by Professor E. L. Glaser, Case-Western Reserve University.

Probability and real decision making, by Professor W. T. Morris, The Ohio State University.

Consequences of continuity, by Professor R. P. Boas, Northwestern University, President of the MAA.

The invited addresses were followed by swap sessions on the calculus course and service courses. During the evening the following committees met: Committee on Teacher Training and Certification, Committee on Curriculum, Committee on Cooperation between Colleges and Universities, and Committee on By-Laws. The Friday session concluded with a meeting of the Executive Committee of the Section.

The program, on Saturday morning, consisted of a panel discussion on Individualized Instruction-Procedures, Results, and Evaluation. Participants on the panel included Professors Anna L. Shrider and William Cat, Eastern Campus of Cuyahoga County Community College; Professor Stephen Slack, Kenyon College; Professors David Mader and Larry Elbrink, The Ohio State University; Professor Samuel Goldberg, Oberlin College; and Professors W. H. Beyer and W. W. Hokman, University of Akron. A question and answer session followed the presentation by the panelists.

Commercial exhibits of textbooks and other instructional materials were made available during the meeting.

R. H. Rolwing, *Secretary-Treasurer*

NOVEMBER MEETING OF THE SEAWAY SECTION

The Fall Meeting of the Seaway Section of the MAA was held at Genesee Community College, Batavia, N. Y., on November 3, 1973, with an attendance of 128 people, including 109 members of the Association. Professor W. C. Stone of Union College, Chairman of the Section, presided.

At the Saturday morning session the participants received Greetings from the College, extended by Dr. C. V. Robbins, President of Genesee Community College.

Professor G. S. Young of the University of Rochester spoke on "The Role of the Mathematical Association of America in the Two-Year College."

An invited address was given by Professor Tom Storer, State University of New York at Buffalo, whose topic was "Interrelations between Combinatorial Designs."

During the afternoon the following contributed papers were presented:

The Development of Perspective in Western Art, by D. L. Farnsworth, Eisenhower College.

A Class of Integer Identities, by Robert DeCarli, Rosary Hill College.

Construction of Rings on the Cartesian Product of the Integers and an Ideal, by Margaret W. Groman, State University College at Oswego.

The Trouble with Statistics, by H. A. Still, Queen's University.

Applied Mathematics — An Introduction via Models, by R. F. Barnes, State University College at Brockport.

Teaching Calculus via Self-paced Instruction: Advantages (many) and Disadvantages (few), by W. H. Reynolds, State University College at Cortland.

A New Approach to Leibnitzian Differentials, by J. L. Delkin, University of Western Ontario.

A Characterization of Noncommutative Quaternion Rings, by C. W. Kohls, Syracuse University.

APL Graphics for Undergraduate Students, by Kimyong Kim, State University College at Brockport.

APL in Teaching Numerical Analysis, by J. E. McKenna, State University College at Fredonia.

Non-Equivalence of the Theorems of Helly and Radon in General Convexity Theory, by R. E. Reed, State University College at Oneonta.

2-*Normed Spaces and Spaces with Euclidean Triangles*, by Charles Diminnie and Albert White, St. Bonaventure University.

Emmet Stopher, *Secretary-Treasurer*

CALENDAR OF FUTURE MEETINGS

Summer Meeting 1974: There will be no joint summer meeting in 1974, in order that mathematicians may attend the International Congress of Mathematicians to be held in Vancouver, British Columbia, August 21–29, 1974.

Fifty-eighth Annual Meeting, Washington, D. C., January 25–27, 1975.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Editorial Director.

ALLEGHENY MOUNTAIN
FLORIDA
ILLINOIS
INDIANA
IOWA
KANSAS, Ottawa University, Ottawa, Spring 1974.
KENTUCKY
LOUISIANA-MISSISSIPPI
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA
METROPOLITAN NEW YORK
MICHIGAN
MISSOURI
NEBRASKA
NEW JERSEY, Princeton University, Princeton, October 12, 1974.
NORTH CENTRAL
NORTHEASTERN, Lowell Technological Inst. Lowell, Massachusetts, November 30, 1974.
NORTHERN CALIFORNIA, Chabot College, Hayward, February 1975.
OHIO
OKLAHOMA-ARKANSAS
PACIFIC NORTHWEST, University of British Columbia, Vancouver, August 21–24, 1974 (business meeting only — no general meeting).
PHILADELPHIA
ROCKY MOUNTAIN
SEAWAY, St. John Fisher College, Rochester, N. Y., November 1–2, 1974.
SOUTHEASTERN
SOUTHERN CALIFORNIA
SOUTHWESTERN
TEXAS
WISCONSIN

FUTURE MEETINGS OF OTHER ORGANIZATIONS

AMERICAN ASSOCIATION FOR THE ADVANCEMENT OF SCIENCE
AMERICAN MATHEMATICAL SOCIETY, Washington, D. C., January 23–26, 1975.
AMERICAN SOCIETY FOR ENGINEERING EDUCATION, Rensselaer Polytechnic Institute, Troy, New York, June 17–20, 1974.
ASSOCIATION FOR COMPUTING MACHINERY, San Diego, California, November 11–13, 1974.
ASSOCIATION FOR SYMBOLIC LOGIC
FIBONACCI ASSOCIATION
INSTITUTE OF MATHEMATICAL STATISTICS
MU ALPHA THETA, University of Arkansas, Fayetteville, August 4–7, 1974.
NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS, Washington, D. C., January 25–26, 1975 (joint meeting with MAA).
OPERATIONS RESEARCH SOCIETY OF AMERICA, San Juan, Puerto Rico, October 16–18, 1974.
PI MU EPSILON, Western Michigan University, Kalamazoo, August 19–20, 1975.
SCHOOL SCIENCE AND MATHEMATICS ASSOCIATION, Sheraton-Gibson Hotel, Cincinnati, November 7–9, 1974.
SOCIETY FOR INDUSTRIAL AND APPLIED MATHEMATICS, Montana State University, Bozeman, June 24–26, 1974.

THE AMERICAN
MATHEMATICAL MONTHLY

(FOUNDED IN 1894 BY BENJAMIN F. FINKEL)

THE OFFICIAL JOURNAL OF

THE MATHEMATICAL ASSOCIATION OF AMERICA

VOLUME 81 NUMBER 6

CODEN: AMMYAE

CONTENTS

(*Continued on inside cover*)

JUNE-JULY 1974

NOTICE TO AUTHORS

Research papers per se are unsuitable; see statement of policy (Vol. 81, p. 1).

Please follow the format in current issues of the MONTHLY. Manuscripts must be legibly typewritten or reproduced from typewritten copy, double spaced with wide margins. Three copies should be submitted to the appropriate editor and one kept by the author as protection against loss. The author's full address *must* appear at the end of the manuscript.

Backlog: Main Articles 11 months, Math. Notes 23 months, Research Problems 13 months, Classroom Notes 22 months, Math. Education 18 months.

EDITORIAL CORRESPONDENCE AND MAIN ARTICLES: to ALEX ROSENBERG, Department of Mathematics, Cornell University, Ithaca, N.Y. 14850; NOTES, etc.: to the corresponding Associate Editor; ADVERTISING CORRESPONDENCE: to RAOUL HAILPERN, Mathematical Association of America, SUNY at Buffalo, Buffalo, N. Y. 14214; CHANGE OF ADDRESS and SUBSCRIPTIONS: to A. B WILLCOX, Mathematical Association of America, 1225 Connecticut Ave., N.W., Washington, D.C. 20036.

Annual dues for members of the Association (including a subscription to the American Mathematical Monthly) are $14.00 for each of the first two years of membership and $18.00 thereafter. Student Membership is available with annual dues of $9.00. For nonmembers the subscription price is $18.00.

PUBLISHED BY THE ASSOCIATION at Washington, D. C., and Menasha, Wisconsin, during the months of January, February, March, April, May, June–July, August–September, October, November, December.

Second-class postage paid at Washington, D. C., and additional mailing offices.

NOTICE TO OUR CONTRIBUTORS

The following requirement is made necessary because of rapidly increasing costs. The authors of material accepted for the MONTHLY which contains figures or diagrams MUST supply these in a form suitable for photographic reproduction. The figures must be fully lettered in a style consistent with that used in the rest of the article. The figures may be larger than the form in which they will finally appear.

NOTICE: RESOURCE MATERIALS FOR GRADES 7–12

With the financial support of the National Science Foundation, the National Council of Teachers of Mathematics and the Mathematical Association of America through its Committee on the Undergraduate Program in Mathematics, are engaged in producing resource materials in all the various applications of mathematics suitable for use by both teacher and student in mathematics instruction for grades 7–12, i.e., the last six years of secondary school. Applications of arithmetic, elementary and advanced algebra, geometry, computing, and other more advanced topics are being worked on. In addition to the uses of mathematics in other disciplines, applications of mathematics in daily life and to skilled trades will be especially emphasized. The readership of this journal is hereby requested to send suggestions regarding this project, sample problems, references, or any other suitable materials ranging from simple exercises to extended model building and mathematical development to

CUPM
P.O. Box 1024
Berkeley, California 94701
USA.

The readership is reminded that through hobbies or previous employment it may know of special applications that might otherwise escape notice.

HUGO STEINHAUS — A REMINISCENCE AND A TRIBUTE

MARK KAC

1. Hugo D. (for Dyonizy, a name he disliked and seldom used) Steinhaus who died on February 25, 1972, at the age of eighty-five years was unique as a mathematician and as a man.

His long and distinguished service to science and to his country was interrupted only during the years of the Second World War when he was forced to hide from the Nazi barbarians. And even then his sharp restless mind was at work on a multitude of ideas and projects.

He was one of the architects of the school of mathematics which flowered miraculously in Poland between the two wars and it was he who, perhaps more than any other individual, helped to raise Polish mathematics from the ashes to which it had been reduced by the Second World War to the position of new strength and respect which it now occupies.

He was a man of great culture and in the best sense of the word a product of Western civilization.

He was fluent in German, French, and English, knew Latin and Greek, and his knowledge of Polish (the language he truly loved) was as nearly perfect as is humanly possible.

He read prodigiously, had a phenomenal memory (I swear that he knew *Faust* by heart, since he could always be counted on to produce an appropriate quotation), and was intolerant of any kind of sloppiness.

He was convinced that no one could be considered to have been really educated without a thorough knowledge of Latin and was genuinely distressed when he discovered that I did not know Latin at all. The school I went to, the Lycée of Krzemieniec, though probably the best known of all Polish secondary schools, inexplicably was in my days of the type which did not offer or require Latin. He finally forgave me when he met my father and discovered that he not only had a commanding knowledge of Greek and Latin but was fluent in Hebrew and knew old church Slavonic to boot. On the basis of some kind of averaging, I was acceptable.

2. Except for one year in Lwów and a term in Munich, Steinhaus received his mathematical education in Goettingen where in 1911 he was granted the doctor's degree *summa cum laude* for a dissertation "Neue Anwendungen des Dirichlet'schen Prinzips" written under Hilbert.

His own work did not bear the imprint of Goettingen. I don't believe that following his doctoral dissertation he ever wrote anything directly connected with differential and integral equations or calculus of variations. He was at one time,

though, interested in foundations of geometry, and one of the doctoral dissertations written under his direction was on this very subject. The influence of that great school on his overall mathematical outlook, however, was crucial—and no wonder. For during his student years he came in close contact not only with Hilbert and Klein but also with F. Bernstein, Carathéodory, Courant, Herglotz, Koebe, Landau (who succeeded Minkowski, who died a little earlier), Runge, Toeplitz, and Zermelo, not to mention scores of physicists, astronomers, and philosophers who also gravitated to Goettingen.

An episode from this period is worth recording. In the spring of 1911 Albert A. Michelson was invited to Goettingen, and he stayed with his family in the pension in which Steinhaus lived. The two men became acquainted and Michelson offered Steinhaus a job as his mathematical assistant.

The Goettingen experience strongly influenced his teaching: he was, for example, the first (and between the two wars the only one) in Poland to lecture regularly on numerical and graphical methods, an interest which is directly traceable to Runge's lectures on this subject. He also emulated Klein and offered from time to time a one-semester course on elementary mathematics (mainly geometry) from an advanced point of view.

But the decisive influence on his own scientific work did not come from Goettingen. It came from Paris where Henri Lebesgue, as yet unrecognized outside his native France, was setting mathematics on a new course which it would travel for many decades to come.

I do not know exactly when Steinhaus became aware of the work of Lebesgue. In the first installment of his memoirs (published in 1970 in the Polish monthly *Znak*), there is a casual mention that he was reading "Leçons sur les séries trigonométriques" in 1912 (or perhaps early in 1913), but there seems to be no mention of "Leçons sur l'intégration" which, of the two books, was certainly the more fundamental.

There is, however, no doubt that he mastered the new theory, for in 1918 he proved that every linear functional on $L(0,1)$ is of the form

$$\Lambda(f) = \int_0^1 f(x)m(x)\,dx, \qquad f \in L(0,1),$$

where $m(x)$ is bounded almost everywhere, thus completing a series of investigations on the forms of linear functionals which was begun by Fréchet and F. Riesz.

In 1920 he proved the beautiful theorem that the set of distances of a set of positive Lebesgue measure contains an interval; at about the same time he also became an extraordinary (Associate) Professor at the University of Lwów and offered a course on Lebesgue's theory, one of the earliest such courses given outside of France.

Somewhat earlier he had devoted himself to the theory of trigonometric and Fourier series and became, almost at once, a recognized master of this field.

He gave the first example of an everywhere divergent trigonometric series whose coefficients tend to zero and, more remarkably, an example of a trigonometric series which converged in one interval and diverged in another. This example led Alexander Rajchman to the creation of the theory of formal multiplication of trigonometric series, and it was Rajchman who in turn interested Zygmund in the field.

Steinhaus also constructed a trigonometric series which converges everywhere without being uniformly convergent in any interval and made numerous other contributions to the anatomy and pathology of trigonometric series.

His interest in trigonometric series led naturally to an interest in more general orthogonal series and to this field Steinhaus also made numerous important contributions culminating in the classic monograph *Theorie der Orthogonalreihen* (Monografje Matematyczne) written jointly with S. Kaczmarz which appeared in 1937.

The theory of orthogonal series had, from the start, been closely related to the theory of linear operators and linear spaces, but the full exploitation of the intimate relation between the two theories came as a result of the pioneering work of Banach.

Steinhaus met young Banach by chance when strolling one day in 1916 through a park in Cracow, he overheard a snatch of a conversation in which the term "Lebesgue integral" was used. Startled, he introduced himself to the two young men who were discussing this unlikely subject, and one of them was Banach. The other was Otton Nikodym.

Steinhaus and Banach wrote only two joint papers, but one of these "Sur le principe de la condensation des singularités" (*Fund. Math.* 9, 1927, pp. 50–61) became a classic and is known to every student of functional analysis.

The principal result is that if $\{F_{pq}(x)\}$ is a double sequence of linear operators on a Banach space E (with values in another Banach space F) such that for each $p = 1,2,\cdots$

$$\limsup_{q\to\infty} \| F_{pq} \| = \infty$$

($\| F \|$ is the norm of the operator F) there exists a set $X \subset E$ of the second category such that for every $x \in X$

$$\limsup_{q\to\infty} \| F_{pq}(x) \| = \infty, \qquad p = 1,2,\cdots,$$

($\| F(x) \|$ is the norm of the element $F(x) \in F$).

The collaboration between Banach and Steinhaus was however much closer than the two joint papers would indicate.

Many of Banach's early ideas were developed and first tested in a seminar

conducted by Steinhaus, and the two men jointly founded *Studia Mathematica*, which became one of the major mathematical journals of the world.

Banach was Steinhaus' first doctoral student, and Steinhaus joked later on that Banach was his most important mathematical discovery.

In 1938 when the threat of war hung heavily over Poland and Steinhaus presided over the award of an honorary doctorate to Henri Lebesgue, he said to me after the ceremony, "It will not be a bad record to leave behind, to have had Banach as the first and Lebesgue as the last doctoral candidate."

3. In 1923 Steinhaus published in *Fundamenta Mathematica* (4, pp. 283–310) a remarkable memoir under the title "Probabilités dénombrables et leur rapport à la théorie de la mesure." It was the first rigorous axiomatic treatment of coin tossing based on measure theory. (However, almost simultaneously a similar axiomatization was given by A. Łomnicki, a Professor of Mathematics at the Engineering School (Polytechnicum) of Lwów. It should also be mentioned that a nephew of A. Łomnicki, Z. Łomnicki jointly with Ulam already gave in 1932 a most general axiomatic treatment of independence.) It contained the seeds of future developments, and it certainly influenced Kolmogorov's definitive axiomatization, which came ten years later.

It was a truly pioneering work for, at the time it was written, probability theory was not even of peripheral concern to most mathematicians and, in fact, was not generally considered to be a part of mathematics.

It was typical of the way Steinhaus approached mathematics that the main point of the paper was not the axiomatization, for he disliked axiomatization for its own sake, but the concrete question of what the probability is that a series

$$\sum_{k=1}^{\infty} \pm c_k,$$

with $\pm$ signs chosen "at random," converges.

Within the framework of the classical Laplacian probability theory, such a question cannot be properly formulated, and Steinhaus set himself the goal of extending and modifying this framework to make such questions well-posed.

He did it by constructing a measure on the set of all infinite sequences

$$(r_1, r_2, r_3, \cdots),$$

where each r_k is either $+1$ or -1, the measure reflecting the assumptions that the tosses are independent, and that in each toss the two alternatives are equiprobable.

This he accomplished by showing that the desired measure maps into the ordinary Lebesgue measure on (0, 1) by the map

$$(r_1, r_2, \cdots) \to t = \sum_{k=1}^{\infty} \frac{(2r_k + 1)}{2^{k+1}}.$$

Writing t as a binary expansion

$$t = \sum_{k=1}^{\infty} \frac{\varepsilon_k(t)}{2^k}$$

and setting

$$r_k(t) = \frac{2\varepsilon_k(t) - 1}{2},$$

we see at once that the question of the probability of convergence of the series $\sum \pm c_k$ becomes equivalent to the question of the measure of the set of convergence of the series

$$\sum_{k=1}^{\infty} c_k r_k(t).$$

This question was partially answered by Rademacher who had proved a few years before that if

$$\sum c_k^2 < \infty,$$

the series converges almost everywhere, and hence the desired probability is one.

Somewhat later Paley and Zygmund proved that if $\sum c_k^2 = \infty$, the probability in question is zero.

These results were clear forerunners of Kolmogorov's famed "three-series theorem" concerning convergence of sums of independent random variables.

To the same circle of ideas belongs also the result (see "Über die Wahrscheinlichkeit dafür, dass der Konvergenzkreiss einer Potenzreihe ihre natürliche Grenze ist," *Math. Zeit.*, 31 (1929) pp. 408–416) that if the power series

$$\sum_{1}^{\infty} c_n z^n$$

has radius of convergence r $(0 < r < \infty)$ then with probability one the circle $|z| = r$ is a natural boundary for the series

$$\sum_{1}^{\infty} c_n e^{2\pi i \theta_n} z^n,$$

where the θ_n are independent and uniformly distributed in $(0, 1)$.

In 1929 it was far from clear what "independent and uniformly distributed" meant, and Steinhaus had first to make these concepts precise. This he did by constructing a product measure in the Hilbert cube (i.e., the direct product of denumerably many unit intervals) with uniform measure on each component.

4. Let me now jump many years ahead and speak briefly of yet another of Steinhaus' achievements.

Logic and foundations were not among his main interests, and, as a matter of fact, his attitude toward them was mildly critical. He did as a young man write a paper in which he axiomatized the concept of limit, and he was, as mentioned above, in the midtwenties interested in foundations of geometry, but that was about all. Thus he did not appear to be a likely candidate to challenge the axiom of choice and propose a substitute. But this is exactly what he did, and in a manner characteristic of the way in which he approached mathematics, and it had to do with his interest in games and the problems they pose.

Games always fascinated Steinhaus, and he was among the first to define and discuss the concept of strategy. This he did in a little-known note in Polish, published in an obscure non-scientific Polish journal, in 1925, fortunately saved for posterity through a translation into English ("Definitions for a theory of games and pursuits" in *Naval Research Logistics Quarterly* 7, 1960, pp. 105–108). He was aware of the following theorem: Let G be a two-person game with perfect information, terminating in a *finite number* of moves in a win by one of the players. Then there must exist a winning strategy for either one or the other adversary.

The proof which I heard him give in a lecture at Rockefeller University in 1962 is unforgettably simple:

Denoting the players by A and B and their moves by $x_1, x_2, \cdots$ and $y_1, y_2, \cdots$ respectively, we can express the fact that A has a winning strategy symbolically as follows (we assume that A starts)

$$(\exists x_1)(y_1)(\exists x_2)(y_2)\cdots(\exists x_n)(y_n) \qquad A \text{ wins.}$$

The negation of this statement is obtained by the familiar rule of De Morgan, and it reads

$$(x_1)(\exists y_1)(x_2)(\exists y_2)\cdots(x_n)(\exists y_n) \quad A \quad \text{does not win.}$$

This however is clearly the statement that B has a winning strategy, and the proof is thus complete!

Steinhaus now proposed that this simple theorem be made into an axiom by removing the restriction that n is finite, i.e., that the game must terminate in a finite number of moves.

It is here that one runs afoul of the axiom of choice.

In the nineteen-thirties, Banach and Mazur (with much help from Ulam) had considered a class of infinite games of which the following is typical:

Let S be a subset of the interval $(0,1)$ and let the players A and B choose binary digits x_1, y_1, x_2, $y_2, \cdots$ (actually in the original Banach-Mazur version the players were allowed to pick an arbitrary finite number of digits) defining the number ξ,

$$\xi = \frac{x_1}{2} + \frac{y_1}{2^2} + \frac{x_2}{2^3} + \frac{y_2}{2^4} + \cdots .$$

Player A wins if $\xi \in S$ and B if $\xi \notin S$. Using the axiom of choice one can "exhibit" (as was shown by Mycielski on the basis of an older result of Banach-Mazur-Ulam) a set S for which neither A nor B had a winning strategy, and therefore the Steinhaus axiom (which came to be known as the axiom of determinacy) contradicted the axiom of choice. I recall vaguely that the Banach-Mazur-Ulam result was used to "prove" that there could be at most one God, since if there were two, they could be made to play the game, with a Banach-Mazur-Ulam set giving rise to the inevitable difficulty that neither could be considered omniscient. Steinhaus felt that his axiom was closer to "reality" than the axiom of choice, for he insisted that intuition demands that for each set one of the players should have a winning strategy.

The brief note (written jointly with J. Mycielski, who also contributed much to the clarification of the role of the new axiom) "A mathematical axiom contradicting the axiom of choice" (*Bull. Ac. Pol. Sc. Série des sci. math., astr. et phys,.* 10 (1962) pp. 1–3) attracted considerable attention among contemporary logicians and is responsible for an ever-growing body of work.

I mention this unique excursion into set theory because it is so characteristic of Steinhaus' mathematical style—a style in which superb intelligence was combined with rare wit and an unerring instinct for what was essential and promising, with an eye for the unusual and striking.

His wit was not confined to mathematics and many of his repartees became legendary. When, about fifteen years ago, after failing to attend an important meeting of a committee of the Polish Academy of Sciences, he received a letter chiding him (along with several others) for not having "justified his absence," he wired the President of the Academy that, "as long as there are members who have not yet justified their *presence,* I do not have to justify my absence."

He was also a master of mathematical paraphrase. When he first heard the statement of the Borsuk-Ulam theorem, he immediately said, "It means that at any given time there is at least one pair of antipodal points on the surface of the earth at which the temperatures and pressures are the same."

It should be also recorded that it was Steinhaus who proved and invented the name "ham sandwich theorem" and that he liked to explain jokingly why π appears so often in probability theory by quoting a Polish proverb *Fortuna kołem się toczy* ("Fortune moves in circles" in a translation which unfortunately leaves much to be desired).

I cannot resist giving one more example of Steinhaus' quick mathematical intelligence. It has to do with his estimate of the casualties of the German army in 1944, and it should be borne in mind that he was then in hiding and completely cut off from any source of reliable news.

He noticed that some of the obituaries of German soldiers which were published in the rigidly controlled local newssheet mentioned that the dead was the second or even third member of his family to have fallen in the war, and this was information enough!

For by dividing the percentage of obituaries of second, third, etc. sons by the (conditional) probability that a family with at least one son will have more than one, an estimate of casualty percentage can be obtained. Disregarding the age factor (some sons may be too young to be drafted), all one needs is the average number of sons in a family (easily estimable) and the knowledge that the number of sons obeys the Poisson distribution.

5. I met Steinhaus for the first time in the Spring of 1932 at the end of my first year at the University of Lwów, when I drew him as the oral examiner in Analysis I. There were four Professors of mathematics, and to insure equitable distribution of examination fees, examiners were chosen by lot. He had a reputation of being very tough.

This reputation was not all that well deserved, as the following anecdote indicates. A girl student who was not terribly good drew Steinhaus as the examiner in Analysis II, by far the most difficult of all examination subjects. We were all surprised when she emerged with a B, and I asked him later on how it happened. "Well," he said, "I asked her to describe the Riemann surface of $\sqrt{z}$, and she said that she had one in her purse and after a brief search produced a rather nice model. Don't you think that any young lady who carries a Riemann surface in her purse deserves at least a B?" I couldn't argue the point.

He asked me, as I recall, two very simple questions and gave me an A. Before I left his office, I asked permission to attend his Seminar (which he conducted jointly with Mazur). Permission was needed, since I would be jumping into the second year of a two year cycle. He allowed me to register, and thus there began my mathematical life.

For a while my contacts with him were confined only to the Seminar, in which I learned more mathematics than in retrospect seems possible, but sometime in 1934, I believe, he presented me with a definition of independent functions (in the statistical sense) and suggested that I try to do something with it. We began to collaborate and from about 1935 until November 1938 when I left Poland, we were almost inseparable. For a time he employed me as a private assistant, and in this capacity I helped him with *Mathematical Snapshots* (2nd Ed., Oxford University Press, 1950) but only in mundane matters of routine. All the ideas that went into this remarkable book were his.

I shall not attempt to evaluate the role which our collaboration on independent functions played in some of the developments in probability theory of the last forty years. Suffice it to say that when I dedicated my Carus Monograph *Statistical*

Independence in Probability, Analysis and Number Theory to him, it was not merely the sentimental gesture of a grateful pupil. Nearly everything in that book is traceable to the years of our collaboration, to long walks through the streets and parks of Lwów, to interminable discussions and arguments, even to occasional minor battles. Were it not for the war and separation, the book might have been a joint undertaking. In a way it is, although only my name appears on the cover.

I should like to go back to the *Snapshots*, because to understand and appreciate Steinhaus' mathematical style, one must read (or rather look at) *Snapshots*.

Written in 1937 and designed to appeal to "the scientist in the child and to the child in the scientist," it has gone through uncountably many editions, has been translated into fourteen languages, and is still among the best selling "popular" books on mathematics. It is a book unlike any other, and it expresses, not always explicitly and at times even unconsciously, what Steinhaus thought mathematics is and should be.

To Steinhaus mathematics was a mirror of reality and life much in the same way as poetry is such a mirror, and he liked to "play" with numbers, sets, and curves, the way a poet plays with words, phrases, and sounds.

His approach to mathematics was largely visual and only seldom abstract. He liked objects and facts and was suspicious of most generalizations and extensions. "A statement about curves is not interesting unless it is already interesting in the case of a circle," he told me years ago, and this sums up well his fundamental belief that real insights are gained from contemplating the simplest and most elementary things.

Steinhaus deplored the growing professionalization of mathematics, the ever-increasing specialization, the flight from robust reality into the murky clouds of uncontrolled abstraction.

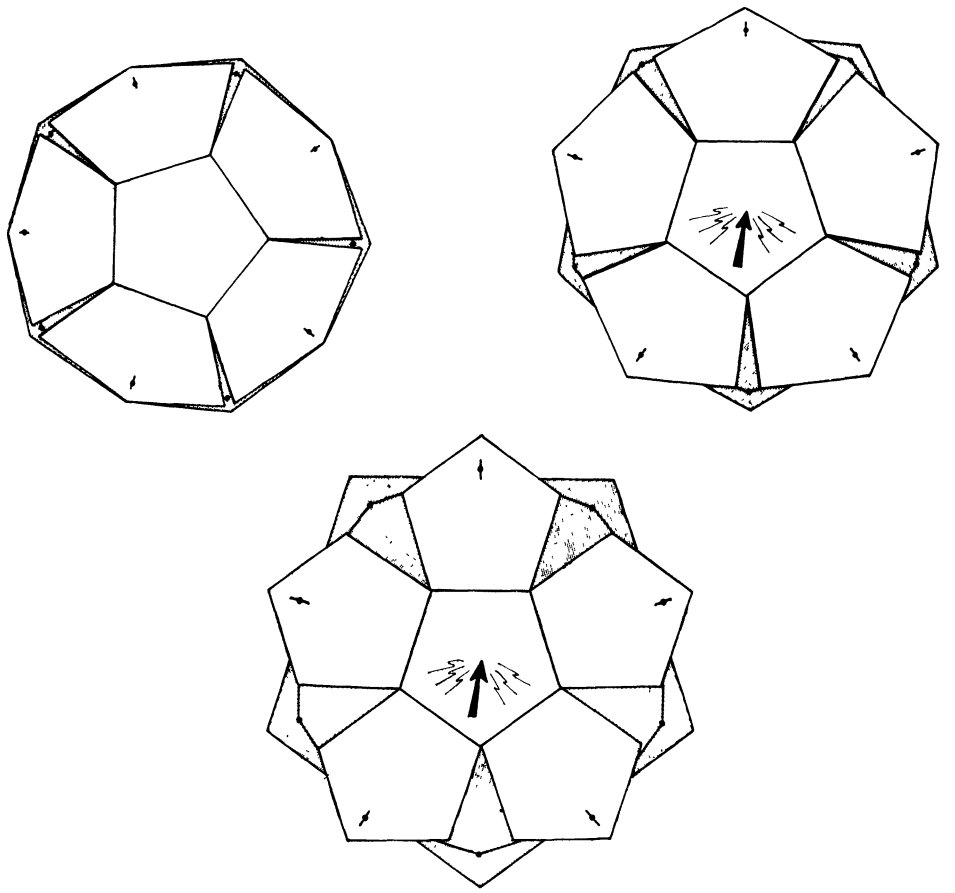
He spent a significant portion of his scientific life in collaborating with physicians (he was awarded an honorary degree of Doctor of Medicine in recognition of his contributions toward applying mathematics to a wide variety of biological and medical problems), engineers, oil prospectors, geologists, without a thought entering his mind that he might be engaging in a different sort of activity than that which led him and Banach to formulate the principle of condensation of singularities.

Mathematics to Steinhaus was mathematics, and he was scornful of labels such as "pure," "applied," "concrete," "useful," etc. He liked clear sharp points and was impatient with long-winded discourses. "Wo ist der Witz?" ("Where is the joke?"), he liked to ask in an attempt, not always successful, to cut through the fog.

He had a life-long love affair with elementary mathematics and could spot new wonders in the simplest and most familiar objects.

He was particularly proud (and justly so) of having invented the self-folding dodecahedron, and each copy of the first edition of *Snapshots* was provided with a handmade model (this had to be abandoned in later editions because of prohibitive cost).

The three sketches below are based on one of the few remaining models.



The barely visible line represents a rubber band which when unstretched is a geodesic of the regular dodecahedron, and as pressure is applied to the upper face, the band stretches. As the pressure is released, the tendency of the rubber band to snap back has the effect of restoring the "squashed" dodecahedron to its original Platonic form. Years ago in Lwów I remember Steinhaus telling me: "Everything I have done could have been done by someone else, but only I could have invented the self-folding dodecahedron."

He was wrong in the first premise but right in the second. Only he could have done it.

Acknowledgment: I wish to thank Professors D. A. Martin and J. Mycielski for setting me straight on a number of points concerning the axiom of determinacy, and Mr. Jonothan Logan for many suggestions and improvements.

DEPARTMENT OF MATHEMATICS, ROCKEFELLER UNIVERSITY, NEW YORK, NEW YORK 10021.

THREE KAKEYA PROBLEMS

F. CUNNINGHAM, JR.

This paper is dedicated to Professor I. J. Schoenberg on the occasion of his seventieth birthday.

The original Kakeya problem was to find the minimum area of a plane set K in which a line segment of length 1 can be continuously moved so as to return to its starting position with its direction reversed. A. S. Besicovitch solved this problem [**1**] by showing that it is not really a minimum problem at all; rather K can have area as small as you please. This result and its proof have been improved in various ways [**2**, **5**, **10**] and the fact needed in this paper is that one can at the same time require K to be contained inside a unit circle [**5**]. The surprise of Besicovitch's solution does not wear off, but is rather renewed by further shocks to the intuition as its applications to other similar geometric problems unfold. The three results presented here were triggered by the Kakeya problem for the sphere, which was put to me in a letter by I. J. Schoenberg, a longtime contributor to Kakeya theory. Here the action takes place on the surface of a unit sphere, and the moving segment is replaced by an arc of great circle. Because this geometry is bounded, the answer will depend on the length of the moving arc. In Section 1 below it is shown that this dependence is in fact a step function.

The solution to the spherical Kakeya problem is applied in Section 2 to another problem in the plane. Several authors [**3**, **11**, **7**] have given examples of "thin sets of circles," that is, sets of measure 0 which contain circles of every radius. These examples are reminiscent of Besicovitch's example of a plane set of measure 0 which contains lines in all directions, solving a problem which Besicovitch called the twin of the Kakeya problem. The Kakeya twin to the thin set of circles would be the following problem: Given $a < 2\pi$, what is the greatest lower bound of areas of plane sets within which a circular arc of a radians can be shrunk to a point, allowing continuous motions with variable radius but constant radian measure? The answer is 0 for all $a < 2\pi$.

In the third problem to be discussed the moving figure is a line segment marked off into three parts. Figuratively representing a bird, the small central segment is called the **body**, and the long end segments, the **wings**. The object is no longer to turn the figure around, but to move it continuously in such a way that the body will touch every point of a given set, while the wings sweep over a set of small area. Closely related problems, sometimes for a one-winged bird, have arisen naturally for applications in various parts of analysis [**4**, **6**, **9**]. This problem attracted my attention because a negative answer to it would give a negative answer to the spherical Kakeya problem for a slightly less than π. But the answer (to both problems!) is affirmative, (Theorem 3).

As the Besicovitch twin to this result, I also give, in Section 3, a startling theorem

in which the bird's motion is not required to be continuous, but in which, for revenge, the bird has become a whole infinite line, its body has become a point on the line, the set to be covered by the body-positions is the whole plane, and the wings stay in a set of measure 0. After submitting the first version of this paper I learned from Roy O. Davies that he has proved substantially the same result in [**6**]. Some remarks on this situation will be found at the end of Section 3, and Section 4 outlines some further problems.

To save space and gain readability I give the proofs somewhat informally, inviting the skeptical reader to share in the creative task of verifying many pesky but elementary details.

1. The spherical Kakeya Problem. Schoenberg's question asks for the value of k_a, the greatest lower bound of areas of sets on the unit sphere in which a great circle arc of length a, hereafter referred to as the **needle**, can be continuously turned around. Knowing the answer to the plane Kakeya problem, one expects the answer here to be $k_a = 0$ at least for small values of a. This is in fact the case, as has been shown by Wilker [**12**] for $a < \pi/2$. At the other extreme where $a = 2\pi$ it is clear from topological reasons that the answer is 4π, the area of the whole sphere. This is because a continuous turning of a whole great circle in any set which is not the whole sphere would be topologically equivalent to a one-to-one continuous reversal of a circle in the plane, which is impossible. This remark also provides a trivial proof that $k_\pi = 2\pi$. Indeed, the problem here is to turn a great semicircle around. Suppose this can be done in a set K, and let K' be the antipodal image of K. As one half of the great circle moves in K, the other half simultaneously moves in K', so that the whole great circle gets turned around in $K \cup K'$, which must therefore be the whole sphere. Since K and K' are congruent, each has area at least half the area of the sphere, showing that $k_\pi \geqq 2\pi$. Actually $k_\pi = 2\pi$, since this minimum area is achieved by a rotation. What is the situation for $\pi/2 \leqq a < \pi$?

THEOREM 1. *With k_a as defined above for $0 < a \leqq 2\pi$, we have*

$$k_a = \begin{cases} 0 & \text{for } a < \pi \\ 2\pi & \text{for } \pi \leqq a < 2\pi \\ 4\pi & \text{for } a = 2\pi. \end{cases}$$

The lower bound is achieved when $a = \pi$ or 2π, not otherwise.

The proof draws on [**5**] and [**12**].

For the first step let $\varepsilon_1 > 0$ and $0 < a_1 < \pi/2$ be given; we shall produce a set K_1 of area $< \varepsilon_1$ on the sphere, which is a Kakeya set for needle length a_1, that is, a great circle arc of length a_1 can be turned around in K_1. To do this, first apply the technique of [**5**] to obtain a plane Kakeya set K of area $< \varepsilon_1$, which works for needles of length a (some large number), and which is contained in a circle of

radius a. Now place the unit sphere tangent to the plane, with the North pole at the center O of the circle, and project from the plane to the northern hemisphere by lines through the center of the sphere. Let K_1 be the image of K under this projection. The area of K_1 is small enough, because the projection decreases areas. The projection takes line segments in the plane to great circle arcs on the sphere, so that to each needle position in K there corresponds a needle position in K_1, though unfortunately shorter. While it is not easy to make a precise estimate for a lower bound of these lengths, if we take into account that all needle positions in K lie on lines which can be kept arbitrarily close to O and have one end near O, it appears that for a large a we can make a_1 approach $\frac{1}{2}\pi$. (Actually, a smaller value of a_1 would suffice for the application of this step in what follows.) This step, due to Wilker, proves that $k_a = 0$ for $a < \frac{1}{2}\pi$. The method even works for $a = \frac{1}{2}\pi$, but not beyond.

The second step is to show how to enlarge a given spherical Kakeya set to make it work for a longer needle. Specifically, given on the sphere a set K_1 of area less than ε_1 in which an arc of length a_1 can be turned around, and given $\varepsilon_2 > 0$, we shall show how to make K_2 of area less than $\varepsilon_1 + \varepsilon_2$ in which an arc of length a_2 can be turned around; if $a_1 = \pi - d$, we can achieve almost $a_2 = \pi - \frac{1}{2}d$. In this step we use the fact that the plane Kakeya sets we are using are finite unions of very acute isosceles triangles of height a. Each such triangle serves as a **partial Kakeya set** in which the needle turns from one of the long sides to the other. We therefore suppose that K_1 is the union of n acute isosceles spherical triangles of height a_1, each of which provides by rotation the needle's transition from one great circle to another. If we can enlarge each of these triangles at a cost of only ε_2/n in additional area to make a partial Kakeya set for the same pair of great circles but for needle length a_2, then the union of these new sets will serve for K_2.

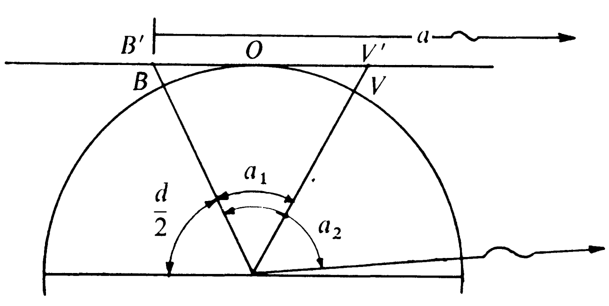


FIG. 1

Consider then one such triangle with base B and vertex V. Take as temporary North pole the midpoint O of its altitude BV, place a plane tangent to the sphere at O, and map the triangle to the plane by the inverse of the projection used in step 1. This gives a plane isosceles triangle with base B' and vertex V', the midpoint of the altitude $B'V'$ being O (Figure 1). Starting with this triangle as "nucleus," and for

a very large needle length a, use the replacement technique of [5, Section 4] to build a partial Kakeya set K' for the two sides (extended) of triangle $B'V'$, the area of K' being only slightly greater than the area of the triangle. Finally, project K' back to the sphere to get a set K enlarging the initial triangle BV. The area of K is small enough. As in step 1 estimating just how long a needle will turn in K is awkward, but when we remember that needle positions in K' all lie almost on the altitude $B'V'$ (extended) and reach a distance a either to the right from B' or to the left from V', it is apparent from Figure 1 that by taking a large we can make a_2 approach $a_1 + \frac{1}{2}d = \pi - \frac{1}{2}d$.

It is now easy to prove the theorem for the case $a < \pi$. Given $\varepsilon > 0$, first apply step 1 to make a set K_1 of area less than $\varepsilon_1 = \frac{1}{2}\varepsilon$ which is a Kakeya set for some needle length $a_1 = \pi - d < \frac{1}{2}\pi$. Now apply step 2 repeatedly, allowing area increases $\varepsilon_2 = \frac{1}{4}\varepsilon$, $\varepsilon_3 = \frac{1}{8}\varepsilon, \cdots$ to make a sequence of sets $K_1, K_2, K_3, \cdots$, where K_n is a Kakeya set for some needle length a_n which is almost $\pi - d2^{-n}$. As n increases a_n approaches π as limit, so that eventually $a_n > a$, and then K_n answers to the requirements of the theorem.

It remains to consider the case $\pi < a < 2\pi$. Let $b = a - \pi$ so that $0 < b < \pi$. Given $\varepsilon > 0$, let K be a Kakeya set of area less than $\frac{1}{2}\varepsilon$ for needle length b. (We now know that such a set exists.) We shall make from K a Kakeya set for needle length a having area less than $\pi + \varepsilon$. Scrutiny of the preceding steps reveals that K has a center O such that the distance from O to the moving needle is never more than some $\delta > 0$ which we may make as small as we please. Take, in fact, δ so small that when a hemisphere is enlarged by a band of width δ around its boundary the added area is $\frac{1}{2}\varepsilon$. Now divide the sphere into two hemispheres H_1 and H_2 by an arbitrary great circle through O, and enlarge one of them, say H_2, as indicated above to make a set H_δ. Let $K_1 = K \cap H_1$ and $K_2 = K \cap H_2$, and let K_2' be the set antipodal to K_2. Then the promised set is $L = K_1 \cup H_\delta \cup K_2'$. The area of L is less than the sum of the areas of K and H_δ, so less than $2\pi + \varepsilon$. To see that L is a Kakeya set for needle length a, observe that for every position in K of a needle of length b there is a corresponding position in L of a needle of length $a = b + \pi$. For of the short needle in K one keeps the part in K_1, replaces the part in K_2 by its antipodal image in K_2', and joins the two pieces together by an arc of length π lying in H_2. (The δ-band added to H_2 is to take care of needle positions which do not meet H_2.)

2. Shrinking a circular arc. Again (for topological reasons), a circle of radius 1 cannot be continuously shrunk to a point in the plane in any set of area less than π. As soon as a circular arc is less than the whole circle, however, the situation is as different as possible.

THEOREM 2. *Given $a < 2\pi$ and $\varepsilon > 0$, there is a plane set K of area less than ε within which a circular arc of a radians and unit radius can move continuously, with variable radius but fixed radian measure, so as to shrink to a point.*

The proof uses the stereographic projection familiar from complex variable theory. Let Σ be a sphere with center O, equator E, and North and South poles N, S. Let Π be the equatorial plane containing O and E. Then each line through N not parallel to Π meets $\Sigma - \{N\}$ and Π just once each, so that we get a bijection $\sigma: \Sigma - \{N\} \to \Pi$ taking S to O and E to itself. It is well known that σ takes circles on Σ to circles in Π, except that circles through N go to straight lines. The images of meridian circles (great circles through N and S) are straight lines through O; otherwise great circles go to circles in Π of various radii which meet E at antipodal points. Areas and the radian measure of arcs are not preserved by σ, those in the northern hemisphere being increased, and those in the southern hemisphere being decreased. On a fixed great circle, not E, a small arc is magnified most by σ when it is nearest N, the extent of this maximum magnification being greatest when the great circle is most inclined to E. Areas also are most magnified when they are near N. If we restrict our sets on Σ to stay a positive distance δ away from N, then there will be a constant M such that the radian measure of an arc and the area of a set will at most be multiplied by M when the figure is taken by σ to the plane.

Now let C be any circle in the plane, and let $a < 2\pi$ and $\varepsilon > 0$ be given. Choose on C two points so close together that the circle E with the chord joining these two points as diameter encloses an area less than $\frac{1}{2}\varepsilon$. Let Σ be the sphere having E as equator, and invoke all the notations of the preceding paragraph. Then C is the image of a great circle C' on Σ, not through N. Let δ be half the distance from N to C', and, using the technique of Theorem 1, construct a set K on Σ with the following properties:

(i) *K is a partial Kakeya set in which a great circle arc of radian length a' ($a' < 2\pi$ to be specified presently) can move from the great circle C' to the great circle E.*

(ii) *K contains the southern hemisphere.*

(iii) *The part of K in the northern hemisphere has small area (how small to be specified presently).*

(iv) *The distance from N to K is at least δ.*

Then $\sigma(K)$ will be the plane set required in the theorem. Indeed, because of (iv) arcs and areas in K are magnified at most by a factor M (depending on δ). Therefore, if in (iii) we make the area of K small enough, the area of $\sigma(K)$ will be the area inside E plus an area less than $\frac{1}{2}\varepsilon$, making a total less than ε. Also every position of a needle of radian length $a' = 2\pi - d$ in K corresponds to a circular arc in $\sigma(K)$ of radian measure at least $2\pi - Md$. Taking d small enough in (i) will make $2\pi - Md \geqq a$. Then as the needle moves in K its image does in $\sigma(K)$ a motion of the kind prescribed in the theorem. When the needle at last arrives on E, its image is on E and shrinks the rest of the way in the disk.

3. The shadow of a gliding bird. Recall that the bird is a line segment made up of three sections, a body of length $2b$ in the middle flanked by wings each of length w.

One might expect that as the bird glides about the plane so that its body sweeps out a large area the wings would necessarily sweep out a considerable area as well. This is not the case. Two answers to this question are both based on the following lemma, which is essentially Lemma 6 of [**6**]. Departing for the moment from the bird metaphor, it has the following interpretation: it is possible to thatch a hut standing in the middle of a garden with straws which are much too long (projecting beyond the garden on both sides) in such a way that the roof has no leaks and yet an arbitrarily small part of the garden is shaded by the eaves.

The data for the lemma consist of the following configuration. There is a rectangle R, which we take to be $A \cap B$, where A is the vertical strip $|x| \leqq a$ and B is the horizontal strip $|y| \leqq b$ $(a, b > 0)$. There is also an H-shaped region H, which is the union of A and the complement of a wider horizontal strip W: $|y| \leqq b + w$ (Figure 2). The object is to cover R with strips contained in H. (All strips in this section are closed sets bounded by a pair of parallel lines.)

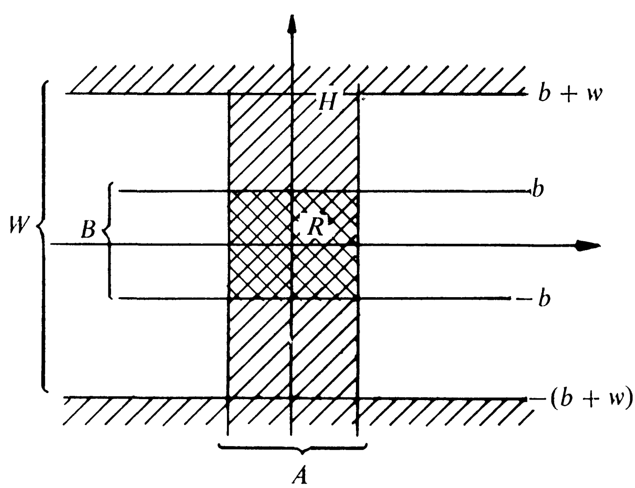


FIG. 2

LEMMA 1. *Given the above configuration, and given* $\varepsilon > 0$, *there exists a finite collection* $\{A_1, \cdots, A_n\}$ *of strips satisfying the following conditions*:

(i) $A_i \subset H$ *for* $i = 1, \cdots, n$.

(ii) $\{A_1, \cdots, A_n\}$ *covers* R.

(iii) *The area of* $(\bigcup_{i=1}^{n} A_i) \cap (W - B)$ *is less than* ε.

Proof. We shall show that this "two-wing" lemma follows from an easier one-wing lemma which reads the same, except that in (iii) $W - B$ is replaced by only its upper half, the strip $b \leqq y \leqq b + w$. This is done by applying the one-wing lemma in two stages. With the first application we can get strips $\{A_1, \cdots, A_n\}$ whose union has in the upper strip $b \leqq y \leqq b + w$ an area less than $\frac{1}{2}\varepsilon$ but whose area in

the lower half is uncontrolled. In the second stage the one-wing lemma is applied upside down n times, once for each of the new strips. More explicitly, temporarily fixing the index i, keep B and W as they are, replace A by A_i, R by $R_i = A_i \cap B$, H by H_i (the union of A_i and the complement of W), and ε by $\varepsilon/2n$. Since A_i need not be vertical, introduce an *ad hoc* coordinate system (using a horizontal shear) which makes it look vertical, and which at the same time reverses the sign of y. Then the one-wing lemma gives a finite collection of strips contained in H_i (hence also in H), covering R_i, whose union has in the horizontal band $-(b+w) \leqq y \leqq -b$ an area less than $\varepsilon/2n$. Combining these n collections makes one collection covering R and otherwise satisfying the conditions of the two-wing lemma.

It remains to prove the one-wing lemma. We begin the construction of $\{A_1, \cdots, A_n\}$ by taking as our first two strips vertical ones $A_1 = \{-a \leqq x \leqq -a+\eta\}$ and $A_2 = \{a-\eta \leqq x \leqq a\}$, taking $\eta > 0$ small enough to make the area of $(A_1 \cup A_2) \cap \{b \leqq y \leqq b+w\}$ less than $\frac{1}{4}\varepsilon$. Next cover the remaining rectangle $R' = \{|x| \leqq a-\eta\} \cap B$ by a collection of triangles $\{T\}$ with bases on the lower edge of R' and vertices on the segment $|x| \leqq a-\eta$, $y = b+\delta$, taking δ so small that the combined area of the projecting tips of all these triangles above $y = b$ is less than $\frac{1}{4}\varepsilon$. Moreover, make these triangles so thin that their sides extended both upwards and downwards stay in H. It is possible to do this with a finite number k of triangles (Figure 3). Next sprout each triangle T upwards [5, Section 3] to form

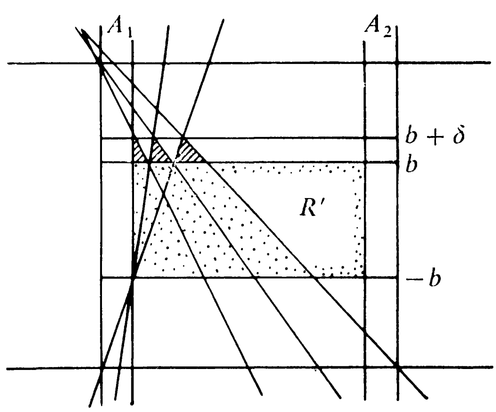


FIG. 3

a tree of height more than $b+w$ and new area (exclusive of T) less than $\varepsilon/2k$. Each of these trees is a finite union of triangles t with vertices in A above $y = b+w$ whose downward extensions stay in H. Taken together, for all T, they cover R'. Finally, replace each t by a finite collection of strips. (Each strip will, from $y = b+w$ downward, be contained in t extended, and they will cover t.) It is easy to check that the collection consisting of all these strips (for all T and all t) and A_1 and A_2 will answer the requirements of the one-wing lemma, thereby completing the proof of Lemma 1.

This lemma leads immediately to a bird theorem of Kakeya type.

THEOREM 3. *Given a bird as described above and any bounded set* S, *and* $\varepsilon > 0$, *there exists a continuous motion of the bird such that its body passes over every point of* S *while both its wings stay in a set* K *of area less than* ε.

Proof. Cover S by a finite number n of rectangles of height somewhat less than $2b$. Using $\varepsilon/2n$, apply Lemma 1 to each of them to get altogether a finite number k of strips. In each of these strips the bird can execute a short flight so that during all of them collectively its body passes over every point of S while its wings stay in a set K_1 of area less than $\frac{1}{2}\varepsilon$. It remains to provide for continuous transitions from each strip to the next. This can be done at an additional cost of $\frac{1}{2}\varepsilon$ in area by adjoining to K_1, $k-1$ partial Kakeya sets in which the bird as a whole can turn from one strip into another.

The twin way of interpreting the bird problem is not to ask for a continuous motion, but only for a collection of bird positions. In this interpretation the sets considered need not be finite unions of rectangles or strips. This allows us to take a limit of repeated applications of Lemma 1 to get the following stronger theorem.

THEOREM 4. *There is a plane set* K *of measure* 0 *such that for every point* p *in the plane there is a line through* p *which is contained in* $\{p\} \cup K$.

To prove Theorem 4, it suffices to produce a set K of measure 0 which works for all p in some rectangle R, because then a countable union of such sets will work for the whole plane. It is easy to see how the same kind of argument which proves the two-wing lemma from the one-wing one gives also the following:

LEMMA 2. *Let* B *and* W *be horizontal strips with* $B \subset W$, *and let* R *be a rectangle contained in* B. *Given a finite collection* $\mathscr{A}_1$ *of strips covering* R *and* $\varepsilon > 0$, *there is another finite collection* $\mathscr{A}_2$ *of strips such that*

(i) *For every strip* A *in* $\mathscr{A}_2$, $A \cap W$ *is contained in some strip belonging to* $\mathscr{A}_1$.

(ii) $\mathscr{A}_2$ *covers* R.

(iii) *If* K *is the union of all* $A - (A \cap B)$ *for* $A \in \mathscr{A}_2$, *then* $K \cap W$ *has area less than* ε.

Now let $\{W_n\}$ be an increasing sequence of horizontal strips whose union is the whole plane, and let $\{\varepsilon_n\}$ be a decreasing sequence of positive numbers tending to 0. We shall define recursively a sequence $\{\mathscr{A}_n\}$ of finite collections of strips by repeated use of Lemma 2. Starting with a given rectangle R, Lemma 1 gives us a first collection $\mathscr{A}_1$ of strips, using W_1 and ε_1. Next divide B (and at the same time R) into two equal horizontal strips, and for each of them in place of B, apply Lemma 2 to $\mathscr{A}_1$, using W_2 and $\varepsilon_2/2$. Let $\mathscr{A}_2$ be the union of the two collections of strips which result. In general, when you have $\mathscr{A}_n$, to make $\mathscr{A}_{n+1}$, B and R are cut horizontally into 2^n equal parts and Lemma 2 is applied to $\mathscr{A}_n$ on each part, using W_{n+1} and $\varepsilon_{n+1}/2^n$. Then $\mathscr{A}_{n+1}$ is the union of the 2^n collections of strips which you get. This sequence of collections of strips has the following properties.

(i) *For every strip A in $\mathscr{A}_{n+1}$ $A \cap W_{n+1}$ is contained in some strip belonging to $\mathscr{A}_n$.*

(ii) *$\mathscr{A}_n$ covers R for each n.*

In more detail each $\mathscr{A}_n$ is the union of 2^{n-1} subcollections, each of which covers a particular horizontal slice through R. We define for each strip A an **interrupted strip** $\hat{A}$ by removing from A the parallelogram where it crosses the slice relevant to the subcollection to which A belongs. We then define a set K_n to be the union of all $\hat{A}$ for $A \in \mathscr{A}_n$ and we have

(iii) *The area of $K_n \cap W_n$ is less than ε_n.*

Finally, define K to be the limit inferior of the sequence $\{K_n\}$. This means K is the union of all intersections of tails of the sequence; $K = \bigcup_n \bigcap_{i>n} K_i$. It also means that a point q belongs to K if and only if $q \in K_n$ for all sufficiently large n. We shall see that K has the two properties claimed for it in the theorem.

1. K has measure 0. Since $K = \bigcup_N (K \cap W_N)$, it suffices to prove that $K \cap W_N$ has measure 0 for each N. Further, since K is the countable union of sets $T_n = \bigcap_{i>n} K_i$, it suffices to prove that each $T_n \cap W_N$ has measure 0. But for every index i greater than both n and N we have $T_n \cap W_N \subset T_i \cap W_i$, which has area ε_i, tending to 0 as $i \to +\infty$.

2. For every point p in R there is a line L through p contained in $\{p\} \cup K$. First, there is a nested sequence $\{B_n\}$ of horizontal strips, obtained from B by successive bisections, which converges to the horizontal line through p. Next, for each n, $R \cap B_n$ is covered by one of the 2^{n-1} subcollections of $\mathscr{A}_n$, so that there is a strip A_n in this subcollection containing p. Each A_n contains a line L_n through p, and it can be seen from (i) that this sequence of lines converges to some line L. (If it were not so, we could still pass to a convergent subsequence.) We shall prove that L is the desired line, that is, if $q \in L$, and $q \neq p$, then $q \in K$. Since $q \neq p$ we can find N so large that q is not in the closed set B_N; at the same time make sure that q is in the interior of W_N. Now choose a sequence of points $\{q_n\}$ converging to q such that $q_n \in L_n$ for all n. We can suppose that $q_n \in W_N - B_N$ for all n. Then for $i > n > N$ we have from (i) that $q_i \in L_i \cap W_N \subset A_i \cap W_n \subset A_n$. Therefore, since $q_i \notin B_N \supset B_n$, $q_i \in \hat{A}_n$. This being so for all large i, and $\hat{A}$ being closed, $q \in \hat{A}_n \subset K_n$. Since this holds for all $n > N$, $q \in K$ by definition.

4. Remarks. In [6] Davies proved the existence of a plane set of full measure, every point of which is linearly accessible, meaning that some line through that point meets the set in only that point. Clearly the complement of the set K of measure 0 constructed in Theorem 4 has the properties of Davies' theorem and a little more. Namely *all* points, in K as well as in its complements, are linearly accessible in K; put otherwise, in addition to having a linearly accessible complement, K is a union of lines. Actually, Davies' example also has this extra property, but Davies was not looking for it. Moreover, Davies goes further: he gives a version in which each

point is covered by not just one line in K, but by uncountably many in every angle. Using his idea Theorem 4 could be strengthened in the same way.

The power of Theorem 4 is revealed by the fact that it has as an easy consequence, with apparently plenty left over, the theorem of Besicovitch to the effect that there exists a plane set of measure 0 containing lines in every direction. To see this let L be any straight line. Through every point p on L (let alone all the other points in the plane!) there passes a line contained in $\{p\} \cup K$. Now subject K to a projective transformation which takes L to the line at infinity, and adjoin to the image of K the image of the line at infinity. The new set has measure 0 and contains a line in every parallel pencil.

5. Other problems. Besicovitch started the ball rolling by solving at the same time a pair of twin problems, the Kakeya problem and his own which differs from Kakeya's in not requiring continuity of the motion. The solution to the Besicovitch problem is the stronger in two respects: measure 0 can be achieved for the set in question, and the moving figure is a line instead of only a segment. I have pointed out earlier the same twinship, both as to problems and solutions, between Theorem 2 and the existence of thin sets of circles, and again between Theorem 3 and Theorem 4.

Formulation of the twin to a given problem is not always obvious, however, and there is no guarantee that problems which look like twins will have matched solutions. For example, J. M. Marstrand and, independently, Davies [**8**] have proved the existence of a set of measure 0 containing a translate of every polygonal arc, a result of Besicovitch type. Moreover, Davies shows that the Kakeya twin of this result is false: a set containing two non-parallel segments cannot move continuously in a set of arbitrarily small area.

There is no perfect twin to the spherical Kakeya problem for lack of a parallelism for great circles. The best one can do is take a reference circle, the equator, and ask: does there exist a set of measure 0 containing a great circle with every inclination with respect to the equator? I do not know.

Theorem 4 can easily be made into an analogous theorem for the projective plane, and then answers a corresponding problem on the sphere, because the sphere maps two-to-one onto the projective plane. The spherical bird in this interpretation is a great circle with a pair of antipodal points as body and the rest wings. The Kakeya version of the bird problem, and the shrinking circular arc problem and its twin, can all be asked for the sphere. I believe their solutions will turn out to be similar to those of their planar analogue.

Perhaps more interesting is the Kakeya problem in the plane for circular arcs of fixed radius and radian measure: can an arc be continuously moved by a rigid motion from one circular to another in a small area? How should a Besicovitch twin to this problem be formulated?

Of course what makes the problems dealt with in this paper easy is that they have affirmative solutions. In the end, all that is required is a vigorous exploitation

of the fundamental technique invented by Besicovitch and perfected by Perron and Schoenberg.

References

1. A. S. Besicovitch, On Kakeya's problem and a similar one, Math. Z., 27 (1928) 312–320.
2. ———, The Kakeya problem, this MONTHLY, 70 (1963) 697–706.
3. ——— and R. A. Rado, A plane set of measure zero containing circumferences of every radius, J. London Math. Soc., 43 (1968) 717–719.
4. H. Busemann and W. Feller, Differentiation der L-Integrale, Fund. Math., 22 (1934) 226–256.
5. F. Cunningham, Jr., The Kakeya problem for simply connected and for star-shaped sets, this MONTHLY, 78 (1971) 114–129.
6. R. O. Davies, On accessibility of plane sets and differentiation of functions of two real variables, Proc. Cambridge Philos. Soc., 48 (1952) 215–232.
7. ———, Another thin set of circles, J. London Math. Soc., (2) 5 (1972) 191–192.
8. ———, Some remarks on the Kakeya problem, Proc. Camb. Philos. Soc., 69 (1971) 417–421.
9. Charles Fefferman, The multiplier problem for the ball, Ann. of Math., 94 (1971) 330–336.
10. B. Fisher, On a problem of Besicovitch, this MONTHLY, 80 (1973) 785–787.
11. J. R. Kinney, A thin set of circles, this MONTHLY, 75 (1968) 1077–1081.
12. J. B. Wilker, The non-Euclidean Kakeya problem, Proceedings of the 25th Summer Meeting of the Canadian Mathematical Congress, (1971) 603–607.

DEPARTMENT OF MATHEMATICS, BRYN MAWR COLLEGE, BRYN MAWR, PA 19010.

ON USING A DIFFERENTIAL EQUATION TO GENERATE POLYNOMIALS

TYRE A. NEWTON

1. Introduction. Two articles in this MONTHLY, [3] and [5], are concerned with the differential equation

$$xy'' - (x + N)y' + Ny = 0, \tag{1}$$

where N is a nonnegative integer, and the fact that it has the transcendental solution

$$y = e^x \tag{2}$$

and the polynomial solution

$$y_N(x) = 1 + \frac{1}{1!}x + \frac{1}{2!}x^2 + \cdots + \frac{1}{N!}x^N, \tag{3}$$

the Maclaurin approximation to e^x. Due to the author's interest in using the electronic analog computer to illustrate mathematical concepts (see, e.g., [6], [7], [8], and [9]), he found these articles to be of interest.

To generate an Nth degree polynomial such as (3) direct on the analog computer would require a sequence of up to N integrators and/or multipliers, as well as additional summing amplifiers. Thus N need not be very large to soon exceed the modest capacity of many analog computer facilities. So, it appears that if one is to get a computer generated plot of an Nth degree polynomial he must turn to the

digital computer. However, in (1) the order of the equation is 2, independent of N, the degree of the polynomial solution, $y_N(x)$. Thus it would appear that if we simulate equation (1) with the electronic analog computer, then we can plot not only e^x, but its Nth degree polynomial approximation for any N by merely changing initial conditions and coefficients in (1).

The aims of this paper are to determine other equations which can be used to not only illustrate a given function, but to illustrate various polynomial approximations to that function, and to consider some of the problems that arise in their simulation with the electronic analog computer.

2. A class of differential equations. We first note that equation (1) can be written in the form

$$(xD - N)(D - 1)y = 0. \tag{1$'$}$$

This leads us to the consideration of equations of the form

$$(xD - N)[P(x)D^2 + Q(x)D + R(x)]y = 0, \tag{4}$$

where P, Q, and R are analytic at $x = 0$. It is immediate that solutions of

$$[P(x)D^2 + Q(x)D + R(x)]y = 0 \tag{5}$$

are also solutions of (4), and that if $u = [P(x)D^2 + Q(x)D + R(x)]y$ is a solution of $(xD - N)\,u = 0$ then y will be a solution of (4).

For example, since $(D - 1)e^x = 0$, and since both

$$(D - 1)\sum_{n=0}^{N} \frac{1}{n!}x^n = -\frac{1}{N!}x^N$$

and

$$(xD - N)\left(-\frac{1}{N!}x^N\right) = 0$$

it follows that both e^x and $y_N(x)$ as defined by (3) are solutions of (1).

Consider now the differential operator

$$\mathscr{L} = P(x)D^2 + Q(x)D + R(x), \tag{6}$$

where P, Q, and R are analytic at $x = 0$. Then for some $r > 0$ there exists sequences $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ such that for $|x| < r$,

$$P(x) = \sum_{n=0}^{\infty} \alpha_n x^n, \quad Q(x) = \sum_{n=0}^{\infty} \beta_n x^n, \quad R(x) = \sum_{n=0}^{\infty} \gamma_n x^n.$$

Assume that there exists a solution of

$$\mathscr{L}y = 0 \tag{7}$$

which is analytic at $x = 0$. Thus for sufficiently small x, we can express this solution as

$$(8) \qquad y(x) = \sum_{n=0}^{\infty} c_n x^n.$$

Substituting the latter into (7) and equating coefficients of x^n to zero, we find that

$$(9) \qquad \sum_{j=0}^{n} [(j+1)(j+2)c_{j+2}\alpha_{n-j} + (j+1)c_{j+1}\beta_{n-j} + c_j\gamma_{n-j}] = 0$$

for $n = 0, 1, 2, \cdots$. Applying (9), we find that

$$(10) \qquad \mathscr{L}\left(\sum_{n=0}^{N} c_n x^n\right) = p_N\, x^{N-1} + q_N x^N + \sum_{n=N+1}^{\infty} r_n x^n,$$

where

$$p_N = -N(N+1)c_{N+1}\alpha_0,$$

$$q_N = -(N+1)[(N\alpha_1 + \beta_0)c_{N+1} + (N+2)\alpha_0 c_{N+2}],$$

and for $n = N+1, N+2, \cdots$,

$$r_n = \sum_{j=0}^{N-2} (j+1)(j+2)\, c_{j+2}\alpha_{n-j} + \sum_{j=0}^{N-1} (j+1)c_{j+1}\beta_{n-j} + \sum_{j=0}^{N} c_j\gamma_{n-j}.$$

It now follows that if $(xD - N)$ is an annihilator of the right side of (10), then both the expansion (8) and its Nth partial sum

$$y_N(x) = \sum_{n=0}^{N} c_n x^n$$

will be a solution of (4).

Consider now the special case of (6), $\mathscr{L} = D^2 + \beta_0 D + \gamma_0$. The corresponding special case of (10) is

$$\mathscr{L}\left(\sum_{n=0}^{N} c_n x^n\right) = -N(N+1)c_{N+1}x^{N-1} - (N+1)[\beta_0 c_{N+1} + (N+2)c_{N+2}]x^N.$$

But with $n = N$ in (9), we find that $(N+1)(N+2)c_{N+2} + (N+1)\beta_0 c_{N+1} = -c_N\gamma_0$ hence

$$\mathscr{L}\left(\sum_{n=0}^{N} c_n x^n\right) = -N(N+1)c_{N+1}x^{N-1} + c_N\gamma_0 x^N.$$

It now follows that

$$(xD - N)\mathscr{L}\left(\sum_{n=0}^{N} c_n x^n\right) = N(N+1)c_{N+1}x^{N-1}.$$

We now conclude that if a solution of $(D^2 + \beta_0 D + \gamma_0)y = 0$ has an expansion (8) in which $c_{N+1} = 0$ for some N, then both (8) and its Nth partial sum will satisfy

$$(xD - N)(D^2 + \beta_0 D + \gamma_0)y = 0.$$

For example, recall that the null space of $D^2 + 1$ is spanned by $\cos x$ and $\sin x$. Say that we choose $y(x) = \sin x$. Then in the expansion

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$$

we see that $c_{2N} = 0$ for $N = 0, 1, 2, \cdots$, and hence both $y = \sin x$ and

$$y_{2N+1}(x) = \sum_{n=0}^{N} \frac{(-1)^n}{(2n+1)!} x^{2n+1} \tag{11}$$

are solutions of

$$[xD - (2N+1)](D^2 + 1)y = 0. \tag{12}$$

As a second example, consider Bessel's equation

$$x^2 y'' + xy' + (x^2 - m^2)y = 0 \tag{13}$$

for integer m, having as a solution the Bessel function

$$J_m(x) = \sum_{k=0}^{\infty} \frac{(-1)^k (x/2)^{m+2k}}{k!\,(k+m)!}. \tag{14}$$

In this case (6) becomes $\mathscr{L} = x^2 D^2 + xD + (x^2 - m^2)$ and it follows from (10) that

$$\mathscr{L}\left(\sum_{n=0}^{M} c_n x^n\right) = c_{M-1} x^{M+1} + c_M x^{M+2}.$$

In (14), $c_{2k+m-1} = 0$, hence for $M = 2N + m$, it follows that

$$\mathscr{L}\left(\sum_{k=0}^{N} \frac{(-1)^k (x/2)^{m+2k}}{k!\,(k+m)!}\right) = \frac{(-1)^N (x/2)^{2N+m+2}}{N!\,(N+m)!}.$$

Since $[xD - (2N + m + 2)]$ is an annihilator of the term on the right, it follows that both $J_m(x)$ and its $(2N + m)$-th partial sum is a solution of

$$[xD - (2N+m+2)][x^2 D^2 + xD + (x^2 - m^2)]y = 0. \tag{15}$$

3. The electronic realization. The direct electronic realization of equations such as (1), (12), and (15) still presents a problem. In each case, $x = 0$ is a singular point; direct simulation will first entail division by a power of x, the coefficient of the highest ordered derivative. This usually introduces realization difficulties in a neighborhood of $x = 0$. However, we can get around this difficulty by using the technique of introducing new dependent variables and a new independent variable (see Hausner [**2**, pp. 36–52], and Korn and Korn [**4**, pp. 56–57]).

For equation (12), let

$$u = (D^2 + 1)y = \frac{d^2y}{dx^2} + y$$

and

$$\text{(16)} \qquad \frac{dy}{dx} = v.$$

Then

$$\text{(17)} \qquad \frac{dv}{dx} = u - y.$$

But, for the second solution, we want $[xD - (2N + 1)]u = 0$, hence

$$\text{(18)} \qquad \frac{du}{dx} = \frac{(2N + 1)u}{x}.$$

Now say that x, y, u, and v are functions of t and that $dx/dt = \xi x$ for non-zero constant ξ. It follows from equations (16), (17), and (18) that any orbit of

$$\text{(19)} \qquad \frac{d}{dt}\begin{bmatrix} x \\ y \\ u \\ v \end{bmatrix} = \xi \begin{bmatrix} x \\ xv \\ (2N + 1)u \\ x(u - y) \end{bmatrix}$$

will project onto a solution of equation (12) in the x-y plane. In particular, that solution of (19) satisfying the initial condition

$$\text{(20)} \qquad \begin{bmatrix} x(0) \\ y(0) \\ u(0) \\ v(0) \end{bmatrix} = \begin{bmatrix} x \\ y(x) \\ (D^2 + 1)y(x) \\ Dy(x) \end{bmatrix}_{x=x_0}$$

projects onto that solution of (12) in the x-y plane that satisfies the given initial values $y(x_0)$ and $y'(x_0)$. Notice that there is no division by x in (19).

The sign of ξ in (19) determines the direction that the point $(x(t), y(t), u(t), v(t))$ travels along the orbit of (19) as t increases. Another interpretation of ξ is that of a time scale factor; its sign determines whether we are going forward or backward in time and its magnitude determines the speed.

Figure 1 shows plots of projections of solutions of (19) onto the x-y plane for initial conditions defined by (20) with $y(x) = y_{2N+1}(x)$ as defined by (11) for

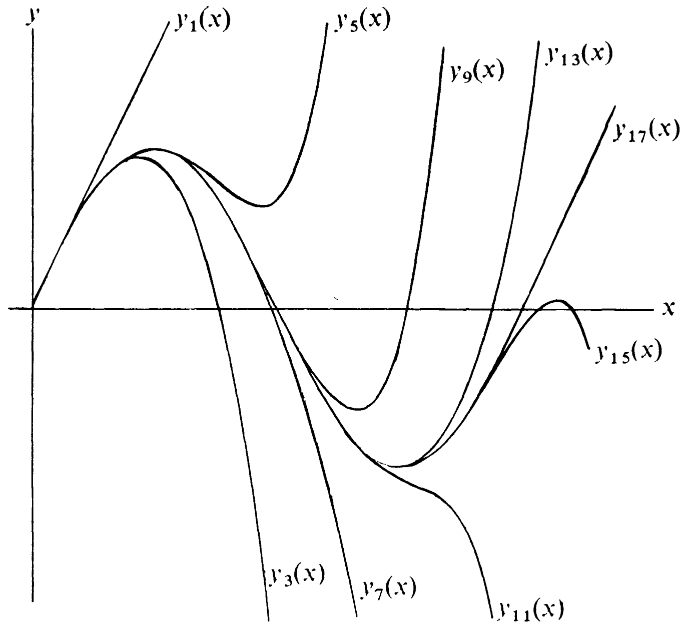


FIG. 1

$N = 0, 1, 2, 3, 4, 5, 6, 7$, and 8, and increasing x_0. For $N = 0$, plots were made for $\xi < 0$ and $\xi > 0$; the remaining plots were made for $\xi > 0$. Thus we are simulating polynomial approximations to sin x of degrees 1 through 17 by varying only one coefficient in the 4 dimensional system (19) and by varying the initial conditions (20).

In a similar manner for equation (15), let

$$u = [x^2D^2 + xD + (x^2 - m^2)]y. \tag{21}$$

Then $[xD - (2N + m + 2)]u = 0$, hence

$$\frac{du}{dx} = \frac{(2N + m + 2)u}{x}.$$

It follows from (21) that

$$\frac{d}{dx}\left(x\,\frac{dy}{dx}\right) = \frac{u + (m^2 - x^2)y}{x}.$$

Let $v = x(dy/dx)$ and

$$\frac{dv}{dx} = \frac{u + (m^2 - x^2)y}{x}.$$

We then introduce the parameter t by considering the equation

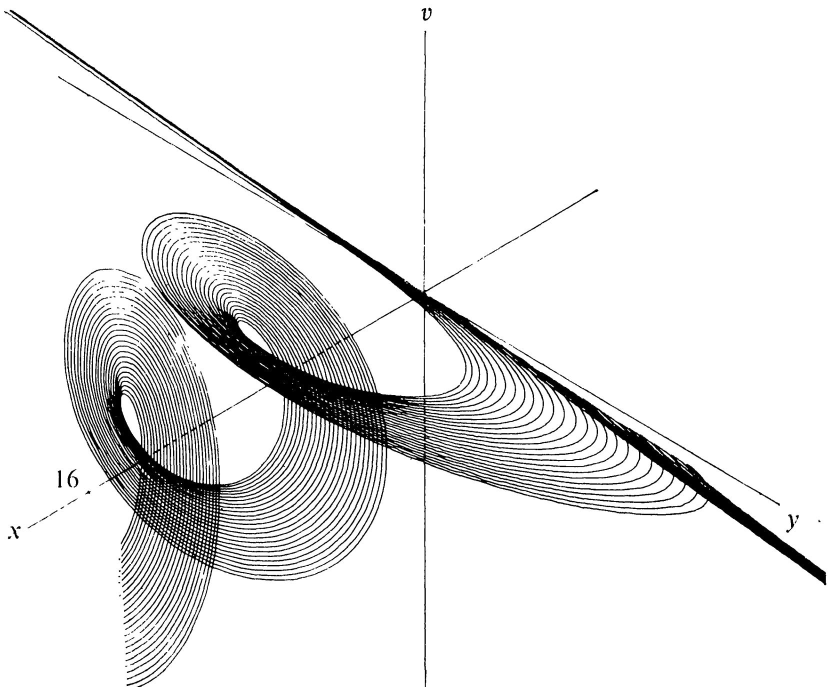


FIG. 2

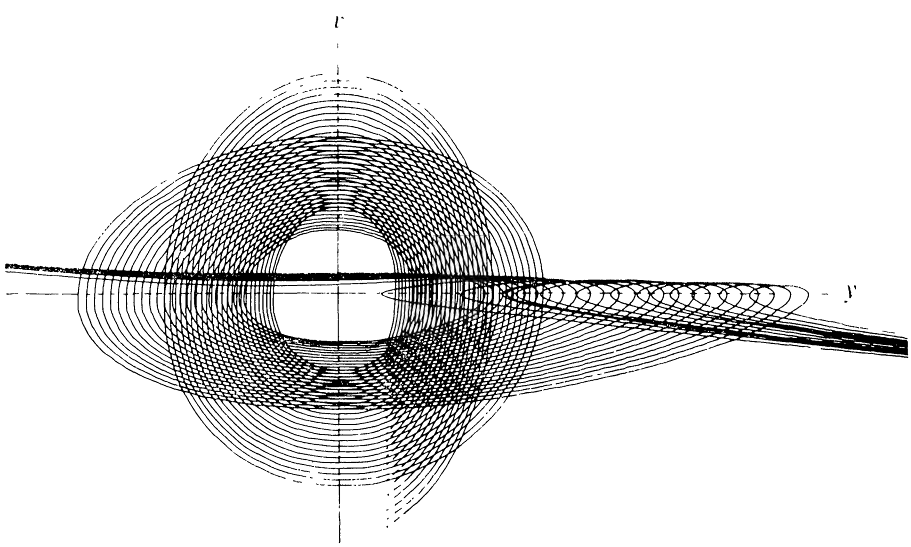


FIG. 3

$$\frac{d}{dt}\begin{bmatrix} x \\ y \\ u \\ v \end{bmatrix} = \xi \begin{bmatrix} x \\ v \\ (2N+m+2)u \\ u+(m^2-x^2)y \end{bmatrix} \tag{22}$$

for non-zero constant ξ, subject to the initial conditions

$$\begin{bmatrix} x(0) \\ y(0) \\ u(0) \\ v(0) \end{bmatrix} = \begin{bmatrix} x \\ y(x) \\ [x^2D^2+xD+(x^2-m^2)]y(x) \\ xDy(x) \end{bmatrix}_{x=x_0} \tag{23}$$

Note that if we wish to use the x-y projection of (22) to describe $J_m(x)$, then in (23), $y(x) = J_m(x)$, $u(0) = 0$, and hence $u(t) = 0$. We then need only to consider the first, second, and third entries of vector equation (22) with $u = 0$. Figures 2, 3, and 4

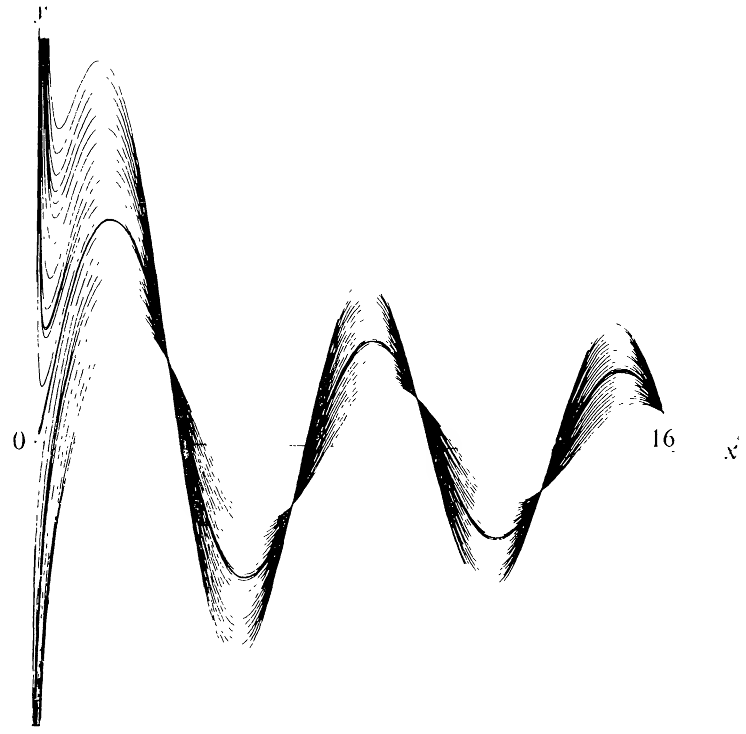


FIG. 4

show views of some of these solutions for $m = 1$. In Figure 2, we see orbits in x-y-v space for $x(0) = 16$, $y(0) = J_1(16)$, $v(0)$ taking on values between $-.450$ and $-.105$, and $\xi < 0$ so that $x(t) \to 0$ as $t \to +\infty$. In Figure 3 we view these orbits from down the positive x axis. Figure 4 shows the projections of these orbits onto the x-y plane. The darker curve in Figure 4 is the graph of $J_1(x)$. Figure 5 shows the projection of the orbits of (22) onto the x-y plane for $m = 1$, where in the initial conditions (23),

$$y(x) = y_{2N+1}(x) = \sum_{k=0}^{N} \frac{(-1)^k (x/2)^{2k+1}}{k!\,(k+1)!}$$

for $N = 3, 6, 9, 12, 15$, and 18. Thus, in making Figure 5, we have used only four integrations, and two multiplications of time dependent variables in order to illustrate polynomials of degree up to 37.

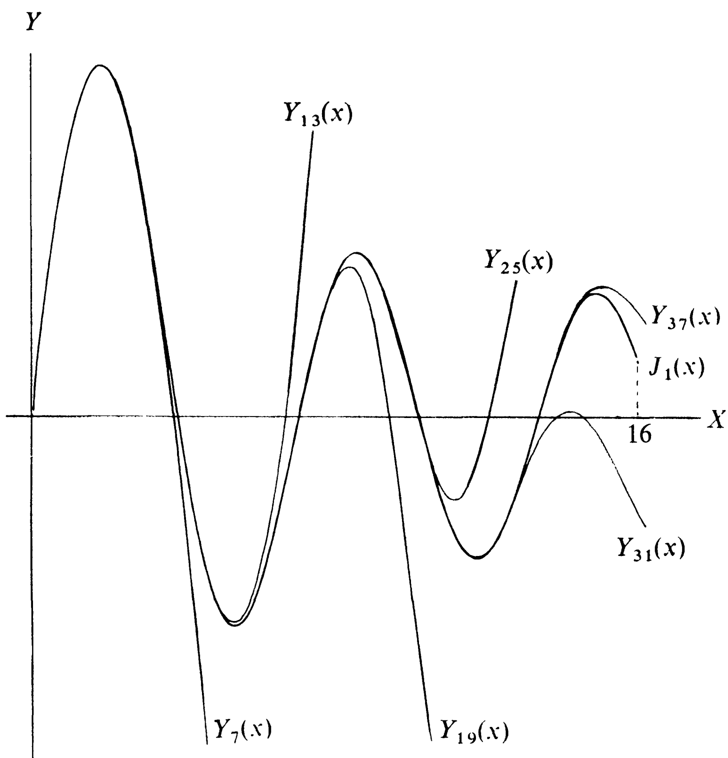


FIG. 5

Recall that the orbits in Figures 2, 3, and 4 came from the first, second, and third entries of (22) with $u = 0$ and $m = 1$. We rewrite the resulting three dimensional equation in the quasilinear form

$$(24)\qquad \frac{d}{dt}\begin{bmatrix} x \\ y \\ v \end{bmatrix} = \begin{bmatrix} \xi & 0 & 0 \\ 0 & 0 & \xi \\ 0 & \xi & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ v \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ -\xi x^2 y \end{bmatrix}$$

Since the eigenvalues of the constant matrix on the right are $\lambda = \xi, \xi, -\xi$, it follows from a theorem of Coddington and Levinson [**1**, Theorem 4.1] that for $\xi < 0$, there is a two dimensional manifold S in the x-y-v space containing the origin with the property that any solution of (24) originating on S will tend to the origin as $t \to +\infty$, and any solution originating off of S will be bounded away from the origin. Figures 2, 3, and 4 illustrate this theorem in that the initial points $(16, J_1(16), v(0))$ for each curve vary over a vertical line which passes through S.

References

1. E. A. Coddington and N. Levinson, Theory of Ordinary Differential Equations, McGraw Hill, New York, 1955.

2. A. Hausner, Analog and Analog/Hybrid Computer Programming, Prentice-Hall, Englewood Cliffs, N. J., 1972.

3. J. P. Hoyt, Remarks on a second order differential equation, this MONTHLY, 74 (1967) 436.

4. G. A. Korn and T. M. Korn, Electronic Analog and Hybrid Computers, McGraw-Hill, New York, 1964.

5. W. Leighton, On the differential equation $xy'' - (x+n)y' + ny = 0$, this MONTHLY, 76 (1969) 52–53.

6. T. A. Newton, A mathematician uses the analog computer to illustrate mathematical concepts, Analog/Hybrid Computer Educational Society Transactions, 4 (1972) 183–193.

7. ———, On using the analog computer to illustrate space curves, Proceedings of the Second Annual Houston Conference on Circuits, Systems, and Computers, April 20–21, 298–307.

8. ———, An analog prepatch panel for the two-dimensional quasilinear differential system, IEEE Transactions on Education, E-14 (1971) 57–62.

9. ———, On using the electronic analog computer in teaching calculus and differential equations, Proceedings of a Conference on Computers in the Undergraduate Curricula, June 16–17, 1970, The University of Iowa, 4.45–4.55.

DEPARTMENT OF MATHEMATICS, WASHINGTON STATE UNIVERSITY, PULLMAN, WA 99163.

AN ELEMENTARY PROOF OF THE KRONECKER-WEBER THEOREM

M. J. GREENBERG

A recently published textbook on Galois theory [Gaal] attempted unsuccessfully to give a proof of the Kronecker-Weber theorem without using any number theory (the book is quite useful for students despite this flaw). This famous theorem asserts that any Galois extension K of the field $\boldsymbol{Q}$ of rational numbers whose Galois group is *abelian* must be a subextension of a field obtained from $\boldsymbol{Q}$ by adjoining roots of

unity — in brief, *every abelian extension of $\boldsymbol{Q}$* is *cyclotomic*. In teaching an introductory course in algebraic number theory, I decided to present a proof of this theorem, using as little machinery as possible, so that the students could see a substantial result early in the game. The idea for such a proof goes back to Hilbert, with later simplifications by Weber and Speiser. Since all recent number theory texts give the impression that class field theory is needed to prove the K-W theorem, it seems worthwhile to revive this more elementary proof. The key tool is ramification theory. We shall quickly review the facts needed (the main reference is [Zariski-Samuel], abbreviated [Z-S]).

Let k be a finite extension field of $\boldsymbol{Q}$, K a finite Galois extension of k of degree n, G its Galois group. Let A be the ring of all algebraic integers in K, i.e., the ring of all numbers in K which satisfy a polynomial equation with ordinary integer coefficients and highest coefficient one. If $\mathfrak{P}$ is a prime ideal of A, its intersection with k is a prime ideal $\mathfrak{p}$ in the ring $\mathfrak{O}$ of integers of k. Let $\bar{K} = A/\mathfrak{P}$, $\bar{k} = \mathfrak{O}/\mathfrak{p}$ be the residue fields, which are finite fields of orders q^f, q respectively.

Fact 0. The ideal $\mathfrak{p}A$ generated by $\mathfrak{p}$ is equal to a power product

$$(\mathfrak{P}_1\mathfrak{P}_2\cdots\mathfrak{P}_g)^e,$$

where $\mathfrak{P}_1 = \mathfrak{P}$ and the other $\mathfrak{P}_j$'s are images of $\mathfrak{P}$ under automorphisms in G. The numbers e, f, g, n satisfy [Z-S, p. 289]

$$n = efg.$$

The exponent e is called the **ramification index** of $\mathfrak{P}$ over $\mathfrak{p}$. If $e = 1$ (resp., $e = n$) we say $\mathfrak{p}$ is **unramified** (resp. **totally ramified**) in K. The **decomposition group** Z of $\mathfrak{P}$ consists of all $\sigma \in G$ which leave $\mathfrak{P}$ invariant: $\sigma(\mathfrak{P}) = \mathfrak{P}$. It has a descending chain $T = V_0 \geqq V_1 \geqq \cdots \geqq V_j \geqq \cdots$ of normal subgroups defined as follows: If A is the ring of integers in K, each $\sigma \in Z$ induces an automorphism σ_j of the ring $A/\mathfrak{P}^{j+1}$; the assignment $\sigma \to \sigma_j$ is a homomorphism of Z, and V_j is by definition its kernel. T is called the **inertia group of** $\mathfrak{P}$ and the other V_j are **the higher ramification groups** of $\mathfrak{P}$.

Fact 1. For $j = 0$, the homomorphism $\sigma \to \sigma_0$ induces an isomorphism of Z/T onto the Galois group $\bar{G}$ of the residue extension $\bar{K}/\bar{k}$ [Z-S, p. 292]. In particular, Z/T is cyclic, generated by the coset of an automorphism σ such that

$$\sigma x \equiv x^q \pmod{\mathfrak{P}}$$

for all $x \in A$ [Lang, p. 17].

Fact 2. Let e be the ramification index of $\mathfrak{P}$ over $\mathfrak{p}$. Then e is the order of the inertia group T. If $g = (G\colon Z)$, then $efg = n$. If K_T is the fixed field of T, and $\mathfrak{P}_T = \mathfrak{P} \cap K_T$, then $\mathfrak{P}_T$ has ramification index 1, i.e., is unramified over $\mathfrak{p}$, whereas $\mathfrak{P}$ is totally ramified over $\mathfrak{P}_T$ [Z-S, pp. 291–2].

Fact 3. T/V_1 is isomorphic to a subgroup of the multiplicative group $\bar{K}^*$ of $\bar{K}$, hence is cyclic and its order divides $q^f - 1$. For each $j \geqq 1$, V_j/V_{j+1} is isomorphic to a subgroup of the additive group of the residue field $\bar{K}$; hence if $\bar{k}$ has characteristic p, then V_j/V_{j+1} is either trivial or a direct product of cyclic groups of order p. For j sufficiently large, V_j itself is trivial [Z-S, pp. 83 and 295].

Fact 4 (Minkowski's Theorem). For every finite extension K of the field $\boldsymbol{Q}$ of rational numbers, $K \neq \boldsymbol{Q}$, there exist primes which ramify in K [Weiss, p. 215; Lang, p. 120], and there are only finitely many ramified primes [Z-S, p. 303].

Fact 5. If m is a positive integer, let $\zeta(m)$ denote a primitive mth root of unity. A **cyclotomic extension** of $\boldsymbol{Q}$ is by definition a subfield of $\boldsymbol{Q}(\zeta(m))$ for some m. The composite of any finite number of cyclotomic extensions is cyclotomic. $\boldsymbol{Q}(\zeta(m))$ is an abelian extension of $\boldsymbol{Q}$ whose Galois group is isomorphic to the multiplicative group of units in the ring $\boldsymbol{Z}/m\boldsymbol{Z}$ of integers mod m. If p is an odd prime, then for all r, $\boldsymbol{Q}(\zeta(p^r))$ is cyclic of order $p^{r-1}(p-1)$, whereas for $r \geqq 3$, $\boldsymbol{Q}(\zeta(2^r))$ is the direct product of two cyclic groups, one of order 2^{r-2} and the other of order 2 generated by the automorphism $\zeta \to \zeta^{-1}$. For any prime p and any r, p is the only ramified prime in $\boldsymbol{Q}(\zeta(p^r))$ and it is totally ramified (except when $p = 2$, $r = 1$). For any $m > 2$, the ramified primes in $\boldsymbol{Q}(\zeta(m))$ are the primes dividing m [Weiss, pp. 256–263].

We also need the following fact from Galois Theory, which is an easy consequence of the Theorem on Natural Irrationalities.

Fact 6. If K, L are Galois extensions of a field k with Galois groups G, H, and KL is the compositum of K and L, then KL is a Galois extension of k, whose Galois group is canonically isomorphic to the subgroup of $G \times H$ consisting of those pairs (σ, τ) such that σ and τ have the same restriction to $K \cap L$. The isomorphism mentioned assigns to an automorphism ρ of KL over k its pair of restrictions $(\rho|K, \rho|L)$.

We now begin the proof, proper with the following improvement of Fact 3.

LEMMA 1. *If Z/V_1 is abelian, then T/V_1 is cyclic of order dividing $q-1$.*

Proof. Localizing if necessary, we may assume $\mathfrak{P}$ is a principal ideal generated by some element π. Then for each $\sigma \in Z$, $\sigma\pi = a_\sigma \pi$, where a_σ is an integer of K not divisible by $\mathfrak{P}$. If $\bar{a}_\sigma$ is its residue mod $\mathfrak{P}$, then the assignment $\sigma \to \bar{a}_\sigma$ induces the isomorphism of T/V_1 into the multiplicative group $\bar{K}^*$ mentioned in Fact 3 [Z-S, p. 295]; let $\tau \in T$ be such that the coset of τ mod V_1 generates T/V_1. Let $\sigma \in Z$ induce the Frobenius automorphism $\xi \to \xi^q$ of $\bar{K}/\bar{k}$ (Fact 1). For simplicity, write

$$\sigma\pi = a\pi, \quad \tau\pi = b\pi, \quad \sigma\tau\sigma^{-1}\pi = c\pi$$

and notice that $\sigma^{-1}\pi = \sigma^{-1}(a)^{-1}\pi$. The hypothesis that Z/V_1 is abelian tells us that $\bar{c} = \bar{b}$. We compute c:

$$\sigma\tau\sigma^{-1}\pi = \sigma\tau(\sigma^{-1}(a)^{-1}\pi) = \sigma(\tau\sigma^{-1}(a)^{-1}b\pi) = \sigma\tau\sigma^{-1}(a)^{-1}\sigma(b)\,a\pi$$

so $c = \sigma\tau\sigma^{-1}(a)^{-1}\,\sigma(b)a$. Reduce this equation mod $\mathfrak{P}$, remembering that τ_0 is the

identity and $\overline{\sigma(\bar{b})} = \sigma_0(\bar{b}) = \bar{b}^q$; we get $\bar{c} = \bar{b}^q$, hence $\bar{b}^{q-1} = 1$, which proves Lemma 1.

LEMMA 2. *If the Kronecker-Weber theorem holds for cyclic extensions of prime power order, then it holds for all abelian extensions.*

Proof. By the fundamental theorem on abelian groups, the Galois group G of the abelian extension K of $\boldsymbol{Q}$ is the direct product of subgroups G_i each cyclic of prime power order. If K_i is the fixed field of $\prod_{j \neq i} G_j$, then $K_i/\boldsymbol{Q}$ has Galois group isomorphic to G_i, and K is the composite of all the K_i's. So if each K_i is cyclotomic, K is also (Fact 5).

LEMMA 3. *Suppose K is an abelian extension of $\boldsymbol{Q}$ of prime power degree λ^m. It suffices to prove that K is cyclotomic under the additional assumption that every prime $p \neq \lambda$ is unramified in K.*

Proof. Suppose $p \neq \lambda$ is ramified in K, and $\mathfrak{P}$ is a prime ideal of K lying over p. Then p does not divide the order of any quotient of subgroups of G, so by Fact 3, all higher ramification groups V_j, $j \geqq 1$, of $\mathfrak{P}$ are trivial. The order of T is a power λ^u of the prime λ. Since $k = \boldsymbol{Q}$, $q = p$, so by Lemma 1,

$$p \equiv 1 \pmod{\lambda^u}.$$

Since $\boldsymbol{Q}(\zeta(p))$ is cyclic of degree $p-1$ (Fact 5), it has a unique subfield L which is cyclic of degree λ^u over $\boldsymbol{Q}$.

Furthermore, p is totally ramified in $\boldsymbol{Q}(\zeta)$, hence p is totally ramified in L, and no other prime is ramified in $\boldsymbol{Q}(\zeta)$—*a fortiori*, in L (Fact 5).

We now form the composite extension KL of $\boldsymbol{Q}$, which has degree λ^{m+v}, $v \leqq u$. Let $\mathfrak{P}'$ be a prime ideal of KL lying over $\mathfrak{P}$, T' the inertia group of $\mathfrak{P}'$ over p, H the Galois group of $L/\boldsymbol{Q}$. Restriction to K maps T' into T, so by Fact 6, $T' \leqq T \times H$. The order of T' is at least λ^u (since the ramification index of $\mathfrak{P}'$ over p is at least that of $\mathfrak{P}$ over p). As before, the higher ramification groups of $\mathfrak{P}'$ are trivial, so T' is cyclic (Fact 3). Since no element of $T \times H$ has order $> \lambda^u$, T' must have order exactly λ^u. Let K' be the fixed field of T', $\mathfrak{P}'' = \mathfrak{P}' \cap K'$. Then $\mathfrak{P}''$ is unramified over p (Fact 2). Moreover $K' \cap L = \boldsymbol{Q}$, since $\mathfrak{P}'' \cap L$ is both unramified and totally ramified over p. Since $[KL: K'] = \lambda^u = [L:\boldsymbol{Q}]$, it follows that $[K'L:\boldsymbol{Q}] = [K':\boldsymbol{Q}]\,[L:\boldsymbol{Q}] = [KL:\boldsymbol{Q}]$, so $K'L = KL$. Thus K would be cyclotomic if it could be shown that K' is. The advantage of K' is that p no longer ramifies in K'. Moreover, no new primes ramify in K', since such a prime would be ramified in KL, yet its inertia group in KL is contained in the product of its inertia groups in K and L (Fact 6), both of which are trivial. So by repeating this process, we shall eventually eliminate the finitely many primes $p \neq \lambda$, which are ramified.

COROLLARY 1. *Let K be an abelian extension of $\boldsymbol{Q}$ of prime power degree λ^m, and suppose that $p \neq \lambda$ is the only prime ramified in K. Then p is totally*

ramified in K,

$$p \equiv 1 \pmod{\lambda^m},$$

and K *is the unique subfield of* $Q(\zeta(p))$ *of degree* λ^m; K/Q *is therefore cyclic.*

Proof. The field K' constructed above is unramified over Q, hence $K' = Q$ (Fact 4), so $K = L$.

COROLLARY 2. *If* K *is an abelian extension of* Q *of odd degree, then* 2 *is unramified in* K.

Proof. For the argument above showed V_1 trivial and Lemma 1 implies T is also trivial ($q = 2$).

We are therefore reduced to the case of degree λ^m with λ the only ramified prime. This case separates into two subcases depending on whether λ is odd or $\lambda = 2$.

LEMMA 4. *Let* K *be an abelian extension of* Q *of degree* λ^m, λ *an odd prime, in which* λ *is the only ramified prime. Then* K/Q *is cyclic.*

Proof. If T is the inertia group of a prime Λ lying over λ, then λ is unramified in the fixed field of T (Fact 2), so by Minkowski's Theorem (Fact 4), T is the entire Galois group, hence λ is totally ramified. Therefore, $q = \lambda$, $f = 1$, and $\bar{K}$ is the finite field with λ elements. Since a power of λ does not divide $\lambda - 1$, Fact 3 tells us that $V_1 = T$ and for each $j \geqq 1$, V_j/V_{j+1} is either trivial or cyclic of order λ.

SUBLEMMA 1. *If* $m = 1$, *i.e.,* $[K:Q] = \lambda$, *then* V_2 *is trivial.*

Proof. Localizing, we may assume the prime ideal Λ to be principal, generated by π. Let $f(X)$ be the minimal polynomial of π over Q; let v be the valuation of K associated to Λ. Say V_{j+1} is the first trivial ramification group, so V_j is the whole Galois group G. We claim

$$v(f'(\pi)) = (j+1)(\lambda - 1). \tag{1}$$

Namely, $f'(\pi)$ is the product of all $\pi - \sigma\pi$ as σ runs through all automorphisms other than the identity, i.e., $\sigma \in V_j - V_{j+1}$, and $v(\pi - \sigma\pi) = j + 1$ [Z-S, p. 296].

On the other hand,

$$f'(\pi) = \lambda\pi^{\lambda-1} + a_{\lambda-1}(\lambda-1)\pi^{\lambda-2} + \cdots + a_1.$$

The a_i are integers, and since λ is totally ramified in K, $v(\lambda) = \lambda$, so

$$v(a_i) \equiv 0 \pmod{\lambda}.$$

Therefore the valuation v_i of the term involving $\pi^{\lambda-i}$ in $f'(\pi)$ satisfies

$$v_i \equiv \lambda - i \pmod{\lambda},$$

whence all these terms have different valuations and $v(f'(\pi))$ equals the minimum of

the v_i. Thus

$$2\lambda - 1 = v(\lambda\pi^{\lambda-1}) \geqq v(f'(\pi)). \tag{2}$$

Combining (1) and (2) yields $2\lambda - 1 \geqq (j+1)(\lambda - 1)$. Since $\lambda > 2$, the only $j \geqq 1$ satisfying this inequality is $j = 1$, so V_2 is trivial.

Returning to the case $m > 1$, we will show that K/Q is cyclic by showing V_2 is the unique subgroup of the Galois group $G = V_1$ of index λ.

Let H be any subgroup of index λ in G, K' its fixed field, $G' \cong G/H$ the Galois group of K' over Q, V_j' the jth ramification group of K'. By restriction to K', V_j maps into V_j'. According to the sublemma, V_2' is trivial. Hence $V_2 \leqq H$. Applying this, in particular to the case where $H = V_j$ is the first ramification group which is not all of G, we see that $j = 2$ and V_2 has index λ. Hence V_2 is the unique subgroup of index λ.

LEMMA 5. *The Kronecker-Weber theorem holds for abelian extensions of Q of degree λ^m, where λ is an odd prime.*

Proof. By Lemma 3 and Fact 4, we may assume λ is the only ramified prime in K. Let ζ be a primitive λ^{m+1}-st root of unity, and let K' be the unique subfield of $Q(\zeta)$ which has degree λ^m over Q (recall that $Q(\zeta)$ is cyclic of order $\lambda^m(\lambda - 1)$ over Q—Fact 5). Then λ is the only ramified prime in K' (Fact 5).

We claim $K = K'$: Otherwise λ would be the only ramified prime in the composite abelian extension KK', which has degree $> \lambda^m$. By Lemma 4, KK' is cyclic over Q, but by Fact 6, no element of its Galois group has order greater than λ^m—contradiction.

COROLLARY. *If K is an abelian extension of Q of degree λ^m, λ an odd prime, in which λ is the only ramified prime, then K is the unique subfield of $Q(\zeta(\lambda^{m+1}))$ of degree λ^m.*

LEMMA 6. *Every quadratic extension of Q is cyclotomic.*

Proof. One reduces immediately to the case of $Q(\sqrt{\pm p})$, p prime. For $p = 2$, this field is contained in $Q(\zeta(8))$, as follows from the equation

$$(1 + i)^2 = 2i.$$

For p odd, let $\zeta = \zeta(p)$. Consider the pth cyclotomic polynomial

$$F(X) = X^{p-1} + X^{p-2} + \cdots + X + 1$$

and its discriminant

$$\Delta = \prod_{1 \leqq i < j \leqq p-1} (\zeta^i - \zeta^j)^2,$$

which is a square in $Q(\zeta)$. An easy calculation [Weiss, p. 265] shows that

$\Delta = (-1)^{(p-1)/2}p^{p-2}$. Hence

$$\sqrt{\pm p} \in \boldsymbol{Q}(\zeta, \sqrt{-1}) = \boldsymbol{Q}(\zeta(4p)).$$

LEMMA 7. *Every cyclic extension K of $\boldsymbol{Q}$ of degree 2^m is cyclotomic.*

Proof. By induction on m: The induction starts with Lemma 6. Assume $m > 1$. By Lemma 3 and Fact 4, we may assume 2 is the only ramified prime in K. We may also assume K embedded in the field of complex numbers. Complex conjugation restricted to K is either the identity or is an automorphism of order 2, so its fixed field is real and has degree at least 2^{m-1} over $\boldsymbol{Q}$. Since $K/\boldsymbol{Q}$ is cyclic, it has a unique quadratic subfield K' which must be real. Since furthermore 2 is the only ramified prime in K', $K' = \boldsymbol{Q}(\sqrt{2})$ [Weiss, p. 235].

Let ζ be a primitive $4n$th root of unity, where $n = 2^m = [K:\boldsymbol{Q}]$. Then the subfield

$$L = \boldsymbol{Q}(\zeta + \zeta^{-1})$$

of $\boldsymbol{Q}(\zeta)$ is cyclic of degree n over $\boldsymbol{Q}$ with 2 as its only ramified prime (Fact 5). Again its unique quadratic subfield is $\boldsymbol{Q}(\sqrt{2})$. Hence the degree of KL over $\boldsymbol{Q}$ is less than n^2 and the Galois group Γ of KL over $\boldsymbol{Q}$ is a proper subgroup of $G \times H$, $G = \mathrm{Gal}(K/\boldsymbol{Q})$, $H = \mathrm{Gal}(L/\boldsymbol{Q})$ (Fact 6). Choose generators σ, τ of G, H respectively which agree on $L \cap K$. Then $(\sigma, \tau) \in \Gamma$ and (σ, τ) generates a subgroup Δ of order n. The fixed field F of Δ has degree 2^r over $\boldsymbol{Q}$, where $r < m$, and 2 is still the only ramified prime in F. By inductive hypothesis, F is cyclotomic. Moreover, only the identity automorphism in Δ restricts to the identity on L. Thus $FL = KL$ and K is cyclotomic.

Our sequence of lemmas proves the Kronecker-Weber theorem.

Note: Richard Niles has pointed out to me that Lemma 4 and Corollary to Lemma 5 are valid under the weaker assumption that $K/\boldsymbol{Q}$ is Galois — it is not necessary to assume $K/\boldsymbol{Q}$ is abelian. For the proof of Lemma 4 shows that the Galois group G has a unique normal subgroup of index λ. Since G is a λ-group, this implies G is cyclic [Hall, p. 176].

References

1. L. Gaal, Classical Galois Theory with Examples, Markham, Chicago, 1970.
2. M. Hall, Jr., The Theory of Groups, Macmillan, New York, 1959.
3. S. Lang, Algebraic Number Theory, Addison-Wesley, Reading, Mass., 1970.
4. A. Speiser, Die Zerlegungsgruppe, J. Reine Angew. Math., 149 (1919) 174–188.
5. E. Weiss, Algebraic Number Theory, McGraw-Hill, New York, 1963.
6. O. Zariski and P. Samuel, Commutative Algebra, Vol. I, Van Nostrand, Princeton, N. J., 1958.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA, SANTA CRUZ, CA 95060.

PROGRAMMING AS A DISCIPLINE OF MATHEMATICAL NATURE

E. W. DIJKSTRA

In this article I intend to present programming as a mathematical activity without undertaking the arduous task of supplying a definition of "mathematics" that will please all mathematicians, nor of defining "programming" in a way that is palatable to all programmers.

With respect to mathematics I believe, however, that most of us can agree upon the following characteristics of most mathematical work:

(1) Compared with other fields of intellectual activity, mathematical assertions tend to be unusually precise.

(2) Mathematical assertions tend to be general in the sense that they are applicable to a large (often infinite) class of instances.

(3) Mathematics embodies a discipline of reasoning allowing such assertions to be made with an unusually high confidence level.

The mathematical method derives its power from the combination of all these three characteristics; conversely, when an intellectual activity displays these three characteristics to a strong degree, I feel justified in calling it "an activity of mathematical nature," independent of the question whether its subject matter is familiar to most mathematicians. In other words, I grant the predicate "mathematical nature" rather on the *quo modo* than on the *quod.*

A programmer designs algorithms, intended for mechanical execution and intended to control existing or conceivable computing equipment. These — usually electronic — devices derive their power from two basic characteristics. First, the amount of information they can store and the amount of processing that they can perform, in a reasonably short time, are both large beyond imagination. And as a result, what the computer can do for us has outgrown its basic triviality by several orders of magnitude. Second, as executors of algorithms, computers are reliable and obedient, again beyond imagination: as a rule they indeed behave exactly as instructed.

This obedience makes heavy demands on the accuracy with which the programmer instructs the machine: if the instructions were to produce non-sense, the machine will produce non-sense. Inexperienced programmers often blame the machinery for its strict obedience, for the impossibility to appeal to the machine's "common sense;" more experienced programmers realize that it is exactly its strict obedience that enables us to use it reliably, to forge it into a sophisticated tool that we would never be able to build if the executor of our algorithm had the uncontrolled freedom to interpret our instructions in "the most plausible way." As a result, the competent programmer does not regard precision and accuracy as mean virtues: he knows that he could not work without them.

The programmer's work is also always "general" in the sense that each program is able to evoke as many different computations as we can supply it with different input data. An assertion about a program is an assertion about the whole class of possible computations that could be evoked under control of it and "designing an algorithm" is nothing more nor less than "designing a whole class of computations." So the programmer's work also shares the second characteristic.

Finally, what about the confidence level of his work? Well, it should be very high for two reasons. Firstly, a large sophisticated program can only be made by a careful application of the rule "Divide and Conquer" and as such consists of many components, say N; if, however, p is the probability that an individual component is correct, then the probability P of the whole aggregate being correct satisfies something like $P \leqq p^N$. In other words, unless p is indistinguishable from 1, for large N the value P will be indistinguishable from zero! If we cannot design the components sufficiently well, it is vain to hope that their aggregate will work at all. Secondly, its confidence level should be very high if we, as society, would like to rely upon the performance of the algorithm. And we do rely on algorithms when we use machines for air traffic control, banking, patient care in hospitals or earthquake prediction.

Now honesty compels me to admit that today, on the average, the confidence level reached by the programming profession is not yet what it should be.

From a historical point of view, this sorry state of affairs is only too understandable. The tradition of programming is very young and has still many traceable roots in the recent past, when machines were still rather small and programming was not yet such a problem. (Before we had machines, programming was no problem at all!) But in the last ten to fifteen years, the power of available machinery has grown at least with a factor of a thousand, thereby completely changing the scope of the programming profession.

The old technique was to make a program and then to subject it to a number of testcases where the answer was known; and when the testruns produced the correct result, this was taken as a sufficient ground for believing the program to be correct.

But with growing sophistication, this assumption proved more and more to be unjustified until, some five years ago, it surfaced in the form of "the software crisis." One of the first considerations of what was later to emerge as "programming methodology" was this question of the confidence level: "How can we rely on our algorithms?"

An analysis of the situation quite forcibly showed that program testing can be used very convincingly to show the presence of bugs, but never to demonstrate their absence, because the number of cases one can actually try is absolutely negligible compared with the possible number of cases. The only way out was to prove the program to be correct.

The suggestion that the correctness of programs could and should be established by proof was met with a great amount of scepticism. (In the mean time, many older people had already accepted as a Law of Nature, that each program is bound to contain bugs!) The scepticism, however, was not without reason.

To start with, it was not clear what form such correctness proofs could have. You cannot build a proof on quicksand; you must have axioms, in this case an axiomatic definition of the semantics of the programming language in which the program has been expressed. It was only after a few efforts that a technique for semantic definition emerged that could serve as a possibly practical starting point for correctness proofs.

When people then tried to give correctness proofs for existing programs, the result of that effort was very disappointing: the proofs were so long, intricate and clumsy, that they failed to convince. And also this disappointment can still be traced in the scepticism of many. Three discoveries have changed the scene.

The first discovery was that the amount of formal labour needed to prove the correctness of a program could depend very heavily on the structure of the program.

The second discovery was that of a few useful theorems about program constructs and thanks to them we no longer needed to go all the way back to the axioms all the time.

The most drastic discovery, however, was the last one, that what we then tried, *viz.*, to prove the correctness of a given program, was in a sense putting the cart before the horse. A much more promising approach turned out to be letting correctness proof and program grow hand in hand: with the choice of the structure of the correctness proof one designs a program for which this proof is applicable. The fact that the correctness concerns act as an inspiring heuristic guide is an added benefit.

If I ended this article with the above optimistic note I could create the wrong impression that now the intrinsic difficulties of programming have been solved, but this is not true: the best I can say is that now we have a better insight into the nature of the difficulties of the programming task. In my closing paragraphs I hope to convey this nature, at the same time sketching the intellectual demands made upon the competent programmer.

A programmer must be able to express himself extremely well, both in a natural language and in formal systems. The need for exceptional mastery of a natural language is twofold. First it is not uncommon that e. g., English is the language in which the problem is communicated to him and in which he must describe his interpretation or modification of the problem. This circumstance has been a source of many misunderstandings to the extent that there is a wide-spread belief that, e. g., English by its very nature is inadequate for that communication task. I don't believe it (although sloppy English certainly is!). On the contrary: I always have the feeling that our natural language is so intimately tied with what we call understanding that we must be able to use it to express what we have understood. Secondly, we should not close our eyes for the fact that formalization, in a sense, is always "after the fact" and that therefore natural language is an indispensable tool for thinking, in particular when new concepts have to be introduced. And this is what a programmer has to do all the time: he has to introduce new concepts — not

occurring in the original problem statement — in order to be able to find, to describe and to understand his own solution to the problem. For instance, when asked to construct a detector, analysing a string of characters for the occurrence of an instance of the — nicely formally defined — syntactic category "sentence," he may find himself led to the introduction of a completely new syntactic category "proper begin of a sentence," i. e., a string of characters that is admissible as the opening substring of a sentence but not yet a complete sentence itself. After having established that this is indeed a useful concept for the characterization of some intermediate stages of the computational process, he will proceed by manipulating the given formal syntax in order to derive the formal definition of this new syntactic category. In other words, given the problem, the programmer has to develop (and formulate!) the theory necessary to justify his algorithm. In the course of this work he will often be forced to invent his own formalism.

Such demands, of course, are common to most mathematical work, but there are reasons to suppose that in programming they are more heavy than anywhere else. Besides the need for precision and explicitness, the programmer is faced with a problem of size that seems unique to the programming profession. When dealing with "mastered complexity," the idea of a hierarchy seems to be a key concept. The notion of a hierarchy implies that what at one level is regarded as an unanalyzed unit, is regarded as a composite object at the next lower level of greater detail, for which the appropriate grain (say, of time or space) is an order of magnitude smaller than the corresponding grain appropriate at the next higher level. As a result, the number of levels that can meaningfully be distinguished in a hierarchical composition is approximately proportional to the logarithm of the ratio between the largest and the smallest grain. In programming, where the total computation may take an hour, while the smallest time grain is in the order of a microsecond, we have an environment in which this ratio can easily exceed 10^9 and I know of no other environment in which a single technology has to encompass so wide a span.

It seems to be the circumstance sketched in the above paragraph that gives programming as an intellectual activity some of its unique flavour. The concepts he introduces must be highly effective tools for bringing the necessary amount of reasoning down to an amount that can be done. And also the formalism he chooses must be such that his formulae do not explode in length, a regrettable phenomenon that is bound to occur unless the programmer pays conscious care to measures for avoiding that explosion. A final consequence of the hierarchical nature of his artefacts is the competent programmer's agility with which he switches back and forth between various semantic levels, between global and local considerations, between macroscopic and microscopic concerns, an ability that has been described as "a mental zoom lens." This agility is bewildering for those unaccustomed to it.

If I contrast the preceding concept of programming with my impression of "the standard mathematical curriculum" (I hope I am fair), I observe the following differences in stress:

(1) In the standard mathematical curriculum the student becomes familiar (sometimes even very familiar!) with a standard collection of mathematical concepts, he is less trained in introducing new concepts himself.

(2) In the standard mathematical curriculum the student becomes familiar (sometimes even very familiar!) with a standard set of notational techniques, he is less trained in inventing his own notation when the need arises.

(3) In the standard mathematical curriculum the student often only sees problems so "small" that they are dealt with at a single semantic level. As a result many students see mathematics rather as the art of organizing their symbols on their piece of paper than as the art of organizing their thoughts.

If, on the other hand, I have given some of my readers the first germs of the feeling that to an inventive and effective mathematician the field of programming may provide the area par excellence in which to find his challenge and bring his abilities to bear, then I have fulfilled one of my most cherished aims.

References

1. N. Wirth, Systematic Programming, Prentice Hall, Englewood Cliffs, N. J., 1973.
2. O. J. Dahl, E. W. Dijkstra, C. A. R. Hoare, Structured Programming, Academic Press, New York, 1972.

Department of Mathematics, Technological University, Eindhoven, Netherlands.

THERE'S A DELTA FOR EVERY EPSILON (Calypso)

Words and Music by Tom Lehrer

There's a delta for every epsilon,
It's a fact that you can always count upon.
There's a delta for every epsilon
And now and again,
There's also an N.

But one condition I must give:
The epsilon must be positive
A lonely life all the others live,
In no theorem
A delta for them.

How sad, how cruel, how tragic,
How pitiful, and other adjec-
Tives that I might mention.
The matter merits our attention.
If an epsilon is a hero,
Just because it is greater than zero,
It must be mighty discouragin'
To lie to the left of the origin.

This rank discrimination is not for us,
We must fight for an enlightened calculus,
Where epsilons all, both minus and plus,
Have deltas
To call their own.

QUERIES

Edited by A. C. Zitronenbaum

This Department welcomes queries from readers about mathematics at the collegiate level, such as sources for exposition of a particular topic from a special point of view, references to vaguely remembered articles, descriptions of special kinds of courses or teaching methods, and methods for constructing illustrative examples for exercises of particular kinds (questions on research topics should, in general, be addressed to the "Queries Department" of the Notices of the American Mathematical Society). Replies will be forwarded to the questioner and may also be edited into a composite answer for publication in this Department. Consequently all items submitted for consideration for possible publication should include the name and complete mailing address of the person who is to receive the reply. Queries and answers should be sent to A. C. Zitronenbaum, Department of Mathematics, Cornell University, Ithaca, NY 14850.

Replies to Query 1. Treatments of the problem can be found in D. St. P. Bernard, *Adventures in Mathematics*, Funk and Wagnalls, New York 1968, Ch. VIII (W. Bartlett, J. D. E. Konhauser, T. C. Wales), and L. A. Graham, *Ingenious Mathematical Problems and Methods*, Dover, New York 1959, pp. 66–68 (R. Mansfield).

Replies to Query 2. A proof may be found in S. S. Cairns, *Introductory Topology*, Ronald Press, New York 1961, pp. 4–6 (D. E. Christie, J. Di Paola) or in Section 2.7 of L. R. Ford, Jr. and D. R. Fulkerson, *Flows in Networks*, Princeton University Press, Princeton, 1962 (D. Rubin).

MATHEMATICAL NOTES

Edited by David Roselle

Material for this Department should be sent to David Roselle, Department of Mathematics, Louisiana State University, Baton Rouge, LA 70803.

MORE ON INVOLUTIONS OF A CIRCLE

W. F. Pfeffer

An elementary proof that two free involutions of a circle have a coincidence point was given in [4]. The purpose of this note is to show that the existence of a coincidence point can actually be established for any two involutions of which one is free and the other effective. It will also be shown that an involution of a circle has a fixed point if and only if it commutes with a free involution. The methods of proof are as elementary as those of [4].

In a complex plane C we shall consider the unit circle $S = \{z \in C: |z| = 1\}$ and the open unit disc $D = \{z \in C: |z| < 1\}$. A continuous map $\sigma: S \to S$ is called

an **involution** of S if $\sigma^2 = \sigma \circ \sigma$ is the identity map of S. An involution σ of S is called **effective** if it is different from the identity map, i.e., if $\sigma(x) \neq x$ for some $x \in S$; it is called **free** if it has no fixed points, i.e., if $\sigma(x) \neq x$ for all $x \in S$.

If $z_1, z_2 \in C$ and $z_1 \neq z_2$, we shall denote by (z_1, z_2) the open line segment connecting z_1 and z_2, and by (z_1, z_2, ∞) the open half-line from z_1 through z_2. Hence $(z_1, z_2) = \{tz_2 + (1-t)z_1 : 0 < t < 1\}$ and $(z_1, z_2, \infty) = \{tz_2 + (1-t)z_1 : t > 0\}$. If $z_1 = z_2$ we set $(z_1, z_2) = \emptyset$ while (z_1, z_2, ∞) we leave undefined. For $w \in D$ and $z \in S$ we define $\alpha_w(z)$ to be the unique intersection point of S and (z, w, ∞). Clearly, for all $w \in D$, α_w is a free involution of S. In particular, $\alpha_0(z) = -z$ and so α_0 is the **antipodal** map.

We state a slightly stronger version of a lemma which was proved in [**4**]. Its proof is quite analogous to that given in [**4**] and it is therefore left to the reader.

LEMMA. *Let σ be an involution of S. If σ is free then $(x, \sigma(x)) \cap (y, \sigma(y)) \neq \emptyset$ for all x, $y \in S$. If σ is not free then $(x, \sigma(x)) \cap (y, \sigma(y)) = \emptyset$ for all $x, y \in S$ and $x \neq y$, $\sigma(x) \neq y$.*

COROLLARY 1. *Let σ_1 and σ_2 be two different free involutions of S. Then $\sigma_1 \circ \sigma_2 \neq \sigma_2 \circ \sigma_1$.*

Proof. Since $\sigma_1 \neq \sigma_2$ there is a $z \in S$ such that $\sigma_1(z) \neq \sigma_2(z)$. It follows immediately from the lemma that $\sigma_2[\sigma_1(z)] \neq \sigma_1[\sigma_2(z)]$ (draw a picture!).

COROLLARY 2. *Let σ be an effective involution of S. Then σ is either free or it has precisely two distinct fixed points.*

Proof. There is a $z \in S$ such that $\sigma(z) \neq z$. If σ is not free, then by the lemma, in each open arc determined by points z and $\sigma(z)$ there is at least one fixed point of σ. An obvious connectivity argument will show that these fixed points are unique.

PROPOSITION 1. *Let σ be an involution of S. Then σ has a fixed point if and only if there is a free involution τ of S such that $\tau \neq \sigma$ and $\tau \circ \sigma = \sigma \circ \tau$.*

Proof. If such a τ exists then, by Corollary 1, σ has a fixed point. Hence suppose that σ has a fixed point. We may assume that σ is effective, for otherwise it suffices to set, e.g., $\tau = \alpha_0$. By Corollary 2, σ has precisely two distinct fixed points, say x and y. Choose $z \in S - \{x, y\}$. From the lemma it follows that

$$(x, y) \cap (z, \sigma(z)) \neq \emptyset$$

and we denote the intersection point by w. Let A and B be those closed arcs determined by points x, z and x, $\sigma(z)$, respectively, which do not contain point y. For $u \in A \cup \alpha_w(A)$ set $\tau(u) = \alpha_w(u)$ and for $u \in B \cup \alpha_w(B)$ set $\tau(u) = \sigma \circ \alpha_w \circ \sigma(u)$. It follows immediately that τ is a free involution of S (draw a picture and use the lemma!). Moreover, $\sigma \circ \tau = \tau \circ \sigma = \sigma \circ \alpha_w$ on $A \cup \alpha_w(A)$ and $\sigma \circ \tau = \tau \circ \sigma = \alpha_w \circ \sigma$ on $B \cup \alpha_w(B)$.

PROPOSITION 2. *Let σ_1 and σ_2 be two involutions of S. If σ_1 is free and σ_2 is effective, then there is a $z \in S$ such that $\sigma_1(z) = \sigma_2(z)$.*

Proof. We may assume that σ_2 is not free, for otherwise the proposition follows from [4]. Since σ_2 is effective there is an $x \in S$ such that $\sigma_2(x) \neq x$. Suppose $\sigma_1(x) \neq \sigma_2(x)$ (for if $\sigma_1(x) = \sigma_2(x)$ there is nothing to prove) and consider the closed arc A determined by points x, $\sigma_1(x)$ and $\sigma_2(x)$. Let B be the closed subarc of A determined by points $\sigma_1(x)$ and $\sigma_2(x)$. According to the lemma we have either $\sigma_1(B) \subset \sigma_2(B)$ or $\sigma_2(B) \subset \sigma_1(B)$ (draw a picture!), and so either $\sigma_2 \circ \sigma_1$ or $\sigma_1 \circ \sigma_2$ maps B into itself. Because B is homeomorphic to the unit interval $[0,1]$, there is a $z \in B$ such that $\sigma_1(z) = \sigma_2(z)$.

By z^* we denote the complex conjugate of $z \in C$. Letting $\sigma_1(z) = z^*$ and $\sigma_2(z) = -z^*$ for all $z \in S$, we have defined two effective involutions of S with fixed points but no coincidence point (for $\sigma_1 \circ \sigma_2 = \sigma_2 \circ \sigma_1 = \alpha_0$). Since a free involution has clearly no coincidence point with the identity map we see that Proposition 2 cannot be improved.

COROLLARY 3. *Let σ be an involution of S. Then either $\sigma(z) = -z$ for some $z \in S$ or $\sigma(z) = z$ for all $z \in S$.*

The reader should compare this with the analogous corollary in [4].

COROLLARY 4. *If σ is an effective involution of S, then*

$$D = \bigcup_{z \in S}(z, \sigma(z)).$$

Proof. Clearly $(z, \sigma(z)) \subset D$ for all $z \in S$. Choose $w \in D$. Then by Proposition 1 there is $z \in S$ such that $\sigma(z) = \alpha_w(z)$. But this implies that $w \in (z, \sigma(z))$.

REMARK. Corollaries 1 and 2 are special cases of general results due to P. A. Smith (see [3], Appendix B, and [5]). The author does not know if Proposition 1 and 2 can be generalized to higher dimensional spheres. In [1], 36.6, p. 89, Proposition 2 has been proved for arbitrary spheres provided both involutions were free. However, the argument used there does not directly apply if one of the involutions is only effective. For an arbitrary sphere Corollary 4 was proved in [2] for a free involution and in [6] for any effective involution.

We note that all these generalizations are obtained by fairly sophisticated methods of algebraic topology.

A careful analysis of the arguments employed in [4] and in this paper yields an interesting generalization.

Let $f: S \to S$ and let $z \in S$. If $f(x) \neq z$ and $f[f(z)] = z$, then z is called an **involutory point** of f. If, in addition, $f(y) \in \{z, f(z)\}$ implies $y \in \{z, f(z)\}$, then z is called a **simple** involutory point of f. Clearly, all involutory points of an injective map are simple.

PROPOSITION 3. *Let $f: S \to S$ be a continuous map with an involutory point*

and suppose that f is either free or an involutory point of f is simple. If σ is a free involution of S, then there is a $z \in S$ such that $f(z) = \sigma(z)$.

Proof. If f is free we can repeat verbatim the proof of the Proposition in [4]. Hence suppose that f has a simple involutory point $x \in S$ and that $f(z) \neq \sigma(z)$ for all $z \in S$. Let $S - \{x, f(x)\} = A \cup B$, where A and B are disjoint open arcs and let A^- and B^- be closures of A and B, respectively. Orient S and denote by $\prec$ the induced orderings on A^- and B^-. Without loss of generality we may assume that $\sigma(x) \in A$ and that $\sigma(x) \prec \sigma[f(x)]$; note that this has meaning, for by the lemma $\sigma[f(x)] \in A$ (draw a picture!). Clearly either $f(A) \subset A$ or $f(A) \subset B$. If $f(A) \subset A$ denote by C the closed subarc of A^- determined by points $f(x)$ and $\sigma(x)$. According to the lemma $\sigma(C) \subset A^-$ and so we can define a set $M = \{z \in C : f(z) \prec \sigma(z)\}$ which is open and closed in C. Since C is connected, $\sigma(x) \in M$ and $f(x) \notin M$, we have a contradiction. If $f(A) \subset B$ we denote by D the closed subarc of A determined by points $\sigma(x)$ and $\sigma[f(x)]$. It follows from the lemma that $\sigma(D) = B^-$ and so we can define a set $N = \{z \in D : f(z) \prec \sigma(z)\}$ which is open and closed in D. Because D is connected, $\sigma[f(x)] \in N$ and $\sigma(x) \notin N$, we have again a contradiction and the proposition is proved.

COROLLARY 5. *If $f: S \to S$ is the map from Proposition 3, then*

$$D = \bigcup_{z \in S} (z, f(z)).$$

The proof is the same as that of Corollary 4.

Let $a = \sqrt{2}(1 + \sqrt{-1})/2$ and denote by A and B the smaller arc of S determined by points a, a^* and a, $-a^*$, respectively. By C we shall denote that arc of S determined by points a^*, $-a^*$ which does not contain a. We define a continuous map $f: S \to S$ by setting $f(z) = z\sqrt{-1}$ if $z \in A$, $f(z) = -z^*$ if $z \in B$, and $f(z) = a^3 z^2$ if $z \in C$. This map has two involutory points but it is not free and has no simple involutory points. Since $0 \notin (z, f(z))$ for all $z \in S$, we see that the assumptions of Proposition 3 and Corollary 5 are essential.

NOTE. It would be interesting to generalize Proposition 3 or Corollary 5 to higher dimensional spheres.

References

1. P. E. Conner and E. E. Floyd, Differentiable Periodic Maps, Springer-Verlag, Berlin, 1964.
2. A. Forrester, A theorem on involutory transformations without fixed points, Proc. AMS, 3 (1952) 333–334.
3. S. Lefschetz, Algebraic Topology, AMS, New York, 1942.
4. W. F. Pfeffer, On involutions of a circle, this MONTHLY, 79 (1972) 159–160.
5. P. A. Smith, Permutable periodic maps, Proc. Nat. Acad. Sci. USA, 30 (1944) 105–108.
6. S. Stein, Families of curves, Proc. AMS, 5 (1954) 745–747.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF GHANA, P.O. BOX 62, LEGON, GHANA.

A THEOREM ON k-SOLUBILITY OF LINEAR EQUATIONS

RICHARD RADO

Let D be a set and S a condition on elements $x_1, x_2, \cdots, x_n$ of D. For each positive integer k, we say that the condition S is k-**soluble** in D if, whenever $D = D_1 \cup D_2 \cup \cdots \cup D_k$, there is an index $i \in \{1, 2, \cdots, k\}$ and a choice of elements $x_1, \cdots, x_n$ of D_i such that $S(x_1, \cdots, x_n)$ is true. Van der Waerden [**1**] proved that *for each pair k, n of positive integers, the condition*

$$S_n: x_1 - x_2 = x_2 - x_3 = \cdots = x_{n-1} - x_n \neq 0$$

is k-soluble in some set $\{1, 2, \cdots, f(k, n)\}$.

A seemingly weaker proposition is [**3**]

THEOREM 1. *For each pair k, n of positive integers, the condition S_n is k-soluble in the set of all complex numbers.*

One might hope that Theorem 1 can be proved more easily than van der Waerden's theorem. It is the purpose of this note to establish a general transfer theorem (Theorem 2, stated below) which shows that basically Theorem 1 cannot be proved more easily than van der Waerden's theorem, in view of the fact that Theorems 1 and 2 immediately imply the latter.

For, let us assume Theorem 1 and Theorem 2. In Theorem 2 put $p = 2$. By Theorem 1, the system

$$S: \mathbf{x}_1 - \mathbf{x}_2 = \mathbf{x}_2 - \mathbf{x}_3 = \cdots = \mathbf{x}_{n-1} - \mathbf{x}_n \neq \mathbf{0}$$

is k-soluble in the set of all real two-dimensional vectors. For our purpose we may identify such vectors with complex numbers. By Theorem 2, there is a finite set F of vectors of the form $(x_1, 0)$ with x_1 rational, such that the condition S is k-soluble in F. By multiplying with a common denominator of the numbers x_1, we obtain a finite set F^* of vectors of the form $(y_1, 0)$, where the y_1 are integers and, again, the system S is k-soluble in F^*. But this is the same as van der Waerden's theorem.

Theorem 2 asserts the existence of a finite set of rational numbers which has a certain property. However, it must be admitted that the proof that follows is a pure existence proof, yielding no upper estimate, however crude, for the numerators and denominators of the elements of this finite set.

THEOREM 2. *Let m, n, p be positive integers and let $a_{\mu\nu}$ be rational numbers, for $\mu \in \{1, \cdots, m\}$ and $\nu \in \{1, \cdots, n\}$. Let X be the space of all real p-dimensional vectors and M be a subset of $X \times X \times \cdots \times X$* (*n* factors). *If $p = 1$, then M is an arbitrary open set of real numbers (we identify real numbers with one-dimensional vectors), and if $p > 1$, then M is a set of the following form. Certain pairs*

$$(\beta_1, \gamma_1), (\beta_2, \gamma_2), \cdots, (\beta_r, \gamma_r),$$

where $1 \leqq \beta_\rho \leqq n$, $1 \leqq \gamma_\rho \leqq n$, *are given in advance. Then*

$$M = \{(\mathbf{x}_1, \cdots, \mathbf{x}_n) \mid \mathbf{x}_{\beta_\rho} \neq \mathbf{x}_{\gamma_\rho} \text{ for } 1 \leqq \rho \leqq r\}.$$

Assume that the condition

$$S: \sum_\nu \mathfrak{a}_{\mu\nu}\mathbf{x}_\nu = \mathbf{0},\ 1 \leqq \mu \leqq m,\ (\mathbf{x}_1, \cdots, \mathbf{x}_n) \in M$$

is k-soluble in X. Then there is a finite set F of vectors of the form $(x_1, 0, \cdots, 0)$ *with* x_1 *rational such that the condition S is k-soluble in F.*

Proof. For sets A and B, let the relation $A \Subset B$ express the condition that A is a finite subset of B. In [2] the following selection lemma is proved.

Let A and N be sets and $A_\nu \Subset A$ *for* $\nu \in N$. *Suppose that, for each* $L \Subset N$ *we are given a choice function* $f_L: L \to A$ *such that* $f_L(\nu) \in A_\nu$ *for* $\nu \in L$. *Then there is a choice function* $f^*: N \to A$ *such that, given any* $L \Subset N$, *there is M with* $L \subseteq M \Subset N$ *and* $f^*(\nu) = f_M(\nu)$ *for* $\nu \in L$.

We now deduce from this lemma that there is $F^0 \Subset X$ such that the condition S is k-soluble in F^0. In the lemma put $A = \{1, 2, \cdots, k\}$; $N = X$; $A_\nu = A$ for $\nu \in N$. Assume that there is no set $F^0 \Subset X$ such that the condition S is k-soluble in F^0. Then, for every $L \Subset N$, there is a function $f_L: L \to A$ such that L contains no solution of S on which the function f_L is constant. Consider the function f^* which is given by the selection lemma. By hypothesis, the condition S is k-soluble in N. Hence there is a solution $(\mathbf{x}_1, \cdots, \mathbf{x}_n)$ of S such that $f^*(\mathbf{x}_1) = \cdots = f^*(\mathbf{x}_n)$. Now apply the conclusion of the lemma to the set $L = \{\mathbf{x}_1, \cdots, \mathbf{x}_n\}$. We find a set M with $L \subseteq M \Subset N$, such that $f^*(\mathbf{x}_i) = f_M(\mathbf{x}_i)$ for $1 \leqq i \leqq n$. But then $f_M(\mathbf{x}_1) = \cdots = f_M(\mathbf{x}_n)$, which contradicts the definition of f_M. Thus the condition S is k-soluble in some set $F^0 = \{\mathbf{x}_1^0, \cdots, \mathbf{x}_l^0\} \Subset X$. We remark that this step of the proof is completely non-constructive.

Let $T(\mathbf{y}_1, \cdots, \mathbf{y}_l)$ denote the condition given by the following requirement. Whenever

$$\alpha_1, \cdots, \alpha_n \in \{1, \cdots, l\} \text{ and } \sum_\nu \mathfrak{a}_{\mu\nu}\mathbf{x}^0_{\alpha_\nu} = \mathbf{0}$$

for all μ, then

$$\sum_\nu \mathfrak{a}_{\mu\nu}\mathbf{y}_{\alpha_\nu} = \mathbf{0}$$

for all μ. Since the $\mathfrak{a}_{\mu\nu}$ are rational, the general solution in real vectors $\mathbf{y}_\lambda$ of the equations in T can be obtained in the form

$$(\mathbf{y}_1, \cdots, \mathbf{y}_l) = \sum_{\sigma=1}^{s} t_\sigma(\mathbf{y}_{\sigma 1}, \cdots, \mathbf{y}_{\sigma l}),$$

where the $\mathbf{y}_{\sigma\lambda}$ are fixed vectors with rational components and the t_σ are independent

real parameters. Then

$$(\mathbf{x}_1^0, \cdots, \mathbf{x}_l^0) = \sum t_\sigma^0(\mathbf{y}_{\sigma 1}, \cdots, \mathbf{y}_{\sigma\lambda})$$

for some real t_σ^0.

Let $\varepsilon > 0$. Choose rational numbers t_σ^1 such that $|t_\sigma^1 - t_\sigma^0| < \varepsilon$ for all σ and put

$$(\mathbf{x}_1^1, \cdots, \mathbf{x}_l^1) = \sum t_\sigma^1(\mathbf{y}_{\sigma 1}, \cdots, \mathbf{y}_{\sigma l}).$$

Then the components of the $\mathbf{x}_\lambda^1$ are rational. Make ε so small that $(\mathbf{x}_{\alpha_1}^1, \cdots, \mathbf{x}_{\alpha_n}^1) \in M$ whenever $\alpha_1, \cdots, \alpha_n \in \{1, \cdots, l\}$ and $(\mathbf{x}_{\alpha_1}^0, \cdots, \mathbf{x}_{\alpha_n}^0) \in M$. This is possible since M is open. We also stipulate that $\mathbf{x}_\lambda^0 \neq \mathbf{x}_{\lambda'}^0$ implies $\mathbf{x}_\lambda^1 \neq \mathbf{x}_{\lambda'}^1$.

I claim that the condition S is k-soluble in the set

$$F^1 = \{\mathbf{x}_1^1, \cdots, \mathbf{x}_l^1\}.$$

Let $F^1 = F_1^1 \cup \cdots \cup F_k^1$. Put $F_i^0 = \{\mathbf{x}_\lambda^0 \mid \mathbf{x}_\lambda^1 \in F_i^1\}$ for $1 \leqq i \leqq k$. Then $F^0 = \bigcup F_i^0$, and by definition of F^0, there is an index $i \in \{1, \cdots, k\}$ and a choice of numbers $\alpha_1, \cdots, \alpha_n \in \{1, \cdots, l\}$ such that $\mathbf{x}_{\alpha_1}^0, \cdots, \mathbf{x}_{\alpha_n}^0 \in F_i^0$ and $S(\mathbf{x}_{\alpha_1}^0, \cdots, \mathbf{x}_{\alpha_n}^0)$ is true. Then $\mathbf{x}_{\alpha_1}^1, \cdots, \mathbf{x}_{\alpha_n}^1 \in F_i^1$ and $S(\mathbf{x}_{\alpha_1}^1, \cdots, \mathbf{x}_{\alpha_n}^1)$ is true.

Case 1. $p = 1$. Then the assertion holds for $F = F^1$.

Case 2. $p > 1$. Choose a positive integer u such that all vectors $u\mathbf{x}_\lambda^1$ have integral components. Let v be a positive integer which is greater than every difference between two elements of the set of components of the vectors $u\mathbf{x}_\lambda^1$. Put

$$\mathbf{w} = (1, 2v, (2v)^2, \cdots, (2v)^{p-1}).$$

Then it follows from the basic property of the representation of integers in the scale of $2v$ that, for $\lambda, \lambda' \in \{1, \cdots, l\}$, whenever $x_\lambda^1 \neq x_{\lambda'}^1$ we have $u\mathbf{w}\mathbf{x}_\lambda^1 \neq u\mathbf{w}\mathbf{x}_{\lambda'}^1$ (here $\mathbf{wx}$ denotes the scalar product). We put

$$F^2 = \{(u\mathbf{w}\mathbf{x}_\lambda^1, 0, \cdots, 0) \mid 1 \leqq \lambda \leqq l\}.$$

I claim that the set F^2 has the property required of F. First of all, the $u\mathbf{w}\mathbf{x}_\lambda^1$ are integers. Now let $F^2 = F_1^2 \cup \cdots \cup F_k^2$. Put

$$F_i^1 = \{\mathbf{x}_\lambda^1 \mid (u\mathbf{w}\mathbf{x}_\lambda^1, 0, \cdots, 0) \in F_i^2 \text{ for } 1 \leqq i \leqq k\}.$$

Then $F^1 = \bigcup F_i^1$, and by what has been proved above, there is an index $i \in \{1, \cdots, k\}$ and a choice of numbers $\alpha_1, \cdots, \alpha_n \in \{1, \cdots, l\}$ such that $\mathbf{x}_{\alpha_1}^1, \cdots, \mathbf{x}_{\alpha_n}^1 \in F_i^1$, and $S(\mathbf{x}_{\alpha_1}^1, \cdots, \mathbf{x}_{\alpha_n}^1)$ is true. Then $(u\mathbf{w}\mathbf{x}_{\alpha_\nu}^1, 0, \cdots, 0) \in F_i^2$ for all ν, and also

$$S(u\mathbf{w}\mathbf{x}_{\alpha_\nu}^1, 0, \cdots, 0), \quad 1 \leqq \nu \leqq n,$$

is true. This proves Theorem 2.

I wish to express my thanks to Peter A. Rado for valuable suggestions he made to me when I was writing this note.

References

1. B. L. van der Waerden, Nieuw. Arch. Wisk., 15 (1927) 212–216.
2. R. Rado, J. Combinatorial Theory, 10 (1970) 176–177.
3. L. S. Hahn, Problem 5808, this MONTHLY, 78 (1971) 798.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF READING, READING, ENGLAND.

UPPER BOUNDS FOR THE NUMBER OF LATTICE POINTS OF CONVEX BODIES

J. BOKOWSKI AND J. M. WILLS

Let K be a convex body of the three-dimensional Euclidean space E^3, let $V = V(K)$ be the volume of K, $A = A(K)$ its surface area, $M = M(K)$ its integral of average curvature, $r = r(K)$ and $R = R(K)$ the radii of the insphere and the circumsphere of K, respectively, and $L = L(K)$ the number of lattice points in K (points with integer coordinates). Let $\omega_3 = V(S) = 4\pi/3$ denote the volume of the unit sphere $S = \{x/|x| \leqq 1\}$.

The number of lattice points $L(K)$ and the volume $V(K)$ are closely related. Thus for example $V(K) = \int_W L(K_t)\,dt$, whereby $K_t = K + t$ denotes K translated by $t \in W = \{x/|x_i| \leqq \frac{1}{2};\ i = 1,2,3\}$. One is led then to estimate $L(K)$ in terms of $V(K)$. But this would give us only a very rough estimation. We mention without proof that $L(K) \leqq 6\,V(K) + 3$, provided K contains a convex hull of lattice points which is three-dimensional. For better estimates we need further functionals. It is known, for example, that $V - (A/2) < L$ [1] and

$$L \leqq V + \sqrt{3}\,\frac{F}{2} + \frac{3\pi}{4}\cdot\frac{M}{\pi} + \frac{\sqrt{3}\pi}{2} \quad \text{(Wills [3, (5)])}.$$

The conjecture

$$L \leqq V + \frac{F}{2} + \frac{M}{\pi} + 1$$

(equality holding if and only if K is the convex hull of lattice points and every one-dimensional face of K is parallel to a coordinate axis) (Wills [3, (6)]), remains up to now unproved. We shall show

THEOREM. *The number of lattice points of a convex body K is less than or equal to the volume of the outer parallel body of K at the distance $\lambda = \omega_3^{-1/3} \approx 0.62$:*

$$L(K) \leqq V(K + \lambda S). \tag{1}$$

It is not possible to replace λ by $\lambda' < \lambda$.

COROLLARIES:

$$L \leqq V + \lambda F + \lambda^2 M + 1, \tag{2}$$

$$L \leqq \omega_3 (R + \lambda)^3, \tag{3}$$

$$L \leqq \left(1 + \frac{\lambda}{r}\right)^3 V. \tag{4}$$

Inequalities (3) and (4) are best possible in the same sense as the theorem.

For analogous results and conjectures for all dimensions see Wills [3]. The attempt to carry over the method of proof applied here to higher dimensions appears to fail.

Proof. The corollaries are direct consequences of the following well-known relations:

$$V(K + \lambda S) = V + \lambda F + \lambda^2 M + \lambda^3 \omega_3 \text{ (Steiner's formula)},$$

$$V(K + \lambda S) \leqq \omega_3 (R + \lambda)^3 \text{ (volume of the circumsphere)},$$

$$V(K + \lambda S) \leqq \left(1 + \frac{\lambda}{r}\right)^3 V \text{ (Hadwiger [2, p. 66])}.$$

That λ in the theorem cannot be replaced by a smaller number is trivial (take $K = \{0\}$).

Let E^3 be divided into cubes with edges of length one and corners at lattice points. Let C be such a cube and let P be a corner of C. If $P \in K$, we can construct a set $Q(P, C)$ as follows. Because V and L are invariant under movements of K which transform the lattice into itself, it is enough to explain the construction for $P = P_0 = (0,0,0)$ and

$$C = C_0 = \{(x_1, x_2, x_3)/0 \leqq x_i \leqq 1,\ i = 1,2,3\}$$

after a previous movement B with $B(P) = P_0$ and $B(C) = C_0$. Let

$$P_1 = (1,0,0),\quad P_2 = (0,1,0),\ P_3 = (0,0,1),$$

$$P_1' = (0,1,1),\ P_2' = (1,0,1),\ P_3' = (1,1,0), \text{ and } P_0' = (1,1,1).$$

For $P_0 \in K$ we distinguish four cases:

(a) $P_i \notin K,\ i = 1,2,3,$

(b) $P_i \in K$ for exactly one i, say $P_1 \in K$, and $P_2' \notin K$, $P_3' \notin K$.

(c) $P_i \in K$ for exactly one i, say $P_1 \in K$ and $P_2' \in K$ or $P_3' \in K$,

(d) $P_i \in K$ for at least two i, say $P_1, P_2 \in K$.

For $P_0 \in K$ let $Q = Q(P_0, C_0)$ be the following set:

Case (a): $Q = \{x/|x| \leqq \lambda\} \cap C_0$,

Case (b): $Q = \{x / |x_1| \leqq \frac{1}{2},\ x_2^2 + x_3^2 \leqq \pi^{-1}\} \cap C_0$,

Cases (c) and (d): $Q = \{x / |x_i| \leqq \frac{1}{2},\ i = 1,2,3\} \cap C_0$.

These sets Q have the following properties:

(1) $V(Q) = \frac{1}{8}$,

(2) $V(Q \cap Q') = 0$ *for any two* $Q = Q(P,C)$ *and* $Q' = Q'(P',C')$ *with* $Q \neq Q'$,

(3) $Q \subset K + \lambda S$.

From (1), (2), (3) it follows that $L(K) = \sum_i V(Q_i) \leqq V(K + \lambda S)$. (2) and (3) remain to be proved; (1) is trivial.

Proof of (2): It suffices to compare only sets $Q \subset C_0$ and $Q' \subset C_0$.

Case (a): Then at most $P_i' \in K$, $i = 0,1,2,3$. The distance between P_0 and a P_i' is $\geqq \sqrt{2}$ but $\lambda < \frac{1}{2}\sqrt{2}$. Thus none of the possible sets $Q_i' = Q'(P_i', C_0)$ has a non-empty intersection with $Q = Q(P_0, C_0)$.

Case (b): Then at most P_1' or P_0' belong to K. But $Q'(P_1', C_0)$ and $Q'(P_0', C_0)$ do not intersect $Z = \{x / x_2^2 + x_3^2 \leqq \pi^{-1}\}$.

Case (c): 1. $Q'(P_1', C_0)$ is a cube or $P_1' \notin K$. All the sets $Q \subset C_0$ are then cubes whose intersections are at most two-dimensional.

2. $Q'(P_1', C_0)$ is a part of a sphere. As under (a) we can see that $Q'(P_1', C_0)$ intersects none of the cubes $Q(P_0, C_0)$, $Q(P_1, C_0)$, etc.

Case (d) turns out as case (c) except that P_0' takes over the role of P_1'.

Proof of (3): Case (a) is trivial.

Case (b): $\overline{P_0P_1} \subset K$ and $\lambda > \pi^{-\frac{1}{2}}$ imply $Q \subset \overline{P_0P_1} + \lambda S \subset K + \lambda S$.

Case (c): If $P_3' \in K$, then $P_4 = (\frac{1}{4}, \frac{1}{4}, 0) \in K$, and because $\dfrac{1}{16} + \dfrac{1}{16} + \dfrac{1}{4} < \lambda^2$ ≈ 0.38, we have $Q \subset \{P_4\} + \lambda S \subset K + \lambda S$. Similarly for $P_2' \in K$.

Case (d): $P_i \in K$, $i = 0,1,2$. Then again $P_4 \in K$ and $Q \subset K + \lambda S$.

References

1. J. Bokowski, H. Hadwiger, J. M. Wills, Eine Ungleichung zwischen Volumen, Oberfläche und Gitterpunktanzahl konvexer Körper im n-dimensionalen Raum, Math. Z., 127 (1972) 363–364.

2. H. Hadwiger, Altes und Neues über konvexe Körper, Birkhäuser, Basel, 1955.

3. J. M. Wills, Zur Gitterpunktanzahl konvexer Mengen, Elemente der Math., 28, 3 (1973) 57–63.

FACHBEREICH MATHEMATIK, TECHNISCHE UNIVERSITÄT BERLIN, 1 BERLIN 12, STRASSE DES 17. JUNI 135, W. GERMANY.

$$\|f(x)-f(y)\| \leqq \sum_{i=1}^{k} \|f(\gamma(t_i))-f(\gamma(t_{i-1}))\|$$
$$< k < \frac{K}{\delta}\|x-y\|+1 < \left(\frac{K}{\delta}+\frac{1}{\eta}\right)\|x-y\|.$$

It follows that f satisfies the LCL, and the theorem is proved.

Theorem 2 shows, in particular, that for functions having convex domain uniform continuity and the SCS property are equivalent.

The final example shows that the conditions imposed on the domain of the function in Theorem 2 are sufficient, but not necessary, for the equivalence of uniform continuity and the SCS property.

EXAMPLE 2. Let $D = \{(x,y): 0 < x < 1 \text{ and } -x+\sin 1/x < y < x+\sin 1/x\}$. Since D is bounded, uniform continuity and the SCS property are equivalent for functions on D. However, curves in D from the point $(1/\pi, 0)$ to the points $(1/n\pi, 0)$ are arbitrarily long for large values of n. Thus D does not satisfy the hypotheses of Theorem 2.

The SCS property can be defined for a function from a metric space X to a metric space Y by replacing the Euclidean distances by the corresponding metrics in the definition given. After minor changes, including a reformulation of the hypotheses on D in Theorem 2, the results of this paper are valid in a metric space context.

This work was initiated by Mr. Telste under the direction of Professor Klopfenstein in an NSF sponsored Undergraduate Research Participation Program at Colorado State University, summer, 1971.

Reference

1. D. Paine, Visualizing uniform continuity, this MONTHLY, 75 (1968) 44–45.

DEPARTMENT OF MATHEMATICS, COLORADO STATE UNIVERSITY, FORT COLLINS, CO 80521.
DEPARTMENT OF MATHEMATICS, AUGSBURG COLLEGE, MINNEAPOLIS, MN 55404.

SPECTRAL RADIUS AND RADIUS OF CONVERGENCE

H. K. WIMMER

Let $A = (a_{ij})$ be a complex $m \times m$ matrix with eigenvalues $\lambda_1, \cdots, \lambda_m$. The **spectral radius** $\rho(A)$ is defined as

$$\rho(A) = \max_{1 \leqq k \leqq m} |\lambda_k|.$$

In this note we use the traces of the successive powers of A to derive a formula for (dA). We recall that

$$\operatorname{Tr} A = \sum_{k=1}^{m} a_{kk} = \sum_{k=1}^{m} \lambda_k \quad \text{and} \quad \operatorname{Tr} A^{\nu} = \sum_{k=1}^{m} \lambda_k^{\nu}.$$

THEOREM.

$$\rho(A) = \limsup_{\nu\to\infty} \sqrt[\nu]{|\operatorname{Tr} A^\nu|}. \tag{1}$$

Proof. If $\rho(A) = 0$, (1) is obvious. We assume $\rho(A) > 0$. For $|z| < 1/\rho(A)$ the order of summation can be changed in $\sum_{k=1}^{m}\sum_{\nu=0}^{\infty} \lambda_k^\nu z^\nu$ and the power series

$$\sum_{\nu=0}^{\infty} \operatorname{Tr} A^\nu z^\nu \tag{2}$$

(a power series with coefficients $\operatorname{Tr} A^\nu$) can be transformed in the following way [2, p. 26]

$$\sum_{\nu=0}^{\infty} \operatorname{Tr} A^\nu z^\nu = \sum_{\nu=0}^{\infty} \operatorname{Tr}(zA)^\nu = \operatorname{Tr}\left[\sum_{\nu=0}^{\infty} (zA)^\nu\right] = \operatorname{Tr}(I - zA)^{-1} = \sum_{k=1}^{m} \frac{1}{1 - z\lambda_k}.$$

Let R be the radius of convergence of (2), then

$$R^{-1} = \limsup_{\nu\to\infty} \sqrt[\nu]{|\operatorname{Tr} A^\nu|}.$$

As the singularities of (2) are given by $z = 1/\lambda_k$, $\lambda_k \neq 0$, the radius of convergence can also be expressed as

$$R = \min \frac{1}{|\lambda_k|} = \frac{1}{\max|\lambda_k|} = \frac{1}{\rho(A)}.$$

Thus $R^{-1} = \limsup_{\nu\to\infty} \sqrt[\nu]{|\operatorname{Tr} A^\nu|} = \rho(A)$.

We note that the mapping $A \mapsto \|A\| = |\operatorname{Tr} A|$ assigns to every complex square matrix A a real number $\|A\|$, satisfying the following axioms:

I. $\|A\| \geqq 0$.

II. $\|cA\| = |c| \, \|A\|$ *for any complex number* c.

III. $\|A + B\| \leqq \|A\| + \|B\|$, *if the sum* $A + B$ *exists.*

Let us call such a mapping a **matrix seminorm.** We have proved that for the seminorm $\|A\| = |\operatorname{Tr} A|$

$$\rho(A) = \limsup_{\nu\to\infty} \sqrt[\nu]{\|A^\nu\|} \tag{3}$$

holds. It is obvious that there are seminorms (take $\|A\| \equiv 0$ for a trivial example) for which (3) is not true. If Axiom I is replaced by the stronger axiom

I*. $A \neq 0$ · implies $\|A\| > 0$ · ,

then $\|\cdot\|$ is called a **generalized matrix norm** [1, p. 61]. For any such generalized matrix norm the spectral radius is given by [1]

$$\rho(A) = \lim_{\nu\to\infty} \sqrt[\nu]{\|A^\nu\|}.$$

We point out the following problem: What are the conditions for a matrix seminorm which is not a generalized matrix norm such that (3) holds?

References

1. A. S. Householder, The Theory of Matrices in Numerical Analysis, Blaisdell, New York, 1964.

2. H. Wielandt, Topics in the Analytic Theory of Matrices, Lecture Notes, University of Wisconsin, Madison, Wisconsin, 1967.

MATHEMATISCHES INSTITUT, TECHNISCHE HOCHSCHULE, GRAZ, AUSTRIA.

A PARTITION IDENTITY OF THE EULER TYPE

D. R. HICKERSON

The celebrated Euler identity in the theory of partitions can be stated as follows:

EULER'S THEOREM. *The number of partitions of n of the form* $n = b_0 + b_1 + \cdots + b_s$ *where, for* $0 \leqq i \leqq s-1$, $b_i > b_{i+1}$, *is equal to the number of partitions of n into odd parts.*

Many generalizations of this identity are known (see [**1**] and [**2**]). In this paper we ask the question whether similar identities exist if the condition $b_i > b_{i+1}$ is replaced by $b_i \geqq rb_{i+1}$, where r is a positive integer. The following theorem shows that there exists such an identity for each positive integer r.

THEOREM. *If* $f(r,n)$ *denotes the number of partitions of* n *of the form* $n = b_0 + b_1 + \cdots + b_s$, *where, for* $0 \leqq i \leqq s-1$, $b_i \geqq rb_{i+1}$, *and* $g(r,n)$ *denotes the number of partitions of* n, *where each part is of the form* $1 + r + r^2 + \cdots + r^i$ *for some* $i \geqq 0$, *then*

$$f(r,n) = g(r,n).$$

Combinatorial Proof. A partition counted by $g(r,n)$ may be represented as

$$n = \sum_{i=0}^{s} a_i(1 + r + r^2 + \cdots + r^i)$$

for some nonnegative integers $s, a_0, \cdots, a_s$ with $a_s > 0$. Given such a partition, let, for $0 \leqq i \leqq s$, $b_i = \sum_{j=i}^{s} a_j r^{j-i}$. Then

$$b_0 + b_1 + \cdots + b_s = \sum_{i=0}^{s} \sum_{j=i}^{s} a_j r^{j-i} = \sum_{j=0}^{s} \sum_{i=0}^{j} a_j r^{j-i}$$

$$= \sum_{j=0}^{s} a_j(r^j + r^{j-1} + \cdots + r + 1) = n.$$

Also, for $0 \leqq i \leqq s-1$,

$$b_i = \sum_{j=i}^{s} a_j r^{j-i} \geqq \sum_{j=i+1}^{s} a_j r^{j-i} = r \sum_{j=i+1}^{s} a_j r^{j-(i+1)} = rb_{i+1}.$$

Therefore, $b_0 + b_1 + \cdots + b_s$ is one of the partitions counted by $f(r, n)$.

This constitutes a mapping from the set of partitions counted by $g(r, n)$ into the set of partitions counted by $f(r, n)$. It therefore suffices to show that this mapping is 1-1 and onto.

Note that $a_s = b_s$ and, for $0 \leqq i \leqq s-1$, $a_i = b_i - rb_{i+1}$. Therefore, the b_i's uniquely determine the a_i's. That is, the mapping is 1-1.

Let $b_0 + b_1 + \cdots + b_s$ be a partition counted by $f(r, n)$. For $0 \leqq i \leqq s-1$, let $c_i = b_i - rb_{i+1}$. Let $c_s = b_s$. Then

$$\sum_{i=0}^{s} c_i(1 + r + r^2 + \cdots + r^i) = \sum_{i=0}^{s-1} (b_i - rb_{i+1})(1 + r + \cdots + r^i) + b_s(1 + r + \cdots + r^s)$$

$$= \sum_{i=0}^{s} b_i(1 + r + \cdots + r^i) - r \sum_{i=0}^{s-1} b_{i+1}(1 + r + \cdots + r^i)$$

$$= \sum_{i=0}^{s} b_i(1 + r + \cdots + r^i) - \sum_{i=1}^{s} b_i(r + r^2 + \cdots + r^i) = \sum_{i=0}^{s} b_i = n.$$

Also, since $b_i \geqq rb_{i+1}$ for $0 \leqq i \leqq s-1$, $c_i \geqq 0$ for $0 \leqq i \leqq s-1$. Finally, $c_s = b_s > 0$, so that

$$\sum_{i=0}^{s} c_i(1 + r + r^2 + \cdots + r^i)$$

is one of the partitions of n counted by $g(r, n)$. It is easily seen that, for $0 \leqq i \leqq s$, $b_i = \sum_{j=i}^{s} c_j r^{j-i}$; so the partition $\sum_{i=0}^{s} c_i(1 + r + r^2 + \cdots + r^i)$ is mapped to the partition $b_0 + b_1 + \cdots + b_s$. Therefore, the mapping is onto, and the proof is complete.

Proof Using Generating Functions. For $N \geqq 0$, let $\phi(r, n; N)$ denote the number of partitions of the type enumerated by $f(r, n)$ with the added restriction that each part is $\leqq N$, and, for $N < 0$, let $\phi(r, n; N) = 0$. Let

$$d_r(N, q) = \sum_{n=0}^{\infty} \phi(r, n; N) q^n.$$

Clearly

$$\text{(A)} \qquad d_r(N, q) = d_r(N-1, q) + q^N d_r\left(\left[\frac{N}{r}\right], q\right).$$

Therefore if

$$u_r(t, q) = \sum_{N=0}^{\infty} d_r(N, q) t^N = \sum_{N=0}^{\infty} \sum_{n=0}^{\infty} \phi(r, n; N) q^n t^N,$$

then, by (A),

$$\begin{aligned} u_r(t,q) &= tu_r(t,q) + \sum_{k=0}^{\infty} \sum_{l=0}^{r-1} q^{kr+l} d_r(k,q) t^{kr+l} \\ &= tu_r(t,q) + \sum_{l=0}^{r-1} t^l q^l u_r(t^r q^r, q). \end{aligned}$$

Therefore

$$\begin{aligned} u_r(t,q) &= \frac{(1-t^r q^r)}{(1-t)(1-tq)} u_r(t^r q^r, q) \\ &= \frac{(1-t^{r^2} q^{r^2+r})}{(1-t)(1-tq)(1-t^r q^{r+1})} u_r(t^{r^2} q^{r^2+r}, q) \\ &\;\vdots \\ &= (1-t)^{-1} \prod_{i=0}^{\infty} (1 - t^{r^i} q^{r^i+\cdots+r^2+r+1})^{-1}. \end{aligned}$$

Let

$$\text{(B)} \qquad v_r(t,q) = (1-t)u_r(t,q) = \prod_{i=0}^{\infty} (1 - t^{r^i} q^{r^i+\cdots+r^2+r+1})^{-1}.$$

Then $v_r(t,q) = (1-t) \sum_{N=0}^{\infty} \sum_{n=0}^{\infty} \phi(r,n;N) q^n t^N$; that is,

$$\text{(C)} \qquad v_r(t,q) = \sum_{n=0}^{\infty} \phi(r,n;0) q^n + \sum_{N=1}^{\infty} \sum_{n=0}^{\infty} (\phi(r,n;N) - \phi(r,n;N-1)) q^n t^N.$$

From (B), $v_r(1,q) = \prod_{i=0}^{\infty} (1 - q^{r^i+\cdots+r^2+r+1})^{-1}$. But, from (C),

$$v_r(1,q) = \sum_{n=0}^{\infty} [\phi(r,n;0) + \sum_{N=1}^{\infty} (\phi(r,n;N) - \phi(r,n;N-1))] q^n = \sum_{n=0}^{\infty} f(r,n) q^n.$$

Then

$$\sum_{n=0}^{\infty} f(r,n) q^n = \prod_{i=0}^{\infty} (1 - q^{r^i+\cdots+r^2+r+1})^{-1} = \sum_{n=0}^{\infty} g(r,n) q^n,$$

and the proof is complete.

I wish to thank the referee for suggesting the second proof.

References

1. H. L. Alder, Partition identities — from Euler to the present, this MONTHLY, 76 (1969) 733–746.

2. G. E. Andrews, Partition identities, Advances in Math., 9 (1972) 10–51.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA, DAVIS, CA 95616.

RESEARCH PROBLEMS

EDITED BY RICHARD GUY

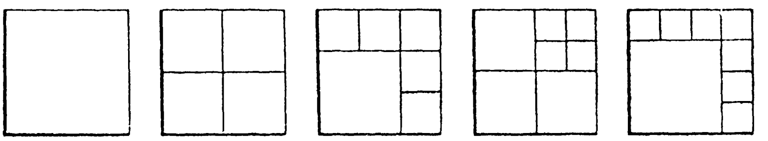
In this Department the Monthly presents easily stated research problems dealing with notions ordinarily encountered in undergraduate mathematics. Each problem should be accompanied by relevant references (if any are known to the author) and by a brief description of known partial results. Manuscripts should be sent to Richard Guy, Department of Mathematics, Statistics, and Computing Science, The University of Calgary, Calgary 44, Alberta, Canada, T2N 1N4.

DECOMPOSITION OF A CUBE INTO SMALLER CUBES

CHRISTOPH MEIER

The following problem is originally due to H. Hadwiger, Bern.

Let I^n ($n \geqq 2$) denote the n-dimensional unit cube. By $D(n)$ we mean the set of numbers $k \in \mathbb{N}$ such that there exists a decomposition (in the sense of elementary geometry) of I^n into k homothetic n-cubes of smaller, not necessarily equal, size. E.g., $D(2)$ consists of all numbers $k \in \mathbb{N}$ except 2, 3 and 5 as shown by the figure:



By cutting one square into four parts we get $k + 3 \in D(2)$ from $k \in D(2)$. By $6, 7, 8 \in D(2)$ we therefore conclude $k \in D(2)$ for $k \geqq 6$. We easily find $2, 3, 5 \notin D(2)$, too.

Let $c(n)$ denote the smallest number with the following property: for every $k \geqq c(n)$, k belongs to $D(n)$. E.g., $c(2) = 6$ and as far as we know $c(3) \leqq 48$. Does $c(3) = 48$ hold?

By a few arguments we prove the existence of $c(n)$:

(1) If $a, b \in D(n)$ and $\nu, \mu \in \mathbb{N} \cup \{0\}$, then

$$c = 1 + \nu(a-1) + \mu(b-1) \in D(n).$$

(This statement can be generalized on a finite set of numbers $a_i \in D(n)$.)

(2) If $p, q \in \mathbb{N}$ with $(p, q) = 1$, then every number $k \geqq (p-1)(q-1)$ is of the form $k = \nu p + \mu q$ with suitable $\nu, \mu \in \mathbb{N} \cup \{0\}$.

(3) 2^n and $(2^n-1)^n - (2^n-2)^n + 1 \in D(n)$; the second number is constructed by decomposition of I^n into $(2^n-1)^n$ n-cubes of edge-length $1/(2^n-1)$ and then re-arranging $(2^n-2)^n$ of them to a n-cube of edge-length $(2^n-2)/(2^n-1)$.

If we combine (1) and (2) we have $c(n) \leqq 1 + (a-2)(b-2)$ if $a, b \in D(n)$, and

$(a-1, b-1) = 1$. (3) yields a and b: of course $(2^n-1, (2^n-1)^n - (2^n-2)^n) = 1$. Therefore the existence of $c(n)$ is proved. Further

$$c(n) \leqq (2^n-2)((2^n-1)^n - (2^n-2)^n - 1) + 1\,.$$

Another somewhat greater upper bound was given earlier by W. Plüss [**2**].

This estimate of $c(n)$ is of order 2^{n^2} and does not seem very good. Actually we believe in the existence of an upper bound of $c(n)$ of the form r^n where $r \in \mathbb{R}$ is a constant, maybe 6 or even smaller, or a function $r = r(n)$ differing not much from a constant function.

Our question is therefore to find a better upper bound for $c(n)$ (or even the exact value, which might be difficult and probably solvable only in small dimensions) or a characterization of the set $D(n)$ or its complement. E.g., we might investigate the gaps of $D(n)$ with respect to $\mathbb{N}$. H. Hadwiger (private communication) showed by a short inductive argument $1 < k < 2^n \Rightarrow k \notin D(n)$ and $2^n < k < 2^n + 2^{n-1} \Rightarrow k \notin D(n)$. In higher dimensions the number of gaps will increase.

The challenge of the problem seems to us to lie in the strange mixture of geometrical and number-theoretical reasoning:

On one hand there is geometry as in statement (1) which can be completed by an old theorem of Dehn [**1**] which states that all ratios of edge-length of the participating cubes must be rational. Therefore, by decomposition of I^n into k^n parts with suitable k great enough and rearranging $k_1^n, k_2^n, \cdots$ of them ($k_i \leqq k$, not too many) to greater parts, every $m \in D(n)$, $m \leqq c(n)$ can be attained. But no system of rearrangement is known to us.

On the other hand, there are diophantine equations with positive solutions as in statement (2) and other number-theoretic problems. E.g., all numbers of the form $k^n - (k-1)^n + 1$, $k > 1$ belong to $D(n)$. By condition $(p, q) = 1$ in (2) we are invited to deal with the factorizations of the quasi-Mersenne numbers $k^n - (k-1)^n$, especially for $k < 2^n - 1$ yielding relatively small and easily describable numbers for our purpose. The numbers $k^n - 1$ will occur, too, if we rearrange n-cubes starting with Dehn's theorem.

References

1. M. Dehn, Über die Zerlegung von Rechtecken in Rechtecke, Math. Ann., 57 (1903) 314–332.

2. W. Plüss, Zerlegung des n-dimensionalen Würfels in homothetische Teilwürfel, Diplomschrift, Bern, 1972.

MATHEMATISCHES INSTITUT, UNIVERSITÄT BERN, CH-3000 BERN, SWITZERLAND.

A COVERING PROBLEM

L. Fejes Tóth

By **plates** we mean congruent replicas of a centro-symmetric convex domain arranged in some way in the Euclidean plane. It is known **[1, 2, 3, 4]** that the density of an arbitrary packing of plates cannot exceed the density of the densest lattice packing of the plates. Probably this theorem has a dual. The density of an arbitrary covering of the plane with plates is never less than the density of the thinnest lattice covering.

This conjecture can be proved under a certain restriction which seems to be automatically fulfilled in the case of an economical covering. The difficulty is to get rid of this restriction.

Two plates are said to **cross** each other if the removal of their intersection causes each plate to fall into disjoint components. It can be proved **[1, 2, 3, 5]** that the density of a covering with non-crossing plates is never less than the density of the thinnest lattice covering. Thus the following problem arises: Can a finite number of plates, covering a convex region, always be rearranged so as to cover the region without any two of them crossing?

Unfortunately the answer is negative. A counterexample, due to A. Heppes, is a square, covered with two hexagons, whose vertices are the midpoints of the sides of the square and a pair of opposite vertices of the square. It can be shown that one of these hexagons cannot cover a connected part of the boundary of the square whose length is one half of the perimeter of the square. Therefore the square cannot be covered without the two hexagons crossing each other.

In the case of two plates the topological and metrical possibilities of a rearrangement are very restricted. Therefore it is imaginable that in the case of more than two plates a similar counterexample does not exist.

As a first approach to the problem mentioned in the introduction we suggest the following more special problem: Can three plates covering a convex region always be rearranged so as to cover the region without crossing each other?

References

1. L. Fejes Tóth, Some packing and covering theorems, Acta Univ. Szeged, Acta Sci. Math., 12/A (1950) 62–67.

2. L. Fejes Tóth, Lagerungen in der Ebene, auf der Kugel und im Raum, Zweite Auflage, Springer-Verlag, Berlin-Heidelberg-New York, 1972.

3. L. Fejes Tóth, Reguläre Figuren, Akadémiai Kiadó, Budapest, 1965.

4. J. W. S. Cassels, An introduction to the geometry of numbers, Springer-Verlag, Berlin-Göttingen-Heidelberg, 1959.

5. R. P. Bambah and C. A. Rogers, Covering the plane with convex sets, J. London Math. Soc., 27 (1952) 304–314.

Mathematical Institute, Hungarian Academy of Sciences, Budapest, Hungary.

CLASSROOM NOTES

EDITED BY DAVID ROSELLE

Material for this Department should be sent to David Roselle, Department of Mathematics, Louisiana State University, Baton Rouge, LA 70803.

TOPOLOGICAL PROPERTIES OF MANIFOLDS

D. B. GAULD

The main purpose of this paper is to collect together some of the folklore of the subject: results which some people suspect but have never proved nor seen in print and which other people seem to be unaware of.

DEFINITION. By an m-**manifold** (or just **manifold**) we mean a space M^m in which each point has an open neighbourhood homeomorphic to either R^m or

$$H^m = \{(x_1, \cdots, x_m) \in R^m : x_m \geqq 0\}.$$

R^m has the usual Euclidean topology and H^m the subspace topology.

Most authors require their manifolds to satisfy further properties: at the least manifolds are usually also paracompact and Hausdorff.

EXAMPLE. Let R be the real line (usual topology) and let S be any set with $R \cap S = \{0\}$. Let $X = R \cup S$. For each $x \in R$, let

$$B(x) = \{N \subset R : N \text{ is a neighbourhood of } x \text{ in } R\},$$

and for each $x \in S$, let

$$B(x) = \{(N - \{0\}) \cup \{x\} : N \text{ is a neighbourhood of } 0 \text{ in } R\}.$$

Topologise X by declaring $\{B(x) : x \in X\}$ to be a system of basic neighbourhoods. The space X is essentially R with $\#(S)$ origins.

X is a separable 1-manifold, the rationals forming a countable dense subset. If S has more than one element, then X is not Hausdorff, since points of S cannot be separated by disjoint open sets. If S is infinite, then X is not paracompact, since if for each $x \in S$ we let

$$U_x = (R - \{0\}) \cup \{x\},$$

then $\{U_x : x \in S\}$ is an open cover of x, having no locally-finite open refinement. Every compact subset of X contains only finitely many members of S, so if S is uncountable, then X is not σ-compact.

THEOREM 1. *Let M^m be a manifold. Then M shares all of the local properties of R^m such as local compactness, local connectedness, local path connectedness, local metrizability, first countability, etc.*

THEOREM 2. *Every paracompact Hausdorff manifold is metrizable.*

Proof. See [**1**, Theorem 2].

THEOREM 3. *Let M be a manifold. Then properties* (i), (ii) *and* (iii) *below are equivalent and each implies property* (iv):

(i) *M is σ-compact*; (ii) *M is Lindelöf*;
(iii) *M is second countable*; (iv) *M is separable.*

Proof. The implications (i) $\Rightarrow$ (ii), (iii) $\Rightarrow$ (ii) and (iii) $\Rightarrow$ (iv) hold in any space. Since the proofs of these implications are straightforward, we omit them.

(ii) $\Rightarrow$ (i) holds in any locally compact space. Indeed, for each $x \in M$, let K_x be a compact neighbourhood of x in M. Then $\{\operatorname{Int} K_x : x \in M\}$ is an open cover of M which, since M is Lindelöf, must have a countable subcover, say $\{\operatorname{Int} K_n : n = 1, 2 \cdots\}$. Then $\{K_1, K_2, \cdots\}$ is a countable collection of compact sets whose union is M; so M is σ-compact.

It remains to verify (ii) $\Rightarrow$ (iii). Since M is Lindelöf, we can find a countable open cover of M, say $\{U_n : n = 1, 2, \cdots\}$, where each U_n is homeomorphic to either R^m or H^m. The spaces R^m and H^m are each second countable, so U_n must also be second countable: for each n, let $\{V_{np} : p = 1, 2, \cdots\}$ be a countable basis for U_n. Then

$$\{V_{np} : n, p = 1, 2, \cdots\}$$

is a countable basis for M. Thus M is second countable.

REMARK. The example shows that (iv) is not in general equivalent to (i) and hence to (ii) and (iii), since in the case where S is uncountable, X is a separable manifold which is not σ-compact. In Theorem 8 below we shall see that (iv) implies the other properties if we also assume the manifold to be paracompact.

THEOREM 4. *A manifold is connected if and only if it is path connected.*

Proof. Every path connected space is connected. Conversely, if M is a connected manifold, then for each $x \in M$, let P_x denote the path component of x in M. The set P_x is open since M is locally path connected. If $y \in M$, then either $P_x \cap P_y = \emptyset$ or $P_x = P_y$. Thus $M - P_x$ is open, being the union of all path components of points not in P_x. Hence P_x is both open and closed, so by connectedness of M we have $P_x = M$. Thus M is path connected.

DEFINITIONS. Let X be a space. A family $\mathscr{F}$ of subsets of X is **star-finite** if for each $F_0 \in \mathscr{F}$ the set $\{F \in \mathscr{F} : F \cap F_0 \neq \emptyset\}$ is finite. The space X is **strongly paracompact** if every open cover of X has a star-finite open refinement.

LEMMA 5. *Every paracompact, locally compact space is strongly paracompact.*

Proof. Let $\mathscr{U}$ be an open cover of the paracompact, locally compact space X. For each $x \in X$, choose an element $U_x \in \mathscr{U}$ with $x \in U_x$. Let C_x be a compact neigh-

bourhood of x, let O_x denote the interior of C_x, and let $K_x = O_x \cap U_x$. Consider $\mathscr{K} = \{K_x : x \in X\}$ which is an open refinement of $\mathscr{U}$. Let $\mathscr{V}$ be a locally-finite open refinement of $\mathscr{K}$. $\mathscr{V}$ is also an open refinement of $\mathscr{U}$.

The family $\mathscr{V}$ is star-finite. Indeed, given $V_0 \in \mathscr{V}$, then there is a $y \in X$ with $V_0 \subset C_y$. Since $\mathscr{V}$ is locally-finite, for each $x \in C_y$ we can find a neighbourhood N_x of x such that $\{V \in \mathscr{V} : V \cap N_x \neq \emptyset\}$ is finite. By compactness of C_y, a finite subfamily, say $\{N_1, \cdots, N_n\}$, of $\{N_x : x \in C_y\}$ covers C_y. For any $V \in \mathscr{V}$, if $V \cap V_0 \neq \emptyset$ then $V \cap N_i \neq \emptyset$ for some i. Hence

$$\{V \in \mathscr{V} : V \cap V_0 \neq \emptyset\} \subset \bigcup_{i=1}^{n} \{V \in \mathscr{V} : V \cap N_i \neq \emptyset\}.$$

The former set is finite since the latter is finite. Hence $\mathscr{V}$ is a star-finite open refinement of $\mathscr{U}$.

LEMMA 6. *Every connected, strongly paracompact space is Lindelöf.*

Proof. Let $\mathscr{U}$ be an open cover of the connected, strongly paracompact space X and let $\mathscr{V}$ be a star-finite open refinement of $\mathscr{U}$. It suffices to find a countable subcover of $\mathscr{V}$. For each non-empty $V \in \mathscr{V}$ and each non-negative integer n, let

$$\begin{aligned} \mathscr{W}(V, n) &= \{W \in \mathscr{V} : \exists V_0, \cdots, V_n \in \mathscr{V} \text{ such that } V_0 = V, \\ V_n &= W \text{ and } V_{i-1} \cap V_i \neq \emptyset \text{ for each } i = 1, \cdots, n\}. \end{aligned}$$

Let $\mathscr{W}(V) = \bigcup_{n=1}^{\infty} \mathscr{W}(V, n)$. It will be shown that $\mathscr{W}(V)$ is a countable cover of X.

(i) *The family $\mathscr{W}(V)$ is countable*: $\mathscr{W}(V, 0) = \{V\}$, so $\mathscr{W}(V, 0)$ is finite. Now assume inductively that $\mathscr{W}(V, n-1)$ is finite for $n > 0$. Then

$$\begin{aligned} \mathscr{W}(V,n) &= \{W \in \mathscr{V} : \exists W' \in \mathscr{W}(V, n-1) \text{ with } W \cap W' \neq \emptyset\} \\ &= \bigcup_{W' \in \mathscr{W}(V, n-1)} \{W \in \mathscr{V} : W \cap W' \neq \emptyset\}. \end{aligned}$$

By star-finiteness of $\mathscr{V}$, for each $W' \in \mathscr{W}(V, n-1)$, the set $\{W \in \mathscr{V} : W \cap W' \neq \emptyset\}$ is finite. Thus $\mathscr{W}(V, n)$ is finite, being a finite union of finite sets. Hence $\mathscr{W}(V)$ is countable.

(ii) *The family $\mathscr{W}(V)$ covers X*: firstly note that for each $V, V' \in \mathscr{V}$, either $\mathscr{W}(V) \cap \mathscr{W}(V') = \emptyset$ or $\mathscr{W}(V) = \mathscr{W}(V')$.

Let $W(V) = \cup \{W \in \mathscr{W}(V)\}$. Then $W(V)$ is open, being a union of open sets. $W(V)$ is also closed since

$$X - W(V) = \cup \{W(V') : \mathscr{W}(V) \cap \mathscr{W}(V') = \emptyset\}.$$

$W(V)$ is non-empty, so by connectedness of X we see that $W(V) = X$, i.e., $\mathscr{W}(V)$ covers X.

THEOREM 7. *Every connected paracompact manifold is Lindelöf and hence also σ-compact, second countable and separable.*

Proof. Manifolds are clearly locally compact, so the result follows from Lemmas 5 and 6 and Theorem 3.

THEOREM 8. *Every paracompact separable manifold is σ-compact, Lindelöf and second countable.*

Proof. A separable locally connected space has only countably many components. By Theorem 7 each component of a paracompact manifold is σ-compact. Thus every paracompact separable manifold is σ-compact and hence also Lindelöf and second countable.

THEOREM 9. *Let M^m be a compact Hausdorff manifold. Then for some n there is an embedding $e: M \to R^n$.*

Proof. By compactness of M we can find an open cover $\{U_1, \cdots, U_l\}$ and, for each i, an embedding $h_i: U_i \to R^m$ with $h_i(U_i) = R^m$ or $h_i(U_i) = H^m$. Consider S^m to be the one point compactification of R^m and write $S^m = R^m \cup \{\infty\}$. For each i define a function $g_i: M \to S^m$ by

$$g_i(x) = \begin{cases} h_i(x) & \text{if } x \in U_i \\ \infty & \text{if } x \in M - U_i. \end{cases}$$

The function g_i is continuous since M is a Hausdorff manifold. Define $g: M \to (S^m)^l$ by letting the ith coordinate of $g(x)$ be $g_i(x)$ for each $x \in M$. Then g is continuous. Moreover, if $x, y \in M$ and $x \neq y$, then $x \in U_i$ for some i. Thus $g_i(x) \neq g_i(y)$ so that $g(x) \neq g(y)$. Hence g is an injective map from the compact space M to the Hausdorff space $(S^m)^l$. Thus g is an embedding. To obtain the desired embedding e, note that $S^m \subset R^{m+1}$, so g determines an embedding from M to $(R^{m+1})^l = R^n$, where $n = (m+1)l$.

Reference

1. J. Nagata, On a necessary and sufficient condition of metrizability, Journal of the Institute of Polytechnics, Osaka City University, series A, 1 (1950) 93–100.

DEPARTMENT OF MATHEMATICS, THE UNIVERSITY OF AUCKLAND, PRIVATE BAG, AUCKLAND, NEW ZEALAND.

ON A CONVERSE OF THE JORDAN CURVE THEOREM

ISTVÁN FÁRY AND ERIC M. ISENBERG

The title refers to the following result, traditionally called a converse of the Jordan curve theorem:

THEOREM 1. *Let C be a compact set in the plane E. If the complement of C in E has two components, and every point of C is an accessible boundary point of both components, then C is a Jordan curve.*

This theorem is due to Schoenflies (*Göttinger Nachrichten*, 1902, p. 185). His proof was very complicated. A similar result was used by Hilbert, and F. Riesz noted that Hilbert's method of proof gives quite easily the Schoenflies theorem (Anhang IV in Hilbert's, *Grundlagen der Geometrie*, pp. 177–184 in the 1913 edition; F. Riesz, *Collected Works*, Volume 1, pp. 27–33). For a thorough discussion, and new proof, see R. L. Wilder, *Topology of Manifolds*, p. 13, and Theorem II 5.38 on p. 67. In a course dealing with elementary topology, we discussed the Jordan curve theorem, and proved it for polygons. We also discussed the Schoenflies theorem, and noticed that it can be formulated and proved for polygons. We give below such a proof in detail.

We shall work in the plane E, and denote $[a, b]$ the (closed straight line) segment with endpoints a, b in E. If $U \subset E$, a continuous map $f: I \to U$, $I = \{t \in R: 0 \leqq t \leqq 1\}$ is called a path connecting $a_0 = f(0)$ with $a_1 = f(1)$ in U. A set $U \subset E$ is called path-connected, if any two points can be connected by a path in U. It is easy to prove that a connected, open set in E is path connected. Let U be a non-empty connected open set in E and b a point not in U. We say that b is an accessible boundary point of U (or that b is accessible from U), if $U \cup \{b\}$ is path connected. An accessible boundary point is, of course, a boundary point $(b \in \bar{U} - U)$. Using the $\sin 1/x$ curve, it is easy to produce not accessible boundary points. We note that the accessible boundary points are everywhere dense on the boundary of a connected open set. Finally, if D is a finite union of segments, and U is a component of $E - D$, then every boundary point of U is accessible from U, though not necessarily from another component of $E - D$. A set in E homeomorphic to I (to a circle S^1) is called a Jordan arc (curve); if it is also the union of a finite number of segments, the adjective "polygonal" is used.

THEOREM 2. *Let C be the union of a finite number of segments in the plane E. If the complement of C in E has two components, and every point of C is boundary point of both components, then C is a polygonal Jordan curve.*

REMARKS. The reader is invited to compare the two theorems, and notice that the second is rigorously a special case of the first, because the word "accessible" can now be omitted as noted above. Theorem 1 would be false, of course, without this condition.

Proof. C is clearly compact, hence any component of $E - C$ is open and path connected. Any circle centered to a point of C, with radius larger than the diameter of C is entirely contained in the same component of $E - C$, called outside of C; the other component will be called inside of C. We have the segments

$$S_1, \cdots, S_n, \tag{1}$$

whose union is C. A non-empty intersection $S_i \cap S_j$, $j \neq i$, is a point or a segment, and in both cases we may subdivide S_i, S_j, so that for the sub-segments obtained

any non-empty intersection is a single point, which is end-point on both segments. Proceeding thus we may replace the originally given segments (1) by a sequence of segments

$$[a_1, b_1], \cdots, [a_m, b_m] \tag{2}$$

such that

$$C = \bigcup_{i=1}^{m} [a_i, b_i], \tag{3}$$

and that

$$\begin{cases} \text{if } c \in [a_i, b_i] \cap [a_j, b_j], \quad j \neq i, \\ \text{then } c \text{ is end-point on both segments.} \end{cases} \tag{4}$$

If all segments in (2) are points, the hypothesis on the complements is not satisfied, hence we suppose $b_1 \neq a_1$, changing the notations, if necessary.

We find now the longest sequence of segments

$$[p_i, p_{i+1}], \quad i = 1, \cdots, k-1, \quad ([p_1, p_2] = [a_1, b_1]), \tag{5}$$

where each $[p_i, p_{i+1}]$ is an $[a_j, b_j]$ and all the endpoints $p_1, \cdots, p_k$ are different. (By our conventions, $[a, b] = [b, a]$ is a set.) Such a sequence (5) exists, although $k = 2$ is not excluded at this point. Two cases are now possible:

(a) The point p_k belongs to the segment $[p_{k-1}, p_k] = [a_i, b_i]$, and to no other (non-degenerate) segment in (2); thus (5) cannot be extended for this reason.

(b) There are segments $[a_i, b_i]$ in (2) and not in (5) such that p_k is a_i (or b_i), but the other endpoint b_i (or a_i, respectively) is one of the points $p_1, \cdots, p_{k-2}$; thus (5) cannot be extended keeping all endpoints different.

Let us prove that (a) is impossible. As p_k belongs to the segment $[p_{k-1}, p_k]$ only, there is an open disc D centered to p_k which does not intersect the other segments in (2). Then $D - C = D - [p_{k-1}, p_k]$ is an open disc less a radius, thus a connected open set containing points both from the inside and from the outside of C, as p_k is a boundary point for both. Furthermore, the inside and the outside of C disect this set, which is a contradiction. As we proved that (a) is impossible, we know now that (b) holds true.

Using the notations of (b), suppose $[a_i, b_i] = [p_k, p_l]$ with $1 \leqq l \leqq k-2$. Define

$$C' = \bigcup_{j=l}^{k-1} [p_j, p_{j+1}] \cup [p_k, p_l].$$

Then C' is a polygonal Jordan curve contained in C. We want to prove $C' = C$. Suppose the contrary, and select q in $C - C'$. There exists then a point $r \notin C$ separated from q by C'. Then any path connecting r to q intersects C', and hence C, in a point $\neq q$, therefore the boundary point q is not accessible from that component of the complement of C that contains r. This contradiction shows $C' = C$; hence the proof of Theorem 2 is complete.

This is not the place, of course, to report on the proof of Theorem 1. However, the main idea of the Hilbert-Riesz proof can be stated in a few words as follows. Given a set C as in Theorem 1, for any two points a, b in C, we connect a to b in the inside of C and also in the outside of C, obtaining thereby a Jordan curve J. We say that two points u, v in C are separated on C by a, b, if J separates u, v. This definition can be then justified by showing that it is independent of J. Finally, for this relation of "separation of points on C" the axioms of cyclic order can be proved. From this a homeomorphism of C onto S^1 can be constructed.

The referee pointed out to us that a more recent (and more common) approach to prove Theorem 1 is to be found in M. H. A. Newman, *Elements of the topology of plane sets of points*, Cambridge (1954), p. 166. In this proof it is shown first that C is connected. Using the Jordan curve theorem for polygons, this is not too difficult. Then, in the notations of the previous paragraph, it is clear that $C - \{a, b\}$ meets both components of the complement of J, hence it is not connected. Now it is a standard theorem that a continuum (connected, compact, separable, metric space), whose connection is destroyed by the removal of any two points, is homeomorphic to a circle.

The first named author gave an upper-division course (Math. 132, Topics in Geometry, Spring, 1972) on elementary point set topology, and suggested that the students may try to prove Theorem 2 directly. The second named author attending the course gave the proof above.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA, BERKELEY, CA 94720.

THE USE OF THE MONODROMY THEOREM AND ENTIRE FUNCTIONS WITH NONVANISHING DERIVATIVE

DAVID STYER AND C. D. MINDA

1. Introduction. Let $\mathscr{E}$ denote the set of all entire functions f such that f' has no zeros in the complex plane $\mathbb{C}$. Clearly $\mathscr{E}$ contains no constant functions so that Picard's little theorem [5, p. 324] guarantees that for any $f \in \mathscr{E}$ the image of $\mathbb{C}$ under f is either $\mathbb{C}$ or $\mathbb{C}$ less exactly one point. This provides a natural partition of $\mathscr{E}$ into two subsets $\mathscr{F}$ and $\mathscr{G}$. $\mathscr{F}$ contains those functions in $\mathscr{E}$ whose image omits precisely one point, while $\mathscr{G}$ consists of all functions in $\mathscr{E}$ which map $\mathbb{C}$ onto itself. Let $\mathscr{F}_0$ be the subfamily of $\mathscr{F}$ containing those functions which omit the origin. In the recent book [3, p. 99] by Krzyż it is claimed that (1) $g \in \mathscr{G}$ implies that g is a similarity transformation; that is, $g(z) = az + b$, where $a, b \in \mathbb{C}$ and $a \neq 0$, and (2) $f \in \mathscr{F}_0$ guarantees that $f(z) = \exp(az + b)$ for some $a, b \in \mathbb{C}$ with $a \neq 0$. The converse of each of these assertions is trivial to establish. Claim (2) is obtained as a consequence of (1). The "proof" of these results is intended to illustrate the idea of analytic continuation and the use of the monodromy theorem. Unfortunately, both of them are false. First we present counterexamples to both assertions. The counterexample for (1)

makes the flaw in the proof of (1) obvious. A closer analysis of the situation permits us to modify the problems and obtain correct formulations.

2. Counterexamples. It is a simple matter to produce a function in $\mathscr{F}_0$ which is not of the form $\exp(az + b)$. It is elementary to verify that $f = \exp \circ \exp$ belongs to $\mathscr{F}_0$ and is not of the prescribed form.

A little more effort is required to exhibit a function in $\mathscr{G}$ which is not a similarity transformation. Set

$$h(z) = \sum_{n=1}^{\infty} \frac{z^n}{n \cdot n!} = \int_0^z \frac{\exp(\zeta) - 1}{\zeta} d\zeta;$$

then h is a transcendental entire function and $1 + zh'(z) = \exp(z)$. The function $g(z) = z\exp(h(z))$ is also a transcendental entire function and a simple computation shows that $g'(z) = \exp[z + h(z)]$. In particular, g' never vanishes. Because g has an essential singularity at ∞, Picard's big theorem [**1**, p. 203] assures us that g assumes every finite complex value infinitely often with at most one exception. Since g takes on the value 0 exactly once, every nonzero complex number is assumed infinitely often by g. Therefore, $g(\mathbb{C}) = \mathbb{C}$ so that $g \in \mathscr{G}$. Evidently, g is not a similarity.

The editor (H. F.) brought to our attention the following attractive example of another function in $\mathscr{G}$ which is not a similarity; he attributed the example to E. Calabi. Let $g(z) = \int_0^z \exp(-\zeta^2) d\zeta$, then $g'(z) = \exp(-z^2)$ never vanishes, $g(0) = 0$ and g is an odd function. If g omits the value w_0, then $w_0 \neq 0$ and by Picard's theorem g cannot omit $-w_0$. But $g(z_0) = -w_0$ implies $g(-z_0) = w_0$, a contradiction. Thus, $g \in \mathscr{G}$ and g is clearly not a similarity.

3. Misuse of the monodromy theorem. In order to pinpoint the error in Krzyż' proof of (1) it is necessary to paraphrase his argument. Suppose $g \in \mathscr{G}$. Since $g(\mathbb{C}) = \mathbb{C}$ and g' never vanishes, any fixed branch of g^{-1} can be continued arbitrarily in $\mathbb{C}$. Since $\mathbb{C}$ is simply connected, the monodromy theorem implies that a single-valued inverse function g^{-1} is defined on $\mathbb{C}$. Hence, g is a univalent entire function and must be a similarity.

The flaw in this argument is apparent for the function $g \in \mathscr{G}$ constructed above in 2. The Riemann surface of g has one sheet over the origin and an infinite number of sheets over all nonzero complex values. Therefore there are many paths through the origin, along which a given branch of g^{-1} cannot be continued. However, given two points in $\mathbb{C}$ there is a path joining them, along which a given branch of g^{-1} may be extended. The monodromy theorem [**4**, p. 217] requires that a function element be continuable along every path in a simply connected region. For a related example see [**2**, p. 49]. As a matter of fact one can show that if Ω is a region in $\mathbb{C}$, f is holomorphic in Ω and f' does not vanish in Ω, then it is not generally true that a branch of f^{-1} can be continued along an arbitrary curve in $f(\Omega)$.

A different perspective on the error in Krzyż' reasoning may be obtained from the viewpoint of the theory of covering surfaces [**6**, pp. 76–79]. The Riemann sur-

face for g^{-1} is simply an example of a smooth, but not unlimited, covering surface of $\mathbb{C}$ with projection equal to $\mathbb{C}$. Krzyż seems to assume that every smooth covering surface is unlimited.

4. Correct formulations. First we recall the definition of an asymptotic value of an entire function. A path is a continuous function $\gamma: [0,1) \to \mathbb{C}$. Suppose f is an entire function. If there is a path γ such that $\lim_{t\uparrow 1}\gamma(t) = \infty$ and a complex number α such that $\lim_{t\uparrow 1} f(\gamma(t)) = \alpha$, then α is called an asymptotic value for f. Now we show that for $g \in \mathscr{E}$ every singular point of g^{-1} corresponds to an asymptotic value of g.

PROPOSITION. *Suppose g is an entire function with nonvanishing derivative. Given $a \in \mathbb{C}$ set $b = g(a)$. Let f_a denote the branch of g^{-1} which is defined in a neighborhood of b and satisfies $f_a(b) = a$. The radius of convergence of the Taylor series expansion of f_a about b is designated by r_a.*

(i) *If $r_a = \infty$, then g is a similarity.*

(ii) *If $r_a < \infty$, then every singular point on the circle of convergence $\{w: |w - b| = r_a\}$ of f_a is an asymptotic value of g.*

Proof. (i) If $r_a = \infty$, then f_a is an entire function which is the inverse function for g. Consequently g is univalent. Since g is a univalent entire function, g is a similarity.

(ii) Let ω be a singular point of f_a on the circle of convergence and $D = \{w: |w - b| < r_a\}$. Initially we will show that

$$(*) \qquad \lim_{\substack{w\to\omega \\ w\in D}} f_a(w) = \infty .$$

If (*) failed to hold, then there would be a sequence $(w_n)_{n=0}^{\infty}$ in D with $w_n \to \omega$ and $z_n = f_a(w_n) \to \zeta \in \mathbb{C}$. Thus, $g(\zeta) = \omega$. The local inverses f_a and f_ζ agree at each point w_n for all n sufficiently large. Select one such w_n. Then both f_a and f_ζ are defined in a neighborhood of w_n and are inverses for the restriction of g to a neighborhood of z_n. Therefore f_a and f_ζ coincide in a neighborhood of w_n and so are equal in $D \cap \{w: |w - \omega| < r_\zeta\}$. But this means that f_ζ is a direct analytic continuation of f_a to a neighborhood of ω which contradicts the fact that ω is a singular point for f_a. Hence, (*) holds. Let Γ be any path in D with $\lim_{t\uparrow 1}\Gamma(t) = \omega$, then $\gamma = f_a \circ \Gamma$ is a path in $\mathbb{C}$, $\lim_{t\uparrow 1}\gamma(t) = \infty$ and $\lim_{t\uparrow 1} g(\gamma(t)) = \lim_{t\uparrow 1}\Gamma(t) = \omega$. This shows that ω is an asymptotic value for g.

COROLLARY 1. *If g is an entire function with nonvanishing derivative, then either g is a similarity or g has at least one finite asymptotic value.*

Proof. This is just a restatement of the conclusion of the proposition.

COROLLARY 2. *If g is an entire function with nonvanishing derivative and $g(\mathbb{C}) = \mathbb{C}$, then either g is a similarity or g has at least two finite asymptotic values.*

Proof. Suppose that $g \in \mathscr{G}$ is not a similarity, then Corollary 1 implies that g has at least one finite asymptotic value, say α. Since $g(\mathbb{C}) = \mathbb{C}$, there is an $a \in \mathbb{C}$ with $g(a) = \alpha$. Let f_a be as in the proposition. Since g is not a similarity, $0 < r_a < \infty$ and there is a singular point of f_a on the circle $\{w: |w - \alpha| = r_a\}$. This singular point is a second finite asymptotic value for g.

Now we are in a position to characterize similarities and the composition of the exponential function with similarities.

PROPOSITION. (i) *An entire function g with a nonvanishing derivative is a similarity if and only if g has no finite asymptotic value.*

(ii) *An entire function f with nonvanishing derivative has the single finite asymptotic value α if and only if $f(z) = \alpha + \exp(az + b)$ where $a, b \in \mathbb{C}$ and $a \neq 0$.*

Proof. (i) The necessity is trivial while the sufficiency follows directly from Corollary 1.

(ii) The necessity is readily established. Suppose that $f \in \mathscr{E}$ and α is the only finite asymptotic value for f. Since $f \in \mathscr{G}$ implies that f is either a similarity or f has at least two finite asymptotic values, we conclude that $f \in \mathscr{F}$. Thus, f omits some complex value. But every value omitted by an entire function is an asymptotic value [**5**, p. 251], so it follows that f must omit α. Then $f - \alpha$ has no zeros so that $f - \alpha = \exp \circ h$ for some entire function h. From $f' = h' \cdot (\exp \circ h)$ and the fact that f' does not vanish, we see that h' has no zeros. If h is a similarity, then we are finished. Otherwise, h has a finite asymptotic value, say A, because $h \in \mathscr{E}$. In this situation $\alpha + \exp(A) \neq \alpha$ is also a finite asymptotic value for f. This is impossible.

Note. Recently, an article by L. Zalcman (this MONTHLY 81 (1974) 115–137) appeared which contains material closely related to our sections 2 and 3; in particular, see section 13 of his article.

References

1. C. Carathéodory, Theory of Functions of a Complex Variable, vol. 2, 2nd ed., Chelsea, New York, 1960.

2. H. Cohn, Conformal Mapping on Riemann Surfaces, McGraw-Hill, New York, 1967.

3. J. G. Krzyż, Problems in Complex Variable Theory, no. 36 Modern Analytic and Computational Methods in Science and Mathematics, American Elsevier, New York, 1971.

4. R. Nevanlinna and V. Paatero, Introduction to Complex Analysis, Addison-Wesley, Reading, Mass., 1969.

5. W. Rudin, Real and Complex Analysis, McGraw-Hill, New York, 1966.

6. G. Springer, Introduction to Riemann Surfaces, Addison-Wesley, Reading, Mass., 1957.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CINCINNATI, CINCINNATI, OHIO 45221.

A MATTER OF DEFINITION

M. C. MITCHELMORE

The investigation to be described below arose out of the following problem*.

PROBLEM. *Find all positive values of x for which*

$$x^{x^{x^{\cdot^{\cdot^{\cdot}}}}} = 2,$$

where the x's continue to infinity.

To save the printer any further headaches, we shall write t for such an "infinite tower" of x's.

Several students solved this problem as follows:

$$\begin{aligned} t &= 2 \\ \Rightarrow x^t &= 2 \\ \Rightarrow x^2 &= 2 \\ \Rightarrow x &= \sqrt{2}. \end{aligned}$$

However, none of them questioned whether an infinite tower of $\sqrt{2}$'s really was equal to 2. In the subsequent discussion, it soon became clear where the real problem lay: We had no definition of the value of an infinite tower of x's, only a vague intuition. We tried two ways to eliminate this shortcoming.

First approach. The only property used in the solution of $t = 2$ above is

(1) $$x^t = t.$$

This property derives from the standard convention that finite towers of exponents

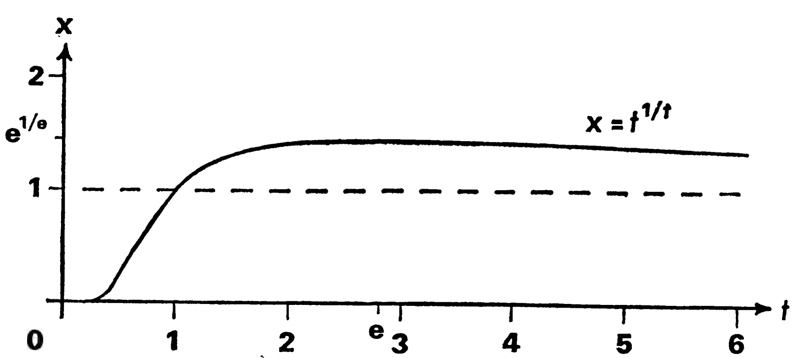


FIG. 1

* See Exercise 12–7 on p. 383 of Apostol's *Mathematical Analysis*, Blaisdell, N. Y. 1961. [ED]

are evaluated "from the top down," a convention which we decided to maintain for infinite towers. (The reader is invited to define infinite towers for other evaluation conventions.) Equation (1) is equivalent to

$$x = t^{1/t}. \tag{2}$$

A graph of this relation is shown in Fig. 1. The derivative at the point (t,x) is $t^{-2+1/t}(1 - \ln t)$. The graph is therefore horizontal at the origin, rises monotonically to a maximum at $(e, e^{1/e})$ $[\approx (2.718, 1.445)]$, and descends asymptotically to $x = 1$ as $t \to \infty$. To sketch this graph required little more than a knowledge of the limits of $(\ln y)/y$ as $y \to 0+$ and as $y \to \infty$.

We saw immediately that t cannot be defined when $x > e^{1/e}$; that t is uniquely defined when $0 \leqq x \leqq 1$ and when $x = e^{1/e}$; and that there are two values of t when $1 < x < e^{1/e}$. For example, $1 < \sqrt{2} < e^{1/e}$ and $\sqrt{2} = t^{1/t}$ has the two solutions $t = 2$ and $t = 4$. Which of these is the value of an infinite tower of $\sqrt{2}$'s? We seemed to be no closer to finding a definition than when we started.

Second approach. Our next idea was to try regarding t as the limit of the sequence

$$x,\ x^x,\ x^{x^x},\ x^{x^{x^x}}, \cdots.$$

Writing t_n for the nth term of this sequence, we have $t_1 = x$ and

$$t_{n+1} = x^{t_n}.$$

If t_n tends to a limit as $n \to \infty$, then this limit is a solution for t in equation (1). Perhaps we could obtain a unique value for t by defining it by this limit.

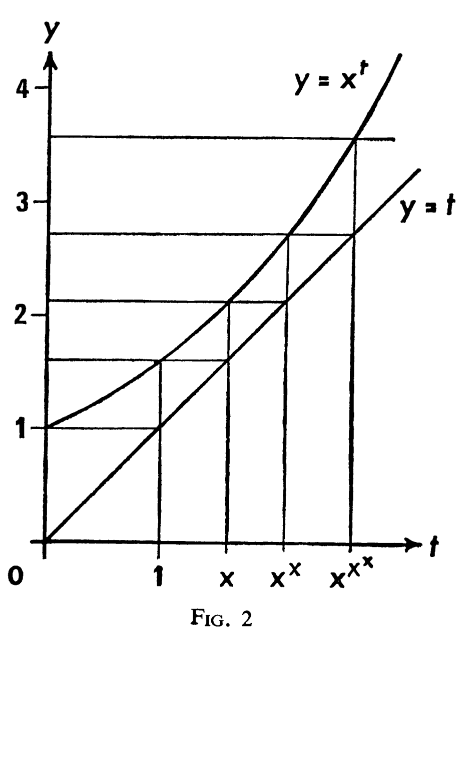


FIG. 2

To find when (t_n) converges, we drew graphs of $y = x^t$ and $y = t$ on the same axes. Fig. 2 and Fig. 3 show two possibilities. In both cases, the zigzag starting at (x,x) gives the successive values of t_n on the axes. In Fig. 2, t_n increases without limit, whereas in Fig. 3, t_n converges to the t-coordinate of the "lower" point of intersection. The convergence of the sequence depends on whether the two curves intersect, which they do if and only if $x^t = t$ has a real solution for t. The t-coordinates of the two points of intersection are therefore given by the two real solutions for t in equation (1).

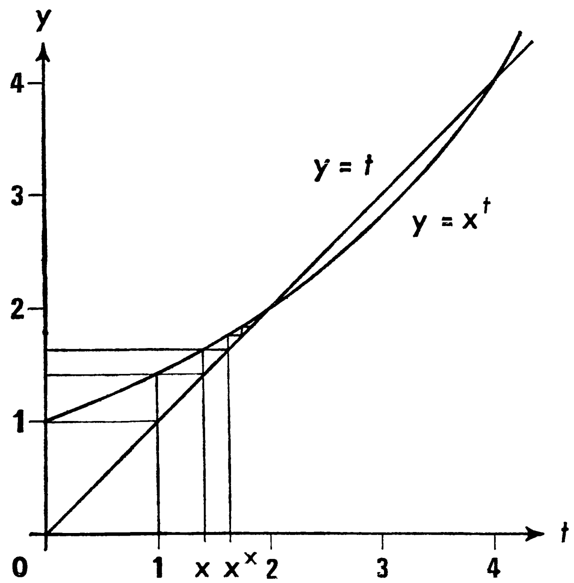


FIG. 3

This showed clearly which of the two possibilities should be chosen when $1 < x < e^{1/e}$. It can be checked rigorously that the zigzag always converges to the point with the smaller coordinates. Thus the value of an infinite tower of $\sqrt{2}$'s is 2, not 4.

Crisis and resolution. We were just about to adopt the definition of t as the limit of t_n, where it exists, when the question was raised: "What happens when $0 < x < 1$?" This did not seem troublesome, because equation (1) has a unique solution in this range; but it was as well to check. When $0 < x < 1$, the graphs of $y = x^t$ and $y = t$ intersect at this solution, as shown in Fig. 4. The sequence (t_n) is now given by a spiral instead of a zigzag.

Put this way, it ceased to be obvious whether (t_n) converges. We found that the gradient of the graph of $y = x^t$ at the point where it intersects $y = t$ is $\ln t$, and this is less than -1 if and only if $x < e^{-e}$ (≈ 0.066). The sequence (t_n) therefore

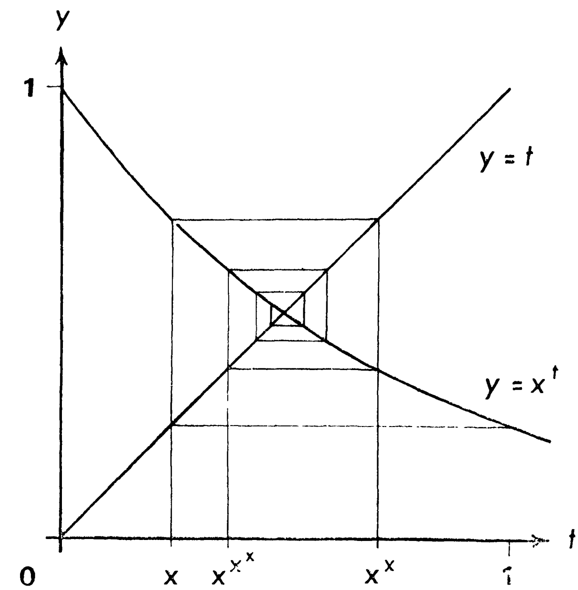


FIG. 4

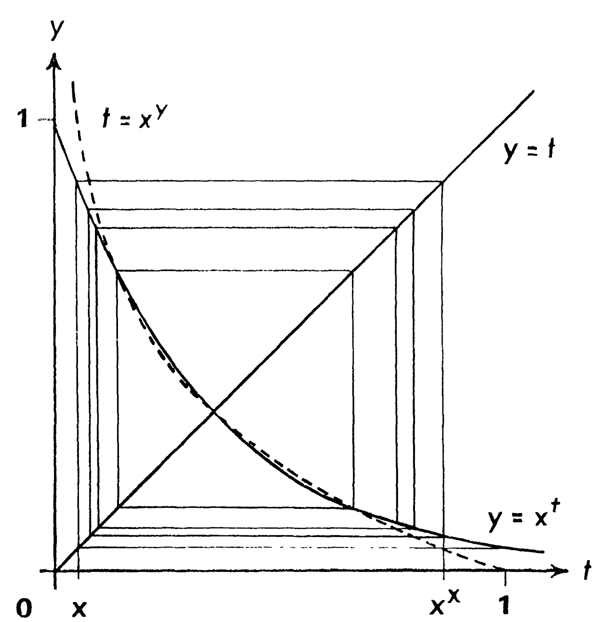


FIG. 5

does not converge when $x < e^{-e}$, even though equation (2) has a unique solution in this interval. (In fact, the sequence now has two limit points, given by the intersections of $y = x^t$ and $t = x^y$ which do not lie on $y = t$. The solution to equation (2) is given by the intersection on $y = t$. This is shown in Fig. 5.) So our second approach also did not work for all values of x.

We were left with the following resolution.

DEFINITION. For all real x such that $0 \leqq x \leqq e^{1/e}$, $x^{x^{x^{\cdot^{\cdot^{\cdot}}}}}$ is the real solution for t of the equation $x^t = t$, and in case this equation has two solutions, then the lesser one.

Conclusion. Here was an investigation with apparently innocuous beginnings but which ranged widely over several concepts and techniques of elementary analysis. It differs from most problems in having a definition as the end-point instead of the starting point. Perhaps problems like this could be used more liberally in college courses as an antidote to the type of mental paralysis which sometimes results from over-exposure to the more dogmatic aspects of the standard expository method.

Acknowledgements. I am grateful to my former colleague, Dr. Petr Liebl, now at Charles University, Prague, for the part he played in the discussion, and for his several constructive criticisms of the first draft of this note; also to my referee for suggesting the inclusion of Fig. 5 and for making me draw the graphs accurately.

DEPARTMENT OF MATHEMATICS, MICO COLLEGE, KINGSTON 5, JAMAICA.

FROM EULER'S FORMULA TO PICK'S FORMULA USING AN EDGE THEOREM

W. W. FUNKENBUSCH

1. Edge Theorem. Given a simple polygon [**1**] form a complete triangularization to obtain a figure which will be a plane graph [**2**]. The number of edges is given by $E = 3I + 2B - 3$, where E is the number of edges, B is the number of boundary vertices, and I is the number of interior vertices.

Proof. Give the completely triangularized plane graph:

(i) If $B = 3$, $I = 0$, then $E = 3$. Check!

(ii) If a new interior vertex is inserted, E is increased by 3. Check!

(iii) If a new boundary vertex is added which results in x old boundary vertices becoming interior vertices, (note: x may be zero) it is obvious that E is increased by $2 + x$. Here we have

$$E + 2 + x = 3(I + x) + 2(B + 1 - x) - 3$$

or

$$E = 3I + 2B - 3. \quad \text{Check!}$$

All possible cases are covered and the theorem is established.

2. Euler's Formula. $V - E + F = 2$ is satisfied for any connected planar graph where V is the number of vertices, E the number of edges, and F the number of regions (faces), one of the regions being the infinite region [**2**].

3. Derivation of Pick's Formula. Let the polygon, given in the Edge Theorem, have lattice points for vertices and let Euler's Formula be applied to the related connected planar graph. We then obtain

$$V = I + B$$

$$E = 3I + 2B - 3$$

$$\frac{F-1}{2} = \text{area of polygon}$$

which when substituted into Euler's Formula gives Pick's Formula, namely that the area of any simple polygon whose vertices are lattice points is given by

$$\text{area} = I + \frac{B}{2} - 1,$$

where B is the number of lattice points on the boundary while I is the number of lattice points inside.

References

1. H. S. M. Coxeter, Introduction to Geometry, Wiley, New York, 1961.
2. C. L. Liu, Introduction to Combinatorial Mathematics, McGraw-Hill, New York, 1968.

DEPARTMENT OF MATHEMATICS, MICHIGAN TECHNOLOGICAL UNIVERSITY, HOUGHTON, MI 49931.

MATHEMATICAL EDUCATION

EDITED BY SHIRLEY HILL AND PAUL MIELKE

Material for this Department should be sent to Shirley Hill, Department of Mathematics, University of Missouri, Kansas City, MO 64110, or to Paul Mielke, Department of Mathematics, Wabash College, Crawfordsville, IN 47933.

MATHEMATICS FOR THE "DISADVANTAGED"

EDDIE R. WILLIAMS, Northern Illinois University

The fact is clear that over the past ten years there has appeared on the University and two-year college scene a "different" student. This student is so different and his number has grown so rapidly, that a special term was created to describe him better.

Consequently on many campuses there are the following student classifications: freshman, sophomore, junior, senior, and "disadvantaged."

The disadvantaged student classification has an interesting history, which need not be developed here. However, it is significant to describe the typical student placed in this classification. The disadvantaged student usually is a student from an inner-city high school, located in some ghetto community. This certainly is a distinguishing fact in itself. Yet, the key here is that in these inner-city high schools, the quality of teachers, teaching facilities, teaching aids, and other such vital elements needed for a good high school education, are inferior to those available at high schools outside this community.

Many more general applications of the term "disadvantaged" student describe him as a student who is deprived culturally or educationally of the same opportunities as other students through no fault of his own. He is that student who, upon his enrollment in high school, is already in many cases two years behind students at other high schools. The emphasis here should be on the phrase, "through no fault of his own," since this phrase indicates that educators in general have failed to educate the necessary elementary and secondary school teachers, and to develop techniques that would establish equal opportunity in education. Further, federal and local governments in many cases have not provided sufficient funds for adequate programs.

In recent years, the two-year and four-year colleges have been required to address the needs of the disadvantaged student. Special programs have appeared overnight. Courses have been developed, and the number of disadvantaged students has significantly increased. (The number of disadvantaged students at Northern Illinois University, for example, has tripled in the last three years, from 280 students in the Fall 1969 to close to 1000 students in the Fall 1972.) In fact, almost every educational institution is faced (or will be faced) with the problem of offering courses for these students. In the past many of the programs and courses that were offered turned out to be empty "fronts" to appease the outside pressures and to attract government funds. In fact, this is unfortunately still the case in many institutions which have not been allowed to completely ignore the situation altogether.

I would like to suggest some basic guidelines as to what I believe a mathematics course for the disadvantaged student should offer, and to present a workable and realistic approach for organizing such a course. The observations made here are based upon results obtained in more than three years' experience in developing, teaching, and refining such a mathematics course.

The most important aspect of organizing and operating such a mathematics course is to face the task in realistic terms. In order to accomplish this end, answers to the following questions are needed:

I. Who are the students to be served by the course? How are these students identified? Where do they come from? (What is their background?)

II. What are their mathematical needs? What should be the objectives of the course?

III. What realistic bounds can be set for the course to assure meeting these needs and objectives, and yet remain in the possible scope of the resources (i.e., faculty assistants, space, funds) available.

IV. What are possible methods for facilitating the learning process in such a course?

I. The students are mainly black, Spanish, and other minorities. They come from inner-city areas, and, in fact, thereby have developed negative attitudes toward mathematics. The theory here is simple. The overall teaching quality in many inner-city schools is very low. Hence classes in the more precise areas, such as general science and mathematics, are usually taught very poorly. This in turn discourages the student from doing work beyond the basic requirements in these areas. This is supported by the fact that minorities generally avoid the sciences and mathematics for a college major or a field of endeavor.

Any course that is constructed must offer special considerations to these negative attitudes. If not, *no matter how well the course is constructed, the benefits to the student will be minimal*, The student enters the course with the pre-determined attitude, "I can't handle mathematics no matter what." He must be convinced that he can succeed in a mathematics course. One way that this can be done is by conducting the course on the level of the individual student. Thus effort on the individual level and reinforcement must be required of both the student and the instructor. This means that some form of individualized attention must be available, and in fact forced on the student. The combination of this individualized attention in conjunction with patient reinforcement will provide the instructor with a powerful tool which will be useful in convincing the student that success is indeed possible.

II. The mathematical needs of these students are very basic. Yet it is at this particular point where many present courses fail. There is no need to establish elaborate theories in mathematics education, or to develop elaborate materials or trick techniques. *These students need basic mathematics*, presented in plain English. (The reader should see the report, *A Course in Basic Mathematics for Colleges*, published by CUPM, P. O. Box 1024, Berkeley, California 94701.) Fractions, percentages, basic algebra head the list. On the other hand, however, it should be carefully noted that the students are not dumb or hard learners. This misconception of labelling the students as dumb and hard learners has caused the development of materials designed for five year olds. It is insulting, and soon turns the student off. In fact, a course such as is proposed here can be taught with some degree of sophistication. The fact is that the majority of the students are in need of basic mathematics because they have been poorly trained. Thus from my experience, I have found the students to be quick to learn when placed in the type of individualized system that is suggested here.

The course should be centered around the goal of providing the student with a working knowledge of mathematics. In light of this goal, I believe that the course objective should be two-fold:

(a) To provide the student with basic mathematics necessary for practical applications (e.g., percentages, fractions, word problems taken from everyday experiences).

(b) To provide the student with a base from which he can step into the normal sequence of courses offered by the given mathematics department. Here topics must be included in the course so that, upon completion of the course, the student is ready to face the next level of courses without entering at a major disadvantage.

The total amount of material covered is naturally limited by time. However, it is my contention (based on a follow-up analysis of students taking such a course; see Table 6) that the student can be brought to a level of achievement whereby he can enter higher level courses without being hopelessly lost.

III. The possibilities of what type of course can be offered naturally depends on the institution. However, a department, in order to avoid a mere token course, must be willing to make certain commitments. First, such a course should not be labelled by the department as a booby-prize, that is, the course every instructor avoids. It is clear that such a course is not attractive in the topics discussed, and that teaching a course of this nature requires many additional hours of work. But it is also clear that only a very special type of instructor will be successful in teaching such a course. In particular, only an instructor who is sensitive to students, concerned with the problems of the disadvantaged, and who is skilled and patient in lecture, will have a chance of succeeding.

Departments must, therefore, place their best teachers on the line, and thus must offer some form of compensation for the additional work involved. (For example, reduced class load, additional student assistants, increased salary.) No other alternative is possible. Poor teaching is usually not corrected by offering the student a course taught by the weakest teachers available.

One grave misconception in forming such a course is the common belief that the teacher must identify directly with the students — a black teacher for black students, a Spanish teacher for Spanish students, etc. In fact, many departments argue against having such a course because the department does not have an instructor of the same ethnic class as the majority of the students. It has been my experience here at Northern Illinois University that such direct identification is completely unnecessary. White teachers have been quite successful at teaching sections of the course which were up to 90% black. (Such success has been measured in many ways including student response and performance on standardized exams, and through direct comparisons with sections taught by black instructors.) Neither is it important for the teacher to assume some false character - trying to be an ultra-liberal who has an understanding heart for the "poor ghetto child." In my opinion, the key to success lies in the instructor's ability to convince the student that he offers the student a course which exhibits good teaching, individualized assistance, useful and significant material, and most important, that he offers the student a course in which the student can be successful.

These conditions are established through a well-planned and structured course, and through an instructor whose primary objective and concern is to reach each student in his course at all cost. Acting has nothing to do with it.

IV. The ultimate, of course, is never possible, but I would like to offer a description of Mathematics 120, the special course now being offered at Northern Illinois University, as at least one realistic approach to this question.

Mathematics 120 is a one-semester, three-credit hour course. At present there are two sections of the course involving approximately 85 students each. Of this number, approximately 80% of the students in the course are identified through a special program known as CHANCE (Complete Help and Assistance Necessary for a College Education). The CHANCE program offers special admission to Northern Illinois University for disadvantaged students mainly from inner-city ghetto areas. Consequently through the CHANCE program as a screening device, students needing Mathematics 120 could be easily identified. Table 1 gives an overview of the educational background of students taking Mathematics 120. Note the weak performance of these students on the national standardized exam (ACT).

TABLE 1 — Background Data on Math 120 Students

Semester Course Taken	Number of students enrolled in Math 120	Number of students from Inner-City High Schools	Average number of years of High School Math beginning with Algebra*		Average Mathematics A.C.T. Score**	
			Inner-City Students	Other	Inner-City Students	Other
Fall 1970	28	26	1.23	1.41	15.4	15.4
Spring 1971	83	71	.876	1.08	13.6	13.6
Fall 1971	151	139	.815	.923	13.2	13.4
Spring 1972	166	152	.764	.729	13.1	13.2

(*) Some high schools offer a course in "General Math" which is below the level of the normal beginning high school Algebra course.
(**) The national mean for the Mathematics section of the A. C. T. for college-bound students has been 19.1 over the years 1968–1971. Normal college admissions standards require a mathematics A. C. T. score of 19.0 in addition to a high class rating.

In addition to screening students, the CHANCE program offers a direct means of feedback on the results of the course. A committee is assigned to each CHANCE student. The counselor works with the student throughout his collegiate career. Thus the overall effects of the course on each student could be monitored very closely. Indeed, it has been this close tie between counselor and student that has allowed the necessary adjustments to be made to the course. The counselors can compile a con-

tinuous flow of data relative to the shortcomings of the course, particular student problems, adequacy of the topics discussed for student usage in other fields, and overall student performance in other mathematics courses. This direct feedback has been crucial in order to properly gear the course to the needs of the student.

The three one-hour course lectures are given through a series of carefully prepared modules of programmed material. The goal is to maximize student's understanding of the lecture material. Using the avenue of student participation in lecture, this maximization of understanding is achieved as follows. When the student enters the lecture hall he receives a number of prepared hand-out sheets. The instructor then places transparencies of these same sheets on an overhead projector, and lectures directly from these pages. In this way, the instructor and the student work on each page together, thus allowing the student to concentrate on the details of the lecture rather than taking extensive notes. The lecture hand-out sheets are arranged with examples and fill-in blanks so that the instructor can at any time require the students to respond to questions or to fill in the blanks on the lecture sheets. The form of lecture format keeps the student's attention, allows maximal time for student concentration, and allows the instructor to cover a larger amount of material more efficiently than the conventional straight lecture method.

In addition to the three lectures per week, the course format also includes two one-hour periods of required laboratory. It is my firm conviction that these laboratories are the true learning centers of the course. The laboratories offer the opportunity for individualized attention. The student must perform to convince his instructor that he has mastered the topic of the day. The instructor, on the other hand, takes this opportunity to get to know his students and their special problems. In fact, the instructor must use this time to convey to the student that he is sincerely interested in the student, and that his goal as instructor is to have every student be successful in his course.

The first laboratory period is conducted in large group sessions with all students present. The students are given a laboratory problem assignment sheet which must be completed during the laboratory period. The emphasis is on performance at the individual level. Students enter the laboratory and immediately begin working the problem sheet. The instructor, aided by assistants, then monitors the students' performance, allowing students to ask questions or simply have their work checked.

The laboratory assignment is not graded in the usual sense. The student is aware that the laboratory period is an opportunity for him to check his abilities without penalty. This knowledge, in conjunction with the freedom to ask questions and to receive individual attention, allows the student to work freely and effectively during the laboratory period. Since the emphasis is on the individual student exhibiting a command of the subject matter, the student is not allowed to leave until he has exhibited some achievement during the period. This achievement can be in any form, from completion of the assignment to simply gaining a full understanding of the material.

The second laboratory period is conducted in small group sessions of approximately fifteen to twenty students. These sessions are conducted by undergraduate assistants. (Although assistants having a strong mathematics background would be preferred, I have been successful using undergraduate students who have had this basic course and perhaps one additional course. If for no other reason, such assistants present themselves to their students as "living examples" of the possibility of success in the course, thereby offering an extra push behind the students that the instructor cannot normally provide.) The purpose of these sessions is to give the student an opportunity to ask questions on lecture materials and homework problems. In addition, the student is then required to work additional problems at the blackboard,

TABLE 2 — EXPERIMENTAL MATHEMATICS, MATH 120: Course Outline

0. *Fractions and Percentages*

I. *Set Theory*
   1. Definition of set: $A \cup B, A \cap B, A - B, \tilde{A}, A \subset B, A \subseteq B$
   2. Special symbols: $\forall, \exists, \ni, \in$
   3. Set notation
   4. Multiples (e.g., $\{x \in \mathbb{R} \mid x = 3k$ for some $k\}$)

II. *Real Number System*
   1. Definition of C, I, Q, Irr, R
   2. Laws for the Real Number System (commutative, associative, distributive, cancellation)
   3. Absolute value, inequalities, signed numbers
   4. Number line (graphs of inequalities and sets)

III. *Functions and Relations*
   1. Definition of relation, function, domain, range
   2. Evaluation of functions at points (e.g., $f(x) = 2x, f(3) = 6$)
   3. Graphs (Description of the Cartesian Coordinate Plane, plotting points, graphs of functions)

IV. *Polynomials*
   1. Degree of a polynomial
   2. Arithmetic operations on polynomials ($+, \cdot, -, \div$)
   3. Pascal's triangle, Binomial Theorem

V. *Solutions of Equations*
   1. Definition of equation, solution, solution set
   2. Rules for manipulating equations
   3. Solutions of linear polynomial equations (word problems included)
   4. Solutions of quadratic equations (word problems included)
      a. Via factoring
      b. Quadratic Formula

VI. *Geometry*
   1. Definitions: angle, perimeter, area
   2. Recognition of various geometric figures
   3. Calculation of areas
   4. Distance formula, Pythagorean Theorem

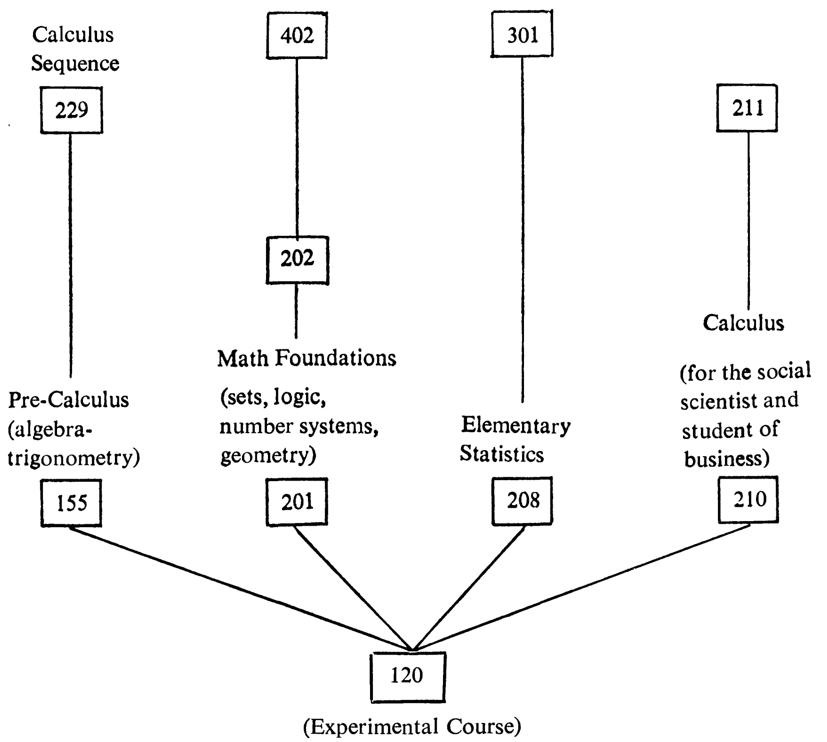
and then explain his solutions to the class. This activity assures the student of his own abilities and presents him with an opportunity to display what he has learned.

In all aspects of the course, the emphasis is on the individual. If the student is unable to perform (in lecture, in laboratory, or on examinations) then the reason for this failure must be explôred and rectified. Thus the course format is an attempt to reach the individual at his own level, providing him with opportunities to learn the material, testing his knowledge and providing him with immediate reinforcement.

The objectives of Mathematics 120 are those mentioned earlier. These objectives

TABLE 3

Division 1	*Division* 2	*Division* 3	*Division* 4
Mathematics (Pure, Applied, Education, Statistics, Computer Science) Physics Chemistry Biology	Elementary Education, Special Education	Sociology Psychology Political Science Economics Geography	Business Psychology Social Science Political Science



The diagram illustrates the four divisions of the undergraduate mathematics program of Northern Illinois University with a list of majors taking each division. Note how certain topics discussed in the beginning courses of each division can be traced to Math 120.

actually dictate the topics to be discussed in the course. Table 2 gives the course outline. The outline indicates that the fundamental topics are covered, in addition to various other topics. These additional topics (e.g., Set Theory, Real Number System) have more relevance to students who will be taking additional mathematics courses. In conjunction with this, note that Table 3 gives an indication of how the topics discussed in Mathematics 120 form a basis from which the student can move into any of the first year courses offered. Table 4 gives an indication of the major field of students taking Mathematics 120. Note that almost 58% of the students taking Mathematics 120 are majoring in areas which require additional mathematics courses. In addition, 69.2% of those Mathematics 120 students who have decided upon majors, have chosen a major requiring additional mathematics.

TABLE 4 — Majors of Students Taking Math 120

Major Field	Fall 1970	Spring 1971	Fall 1971	Spring 1972	Total
Accounting*	1	4	4	4	13
Art	—	—	2	6	8
Biology*	1	2	1	2	6
Business*	3	6	11	13	33
Elementary Education*	4	14	12	16	46
History	—	—	7	6	13
Home Economics	1	2	5	2	10
Mathematics*	1	—	2	2	5
Nursing*	2	8	9	11	30
Physical Education	1	2	11	7	21
Physical Therapy*	—	2	—	—	2
Political Science*	2	—	7	4	13
Pre-Law	—	6	—	2	8
Pre-Medical*	—	3	6	4	13
Psychology*	—	2	4	5	11
Special Education*	3	9	8	13	33
Speech	2	2	5	6	15
Sociology*	2	4	12	7	25
Other	1	4	7	15	27
Undecided	4	13	38	41	96
TOTAL	28	83	151	166	428

(*) denotes that additional mathematics courses are required of students majoring in this field.

By comparing the data given in Table 4, approximately 53.7% of the total number of students taking Math 120 are in fields which require additional mathematics courses (see Table 3). An additional 4% (not reflected in the data of table 4) expressed an interest in taking another mathematics course. Also, 69.2% of the Math 120 students who have decided upon majors are in fields which require additional mathematics.

TABLE 5 — Mathematics Course Performance

Grade Percentages	155	201/202	208/301	210
Percentage of Math 120 students receiving Grade A	—	5%	5%	—
Percentage of Math 120 students receiving Grade B	10%	15%	10%	10%
Percentage of Math 120 students receiving Grade C	40%	45%	60%	50%
Percentage of Math 120 students receiving Grade D	40%	20%	15%	30%
Percentage of Math 120 students receiving Grade F	10%	15%	10%	10%
TOTAL	100%	100%	100%	100%

Table 5 gives an evaluation of the performance of Math 120 students in other math courses (see table 3 for a description of each course numbered above). Although the percentage grade levels are approximate, these figures do indicate a reasonable account of student performance in higher level mathematics courses.

Table 5 gives an indication of the overall performance of Mathematics 120 students in other mathematics courses. In viewing the table, recall the information in Table 1 which describes these students as being far below the national average on the ACT examination. In particular, this fact indicates that these students would ordinarily enter one of the normal beginning level courses with little chance to compete with the other students. Consequently, Table 5 can be interpreted as follows: although the majority of Mathematics 120 students do not perform in the A or B level in other courses, a potential failure has been converted in many cases to passing levels.

Table 6 gives the students' and counselors' reactions to the effectiveness of Mathematics 120 as preparation for higher level mathematics course.

Summary. Mathematics 120 is an attempt to offer more than a token solution to a problem facing our colleges and universities. Although the initial thrust of students characterized by the classification "disadvantaged" has tapered off, this type of student will not disappear from our campuses. Hastily conceived token courses will not be continually tolerated, nor will the luxury of mathematics departments ignoring the needs of these students continue to exist. Arguments to the effect that such students should not be in the University, or that such students do not deserve "special treatment," are invalid as excuses for departments not acting in this area. The students are in the universities and will be in the universities. Thus mathematicians must address themselves to the needs of these students, and face these needs both realistically and effectively. We cannot allow the traditional attitudes and approaches of mathemati-

TABLE 6 — Course Evaluation by Counselors and Students

Area of evaluation of Math 120	Significantly useful and necessary	Useful and necessary	Useful but not necessary	Not useful not necessary	No Opinion
General information for practical usage	Counselors 63% Students 48%	Counselors 15% Students 29%	Counselors 8% Students 15%	Counselors 3% Students 5%	Counselors 11% Students 3%
As background material for Math 155	Counselors 76% Students 69%	Counselors 18% Students 20%	Counselors 4% Students 10%	Counselors — Students 1%	Counsleors 2% Students —
As background material for Math 201	Counselors 41% Students 36%	Counselors 29% Students 25%	Counselors 18% Students 27%	Counselors — Students —	Counselors 9% Students 12%
As background material for Math 208/301	Counselors N.A. Students N.A.	Counselors N.A. Students N.A.	Counselors N. A. Students N.A.	Counselors N.A. Students N.A.	Counselors N.A. Students N.A.
As background material for Math 210	Counselors 53% Students 55%	Counselors 28% Students 31%	Counselors 13% Students 12%	Counselors 2% Students —	Counselors 4% Students 2%

Table 6 gives the evaluation of counselors and students relative to the usefulness and the necessity of Math 120 as it relates to the student. Figures are given in terms of percentages of students taking the course (Math 120) under each category.

cians to limit the scope of the investigation to determine the best overall solution to the problem.

In all of this, there is one fact that stands clear. Results are only obtained when adequate resources are made available. In fact, the price for good teachers properly compensated, assistants, and other vital elements is quite high. But on the other hand, adequate resources used properly will yield results. The theory is simple: reach the individual at his level and work with him through a system of required activities, supported by immediate reinforcement by dedicated instructors. Attitudes must be combatted, and individual performance emphasized.

Mathematics for the "Disadvantaged Student" is no more than mathematics

taught with total consideration given to the student. It is indeed my hope that such "mathematics for the people" will becarne the rule rathei than the exception.

I would like to thank Dr. George Bright for his comments and encouragement.

This study is based on a paper presented at the annual meeting of the National Council of Teachers of Mathematics, April 1972.

This article is based upon the author's experiences working with disadvantaged students at Columbia University and Northern Illinois University. The author is himself a concerned black Mathematician who received his early education in the inner city schools of Chicago.

DEPARTMENT OF MATHEMATICS, NORTHERN ILLINOIS UNIVERSITY, DEKALB, IL 60115.

PROBLEMS AND SOLUTIONS

All problems (both elementary and advanced) proposed for inclusion in this Department should be sent to E. P. Starke, 1000 Kensington Ave., Plainfield, NJ 07060. Proposers of problems are urged to enclose any solutions or information that will assist the editors. Ordinarily, problems in well-known textbooks and results in generally accessible sources are not appropriate for this Department. No solutions (except those accompanying proposals) should be sent to Professor Starke.

ELEMENTARY PROBLEMS

Solutions of Elementary Problems should be sent to Problems Group, Mathematics Department, University of Maine, Orono, ME 04473. To facilitate their consideration, solutions of Elementary Problems in this issue should be typed (with double spacing) and should be mailed before September 30, 1974.

An asterisk (*) means neither the proposer nor the editors supplied a solution.

E2480. *Proposed by M. S. Klamkin, Ford Motor Company*

If $a_i \geqq 0$, $\sum a_i = 1$, and $0 \leqq x_i \leqq 1$ for $i = 1, 2, \cdots, n$, prove that

$$\frac{a_1}{1+x_1} + \frac{a_2}{1+x_2} + \cdots + \frac{a_n}{1+x_n} \leqq \frac{1}{1 + x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n}}.$$

When does equality hold?

E 2481*. *Proposed by Bruce Reznick, California Institute of Technology*

Do the simultaneous equations

$$x^3 + y^3 + z^3 = 1, \qquad x + y + z = 2$$

have a solution in rationals $0 < x < y < z < 1$ other than $x = 1/2$, $y = 2/3$, $z = 5/6$?

E 2482. *Proposed by H. D. Ruderman, Hunter College Campus School*

In the ring of polynomials with coefficients from the integers mod 2, let n_k be the number of positive integers h not exceeding k with the property that $f_h(x) = x^h + x + 1$ is irreducible. Show that

$$\liminf_{k \to \infty} \frac{n_k}{k} > \tfrac{1}{2}.$$

E 2483. *Proposed by M. S. Klamkin, Ford Motor Company*

Let x be nonnegative and let m, n be integers with $m \geqq n \geqq 1$. Prove that

$$(m+n)(1+x^m) \geqq 2n \frac{1 - x^{m+n}}{1 - x^n}.$$

E 2484. *Proposed by Hal Forsey, California State University at San Francisco*

Let S_k be the kth partial sum of the harmonic series. Define k_n to be the least integer k such that $S_k \geqq n$. (For example, $k_1 = 1$ and $k_2 = 4$.) Find

$$\lim_{n \to \infty} \frac{k_{n+1}}{k_n}.$$

E 2485. *Proposed by R. H. Eddy, Memorial University of Newfoundland*

Three numbers are ,chosen at random (without replacement) from the first n natural numbers. What is the probability that they can be the sides of a triangle?

SOLUTIONS OF ELEMENTARY PROBLEMS

Inverse Points, Coincident Points, and Concurrent Lines

E 2419 [1973, 560]. *Proposed by A. W. Walker, Toronto, Canada*

Points G, H, I, O are the centroid, orthocenter, incenter and circumcenter of a scalene triangle $\triangle$, N and P are the midpoints of line segments OH and IH, F is the contact point of the incircle and nine-point circle of $\triangle$, E is the reflection of F in the right bisector of OH, and L is the inverse of E in the circle on GH as diameter. Prove

(a) F and I are inverse in the circle with center N, radius NP;

(b) if $OI = \sqrt{3} \cdot OG$, points L and I coincide:

(c) lines FG, IL, OP concur.

Solution by the proposer. We remark that $\triangle$ may be isosceles, but not equilateral.

(a) The point N is the center of the nine-point circle N' of $\triangle$, F lies on NI produced, and the stated result follows from

$$4(NP)^2 = (OI)^2 = R(R-2r) = 4(NF)(NI),$$

where r, R are the inradius and circumradius of $\triangle$.

(b) Denote by M the midpoint of GH, by M' the orthocentroidal circle on GH as diameter, and by O' the circle with center O and radius $\sqrt{3}\cdot OG$; then O' and M' intersect orthogonally and N is inside O'. Take I inside M' (by part (c) below) and on O', and let IN produced meet O' at Q; then

$$4(QN)(NI) = 4\{(OI)^2-(ON)^2\} = 4(OI)^2 - 9(OG)^2 = (OI)^2 = 4(NF)(NI),$$

so N is the midpoint of QF. If the line MI meets O' again at J,

$$(MI)(MJ) = (OM)^2 - (OI)^2 = 4(OG)^2 - 3(OG)^2 = (MG)^2 = (MN)(MO),$$

so I and J are inverse in M'; also the points I, J, O, N are concyclic (if I does not lie on GH), so the angles QNO, IJO, OIJ, ONJ are equal and J is the reflection of Q in line OGH, coinciding with Q if I is on GH. Hence E and L coincide with J and its inverse I in M'.

(c) Take rectangular cartesian axes with O as origin and line OH as x-axis, the coordinates of G, N, M, H, I being $(2a,0)$, $(3a,0)$, $(4a,0)$, $(6a,0)$, (b,c) where $a>0$ and (without loss of generality) $b^2+c^2=(OI)^2=12$. Then

$$(NI)^2 = (b-3a)^2 + c^2 = 3d, \quad \frac{NF}{NI} = \left(\frac{NP}{NI}\right)^2 = \frac{1}{d},$$

where $d \equiv 3a^2 - 2ab + 4$, $0<d<1$; also

$$(MG)^2 - (MI)^2 = 4a^2 - (b-4a)^2 - c^2 = 4(1-d) > 0,$$

so I lies inside the circle M'. The (x,y) components of the vectors $d(GF)$ and $d(ME)$ are $(ad+b-3a, c)$ and $(-ad-b+3a, c)$, so letting

$$k = 4a^2 - (a^2-1)(7-d)$$

we have

$$d^2(ME)^2 = a^2d^2 + 2ad(b-3a) + 3d = dk, \quad \frac{ML}{ME} = \left(\frac{MG}{ME}\right)^2 = \frac{4a^2d}{k}.$$

Since G lies inside N', it follows that $ME = GF \neq 0$ and $k>0$. The (x,y) components of the vector $k(IL)$ are found to be

$$(a^2-1)\{b(1-d)+6(b-2a)\}, \; c(a^2-1)(7-d).$$

But $3(OG)^2 - (OI)^2 = 12(a^2 - 1)$; also, $1 > d$ and $b > 2a > 0$ since I lies inside M', and it follows that L and I coincide if and only if $OI = \sqrt{3} \cdot OG$, a stronger result than (b). (It can be shown that the meet Y of lines IL and OG divides OG internally as $3R:r$, and that

$$2(3R + r)(IY)(IL) = 3R\{(OI)^2 - 3(OG)^2\},$$

this expression being positive (negative) if L and Y lie on the same side (opposite sides) of I.)

The equations of the lines OP and FG are

$$cx - (6a + b)y = 0, \; cx + \{a(1-d) - (b-2a)\}y - 2ac = 0;$$

multiplying respectively by $(1-d)$ and 6 and adding, we have

$$c(7 - d)x - \{b(1-d) + 6(b - 2a)\}y - 12ac = 0,$$

which is the equation of IL (if $a \neq 1$), so these three lines concur, coinciding if and only if $\triangle$ is isosceles.

Also solved by M. G. Greening (Australia).

Integral Triangles with Numerically Equal Area and Perimeter

E 2420 [1973, 691]. *Proposed by E. T. H. Wang, University of Waterloo, Canada*

Find all triangles with integral sides, each of which has its perimeter numerically equal to its area.

Editor's comment. This problem has been around for a long, long time. References were submitted by the following: H. M. Edgar, A. G. Ferrer (Mexico), Michael Goldberg, M. G. Greening (Australia), J. A. H. Hunter, M. S. Klamkin, Graham Lord, C. L. Morgan, M. R. Murty & V. K. Murty, C. B. A. Peck, Eric Rosenthal, R. W. Sielaff, Alan Wayne, D. P. Wegener, and the proposer, who discovered that the problem was not new only after it was in print.

The answer is that there are but five such triangles: (6, 8, 10), (5, 12, 13), (9, 10, 17), (7, 15, 20), and (6, 25, 29), and the earliest solution appears to be due to B. Yates in 1865. (See L. E. Dickson, *History of the Theory of Numbers*, Vol. II. Stechert, New York, 1934, p. 195.) The problem was also solved by W. A. Whitworth and D. Biddle, Math. Quest. Ed. Times 5 (1904), 54–56 and 62–63; this reference is given in Dickson, *op. cit.*, p. 199. It resurfaced about fifty years later when it was proved as a corollary to Problem E 1168 [1956, 43], and in 1968, it appeared as Problem 694 in MATHEMATICS MAGAZINE. (Interestingly enough, the solution to this problem that was published [1969, 47] was by E. P. Starke, your Problem Editor!) The result was quoted by Subbarao in his paper *Perfect triangles*, this MONTHLY 78 (1971), 384–385, and it can also be found in Shklarsky, Chentzov, and Yaglom, *The USSR Olympiad Problem Book*, p. 29. J. A. H. Hunter comments that he produced a story-teaser entitled *Where the grass grew green*, based strictly on this idea, for the December 1961 issue of his magazine, *Saturday Night;* it later appeared in his book (with co-author J. S. Madachy) *Mathematical Diversions*, Van Nostrand, Princeton, N. J., 1963.

Also solved by Alfred Univ. Math. Group, Leon Bankoff, D. M. Bloom, W. J. Blundon, D. Ž. Djoković, J. H. Driggs, A. H. Foss, Irving Gerst, E. G. Gibson, S. A. Greenspan, Robert Heller,

Sidney Heller, J. A. H. Hunter, Free Jamison, Eleanor Jones, C. D. LaBudde, Harry Lass, O. P. Lossers (Netherlands), Carolyn MacDonald, L. E. Mattics, R. D. Nelson, C. F. Pinzka, Bob Prielipp, L. A. Ringenberg, O. G. Ruehr, Andre Samson & Les Reid, Michael Shimshoni (Israel), John Spellman, O. Strauch (Czechoslovakia), Phil Tracy, E. W. Trost (Switzerland), R. W. Turner, and the proposer.

Power Automorphisms and Commutativity

E 2421 [1973, 691]. *Proposed by R. C. Buck, University of Wisconsin*

(i) Show that if G is a group in which the map $x \to x^3$ is a monomorphism (1:1 homomorphism, but not necessarily onto), then G is abelian.

(ii) Exhibit a non-abelian group for which $x \to x^4$ is an automorphism.

(iii) Are there examples for every other exponent > 4?

Solution by Michael Josephy, McGill University, and P. K. Garlick, Middleton, Wisconsin, (independently). To prove part (i), note that for all $a, b \in G$ we have $a^3b^3 = (ab)^3$, so by the cancellation law it follows that $a^2b^2 = (ba)^2$. Using the result twice, we have $a^4b^4 = (a^2)^2(b^2)^2 = (b^2a^2)^2 = [(ab)^2]^2 = (ab)^4$, which upon cancellation implies $a^3b^3 = (ba)^3$. That is, $(ab)^3 = (ba)^3$; since $x \to x^3$ is injective, necessarily $ab = ba$ and G is abelian.

For parts (ii) and (iii), we show that there exists for each $m > 3$ a non-abelian group G_m in which the mapping $x \to x^m$ is the identity automorphism. If $m = 2^n + 1$ for some $n \geqq 2$, take G_m to be a non-abelian group of order 8; the quaternion group or the dihedral group D_4 will do. Otherwise, let p be an odd prime factor of $m - 1$ and take G_m to be the non-abelian group of order p^3 in which every non-identity element has order p (See Marshall Hall, *The Theory of Groups*, Macmillan, New York, 1959, p. 51). A realization of this group is the set of all matrices of the form

$$\begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix},$$

where a, b, c are taken from the integers mod p.

Also solved by the Alfred Univ. Math. Group, S. Baskaran & B. Subramanian (India), W. D. Blair, C. R. Combrink, S. C. Currier, D. Ž. Djoković, Don Ethington, C. Gardner, Joseph Grear, M. G. Greening (Australia), S. A. Greenspan, Melvin Henriksen, Gary McDonald & Merry McDonald, Ian Macdonald (Scotland), Desmond MacHale (Ireland), H. J. Marcum (Brazil), T. E. Moore, M. Perkel & V. Landazuri, D. A. Robinson, John Shafer, J. G. Sunday, Phil Tracy, E. T. H. Wang, and Qazi Zameeruddin (India). Partial solutions by J. V. Michalowicz, and the proposer.

Editor's comment. Both Moore and MacHale cite G. A. Miller's paper, *Possible α-automorphisms of non-abelian goups*, Proc. Nat. Acad. Sci. 15 (1929), 89–91, which solves all parts of this problem. Henriksen mentions H. F. Trotter, *Groups in which raising to a power is an automorphism*, Canad. Math. Bull., 8 (1965), 825–826, which implicitly solves the problem. MacHale shows that if cubing is assumed to be an epimorphism (i.e., a homomorphism of G onto itself), then G must be abelian. He first shows that $x^2y^3 = y^3x^2$ for all $x, y \in G$. This implies that all squares are in the center of G and

hence $(yx)^2 = x^2y^2 = y^2x^2$, implying $xy = yx$. Baskaran and Subramanian give examples for (ii) and (iii) in which the automorphism in question is not the identity. Shafer notes that (i) holds in cancellative subgroups.

Tiling with Incomparable Rectangles

E 2422 [1973, 691]. *Proposed by E. M. Reingold, University of Illinois, Urbana*

Two rectangles are *incomparable* if neither can be placed inside the other when they are aligned so that corresponding sides are parallel. Prove or disprove: No rectangular region can be tiled with mutually incomparable rectangles.

I. *Solution by P. van Emde Boas and H. W. Lenstra II, Mathematisch Instituut der Universiteit van Amsterdam, Netherlands.* The first rectangle (Fig. 1) contains a solution to our problem. The second rectangle (Fig. 2) represents a solution to the analogous problem which results if the condition of incomparability is replaced by the weaker condition that no subrectangle can be translated inside any other subrectangle. Calling this latter problem the *reduced problem,* we see that each solution to the original problem is also a solution to the reduced problem. Conversely, any solution to the reduced problem yields a solution to the original problem upon

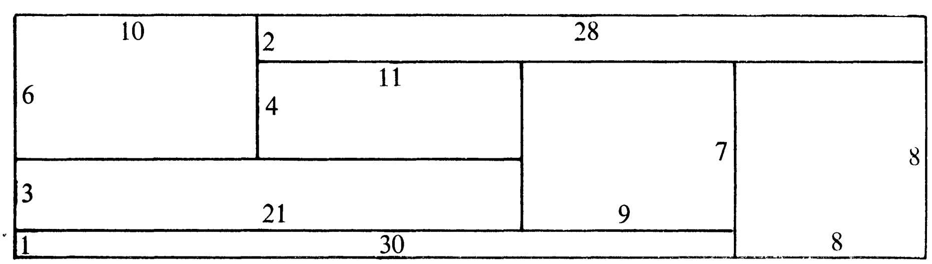


FIG. 1

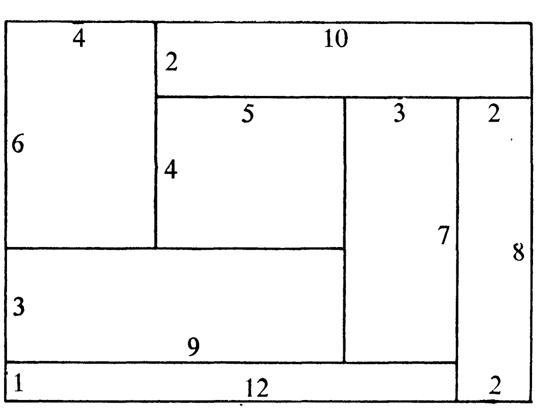


FIG. 2

multiplying one side of the solution by a sufficiently large factor to prevent inclusions of rectangles after rotating them by an angle of 90°.

The above solution to the reduced problem is a minimal one. Each such solution can be shown to need at least seven rectangles; if the solution consists of seven rectangles, its "shape" is as indicated, and the lengths of the sides of the rectangles (if integers) are at least as large as presented. Proof is by trial and error.

II. *Solution by W. A. A. Nuij, Technological University, Eindhoven, Netherlands.* We show more generally that any rectangle can be tiled with mutually incomparable rectangles. We prove this by constructing a covering of a rectangle by seven tiles each with the same area. As the sides of the tiles are unequal, the tiles are clearly incomparable. We now refer to Figure 3.

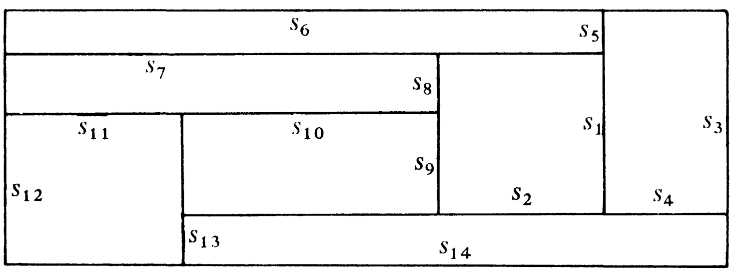


FIG. 3

Let us give all tiles an area equal to unity, and let us make $s_1 = 1$ and $s_3 = a$, where a is a number to be specified later. Computing the length of the sides in the indicated order we find:

$$s_5 = a - 1, \quad s_7 = \frac{2-a}{a-1}, \quad s_9 = \frac{3-2a}{2-a},$$

$$s_{11} = \frac{(2-a)(4-3a)}{(a-1)(3-2a)}, \quad s_{13} = \frac{(3-2a)(4a-5)}{(2-a)(4-3a)}.$$

Since $s_{10} + s_2 + s_4 = s_{14}$ and $s_{10} = 1/s_9$, etc., we have the following equation which determines a:

$$\frac{2-a}{3-2a} + 1 + \frac{1}{a} = \frac{2-a}{3-2a} \cdot \frac{4-3a}{4a-5}.$$

Rewrite this equation in the form

$$15a^3 - 37a^2 + 11a + 15 = 0.$$

This equation has three real roots, only one of which makes all of the sides positive (and unequal). This root is (approximately) $a = 1.2718$. Substituting this value for a solves the problem for one rectangle. As stretching does not change the ratio

of areas, we can fill any rectangle with tiles of equal area. In a finite number of cases, however, the described procedure gives rise to two equal tiles with different orientation. Then we make the width of the covering a little smaller and use the free space for a thin eighth tile, which is incomparable with the other tiles because it has a smaller area but length equal to the length of the whole rectangle.

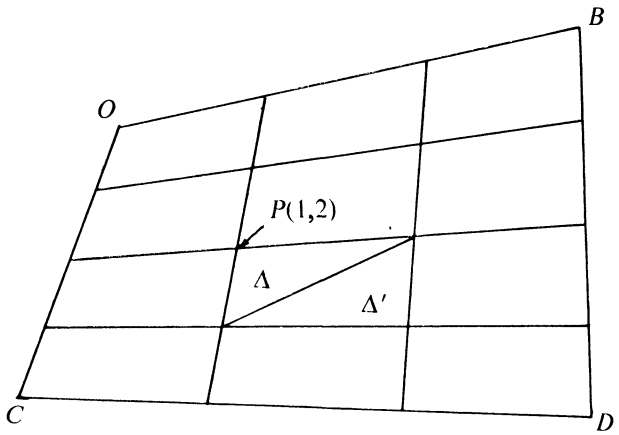
Also solved by Robert Breusch, Michael Goldberg, A. S. Kyle, R. A. Leslie, O. P. Lossers (Netherlands), M. S. Paterson (England), Bill Sands, and P. K. Stockmeyer.

Area Summations in Partitioned Convex Quadrilaterals

E 2423 [1973, 691]. *Proposed by Lyles Hoshek, Monterey Park, California, and B. M. Stewart, Michigan State University*

Let there be given a plane convex quadrilateral of area A. Divide each of its four sides into n equal segments and join the corresponding points of division of opposite sides, forming n^2 smaller quadrilaterals. Prove: (a) the n smaller quadrilaterals in any diagonal (ordinary or broken) have a composite area equal to A/n; (b) The composite area of any row of smaller quadrilaterals and its complementary row (row i and row $n+1-i$) is equal to $2A/n$. (In particular, if n is odd this implies that the composite area of the middle row is A/n.)

Solution by Donald Batman, M. I. T. Lincoln Laboratory, and M. S. Klamkin, Ford Motor Company. We obtain more general results by dividing one pair of opposite sides into n equal segments and the other pair of sides into m equal segments, as shown in the figure.



Denote the given quadrilateral by $OBDC$, where O is the origin. If X is a point in the plane, then we make the usual identification of X with the vector $\mathbf{X}$ from the origin to the point X. Define p, q by

$$\mathbf{D} = (p+1)\mathbf{B} + (q+1)\mathbf{C}.$$

Note that $p, q > -1$ and also $p + q > -1$ since the quadrilateral is convex. The points of division will be denoted by $P(r, s)$, with $r = 0, 1, \cdots, m$ and $s = 0, 1, \cdots, n$; e.g., $P(0,0) = O$ and $P(m,n) = D$. Let $Q(r,s)$ denote the small quadrilateral whose upper left-hand vertex is $P(r,s)$ and partition $Q(r,s)$ into the two triangles $\Delta(r,s)$ and $\Delta'(r,s)$ as shown in the figure.

One can show that for suitable scalars x and y

$$\mathbf{P}(r,s) = \frac{r}{m}\mathbf{B} + x\left\{\mathbf{C} + \frac{r}{m}(\mathbf{D} - \mathbf{C} - \mathbf{B})\right\} = \frac{s}{n}\mathbf{C} + y\left\{\mathbf{B} + \frac{s}{n}(\mathbf{D} - \mathbf{B} - \mathbf{C})\right\}.$$

Since $\mathbf{B}$ and $\mathbf{C}$ are linearly independent, we find that $x = s/n$ and $y = r/m$. Thus

$$\mathbf{P}(r,s) = \frac{r}{m}\left\{1 + \frac{sp}{n}\right\}\mathbf{B} + \frac{s}{n}\left\{1 + \frac{rq}{m}\right\}\mathbf{C}. \tag{1}$$

Since $\mathbf{P}(r+1,s) - \mathbf{P}(r,s)$ and $\mathbf{P}(r,s+1) - \mathbf{P}(r,s)$ are independent of r and s respectively, each segment of the figure is divided into equal parts — m for the "horizontal" segments and n for the "vertical" segments (as shown in the figure).

For the area $|\Delta(r,s)|$ of $\Delta(r,s)$ we have

$$\begin{aligned} 2|\Delta(r,s)| &= |\{\mathbf{P}(r+1,s) - \mathbf{P}(r,s)\} \times \{\mathbf{P}(r,s+1) - \mathbf{P}(r,s)\}| \\ &= \frac{1}{mn}\left\{1 + \frac{sp}{n} + \frac{rq}{m}\right\}|\mathbf{B} \times \mathbf{C}|, \end{aligned} \tag{2}$$

and similarly

$$2|\Delta'(r,s)| = \frac{1}{mn}\left\{1 + \frac{(s+1)p}{n} + \frac{(r+1)q}{n}\right\}|\mathbf{B} \times \mathbf{C}|. \tag{3}$$

Note also that if A is the area of $OBDC$, then

$$2A = (p + q + 2)|\mathbf{B} \times \mathbf{C}|. \tag{4}$$

Look now at any $\Delta(r,s)$ and its centro-symmetric $\Delta'(m-1-r, n-1-s)$. From (2), (3) and (4) we have

$$|\Delta(r,s)| + |\Delta'(m-1-r, n-1-s)| = \frac{A}{mn}. \tag{5}$$

For m, n odd it follows from this that the central small quadrilateral has area A/mn. (The special case $m = n = 3$ was established using a long synthetic proof by B. Greenberg, *That area problem*, Math. Teacher 64 (1971), 79–80.)

If we take any small quadrilateral $Q(r,s)$ and its centro-symmetric quadrilateral $Q(m-1-r, n-1-s)$ we see from (5) that

$$\begin{aligned}|Q(r,s)| + |Q(m-1-r,n-1-s)| &= |\Delta(r,s)| + |\Delta'(r,s)| \\ &\quad + |\Delta(m-1-r,n-1-s)| + |\Delta'(m-1-r,n-1-s)| \\ &= \frac{A}{mn} + \frac{A}{mn} = \frac{2A}{mn},\end{aligned}$$

which proves part (b).

From (2), (3), and (4) we have

$$\begin{aligned}|Q(r,s)| &= |\Delta(r,s)| + |\Delta'(r,s)| \\ &= \frac{A}{mn(p+q+2)}\left\{2 + \frac{(2s+1)p}{n} + \frac{(2r+1)q}{m}\right\}.\end{aligned} \tag{6}$$

Let $m = n$; we can now show that part (a) follows from this formula. In fact we can show that the result holds not only for broken diagonals, but for "generalized diagonals," i.e., for selections of n smaller quadrilaterals with one from each row and each column, as in the individual terms of a matrix expansion. More precisely, let σ be a permutation of $(0, 1, \cdots, n-1)$; an easy computation shows that

$$\sum_{r=0}^{n-1} |Q(r, r\sigma)| = \frac{A}{n},$$

giving the result.

We note that Problem E 1548 [1963, 892] and its generalizations follow from the above results.

Also solved by M. T. Bird, M. G. Greening (Australia), Harry Lass, R. D. Nelson (England), L. A. Ringenberg, and the proposers. Solutions of one part only by Michael Goldberg and Carolyn MacDonald.

The Symmetric Group as a Metric Space

E 2424 [1973, 692]. *Proposed by P. J. Murray, Westminster College, and (independently) by E. T. H. Wang, University of Waterloo*

Let S_n denote the set of all permutations of the first n natural numbers. We can define a metric on S_n as follows: If $\sigma, \tau \in S_n$, then $d(\sigma,\tau) = \sum_{i=1}^{n} |\sigma(i) - \tau(i)|$. What possible numerical values can d assume?

Solution by D. Ž. Djoković, University of Waterloo, Ontario, Canada. Since $d(\rho\sigma, \rho\tau) = d(\sigma,\tau)$ for all $\rho, \sigma, \tau \in S_n$, all of the values of d which are taken on are assumed by $d(\iota, \sigma)$ for some $\sigma \in S_n$, where $\iota \in S_n$ denotes the identity permutation.

Suppose that $d(\iota,\sigma) = m > 0$ for some $\sigma \in S_n$. We shall show that $m - 2$ is also a value of d, implying that, in particular, $m \geqq 2$. It will follow from this that the values of d are all of the even integers from 0 to some maximal value $2t$, which will be determined.

Let $d(\iota,\sigma) = m > 0$. Since $\sigma \neq \iota$, it is not hard to see that there must exist r, s with $1 \leqq r < s \leqq n$ such that $r\sigma > r$, $s\sigma < s$, and $i\sigma = i$ if $r < i < s$. (Notice that necessarily $r\sigma \geqq s$.) If ρ is the cycle $(r, r+1, \cdots, s)$, then an easy computation shows that $d(\iota, \rho\sigma) = m-2$.

It remains to determine t; we claim that $t = [n^2/4]$. Let the "reversing permutation" $\theta \in S_n$ be defined by $i\theta = n + 1 - i$ for all i. By direct calculation (handling the cases of n odd and n even separately), we can show that $d(\iota,\theta) = 2[n^2/4] = [n^2/2]$. If we can show that $d(\iota,\theta) \geqq d(\iota,\sigma)$ for all $\sigma \in S_n$, we will be done.

Let k $(1 \leqq k \leqq n-1)$ be such that $i\sigma = n+1-i$ for $i < k$ but $k\sigma \neq n+1-k$. We shall find a $\sigma' \in S_n$ with the property that $d(\iota,\sigma) \leqq d(\iota,\sigma')$ and which also satisfies $i\sigma' = n+1-i$ for $i \leqq k$. It will then follow by induction that

$$d(\iota,\sigma) \leqq d(\iota,\sigma') \leqq \cdots \leqq d(\iota,\theta).$$

Since $k\sigma \neq n+1-k$ certainly $k\sigma < n+1-k$ since every number greater than $n+1-k$ is the image of some i with $i < k$. But this implies that $r\sigma = n+1-k$ for some $r > k$. Let τ denote the transposition (k, r). Then letting $\sigma' = \tau\sigma$, we have

$$d(\iota,\sigma') - d(\iota,\sigma) = |r - k\sigma| + |n+1-2k| - |k - k\sigma| - |n+1-k-r|.$$

There are two cases to consider, depending on the relative magnitudes of $r, k, k\sigma$, and $n+1-k$. We shall examine one case; the other is similar. Suppose that $k \leqq k\sigma \leqq r \leqq n+1-k$. Then

$$\begin{aligned} d(\iota,\sigma') - d(\iota,\sigma) &= r - k\sigma + n + 1 - 2k - k\sigma + k - n - 1 + k + r \\ &= 2(r - k\sigma) \geqq 0. \end{aligned}$$

Each of the two cases leads to the same conclusion: $d(\iota,\sigma') - d(\iota,\sigma) \geqq 0$, which proves the assertion.

REMARKS. If $n = 2m$ is even, then $d(\iota,\sigma) = 2t$ holds if and only if $i\sigma > m$ for all $i \leqq m$. If $n = 2m+1$ is odd, then $d(\iota,\sigma) = 2t$ holds if and only if either $i\sigma > m$ for all $i \leqq m+1$ or $i\sigma \leqq m+1$ for all $i \geqq m+1$. Let $\Delta(\iota) = \{\sigma \in S_n : d(\iota,\sigma) = 2t\}$. The above remarks imply that $o(\Delta(\iota)) = (m!)^2$ if $n = 2m$ is even and $o(\Delta(\iota)) = n(m!)^2$ if $n = 2m+1$ is odd.

Let G_n denote the group of isometries of the metric space (S_n, d). Since $d(\rho\sigma,\rho\tau) = d(\sigma,\tau)$ it follows that the left translation $L_\rho: S_n \to S_n$ defined by $\sigma L_\rho = \rho\sigma$ is an element of G_n; it is easily checked that the mapping $\rho \to L_\rho$ of S_n into G_n is an anti-monomorphism. Let L_n denote the subgroup of G_n of all these left translations. If $n \geqq 3$, then $o(G_n) = 2n!$ and G_n is a semidirect product of L_n and the cyclic subgroup of order 2 generated by the isometry $\sigma \to \theta\sigma\theta$, where θ is the reversing permutation defined above.

For $\sigma \in S_n$ let $\Delta(\sigma) = \{\tau \in S_n : d(\sigma,\tau) = 2t\}$; this extends the concept $\Delta(\iota)$ defined above. Defining $\sigma \sim \rho$ if $\Delta(\sigma) = \Delta(\rho)$ defines an equivalence relation on S_n;

the equivalence classes for this equivalence relation can be shown to be the left cosets of the subgroup $AB = BA$ of S_n, where A is the subgroup of S_n which fixes $m+1, m+2, \cdots, n$ and where B is the subgroup of S_n which fixes $1, 2, \cdots, n-m$. Note that $o(A) = o(B) = m!$ and since $A \cap B = \{\iota\}$, it follows that $o(AB) = (m!)^2$. When n is even, the left cosets of AB in S_n are precisely the sets $\Delta(\sigma)$; however if $n \geqq 3$ is odd, this is no longer true, since $o(\Delta(\sigma)) = n(m!)^2 \neq o(AB)$.

Also solved by the Alfred University Problem Group, P. K. Garlick, Myron Hlynka, P. W. Lindstrom, J. Choné (France), and the proposer.

ADVANCED PROBLEMS

All solutions of Advanced Problems should be sent to J. Barlaz, Rutgers — The State University, New Brunswick, N.J. 08903. Solutions of Advanced Problems in this issue should be typed (with double spacing) on separate, signed sheets and should be mailed before September 30, 1974.

An asterisk (*) means neither the proposer nor the editors supplied a solution.

5976. *Proposed by Desmond MacHale, University College, Cork, Ireland*

A group element x is said to have trivial centralizer if it commutes only with its powers, i.e., $C_G(x) = \langle x \rangle$, the cyclic subgroup generated by x.

Characterize all finite groups in which every non-identity element has trivial centralizer. *Question*: Do there exist infinite trivial centralizer groups?

5977. *Proposed by J. F. Chew, University of Akron*

Let $(G, \cdot, d)$ be a triple such that $(G, \cdot)$ is a group, (G, d) is a metric space and the function $(x, y) \to x \cdot y$ is a continuous map from $G \times G$ to G. If d is left invariant ($d(a \cdot x, a \cdot y) = d(x, y)$ for all $a, x, y \in G$), then $(G, \cdot, d)$ is a topological group. Can this statement be made without the assumption that d is left invariant?

5978. *Proposed by F. D. Hammer, Berkeley, California*

An exercise in Wilansky, *Topology for Analysis*, (Ginn 1970, p. 59, # 110) asks for a continuous function $f: \mathbb{R} \to \mathbb{R}$ which is 1-1 on the rationals, but not 1-1 everywhere. Can "rational" be replaced with "irrational"?

5979*. *Proposed by Benjamin Volk, Far Rockaway, N.Y.*

Let the function $z + a_2 z^2 + a_3 z^3$ be schlicht in the open unit disk, $|z| < 1$. Then show that $|a_2| \leqq 2\sqrt{2}/3$.

5980. *Proposed by R. M. Cohn, Rutgers University*

Let $M(k)$ be the least upper bound (over distributions) that a sample of size k from a probability distribution on the positive integers, when ordered, is in arithmetic

progression. Does $M(k)$ exceed $k!/k^k$, which is the value of the probability when the distribution assigns probability $1/k$ to the first k integers? Show that $\lim_{k\to\infty} M(k) = 0$.

5981*. *Proposed by R. A. McCoy, Virginia Polytechnic Institute, and State University*

Let i, j, and n be positive integers. Prove or disprove: $i+j \leqq n$ if and only if for every two sets $\{B_1, \cdots, B_i\}$ and $\{D_1, \cdots, D_j\}$ of balls in R^n,

$$\left(\bigcap_{k=1}^{j} D_k\right) \setminus \left(\bigcap_{k=1}^{i} B_k\right)$$

is connected.

SOLUTIONS OF ADVANCED PROBLEMS

Uniform Distribution as a Limit

5906 [1973, 440]. *Proposed by Gérard Letac, University of Clermont, France*

Let $x_0, x_1, \cdots, x_t$ be independent random variables such that $P(x_1 = n) = p_n < 1$ for all t and $n = 0, 1, 2, \cdots$ with $\sum_{n=0}^{\infty} p_n = 1$; let q_n denote $P(x_t < n)$. Find the distribution of

$$q_{x_0} + p_{x_0} q_{x_1} + \cdots + p_{x_0} p_{x_1} \cdots p_{x_{t-1}} q_{x_t} + \cdots.$$

I. *Solution by L. E. Clarke, University of East Anglia, England.* There is no real loss of generality in assuming that $p_0, p_1, p_2, \cdots$ are all > 0. Then any $\xi \in [0, 1)$ can be expressed uniquely in the form

$$q_{n_0} + p_{n_0} q_{n_1} + p_{n_0} p_{n_1} q_{n_2} + \cdots,$$

where $n_0, n_1, n_2, \cdots$ are nonnegative integers. For, determine n_0 so that

$$q_{n_0} \leqq \xi < q_{n_0+1}.$$

Next determine n_1 so that

$$q_{n_1} \leqq (\xi - q_{n_0})/p_{n_0} < q_{n_1+1}.$$

Then $(\xi - q_{n_0} - p_{n_0} q_{n_1})/p_{n_0} p_{n_1} \in [0, 1)$, and so on. Note that if $p = \max(p_0, p_1, p_2, \cdots)$ then $p < 1$ and so

$$p_{n_0} p_{n_1} \cdots p_{n_{t-1}} \leqq p^t \to 0 \text{ as } t \to \infty.$$

Now $s = q_{x_0} + p_{x_0} q_{x_1} + p_{x_0} p_{x_1} q_{x_2} + \cdots$ is $< \xi$ if and only if

$$E_0: x_0 < n_0,$$

or

$$E_1: x_0 = n_0 \text{ and } x_1 < n_1,$$

or $$E_2\colon x_0 = n_0,\ x_1 = n_1 \text{ and } x_2 < n_2,$$

or $\cdots$. Since the events $E_0, E_1, E_2, \cdots$ are mutually exclusive,

$$P(s < \xi) = \sum_{n=0}^{\infty} P(E_n) = q_{n_0} + p_{n_0}q_{n_1} + p_{n_0}p_{n_1}q_{n_2} + \cdots = \xi,$$

and so s is uniformly distributed on $(0, 1)$.

II. *Solution by F. W. Steutel, Enschede, Netherlands.* Putting

$$y = q_{x_0} + p_{x_0}q_{x_1} + \cdots + p_{x_0}p_{x_1}\cdots p_{x_{k-1}}q_{x_k} + \cdots, \tag{1}$$

we have

$$y \stackrel{d}{=} q_{x_0} + p_{x_0}y', \tag{2}$$

where $\stackrel{d}{=}$ denotes equality in distribution, $y' \stackrel{d}{=} y$, and where x_0 and y' are independent. Denoting by F the distribution function of y and by ϕ its Laplace-Stieltjes transform, we have from (2)

$$\phi(s) = \sum_{n=0}^{\infty} p_n \exp\{-s(p_0 + p_1 + \cdots + p_{n-1})\}\phi(p_n s). \tag{3}$$

As we have $0 \leqq y \leqq 1$ with probability 1, the function ϕ is an entire function, which is uniquely determined by its derivatives at zero, i.e., by the moments of y. It is easily seen that, under the conditions given, these moments are recursively and uniquely determined by (2). It follows that (3) has only one solution that is analytic at zero. As, furthermore, the function $(1 - e^{-s})/s$ satisfies (3), the distribution function is given by

$$F(x) = \begin{cases} 0 & \text{if } x < 0 \\ x & \text{if } 0 \leqq x < 1 \\ 1 & \text{if } x \geqq 1, \end{cases}$$

i.e., y is uniformly distributed on $(0,1)$.

Also solved by G. S. Rogers & D. L. Young, J. P. Jordan, S. P. Lloyd, and the proposer.

Min Max $\{|z^n + a_{n-1}z^{n-1} + \cdots + a_1z + 1|;\ |z| = 1\}$

5907 [1973, 440]. *Proposed by J. C. Alexander, University of Maryland*

For $n \geqq 1$, let S_n be the set of polynomials of the form

$$p(z) = z^n + a_{n-1}z^{n-1} + a_{n-2}z^{n-2} + \cdots + a_1z + 1,$$

where $a_1, a_2, \cdots, a_{n-1}$ range through all complex numbers. What is the value of

$$M_n = \min_{p \in S_n}\left(\max_{|z|=1} |p(z)|\right)?$$

Solution by J. P. Jordan, Glen Rock, New Jersey. We prove that $M_n = 2$ for $n \geqq 1$. $M_n \leqq \max_{|z|=1} |z^n + 1| = 2$. If $M_n < 2$ for some n, then there is a $p(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + 1$ such that $\max_{|z|=1} |p(z)| < 2$. In particular, $-2 < \operatorname{Re} p(z) < 2$ at the nth roots of unity given by $r = \cos 2\pi/n + i \sin 2\pi/n$; $r^2, \cdots, r^n = 1$. Adding inequalities, we get $-2n < \operatorname{Re} \sum_{k=1}^n p(r^k) < 2n$, which is a contradiction since $\sum_{k=1}^n p(r^k) = 2n$.

Also solved by Graeme Fairweather, R. Goldstein (England), G. A. Heuer & Karl Heuer (Germany), O. P. Lossers (Netherlands), L. E. Mattics, C. C. Rousseau, Jacob Sturm & Robert Brooks (Israel), Robert Vermes, and the proposer.

Editor's Note. R. Goldstein conjectures that if $p(z) = q(z) + r(z)$, where $q(z)$ and $r(z)$ are polynomials such that the coefficients of $q(z)$ are fixed while the coefficients of $r(z)$ range through all complex numbers, then

$$\min \max \{ |p(z)| ; |z| = 1\} = \max \{ |q(z)| ; |z| = 1\}.$$

Rousseau points out that the theorem of the problem appears in a slightly more general form ($p(z) = Az^n + c_1 z^{n-1} + \cdots + c_{n-1}z + B$, A, B fixed) in V. I. Smirnov and N. A. Lebedev, *Functions of a Complex Variable, Constructive Theory*, Vol. I, p. 353.

Sums of Four Squares of Odd Numbers

5908 [1973, 440]. *Proposed by M. L. Glasser, Battelle Memorial Institute*
Prove

$$\sum_{i,j,k,l} \Delta^{-4} \Big/ \sum_{i,j,k,l} \Delta^{-3} = \pi^2/12,$$

where $\Delta = i^2 + j^2 + k^2 + l^2 + i + j + k + l + 1$ and the summations are over all nonnegative integers.

Solution by N. J. Fine, Pennsylvania State University. We have

$$\sum \Delta^{-k} = \sum_{n=1}^{\infty} f(n) n^{-k},$$

where $f(n)$ is the number of representations of n in the form Δ, or of $4n$ as a sum of four squares of positive odd integers. It is known that $f(n) = \sigma(n)$, the number of divisors of n, for n odd; whereas $f(n) = 0$ for n even. (E. Landau, *Vorlesungen über Zahlentheorie*, Thm. 170.) Since $f(n)$ is multiplicative, the series is equal to the Euler product

$$\prod_p \left(1 + \frac{f(p)}{p^k} + \frac{f(p^2)}{p^{2k}} + \cdots\right).$$

Since $\sigma(p^m) = (p^{m+1} - 1)/(p - 1)$, the factor corresponding to the prime $p \neq 2$ is

$$\sum_{m=0}^{\infty} \frac{p^{m+1} - 1}{(p-1)p^{mk}} = \frac{p^{2k-1}}{(p^{k-1} - 1)(p^k - 1)}.$$

Hence

$$\Sigma\Delta^{-4} \Big/ \Sigma\Delta^{-3} = \prod_{p\neq 2} \frac{p^7}{(p^3-1)(p^4-1)} \cdot \frac{(p^2-1)(p^3-1)}{p^5}$$
$$= \prod_{p\neq 2} \frac{(1-p^{-2})}{(1-p^{-4})} = \frac{5}{4} \prod_{p} \frac{(1-p^{-2})}{(1-p^{-4})} = \frac{5}{4}\frac{\zeta(4)}{\zeta(2)} = \frac{\pi^2}{12}.$$

Also solved by L. Carlitz, L.E. Clarke (England), M. G. Greening (Australia), A. A. Jagers (Netherlands), J. P. Jordan, C. C. Rousseau, and the proposer.

Note. Using a variety of procedures, Jagers, Greening, Rousseau and Jordan obtain the formula

$$\Sigma\Delta^{-s} = (1-2^{-s})(1-2^{-(s-1)})\zeta(s)\zeta(s-1), \qquad \mathrm{Re}(s) > 2,$$

which also follows from the method used in the printed solution.

Unbounded Continuous Function in $\mathbb{R}^n$

5909 [1973, 441]. *Proposed by Gérard Letac, University of Clermont, France*

Let f be a continuous real function on some real, finite dimensional vector space E. For any base $b = (b_1, \cdots, b_n)$ of E, denote by

$$E_b = \{z_1 b_1 + \cdots + z_n b_n;\ z_i \in \mathbb{Z},\ i = 1, \cdots, n\},$$

where $\mathbb{Z}$ is the set of integers. Is it true that f is a bounded function when, for any base b, f restricted to E_b is bounded?

Solution by G. A. Heuer, University of Cologne, Germany. Yes, f is bounded. It will be convenient to deal first with the case $n = 1$. Remark: if $0 < a < b$, the intervals (ta, tb) and $((t+1)a, (t+1)b)$ overlap for all sufficiently large t; thus there exists a number r such that for every $x > r$ there is an interval (c, d) about x and a positive integer m such that $a < c/m < d/m < b$. Assume without loss of generality that f is unbounded above on the positive reals. Since f is continuous, for each positive integer k and each real $r > 0$ there is an interval (c, d), with $c > r$, throughout which $f(x) > k$. Let I_1 be a nondegenerate closed interval on which $f(x) > 1$. By the remark above there is a nondegenerate closed subinterval I_2 of I_1 and an integer m_2 such that $f(x) > 2$ for all x in $m_2 I_2 = \{m_2 y\colon y \in I_2\}$. By an easy induction we obtain a nested sequence of closed intervals $I_1 \supset I_2 \supset I_3 \supset \cdots$, and a sequence of integers $m_2 < m_3 < \cdots$, such that $f(x) > k$ for all x in $m_k I_k$. Thus, if $b \in \cap_k I_k$, f is not bounded on E_b.

For general n, if f is not bounded (above), let $\{p_k\}$ be a sequence of points such that $f(p_k) > k$. Since f is continuous, the set $\{p_k\}$ is unbounded. We claim that there is a basis $(c_1, \cdots, c_n)$ such that the components of the points p_k in the direction of each basis vector are unbounded; i.e., if $p_k = \sum_{i=1}^n \alpha_{ik} c_i$, for each $i = 1, \cdots, n$, the sequence $\{\alpha_{ik}\}_{k=1}^\infty$ is unbounded. For, obviously, at least one of them is unbounded; if say $\{\alpha_{ik}\}$ is unbounded for $1 \leqq i \leqq j$ and bounded for $j+1 \leqq i \leqq n$, let $c_1' = c_1 - \sum_{i=j+1}^n c_i$,

and $c_i' = c_i$ for $i > 1$. Then

$$p_k = \sum_{i=1}^{j} \alpha_{ik} c_i' + \sum_{i=j+1}^{n} (\alpha_{ik} + \alpha_{1k}) c_i',$$

and the components of $\{p_k\}$ in each new basis vector are unbounded, as claimed.

We can, in fact, arrange that they be unbounded above. Choose such a basis. For each point p_k, choose intervals $J_{i,k}, \cdots, J_{n,k}$ in the reals such that

$$u_k = \left\{ x : x = \sum_{i=1}^{n} \beta_{ik} c_i,\ \beta_{ik} \in J_{ik} \right\}$$

is a neighborhood of p_k throughout which $f(x) > k$. Proceeding now as in the one dimensional case, we may choose $b_1 = \gamma_1 c_1$ such that infinitely many integral multiples of b_1 lie in the intervals J_{1k}. Restricting attention now to those k for which $m_k b_1 \in J_{1k}$ for some integer m_k, we may choose $b_2 = \gamma_2 c_2$ such that infinitely many integral multiples of b_2 lie in the intervals $J_{2,k}$. We continue in this fashion, and obtain a basis $b = (b_1, \cdots, b_n)$ for which f is unbounded on E_b.

Also solved by W. R. Emerson, R. J. Evans, Bruce Ferrero, Donald Girod, D. C. Kay, and the proposer.

NOTE. Emerson proves the following more general theorem:

If f is an unbounded continuous function on E and $b = (b_1, \cdots, b_n)$ is any base of E, there exist arbitrarily small perturbations $b' = (b_1', \cdots, b_n')$ of b such that f is unbounded on $E_{b'}$.

$$\boldsymbol{F_n = (n+2)F_{n-1} - (n-1)F_{n-2}}$$

5911 [1973, 564]. *Proposed by Bill Knight, California Institute of Technology*

Let F_n be the nth term of the sequence defined by

$$(1) \qquad F_n = (n+2)F_{n-1} - (n-1)F_{n-2},\ F_1 = a,\ F_2 = b.$$

Find an explicit formula for F_n.

I. *Solution by W. O. Egerland, Ballistic Research Laboratories, Aberdeen Proving Grounds, Maryland.* We observe that for $k \geqq 3$,

$$(k-1)F_k - k^2 F_{k-1} = (k-2)F_{k-1} - (k-1)^2 F_{k-2} = b - 4a.$$

Hence

$$\frac{F_k}{k!\,k} - \frac{F_{k-1}}{(k-1)!\,(k-1)} = \frac{b-4a}{k!\,k(k-1)}.$$

Summation from $k = 2$ to $k = n$ yields

$$F_n = n!\,n \left\{ a + (b-4a) \sum_{k=2}^{n} \frac{1}{k!\,k(k-1)} \right\}$$

or

$$F_n = (3b - 11a)n!\, n + (4a - b)\left(n + 1 + n!\, n \sum_{k=0}^{n-1} 1/k!\right).$$

II. *Solution by Blagoj S. Popov, University of Skopje, Yugoslavia.* Using the Laplace method, let us put $F_n = \int_\alpha^\beta t^{n-1} v(t)\, dt$. From (1) we obtain

$$\frac{dv}{v} = \frac{1 - 3t + t^2}{t - t^2} dt \Rightarrow v = e^{-t} t(t - 1).$$

The limits of the integral are the roots of the equation $t^{n+1}(t-1)e^{-t} = 0$, $t_1 = 0$, $t_2 = \infty$, $t_3 = 1$. Then the general solution is

$$\begin{aligned} F_n &= c_1 \int_0^\infty t^n (t-1) e^{-t} dt + c_2 \int_0^1 t^n (t-1) e^{-t} dt \\ &= c_1 n!\, n + c_2 \left\{ n!\, n \left(1 - \frac{1}{e} \sum_{k=0}^{n} \frac{1}{k!}\right) - \frac{1}{e} \right\} \\ &= (3b - 11a) n!\, n + (4a - b)\left(1 + n!\, n \sum_{k=0}^{n} 1/k!\right). \end{aligned}$$

III. *Solution by Barry Wolk and Roger Kingsley, University of Manitoba.* For $n \geqq 2$, $F_n = -a[n!\, n(11 - 4e)] + b[n!\, n(3 - e)]$, where e is the base of natural logarithms and the brackets indicate the greatest integer function.

Our method of attacking this problem may be of interest. Clearly $F_n = a A_n + b B_n$, where $A_1 = 1$, $A_2 = 0$, $B_1 = 0$, $B_2 = 1$. A computer tabulation of n, A_n, B_n, B_n/n, etc., led to the conjecture $A_n \equiv 4 \pmod{n}$, $B_n \equiv -1 \pmod{n}$. Let $C_n = (4 - A_n)/n$, $D_n = (1 + B_n)/n$. Then a table of C_n, D_n, C_n/C_{n-1}, etc., suggested the relations $C_n = nC_{n-1} - 4$, $D_n = nD_{n-1} - 1$. Assuming that these formulas are valid for all n, we can solve them to get $C_n/n! = 11 - 4S_n$, $D_n/n! = 3 - S_n$, $A_n = -11\ n!\, n + 4 + 4n!\, nS_n$, $B_n = 3n!\, n - 1 - n!\, nS_n$, where $S_n = \sum_{i=0}^{n} (1/i!)$. All these conjectures are proved by verifying that this formula for F_n does indeed solve the problem.

This solution can be rewritten as above by using the relation

$$1/(n+1) < (e - S_n)n! < 1/n.$$

Also solved by W. R. Allaway, Günter Bach (Germany), M. T. Bird, M. T. L. Bizley (England), W. D. Blair, T. S. Bolis, Robert Breusch, R. J. Evans, Margaret J. Hodel, Mourad Ismail, N. L. Johnson, E. L. Koh, Donald La Budde, Harry Lass, O. P. Lossers (Netherlands), L. E. Mattics, M. F. Neuts & E. M. Klimko, David Newman (Israel), R. H. C. Newton, Mauri Orjatsalo (Finland), O. G. Ruehr, F. C. Smith, F. W. Steutel (Netherlands), Jacob Sturm & Michael Steiner (Israel), E. Trost (Switzerland), J. B. van Rongen (Netherlands), David Zeitlin, and the proposer.

REVIEWS

EDITED BY J. ARTHUR SEEBACH, JR. AND LYNN A. STEEN
with the assistance of the mathematics departments of St. Olaf and Carleton Colleges

COLLABORATING EDITOR FOR FILMS: SEYMOUR SCHUSTER, CARLETON COLLEGE

We invite readers to submit reviews of significant recent college-level mathematics books. We especially encourage reviews based on classroom use, or comparative reviews of several related books. Reviews should ordinarily not exceed two pages (per book) typed double spaced. Manuscripts of reviews as well as books submitted for review should be sent to: Book Review Editor, American Mathematical Monthly, St. Olaf College, Northfield, MN 55057.

A Response to Managerial Education, A series of articles by Peter Hilton, J. Myron Atkin, Ernest R. House, Robert M. Exner, Robert B. Davis, Leo Ruth, Peter Braunfeld, Burt A. Kaufman, Vincent Haag, Gail S. Young and Gerald R. Rising, *Educational Technology*, XIII, Number 11 : 11–59, November, 1973. Englewood Cliffs, N.J. Copies $3.00 each. (Telegraphic Review, February 1974.)

Battle lines are drawn for what may turn out to be the educational debate of the century, a debate between the dwellers in two intellectual ghettos: the technology-oriented ghetto of the managers of change in education and the content-oriented ghetto of the humanists. Lawrence Lipsitz, the insightful if not reckless editor of *Educational Technology*, has devoted an entire issue of his magazine to the opposition.

That the chasm that separates the ghettos may defy bridging is clear as one reads the response of the eleven humanist authors. The responders build their case on an age-old distinction between training and education. The managers, say the humanists, unwarrantedly elevate the fulfillment of tasks and the acquisition of skills to a paramount position in education by their obsession with prespecified, measurable objectives and their inattention to unmeasurable objectives and the unpredictable outcomes of education that arise from unexpected opportunities. Values which cannot be prespecified nor accurately measured, such as initiative, imagination, enrichment of the human spirit, and preparation of the individual for achievement of his fullest potential are by-passed. Herein lies the difficulty, since the managers will only reply that if such values exist, behavioral objectives can be generated to measure them, given careful definitions of each, time and effort. The chasm deepens.

We read that the acquisition of skills (training) is deemed "as subordinate to, and part of, the acquisition of an education," (p. 12) and that though education and training are not incompatible a program that concerns itself exclusively with training misses the central issue of education. The responders find problems with an educational system that employs the "simple and appealing strategy" which holds teachers and schools accountable to goals that aim primarily at change in

pupil behavior wrought by the most cost efficient programs of instruction (p. 18). If we really hold teachers accountable, we are told that the result may well be the establishment of an orthodoxy that will leave the children and the country poorer.

The impersonal, highly specific, precise methodology of education that is so closely tied to accountability and the managerial mode in education comes under severe attack as the authors point out that although educational objectives are clearly not testing objectives, the omnipresence of testing in the new programs is distorting education. Hope springs eternal, however, as we are informed about a countermovement in evaluation that is attempting to humanize education by developing evaluation designs that take account of the unspecified outcomes of education, particularly through the judgment of concerned persons.

As for mathematics education, the authors decry the obsession with ends rather than means that comes with the behavioral objectives approach and the attendant atomization of concepts that prevents the cultivation of real understanding of the discipline of mathematics. Following an extended discussion of what is mathematics and what is not, it is argued that the "task of writing sensible behavioral goals for school mathematics is just insurmountably difficult, because most of mathematics simply defies being cast into the language of measurable behavioral outcomes. As a result, curricula based on behavioral goals deal only with the most trivial and dreary part of mathematics. . ." (p. 48). In short, the behavioral objective approach is inadequate in building a mathematics curriculum and in its execution as well.

Concerning teacher education we are reminded that here we are not faced with the question of whether the use of performance criteria in the preparation of teachers is right; rather, educators are being directed by many State Departments of Education to institute competency based teacher education programs, programs that in large measure dehumanize education. The undersigned concur wholeheartedly with the conclusion reached that in teacher preparation as in the classrooms of our children the role of education must prevail over training.

For a long while now, professional societies in the various disciplines have pursued their own courses of action with little or no interchange with each other. Leo Ruth articulates the doubts of the English teaching profession, doubts reflected in a resolution recently adopted at the annual conference of the National Council of Teachers of English. He describes the action taken by a coalition of the English teachers' professional organizations in California and the multi-subject organization, the California Curriculum Correlating Council, which resulted in the defeat of legislation designed to impose a PPB system in all of California's schools. The influence of this coalition produced an alternative program that defined students' progress not by their overt behavior but rather by a mutually agreed-upon list of experiences.

The objections to measurable behavioral objectives raised by Ruth are widely shared by English teachers, at the heart of whose discipline lie "appreciation," "aesthetic responsiveness," "understanding." Unquestionably, certain areas of the

curriculum are easily susceptible to behavioral analysis, the skills of reading and writing in particular. But Ruth rightly fears that the focus on what is trainable and observable and measurable will inevitably skew the curriculum so that only such skills will be valued and taught.

Out of the California controversy came a recommendation to English teachers that is addressed to all the other disciplines concerned with the whole meaning of education: Spend less time talking with yourself in the world of English and rally other subject-matter groups who may not understand that they too will suffer from blanket PPBS legislation that more obviously affects the teaching of English. "To succumb to the folly of specificity and the obligatory postulation of measurable objectives," Ruth concludes, "is to disregard as irrelevant many important human aims." It is better to encourage humanism than ignore it.

Perhaps the strongest sentiment in the entire publication is the recognition that however much improvement can be made in the training level of education by the development of the managerial mode, we must all join together and strongly oppose any attempt to impose any single approach as the model against which all educational efforts must be measured (p. 33).

Every mathematician and educator should read this material. To generalize what Gail Young writes in his closing statement, a failure to maintain an alliance between humanists and managers will have tragic results for our children.

H. W. SHERIDAN, English and Education
P. S. JORGENSEN, Mathematics and Education
Carleton College

Introduction to Graph Theory. By Robin J. Wilson. Academic Press, New York, 1972. viii + 168 pp. \$7.50. (Telegraphic Review, June–July 1973.)

Because of the basic nature of graphs, there are many people, from a variety of fields (and often without substantial mathematics backgrounds), who want to learn about graph theory. This book can be recommended to them as being especially well-suited for an introductory course or for self-study. There are several reasons for this, including (1) the material, which concentrates on basic results, (2) the elementary applications, which stimulate the average reader's interest, and (3) the expository style, which introduces new material well.

Two complementary philosophies appear to be incorporated into the writing. First, along with simply developing graph theory, the author conveys something more on methods of proof. For example, on several occasions he gives contrasting proofs of a theorem. Secondly, he conveys — better than is done in most introductory books — a sense that there is theory in graph theory and that it is more than a collection of interesting results. Duality is the primary device used in achieving this.

Let us turn now to examining some of the contents in more detail. After a good

intuitive introduction in Chapter 1, with numerous illustrations of how graphs serve as models, the formal development begins in Chapter 2. In a subject in which definitions proliferate so, a writer can easily overwhelm a reader with a glossary of terms which he cannot begin to comprehend. Admirably, Wilson is content with just enough fundamental concepts to get the reader started.

In the main, the next six chapters consist of standard topics, but the presentation has some noteworthy aspects. Already in Chapter 3 (Paths and Circuits) some of the similarities between circuits and cut sets are introduced. Chapter 5 (Planarity and Duality) contains an excellent development of duality in planar graphs, perhaps the best exposition of this material available. Chapter 8 is a very substantial one, incorporating as it does a variety of interesting and important results. From the marriage problem as starting point, the reader encounters Hall's theorem on systems of distinct representatives, Menger's theorem on connectivity, the König-Egerváry theorem on independent entries in matrices, and the Ford-Fulkerson theorem on network flows. Although there are far too many variations and extensions of these topics for all to be covered in such a book, one might wish for more on characterizations of non-separable graphs. But all in all, it is an excellent chapter.

The final chapter is devoted to matroids, something not everyone would expect to find in an introduction to graph theory. The best answer to why they are included lies near the end of the book, but unfortunately the average reader will have some difficulty getting there. The author, having done such admirable work with the graph definitions earlier, should have done better here. It is to be hoped that the reader will skim some of this material and get to the later results. Among the applications of matroids to graphs and transversals, he will find that the similarities between cut sets and circuits which were encountered earlier are more than coincidental. He should come away with an interest in matroids both for their own sake and for their bearing on graphs.

The book has a large selection of exercises — some 250 — with the more difficult ones being starred. They cover a wide range: some require only straightforward interpretation of material, some ask for proofs, and others develop new material. Some of my favorites involve applications, which range from simple games to other areas of mathematics. A major weakness (common to most graph theory books) is the lack of a "Hints and Answers" section; almost anyone learning this material will appreciate some assistance with the problems.

In summary then, this book will be of value to anyone who wants to learn or to teach the basics of graph theory. For an elementary course, it is recommended as a text, with the instructor perhaps supplementing with topics from other sources. In a more advanced course, the instructor can benefit from using Wilson's book as a guide to basics and for the approach to certain topics, such as duality.

Finally, a parting comment for the American publishers: Why is the book not available in its original paperback form? Such an edition would be most useful.

L. W. BEINEKE, Purdue University at Fort Wayne

Introduction to Mathematical Statistics, 4th edition. By Paul G. Hoel. Wiley. New York, 1971. x + 409 pp. $11.50. (Telegraphic Review, November 1971.)

The fourth edition differs from the previous edition principally in the organization of material. Although the material has been reorganized so that the first six chapters can be used for a brief course in probability, it is still primarily a classical statistics text.

A one semester course (4 credits, meeting 3 hours per week plus independent study) covered the first ten chapters, which deal with the basics of undergraduate statistics, i. e., combinatorics, discrete and continuous distributions, the binomial, Poisson, normal, χ^2, t and F distributions, point estimation, tests of hypotheses, confidence intervals, correlation and linear regression. The last three chapters are excellent, as they discuss advanced topics such as analysis of variance, statistical design, nonparametric methods and sequential analysis. These chapters can be read by the more advanced and interested students and are suitable for independent honors projects.

The text is well written; the exposition is both readable and lucid and is intended for students with an elementary background in calculus. The author, wisely, omitted many of the more difficult proofs from the main body of the text, and some of these are outlined in the appendix.

Many of the students who used this book mentioned that they felt that more illustrative problems should be worked out in detail in the text. While this may be a valid criticism, there clearly is an abundance of excellent exercises. The exercises are labeled with the number of the section to which they refer, beginning with routine problems and followed by more theoretical ones. It would be helpful if, in addition, there was a set of miscellaneous problems at the end of each chapter so a student could test his mastery of the various topics of a complete chapter, rather than of the individual topics alone. Moreover, a few more difficult and challenging problems would be a helpful supplement.

Chapter 8, titled "General Principles of Statistical Inference" has too much material. This chapter should be divided into two: one part dealing with testing of hypotheses, the other with considering point estimation. The author discusses only two types of point estimates, unbiased and maximum likelihood. A new chapter on estimation should also include consistent and sufficient estimators, which can be studied effectively at the junior or senior level.

There was an enthusiastic student response to this text. I would recommend it to an instructor who is looking for a post-calculus text in probability and mathematical statistics which stresses the basic concepts of these fields. Moreover, a student who has mastered the material in the first ten chapters is prepared for the second examination of the Society of Actuaries.

MURRAY HOCHBERG, Brooklyn College

Advanced Engineering Mathematics, Third Edition. By Erwin Kreyszig. Wiley, New York, 1972. xvii + 866 pp. $14.95. (Telegraphic Review, October 1972.)

Having taught from both the second and third editions of this book, I must say that I found little difference in the third edition. While a new chapter on Numerical Analysis has been added, I see no indication that any other substantial changes were made, except for the rearrangement of parts of a few chapters and the addition of two or three pages of new material on Inner Product Spaces. However, the fact that this does not appear to be a very substantial revision in no way detracts from the overall excellence of the book.

I have taught two different classes in Advanced Calculus from the text, utilizing chapters 5, 7 and 8. The presentation of Green's Theorem and the Divergence Theorem, with numerous examples "scattered all over," gives the students of applied mathematics a chance to see a wide panorama of applications. The section on the Chain Rule for functions of several variables is a little too sketchy, as most students find the problems immediately following it to be rather difficult. A lecturer using this book might help by adding at this point material on the Implicit Function Theorem and its applications.

I have also taught a course from the chapters on Fourier Series and Transforms, Laplace Transforms, and Partial Differential Equations (chapters 4, 9 and 10). The treatment of Laplace Transforms is rather complete; in fact, one of my colleagues picked up the ideas of this chapter and incorporated them into his Differential Equations course in the middle of the semester!

Student reaction to the book has been quite favorable. The style of the text is highly applied, as the title indicates, with no great stress on rigorous proofs. Often proofs are sketched or outlined, while applications are treated extensively. This, I believe, is one of the great assets of the book.

An added bonus of this text is that it also contains an exhaustive treatment of Complex Variables (chapters 11–17), a chapter on Numerical Analysis, and a chapter (19) on Probability and Statistics which includes some non-parametric tests. Former students of mine who used the book in Advanced Calculus have expressed pleasure in discovering that it can be used as a good reference for these other courses.

In many mathematics departments, Advanced Calculus has become a poor sister to an undergraduate course in Real Analysis. However, the renewed interest in applied mathematics by students aware of the present job situation can be encouraged by a good Advanced Calculus course, "done right," and I strongly believe that this book is the ideal choice for such a course.

RONALD F. BARNES, State University of New York, College at Brockport

TELEGRAPHIC REVIEWS

Telegraphic reviews are designed to give prompt notice of new books with sufficient information to assist our readers in deciding whether to order an examination copy or to suggest library purchase. Possible uses are indicated as follows:

T = textbook P = professional reading
S = supplementary reading L = undergraduate library purchase
13 to 18 = freshman to second year graduate level usage
1 to 4 = appropriate time in semesters to cover text
Asterisks (*) or question marks (?) denote special positive or negative emphasis, respectively. Publishers are denoted by standard abbreviations; complete addresses may be found in *Books in Print*.

GENERAL, S**, P, L**, *Mathematical Gems from Elementary Combinatorics, Number Theory, and Geometry.* Ross Honsberger. Dolciani Math. Expos., No. 1. MAA, 1973, xi + 176 pp, $8. Dozens of uncommon mathematical vignettes, each rooted in historical context and developed in a refreshing, informal style that alerts the reader to the role of beauty and surprise in mathematical truth. Every mathematician will find new items of interest throughout this volume, yet most of it is accessible to outstanding high school students. Quite a remarkable achievement! LAS

GENERAL, T(13-14: 2), *Finite Mathematics for Business and Social Science.* William J. Adams. Xerox, 1974, xiv + 354 pp, $10.50. This is an excellent finite mathematics book. Divided into five sections: linear programming, probability (including Markov chains), game theory, matrices, and the mathematics of finance (primarily about interest). Matrices are used in earlier chapters without being formally introduced. The examples and bibliographic references are outstanding. Trade journals in various fields (advertising, economics) appear as often as more scholarly research journals. Ideal for business majors, yet the pure mathematician could also learn a lot about applications. PJM

GENERAL, S, P, *Topologie et Perception: Tome I: Bases Philosophiques et Mathématiques.* Claude Paul Bruter. Doin, 1974, 254 pp, 64F (P). The basic notion here is that points where a table ceases to be a table, i.e., boundary points, are topologically describable. The philosophical basis--theorem and proofs--is reminiscent of Spinoza in form and about as hard to read. The mathematical basis is a good description of Thom's work on the theory of catastrophes. PJM

GENERAL, T(13: 1), S, *Mathematical Logic and Probability with Basic Programming.* William S. Dorn, Herbert J. Greenberg, Sister Mary K. Keller. Prindle, 1973, viii + 216 pp, $7.50 (P). Begins with an elementary introduction to BASIC. Follows with material on the logic of switching circuits and elementary probability, mostly taken from Dorn and Greenberg's *Mathematics and Computing with Fortran Programming* (TR, January 1968), designed to be learned using the computer as an aid. RSK

PRECALCULUS, T(13: 1), *Programmed College Algebra.* J. Bryan Sperry. Holbrook Pr, 1974, xi + 369 pp, $6.95 (P). A well-organized programmed text. Additional exercises and chapter tests follow each unit. LLK

PRECALCULUS, T(13: 1), *College Algebra*. Edward D. Gaughan. Brooks/Cole, 1974, viii + 472 pp, $10.95. The usual topics of college algebra. Each chapter in introduced with a problem which can be solved by the methods of the chapter. Large number of exercises including chapter reviews. LLK

EDUCATION, T*(13-17: 1, 2), *Mathematics: Concrete Behavioral Foundations*. Joseph M. Scandura. Har-Row, 1971, xx + 459 pp, $11.95; *An Algorithmic Approach to Mathematics: Concrete Behavioral Foundations*, vi + 385 pp, $5.95 (P). Non-routine and carefully done. Designed as an intermediate level mathematics text for elementary and middle school teachers (or for liberal arts students). Introduces student to process abilities in mathematics such as the ability to detect regularities (discovery), to construct examples (particularization), to describe mathematical ideas, as well as traditional content such as sets, relations, logic, algebraic systems, and number systems. Methods commentary throughout. Appendices: elementary school geometry and careful analyses of 10 elementary school textbook series. Accompanying workbook presents tasks, rules for solving, examples of solutions, and exercises; intended to aid poorly prepared or less able students. PSJ

EDUCATION, T(13-14, 17: 1), *The Rational Numbers*. NCTM, 1972, vii + 446 pp, $6 (P). A text for pre- or in-service elementary school teachers. Written to accompany 12 teacher-education films (*Elementary Mathematics for Teachers and Students*, NCTM) but can be used separately for study of rational numbers. Stresses content and pedagogy. Glossary. Answers provided for all exercises. PSJ

EDUCATION, P, L, *Algorithmization in Learning and Instruction*. L.N. Landa. Educ. Tech. Pub., 1974, xxxiii + 713 pp, $14.95. Soviet cybernetics applied to pedagogy: a massive exhortation (with supporting research reports) for methods of instruction based on algorithms as a model of human reasoning. Translated from the 1966 Russian original, it blends cybernetics, psychology, linguistics, and logic into a particular educational philosophy and reinforces this philosophy with results of two decades' research in Soviet schools. LAS

EDUCATION, S, P, *Space Mathematics: A Resource for Teachers*. N.A.S.A. Thomas D. Reynolds, Ed. USGPO, 1972, 138 pp, $2 (P). A collection of 138 mathematics problems (with solutions) related to space science for use as supplementary exercises in grades 9-12. Pre-calculus level. Range from easy 9th grade to very challenging 12th grade. Grouped by topics: algebra, probability, geometry, trigonometry, etc. Excellent resource for teachers. For secondary methods courses and curriculum libraries. Bibliography. PSJ

EDUCATION, T(15-17: 1), *Teaching Modern Mathematics in the Elementary School, Second Edition*. Howard F. Fehr, Jo McKeeby Phillips. A-W, 1972, xviii + 515 pp, $9.50. For methods courses for pre- and in-service elementary school teachers. Assumes basic mathematical knowledge. Mainly curriculum and pedagogy. Revision adds chapters on teaching motion geometry and probability and statistics. Other new topics include role of logical words, laboratory teaching, understanding, and the real world and mathematics. Good exercises and chapter references. Updated bibliography. PSJ

NUMBER THEORY, T(18), P, L. *Foundations of a Structural Theory of Set Addition.* G.A. Freiman. Transl. of Math. Mono., V. 37. AMS, 1973, vii + 108 pp, $15.70. An interesting, well-written book. In the first chapter the author defines isomorphisms of subsets of a set with an algebraic operation, and discusses their properties. From his point of view, additive number theory is the study of properties of sets of numbers which are invariant under isomorphic mappings. Chapter II is devoted to a proof of a theorem concerning subsets of the integers mod p and arithmetic progressions mod p. Chapter III centers on density questions for sums of sequences and on the structure of sets of residues mod p. Some exercises included. SG

NUMBER THEORY, P. *Basic Number Theory, Second Edition.* André Weil. Grund. math. Wissenschaften, B. 144. Springer-Verlag, 1973, xviii + 312 pp, $15.30. This second edition contains minor corrections and notes to the first edition (TR, June 1968), along with a few appendices. SG

NUMBER THEORY, P. *Elementary and Analytic Theory of Algebraic Numbers.* Władysław Narkiewicz. PWN, 1974, 630 pp, $30. Algebraic number theory minus class field theory. The first four chapters are classical: rings of integers, the unit theorem, finiteness of class number, relative extensions. The remaining five chapters focus on local and analytic methods, harmonic analysis on local fields (Tate's thesis), Kronecker's theorem, Chebotarev's density theorem, class number formulas, and the Siegel-Brauer theorem. Extensive notes on each chapter. Over 150 pages of bibliography. An exceptionally useful reference work. SG

LINEAR ALGEBRA, T*(13-14: 1), L. *Constructive Linear Algebra.* Allan Gewirtz, Harry Sitomer, Albert W. Tucker. P-H, 1974, xi + 493 pp, $13.95. A refreshing new slant on elementary linear algebra--beginning with linear programming and focusing throughout on the computational and theoretical significance of tableaux and associated pivot operations. With only minor supplements or deletions, this book would be suitable either for courses in finite mathematics or elementary linear algebra. Computer methods are not discussed but the algorithmic spirit stands out in nearly every section. LAS

ALGEBRA, S(17-18), P. *The Algebraic Theory of Modular Systems.* F.S. Macaulay. Cambridge Tracts in Math. and Math. Physics, No. 19. Hafner, 1964, xiv + 112 pp, $4. This work, originally published in 1916, should be of considerable interest to commutative algebraists, and should provide some perspective on what the subject is all about. LCL

ALGEBRA, P*, L. *Reviews on Infinite Groups.* Ed: Gilbert Baumslag. AMS, 1974. *Part 1:* xii + 514 pp, $70 set (P); *Part 2:* 547 pp. Reprints of 4563 reviews on infinite discrete group theory from Vols. 1-40 (1940-1970) of *Mathematical Reviews* arranged under 24 major and 264 minor headings. An invaluable research tool. LAS

CALCULUS, T(13: 1, 2). *Calculus: An Introductory Course.* M.J. Mansfield. Prindle, 1972, ix + 388 pp, $10.95. From preface: "This book has been written for the engineering technician and other students of applied science. It may be short on rigor, but it is not lacking in vigor... Proofs as such are not employed in the book. Nevertheless, informal heuristic discussions are provided to give the student insight into the why of the subject." A programmed guide is available. RBK

CALCULUS, T(13: 1, 2), S, *Programmed Guide to Accompany Calculus: An Introductory Course*. J. Bryan Sperry. Prindle, 1972, 183 pp, $3.95 (P). Programmed exercises for use in conjunction with Mansfield's *Calculus: An Introductory Course*. Branches lead student to a discussion of one of several possible wrong answers. RBK

REAL ANALYSIS, T(16-17: 1), S, *Introduction to Measure and Integration*. S.J. Taylor. Cambridge U Pr, 1966, vi + 266 pp, $5.95 (P). A straightforward classical development of Lebesgue integration theory (via monotone sequences of simple functions) concluding with such topics as the Riesz-Fischer theorem, the Daniell integral and Haar measure. A reprint of the first 9 chapters of Kingman and Taylor's 1966 *Introduction to Measure and Probability*. A good buy. LAS

COMPLEX ANALYSIS, P, *Lecture Notes in Mathematics-336: L'Analyse Harmonique dans le Domaine Complexe*. Ed: E.J. Akutowicz. Springer-Verlag, 1973, v + 169 pp, $8.10 (P). Proceedings, mostly in English, from a meeting at Montpellier, France in September 1972. JAS

COMPLEX ANALYSIS, P, *Introducere în teoria grupurilor Klein*. Dumitru Ivascu. Editura Academiei Rep. Soc. Romania, 1973, 168 pp, (P). Klein groups, automorphisms with respect to Klein groups and Teichmüller spaces; in Rumanian. JAS

DIFFERENTIAL EQUATIONS, T(16-17: 2, 3), L, *Ordinary Differential Equations*. Philip Hartman, P.O. Box 7162, Baltimore, MD 21218. 1973, xiv + 612 pp, $10 (P). Corrected paperback private reprint of 1964 Wiley original--at half the original cost. Includes a substantial "basic course" together with numerous probes into various applications. Diverse exercises; hints, extensive bibliography. LAS

DIFFERENTIAL EQUATIONS, P, *Lecture Notes in Mathematics-309: Topics in Stability and Bifurcation Theory*. David H. Sattinger. Springer-Verlag, 1973, vi + 190 pp, $5.80 (P). A survey of recently developed methods, these notes were the basis of a course at the University of Minnesota. JAS

NUMERICAL ANALYSIS, S(16-17), P, *Sparse Matrices*. Reginald P. Tewarson. Math. in Sci. and Eng., V. 99. Acad Pr, 1973, xv + 160 pp, $11.95. A good exposition of methods of computation for large, sparse matrices. Storage schemes. Elimination and orthogonalization methods. Eigenvalues and eigenvectors. Tearing and bifactorization. Good bibliography. RWN

NUMERICAL ANALYSIS, P, *Topics in Numerical Analysis*. Ed: John J.H. Miller. Acad Pr, 1973, xviii + 348 pp, $19.50. Invited papers from a 1972 conference at University College, Dublin, including an historical discourse on "Computing through the Ages" by C. Lanczos. LAS

FUNCTIONAL ANALYSIS, T(16-17: 1), S, L*, *Functional Analysis: A Short Course*. Edward W. Packel. Intext, 1974, xvii + 172 pp, $10. A carefully selected subset of traditional functional analysis designed to illustrate its beauty, applicability and unifying power to mathematically mature undergraduates. The text is unusually lean, relying exclusively on the formal definition-theorem-proof-exercise style. Would likely be an excellent pre-graduate school primer for the well motivated, but may appear quite dry to the less well motivated. LAS

FUNCTIONAL ANALYSIS, T(18: 1), S, P. *Lecture Notes in Mathematics-355: A Theory of Semigroup Valued Measures.* Maurice Sion. Springer-Verlag, 1973, 140 pp, \$6.20 (P). Theory was developed to yield representation theorems for vector valued functions and has led to a re-examination of the notions of measurable function and integral. Development is via outer measure *à la* Carathéodory although the integral is defined directly. The theory of differentiation is in terms of a limit of ratios using the notion of differentiation basis which is shown to always exist. A sequel will emphasize applications. RBK

OPTIMIZATION, T*(14-17: 2), S, P, L*. *Programming and Probability Models and Operations Research.* Donald P. Gaver, Gerald L. Thompson. Brooks/Cole, 1973, xiii + 683 pp, \$17.95. Comprehensive selection of optimization tools and models, incorporating recent developments. Self-contained: linear algebra and probability are systematically developed, leading to programming (convex sets, linear and non-linear programming) and stochastic models (decision models, waiting-lines, inventory theory, Markov chains, simulation). Plenty of material here! Flow diagrams used to advantage though scope of book excludes computing. Calculus presumed. Clearly written, in a style generous in motivation. Examples lack the punch that might be attained by illustrating with actual usages and real (not "realistic") data. Marred by a sexist example on pp. 4-5. PJC

OPTIMIZATION, T(14-16: 1), S. *Notes on Linear Programming.* M. Sakarovitch. Van N-Rein, 1971, 175 pp, \$3.95 (P). A brief computer-oriented introductory course: duality, simplex method, economic interpretation, parametric programming. LAS

ANALYSIS, S(17-18), P, L. *Recent Results on Function Algebras.* Irving Glicksberg. CBMS Reg. Conf. in Math., No. 11. AMS, 1972, 38 pp, \$3.50 (P). Notes from expository lectures at Fayetteville, Arkansas, June 1971. Highlights of old and new results concerning interpolation, orthogonal measures, rational approximation and C(X). LAS

ANALYSIS, P. *Spectral Theory and Complex Analysis.* Jean Pierre Ferrier. Math. Stud., V. 4. North Holland, 1973, xii + 93 pp, \$7 (P). Application of Waelbroeck's spectral theory of b-algebras to algebras of holomorphic functions on pseudoconvex domains of C^n. Notes from lectures given at Collège de France in 1971, rewritten in English by the author. LAS

ANALYSIS, P. *Analiza.* Krzysztof Maurin. Biblioteka Matematyczna. Tom 38, 41. PWN. *Czesc I, Elementy*, 1973, 486 pp; *Czesc II, Wstep do analizy globalnej*, 1971, 490 pp. Modern analysis with an introduction to global analysis for those who prefer it in Polish. JAS

ALGEBRAIC GEOMETRY, P. *Elliptic Functions.* Serge Lang. A-W, 1973, xii + 326 pp, \$17.50. A valuable exposition of classical and modern aspects of elliptic functions. Part one concerns elliptic curves over C, their morphisms and points of finite order; the j-invariant; modular functions. Part two: complex multiplication; Shinura's reciprocity law; ℓ-adic and p-adic representations. Part three: elliptic curves with non-integral invariant; the Tate parametrization; division points over number fields. Part four: Theta functions; Kronecker limit formulas. An appendix (by Tate) on formulas for

elliptic curves in arbitrary characteristic, and one (by Lang) on traces. Cartier operator, and Hasse invariant. An extremely useful reference. SG

ALGEBRAIC GEOMETRY, P. *Lecture Notes in Mathematics-352: Theta Functions on Riemann Surfaces.* John D. Fay. Springer-Verlag, 1973, 137 pp, \$6.20 (P). Some topics: relations between theta functions and Abelian differentials, theta functions and degenerate Riemann surfaces, Schottley relations for surfaces of special moduli, and theta functions on finite bordered Riemann surfaces. SG

GEOMETRY, T(18: 2). *Elementare Differentialgeometrie.* W. Blaschke, K. Leichtweiss. Springer-Verlag, 1973, x + 369 pp, \$43.20. Called the "fifth edition", this is a major reworking by the second author of a classic text by the now deceased first author. The presentation has a classical (geometric) flavour but uses modern methods and intuition. A substantial text with exercises for the advanced student. JAS

GEOMETRY, T(18: 1, 2), S*(18), P. *Lecture Notes in Mathematics-335: Beweismethoden der Differentialgeometrie im Grossen.* H. Huck, *et al.* Springer-Verlag, 1973, 168 pp, \$8.10 (P). Carefully designed to introduce the advanced student to some of the important methods of modern global differential geometry. The presentation, unbroken by problems, requires experience with introductory differential geometry. Although it lacks an index, it appears to be an excellent supplementary volume to a course. Includes classical motivation, intuitive insight, lots of notation and consciousness of purpose. JAS

GEOMETRY, P. *Discontinuous Groups and Riemann Surfaces.* Ed: Leon Greenberg. Annals Math. Stud., No. 79. Princeton U Pr, 1974, ix + 443 pp, \$12 (P). Proceedings of the May 1973 Conference at U. Maryland. LAS

TOPOLOGY, T*(14-16: 1, 2), S, L. *Topology.* Murray Eisenberg. HR&W, 1974, xiv + 427 pp, \$16. An elementary but serious topology book that is carefully crafted throughout. The needs of analysts and geometric topologists are met with unusual balance. Numerous indices, interrelation charts and notes are provided to help the instructor fit the book to a number of undergraduate course situations. Ordinal numbers and Zorn's lemma are avoided and the topology proper begins with metric spaces in order to limit the pre-requisites to calculus only. JAS

TOPOLOGY, P. *Feuilletages: Résultats Anciens et Nouveaux (Painlevé, Hector et Martinet).* Georges H. Reeb. Pr U Montreal, 1974, 70 pp, \$5 (P). A collection of papers on foliations. PJM

TOPOLOGY, P. *New Developments in Topology.* Ed: Graeme Segal. London Math. Soc. Lect. Notes, No. 11. Cambridge U Pr, 1974, 128 pp, \$7.95 (P). Proceedings of a symposium on algebraic topology at Oxford, June 1972. LAS

PROBABILITY, P. *Probabilistic Methods in Applied Mathematics, V. 1.* Ed: A.T. Bharucha-Reid. Acad Pr, 1968, x + 291 pp, \$16.75. Random eigenvalue problems by W.E. Boyce; Wave propagation in random media by U. Frisch; and Branching processes in neutron transport theory by T.W. Mullikin. (*V. 2*, TR, June/July 1970; *V. 3*, TR, March 1974).LAS

PROBABILITY, P, *Lecture Notes in Mathematics-272; Positive Definite Kernels, Continuous Tensor Products, and Central Limit Theorems of Probability Theory.* K.R. Parthasarathy, K. Schmidt. Springer-Verlag, 1972, vi + 107 pp, \$5.10 (P). Continuous tensor products studied in terms of positive definite kernels with invariance properties under a group action lead to a unified approach to the central limit problems of probability theory, the theory of stochastic processes with stationary increments and the construction of free fields in quantum mechanics. JAS

PROBABILITY, P, *Lecture Notes in Mathematics-296: Probability and Information Theory II.* Ed: M. Behara, K. Krickeberg, J. Wolfowitz. Springer-Verlag, 1973, 223 pp, \$7.40 (P). A sequel to *Lecture Notes in Mathematics-89* (1969) containing nine invited lectures at seminars sponsored jointly by McMaster U. and U. Montreal. LAS

STATISTICS, P*, *The Statistics Cum Index.* James L. Dolby, John W. Tukey. R&D Pr, 1973, xviii + 498 pp, \$28.80. Volume 1 of the Information Access Series, a series designed to provide rapid and effective access to the literature of specialized fields. Gives a cumulative index of 113 selected books in statistics, thus providing a quick list of references, including page numbers, to virtually every standard statistical topic. Usefulness is limited by the selection, which may include many books one does not have available and may omit others one would wish were included. RSK

STATISTICS, P**, *Index to Statistics and Probability: The Citation Index.* John W. Tukey. R&D Pr, 1973, xxx + 1269 pp, \$60. Volume 2 of the information Access Series; first of five volumes which will provide virtually complete coverage of the literature through 1966. This voluminous work provides coded listings of essentially all items (papers, books, reports, etc.), including abstracts and reviews, which cite a given item. RSK

STATISTICS, P, *Symmetric Functions in Statistics.* Ed: Derrick S. Tracy. U of Windsor, 1972, iii + 224 pp, \$6.80 (P). Proceedings of a 1971 symposium in honor of Professor Paul S. Dwyer. Four invited and seven shorter contributed papers. LAS

STATISTICS, T*(16-17: 2), *A First Course in Mathematical Statistics.* George G. Roussas. A-W, 1973, xv + 506 pp, \$16.50. Sophisticated introductory text. Presumes no measure theory or complex analysis, but introduces and uses measurability concepts and the characteristic function, and proves almost all theorems. Many examples and problems, mostly theoretical in nature. RSK

STATISTICS, T(13-14: 1), *A First Course in Probability and Statistics.* Henrick J. Malik, Kenneth Mullen. A-W, 1973, xii + 361 pp, \$9.95. Contains the usual topics, including the analysis of variance and nonparametric statistics. Differs from other non-calculus based texts in its more extensive coverage of probability. Some sophistication is required. RSK

STATISTICS, T(13: 1), *Elementary Statistics.* Robert R. Johnson. Duxbury Pr, 1973, xvi + 480 pp, \$10. Precalculus statistics for non-math majors. Very attractive format: each chapter begins with an outline of topics and goals, ends with numerous exercises, a

problem set and a self-quiz. Each topic is clearly motivated and attractively presented. Extensive use of color to identify important concepts and examples. Less detail and fewer topics than in many similar texts. TAV

STATISTICS, T(13-14: 1, 2). *Fundamental Research Statistics for the Behavioral Sciences.* John T. Roscoe. HR&W, 1969, xv + 336 pp, $11. For students in education and psychology. Non-mathematical presentation of statistics including several non-parametric tests and discussions of psychological measurement and testing. The problem sets are quite short. FLW

STATISTICS, S(13-15), P. *Standard Statistical Calculations, Second Edition.* P.G. Moore, D.E. Edwards, Eryl A.C. Shirley. Halsted Pr, 1972, xi + 123 pp, $6.75. Designed originally as a manual for business and industry, it gives detailed numerical examples illustrating the more common statistical procedures. Theory is omitted, but referenced. Includes some exercises and answers. RSK

STATISTICS, S, P, L. *How to Find Out About Statistics.* Gillian A. Burrington. Pergamon Pr, 1972, ix + 153 pp, $9.25. Miscellaneous and somewhat motley collection of information. First half ranges from careers and training in statistics to how to use a library to sources of books and periodicals. Last half is primarily devoted to sources of social and economic statistical data, mainly in the United Kingdom and the U.S. RSK

STATISTICS, T*(15). *Understanding Statistical Reasoning: How To Evaluate Research Literature in the Behavioral Sciences.* Eleanor Walker Willemsen. Freeman, 1974, xi + 223 pp, $10. The author's goal is to provide the necessary reasoning and logic to enable students who have had an introductory statistics course to use their background effectively in dealing with data both published and raw. Given the limited scope of most precalculus statistics courses, this book seems ideally suited to its audience and its purpose. TAV

STATISTICS, P. *Fitting Equations to Data: Computer Analysis of Multifactor Data for Scientists and Engineers.* Cuthbert Daniel, Fred S. Wood. Wiley, 1971, xiv + 342 pp, $18.25. User's guide to linear least squares including a canned computer program. Concentrates on practical considerations such as selection of the variables and disposition of data points. Brief discussion of nonlinear problems. Many pages of data, graphs and computer output. RWN

STATISTICS, P. *Stochastic Geometry: A Tribute to the Memory of Rollo Davidson.* Ed: E.F. Harding, D.G. Kendall. Wiley, 1974, xiii + 400 pp, $29.95. Various papers by different authors, including several by Rollo Davidson, unified by an opening chapter by Kendall designed to introduce stochastic geometry to the non-specialist. A companion volume to *Stochastic Analysis* (TR, March 1974). LAS

COMPUTER SCIENCE, T(15-16: 2), L. *Standard COBOL: A Problem-Solving Approach.* Marilyn Z. Smith. HM, 1974, 264 pp, $7.95 (P). An excellent textbook for either an elementary or intermediate course in COBOL. Many examples and problems but no solutions. Advanced topics, e.g., file manipulations and sorting, are covered in the last part of the manual and there is a good chapter on flowcharting. RB

COMPUTER SCIENCE, T(14-15: 1), S, *A Course in APL/360 with Applications.* Louis D. Grey. A-W, 1973, xviii + 332 pp, $7.50 (P). APL (A Programming Language) designed by Kenneth Iverson is one of the more mathematically interesting programming languages because of its conciseness and its ability to operate easily on arrays. This text is intended to be a reference and primer. Many of the exercises are applications to scientific problems. Includes primitive functions, array operations, user-defined functions and system commands. RWN

COMPUTER SCIENCE, S, *Scheduling and Control for Industry and Government.* M. Tainiter. Timetable Pr, 1971, 49 pp, $1.85 (P). Informal discussion of the "method of timetables," the Gantt Chart, and critical path methods in business. FLW

COMPUTER SCIENCE, S(13-14), L, *101 BASIC Computer Games.* Ed: David H. Ahl. Digital, 1973, 249 pp, $5 (P). Possible subtitle: 101 ways to waste valuable computer time. Complete listing and sample run of each game (e.g., NIM, QUBIC, LIFE). A good source of interesting programs involving non-numerical techniques. LAS

COMPUTER SCIENCE, S(16-18), P, L, *A Mathematical Theory of Global Program Optimization.* Marvin Schaefer. P-H, 1973, xvii + 198 pp, $9.95. While not a complete survey, this is a very readable and yet mathematically rigorous view of some quite sophisticated methods for compiler-level optimization. Especially enjoyable because of the solid theoretical basis given to intuitive and non-intuitive techniques. Excellent for programmers, theoreticians and computer science students. DK

COMPUTER SCIENCE, T(15-16: 2), S, P, L, *Elementary Computer Programming in Fortran IV.* Boris W. Boguslavsky. Reston, 1974, x + 325 pp, $8.75 (P). This is an exceptionally clear and well written book on a programming language. It gives all the necessary details without becoming too bulky. There are many sample programs and exercises. The section on subprograms is very well written and there are some interesting advanced topics discussed, such as computer graphics, magnetic tape and disks. RB

COMPUTER SCIENCE, T(13-16: 1), S, P, L, *COBOL: A Simplified Approach.* Seymour C. Hirsch. Reston, 1974, xii + 211 pp, $8.95; $6.95 (P). A fine text which covers all the main aspects of COBOL without getting into unnecessary details. There are questions and answers, but few long programming problems. Would be recommended for a short introductory course. RB

COMPUTER SCIENCE, T(13-14: 2), L, *Fortran IV Programming and Applications.* C. Joseph Sass. Holden-Day, 1974, x + 350 pp, $7.95 (P). As far as the many FORTRAN IV textbooks go this one appears better than many. There are answers to problems in the back and sections on WATFOR-WATFIV and keypunches. There are quite a few complete example programs included. This would make a good textbook, but the price is a little high for a paperback. RB

COMPUTER SCIENCE, T(14-15: 1), *Introduction to Computer Science.* Michael Levison, W. Alan Sentance. Gordon, 1968, 159 pp, $12. A brief introduction to several topics: machine language, an extension of Mercury Antocode, Algol, programming techniques and compilation. Exercises. RWN

COMPUTER SCIENCE, T(13-14; 2), L. *Introduction to Computer Programming, Second Edition.* Donald I. Cutler. P-H, 1972, x + 303 pp, $11. One of the main features of this book is EX-1, a hypothetical computer and instruction set, something like MIX in Knuth on a more elementary level. Would be a beautiful text for high school or college students lacking a "hands-on" computer. A wide variety of subjects are covered such as number systems, flow diagrams, programming systems; there are 70 pages of problem solutions in the back. RB

COMPUTER SCIENCE, P. *Combinatorial Algorithms.* Ed: Randall Rustin. Algorithmics Pr, 1973, 126 pp, $8. Papers from the ninth Courant Computer Science Symposium, January 1972. LAS

COMPUTER SCIENCE, P. *Computational Complexity.* Ed: Randall Rustin. Algorithmics Pr, 1973, 268 pp, $10. 15 papers from the seventh Courant Computer Science Symposium, October 1971. Includes a major subject coded bibliography on computational complexity. LAS

COMPUTER SCIENCE, T(13-14), S, L. *Principles of FORTRAN Programming.* DeLos F. DeTar. Benjamin, 1972, ix + 136 pp, $4.95 (P). This FORTRAN manual has some nice features that most such manuals lack, e.g., a chapter on BASIC programming techniques, many complete programs, many programming suggestions. There are two good chapters on Input and Output. The only criticism would be on the small number of questions and answers. RB

COMPUTER SCIENCE, S, P, L. *Structured Programming.* O.-J. Dahl, E. W. Dijkstra, C.A. R. Hoare. APIC Stud. in Data Proc., No. 8. Acad Pr, 1972, viii + 220 pp, $12.50. Three interesting, enlightening essays on data and programming structures. Should be read and readable by any practicing programmer. Envokes an appreciation for the consideration of higher level structures to solve practical programming problems. RWN

COMPUTER SCIENCE, P. *Lecture Notes in Economics and Mathematical Systems-83: Cognitive Verfahren und Systeme.* Springer-Verlag, 1973, viii + 373 pp, $10.40 (P). The papers presented at a special conference on artificial intelligence at Hamburg in April 1973, sponsored by the Gesellschaft für Informatik and the Nachrichtentechnische Gesellschaft. JAS

COMPUTER SCIENCE, S?. *Business Data Processing.* Stuart E. Fink, Barbara J. Burian. Appleton, 1974, xiv + 351 pp, $10.50. Tries to cover a lot of ground yet only the very main points are outlined. Computers are described in general, a few languages (COBOL, FORTRAN, and PL/1) are described and only once chapter on data processing is presented (Inventory Systems). Could not be recommended as a text. but would be good reading for inexperienced personnel. RB

COMPUTER SCIENCE, T(13), S(13-16), P, L. *Computers and Programming Guide for Engineers.* Donald D. Spencer. Howard Sams, 1973, 288 pp, $12.95 (P). An elementary text including questions, solutions and a good outline of elementary FORTRAN and BASIC. The remainder makes interesting reading on such topics as real-time, timesharing, hybrid computers and simulation languages. The price does seem rather high. RB

COMPUTER SCIENCE, P, *Finite Automata: Behavior and Synthesis.* B.A. Trakhtenbrot, Ya. M. Barzdin. Fund. Stud. in Comp. Sci., V. I. North Holland, 1973, xi + 321 pp, $15.50. Includes useful comments by the translator, D. Louvich, and the editors, E. Shamin and L. Landweber. Analyzes automata from a functional rather than structural viewpoint. Contains many important results, especially on existence of algorithms. The first part studies finite-state machines and finite-state operators. A metalanguage is established which is based on monadic second-order predicate logic. The second part studies problems of identification and synthesis. Obtains statistical estimates for the degree of distinguishability, accessibility and reproducibility. RWN

SYSTEMS THEORY, T(17-18: 2), S, P, *Bilinear Control Processes: With Applications to Engineering, Ecology, and Medicine.* Ronald R. Mohler. Acad Pr, 1973, xi + 224 pp, $17.50. The author considers a class of systems whose behavior is governed by a set of differential equations which are linear with respect to the state vector and with respect to the control vector, but not jointly linear as a function of both state and control. The first chapters establish a theoretical base for such systems, while the concluding chapters are concerned with applications. A wide range of applications includes nuclear, ecological, and socioeconomic systems. JJ

SYSTEMS THEORY, P, *Systems Theory Research (Problemy Kibernetiki), V. 23.* Ed: A.A. Lyapunov. Consultants, 1973, vi + 315 pp, $37.50 (P). 17 papers including networks with delays, minimality of circuits, generalized automata and control processes in organisms. RWN

SYSTEMS THEORY, T(16-17: 1), *Modern Control System Theory and Application.* Stanley M. Shinners. A-W, 1973, xiv + 528 pp, $14.95. Especially useful to appliers of control. Approaches control theory using transfer functions and state variables. Includes performance criteria, stability, feedback and optimal control. Mathematical bases are treated lightly. Interesting, diverse applications. Good exercises, half of which have answers in the back. RWN

SYSTEMS THEORY, S(16-17), P, *Foundations of the Theory of Learning Systems.* Ya. Z. Tsypkin. Transl: Z.J. Nikolic. Math. in Sci. and Eng., V. 101. Acad Pr, 1973, xiii + 205 pp, $15. Deeper and narrower than *Adaptation and Learning in Automatic Systems* by the same author. Considers approaches to the design of learning systems related to pattern recognition and learning filters. Algorithms of learning, statistical decision theory, and applications. Good bibliography. RWN

APPLICATIONS (BIOLOGY), P, *Stochastic Processes and Applications in Biology and Medicine.* M. Iosifescu, P. Tautu. Biomathematics. V. 3 & 4. Springer-Verlag, 1973. *V. I: Theory,* 331 pp; *V. II: Models,* 337 pp, $21.80 each. Intended for mathematicians and biologists with strong mathematical backgrounds. Expanded versions of the earlier 1968 Rumanian editions. Volume I treats the theory of stochastic processes of most interest in biology, i.e., Markov chains, diffusion processes, semi-Markov processes. Volume II, surveys, with abundant references, models for population growth, evolutionary processes, epidemics. TAV

APPLICATIONS (COMMUNICATIONS), P, *Proceedings of the Symposium on Computer-Communications, Networks and Teletraffic.* Ed: Jerome Fox. Wiley, 1972, xxii + 664 pp, $30. 60 papers from Brooklyn Polytechnic, April, 1972; includes a lot of diverse mathematical applications. LAS

APPLICATIONS (ENGINEERING), S(15). *Binary Sequences*. G. Hoffmann de Visme. English U Pr, 1971, x + 118 pp, $1.25 (P). Mathematical foundations and applications; linear filters; registers, especially linear feedback shift registers; linear prime polynomials; sampling, noise and drift. Intended for electrical engineering students. RWN

APPLICATIONS (PHYSICS), S*(13-14), L. *Mr. Tompkins in Paperback*. G. Gamow. Cambridge U Pr, 1965, xii + 186 pp, $1.95 (P). 1973 reprint of 1965 edition which combined the 1940 *Mr. Tompkins in Wonderland* and the 1944 *Mr. Tompkins Explores the Atom* with three new stories (on fission and fusion, steady state universe, and elementary particles). An entertaining, informative classic replete with memorable analogies and valuable insights. LAS

APPLICATIONS (PHYSICS), P. *Metode Topologice în Mecanica Clasică*. Andrei Iacob. Editura Academiei Rep. Soc. Romania, 1973, 173 pp, (P). A study of classical mechanics looking at it through modern topology. In Rumanian. JAS

APPLICATIONS (PHYSICS), P. *Causality and Dispersion Relations*. H.M. Nussenzveig. Acad Pr, 1972, xii + 435 pp, $26. An introductory monograph on dispersion relations in classical and non-relativistic quantum theory, with special emphasis on the bearing of this theory on concepts of causality. Uses standard techniques of functional analysis. LAS

APPLICATIONS (PHYSICS), T(16-18), P, L. *Unitary Symmetry and Elementary Particles*. D.B. Lichtenberg. Acad Pr, 1970, xiii + 246 pp, $13. A systematic elementary introduction to that part of group theory (e.g., representations, Lie groups, SU(3), Young tableaux, Clebsch-Gordon coefficients) which is most useful in contemporary particle physics. Concludes with chapters on the Eightfold Way and on the Quark Model. LAS

APPLICATIONS (PHYSICS), P. *Functional Methods and Models in Quantum Field Theory*. H.M. Fried. Mit Pr, 1972, ix + 214 pp, $8.95. A unified presentation of major dynamic models of interacting fields employing the functional methods of Feynman, Schwinger, *et al.* Photocopied from a typed manuscript; correspondingly low price. LAS

APPLICATIONS (PHYSICS), P. *Mathematical Aspects of Statistical Mechanics*. Ed: James C.T. Pool. SIAM-AMS Proc., V. V. AMS, 1972, ix + 89 pp, $8. Seven papers from a 1971 symposium held in New York City. LAS

APPLICATIONS (PHYSICS), P. *Recent Developments in Mathematical Physics*. Ed: Paul Urban. Springer-Verlag, 1973, vi + 610 pp, $57. Proceedings of the XII Internationale Universitätswoches für Kernphysik et Steiermark, Austria, February 1973. 12 papers, all in English. LAS

Reviewers Whose Initials Appear Above

Ralph Bjork, St. Olaf; Paul J. Campbell, St. Olaf; Steven Galovich, Carleton; James Johnson, St. Olaf; Paul S. Jorgensen, Carleton; Lorraine L. Keller, St. Olaf; Roger B. Kirchner, Carleton; Richard S. Kleber, St. Olaf; Dean Krafft, Carleton; Loren C. Larson, St. Olaf; Pierre J. Malraison, Jr., Carleton; R.W. Nau, Carleton; J. Arthur Seebach, Jr., St. Olaf; Lynn A. Steen, St. Olaf; T.A. Vessey, St. Olaf; Frank L. Wolf, Carleton.

NEWS AND NOTICES

EDITED BY RAOUL HAILPERN, SUNY at Buffalo

Readers are invited to contribute to the general interest of this Department by sending news items to Mathematical Association of America, 1225 Connecticut Avenue, NW, Washington, D. C. 20036. Items must be submitted at least two months before publication can take place.

PERSONAL ITEMS

California Institute of Technology: Drs. Joel Anderson and Colin Bennett, Bateman Research Instructors at Caltech, have been appointed Assistant Professors; Dr. Stephen Smith, Oxford University, has been appointed Bateman Research Instructor; Dr. R. A. Dean received an honorary D. Sc. Degree from Denison University in June 1973.

University of Hawaii: Dr. R. C. Olson, University of Washington, has been appointed Assistant Professor; Assistant Professors G. L. E. Csordas and H. M. Hilden have been promoted to Associate Professors.

Middle Tennessee State University: Associate Professors George Beers, Richard McCord, and Thomas Vickrey have been promoted to Professors; Assistant Professors Thomas Forrest and Earl Keese have been promoted to Associate Professors.

University of South Carolina: Assistant Professor W. C. Chewning, Naval Postgraduate School, Monterey, has been appointed Assistant Professor; Dr. C. C. Hughes, North Carolina State University, has been appointed Visiting Assistant Professor; Dr. D. M. Jordan has been appointed Assistant Professor; Visiting Assistant Professor Eugene Norris has been appointed Assistant Professor; Assistant Professors T. L. Markham and J. S. Yang have been promoted to Associate Professors.

Virginia Commonwealth University: Associate Professor R. L. Causey, University of Missouri at Columbia, has been appointed Associate Professor; Dr. W. A. Glynn, Chairman of the Department of Mathematical Sciences, has been promoted from Associate Professor to Professor.

Professor John Baum will be spending three semesters, from January 1974 through June 1975, at the University of New South Wales, Australia, as Honorary Visiting Professor.

Dr. Gordon Raisbeck, Arthur D. Little, Inc., Systems Engineering Section, has been named a Corporate Vice President.

Associate Professor Augustus Frank Bausch, Kalamazoo College, died on June 11, 1973, at the age of 52. He was a member of the Association for seven years.

Mr. Martin Blumberg, Los Altos, California, died on July 23, 1973, at the age of 59. He was a member of the Association for fifteen years.

Professor George C. Caldwell, Georgia Tech, died on October 26, 1973, at the age of 54. He was a member of the Association for thirteen years.

Professor Emeritus Griffith C. Evans, UCB, California, died on December 8, 1973, at the age of 86. He was a member of the Association for fifty-three years.

Dr. Castle W. Foard, Rochester, New York, died on November 6, 1973, at the age of 76. He was a member of the Association for forty-two years.

Dr. Robert L. Poe, Berry College, Georgia, died on May 4, 1973, at the age of 43. He was a member of the Association for fourteen years.

Dr. Charles V. L. Smith, Chevy Chase, Maryland, died on January 9, 1973. He was a member of the Association for thirty-one years.

MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

BUSINESS MEETING OF THE NORTHERN CALIFORNIA SECTION

The annual business meeting of the Northern California Section of the MAA was held in conjunction with the annual joint meetings of the AMS-MAA in San Francisco, California, January 19, 1974. Professor Craig Comstock of the Naval Postgraduate School, Chairman of the Section, presided.

Professor Kenneth Rebman of California State University, Hayward, was elected Vice-Chairman; Professor Newman Fisher of San Francisco State University was elected Secretary-Treasurer to replace Professor G. L. Alexanderson who has resigned. Professor Comstock read a report on the Section's Visiting Secondary School Lecturer Program in the absence of Professor L. F. Klosinski, director of that program. He also reported on the Contest administered in Hawaii by Professor Richard K. Coburn of the Church College of Hawaii.

G. L. ALEXANDERSON
Secretary-Treasurer

1975 SUBSCRIPTION PRICES FOR MAA JOURNALS

In the Fall of 1974 the Association will become the publisher of the TWO-YEAR COLLEGE MATHEMATICS JOURNAL (TYCMJ). The Association will then be the publisher of three journals: THE AMERICAN MATHEMATICAL MONTHLY (MONTHLY), THE MATHEMATICS MAGAZINE (MATH MAG), and the TYCMJ.

These publications will be available at the following prices for members and nonmembers, effective for all subscriptions beginning in 1975. We are proud to continue the Association's long tradition of offering the lowest subscription prices consistent with balanced operating budgets.

	MONTHLY	MATH MAG	TYCMJ
Individual Subscribers:			
MAA Member	Privilege of Membership	\$ 7.00	\$ 5.00
Non-Member	\$25.00	\$10.00	\$ 7.00
Institutions and Libraries:	\$25.00	\$10.00	\$10.00 each, first two copies; \$ 3.00 each, additional copies; all to same address

We extend a special invitation to members of the Association with an interest in the teaching of mathematics at the freshman and sophomore levels, whether in a two-or four-year college, to become subscribers to the TYCMJ.

SUMMER CONFERENCE OF THE MICHIGAN SECTION

There will be a Summer Conference on Algebra and Number Theory at Northern Michigan University on July 29 — August 2, 1974. Lecturers are Irving Kaplansky and Hans J. Zassenhaus. Further information can be obtained by writing to William S. Mutch, Department of Mathematics, Northern Michigan University, Marquette, MI 49855.

CHANGE OF ADDRESS

AMERICAN MATHEMATICAL MONTHLY

OLD ADDRESS

Please affix address label from a recent issue here. The codes it contains are essential for prompt and accurate processing of your requested change.

NEW ADDRESS

NAME

ADDRESS

CITY STATE ZIP

Please return this form to

The Mathematical Association of America
1225 Connecticut Ave., NW
Washington, D.C. 20036

CALENDAR OF FUTURE MEETINGS

Summer Meeting 1974: There will be no joint summer meeting in 1974, in order that mathematicians may attend the International Congress of Mathematicians to be held in Vancouver, British Columbia, August 21–29, 1974.

Fifty-eighth Annual Meeting, Washington, D. C., January 25–27, 1975.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Editorial Director.

ALLEGHENY MOUNTAIN
FLORIDA
ILLINOIS
INDIANA
IOWA
KANSAS
KENTUCKY
LOUISIANA-MISSISSIPPI
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA
METROPOLITAN NEW YORK
MICHIGAN
MISSOURI
NEBRASKA
NEW JERSEY, Princeton University, Princeton, October 12, 1974.
NORTH CENTRAL
NORTHEASTERN, Lowell Technological Institute, Lowell, Massachusetts, November 30, 1974.
NORTHERN CALIFORNIA, Chabot College, Hayward, February 1975.
OHIO
OKLAHOMA-ARKANSAS
PACIFIC NORTHWEST, University of British Columbia, Vancouver, August 21–24, 1974 (business meeting only—no general meeting).
PHILADELPHIA, Swarthmore College, Swarthmore, Pennsylvania, November 23, 1974.
ROCKY MOUNTAIN
SEAWAY, St. John Fisher College, Rochester, New York, November 1–2, 1974.
SOUTHEASTERN
SOUTHERN CALIFORNIA
SOUTHWESTERN
TEXAS
WISCONSIN

FUTURE MEETINGS OF OTHER ORGANIZATIONS

AMERICAN ASSOCIATION FOR THE ADVANCEMENT OF SCIENCE
AMERICAN MATHEMATICAL SOCIETY, Washington, D. C., January 23–26, 1975.
AMERICAN SOCIETY FOR ENGINEERING EDUCATION
ASSOCIATION FOR COMPUTING MACHINERY, San Diego, California, November 11–13, 1974.
ASSOCIATION FOR SYMBOLIC LOGIC, Shoreham Hotel, Washington, D. C., January 23–24, 1975.
FIBONACCI ASSOCIATION
INSTITUTE OF MATHEMATICAL STATISTICS
MU ALPHA THETA, University of Arkansas, Fayetteville, August 4–7, 1974.
NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS, Washington, D. C., January 25–26, 1975 (joint meeting with MAA).
OPERATIONS RESEARCH SOCIETY OF AMERICA, San Juan, Puerto Rico, October 16–18, 1974.
PI MU EPSILON, Western Michigan University, Kalamazoo, August 19–20, 1975.
SCHOOL SCIENCE AND MATHEMATICS ASSOCIATION, Sheraton-Gibson Hotel, Cincinnati, November 7–9, 1974.
SOCIETY FOR INDUSTRIAL AND APPLIED MATHEMATICS, Shoreham-Americana Hotel, Washington, D.C., October 23–25, 1974.

HOLT OFFERS YOU THE BEST IN MATH ON ANY LEVEL

CALCULUS AND ANALYTIC GEOMETRY,

by Abraham Schwartz, *City College of New York*

This extensive revision of an outstandingly successful book is a careful exposition of the necessary topics in calculus. Some features to look for in this new edition are: more emphasis on the definite integral concept; the logarithm is now defined as a definite integral and the exponential function as its inverse function; new material on the triple scalor product and the triple vector product; an explanation of Kepler's Laws of Planetary Motion; addition of proof of the Implicit Function Theorem; new matter on the center of gravity, and a new appendix article on sum notation. Almost all the exercises in each chapter are new and present new ideas and concepts. A detailed Solutions Manual will be available for instructors use.

Spring, 1974/1150 pages (tent.)/$15.00 (tent.)

CALCULUS FOR THE NONPHYSICAL SCIENCES With Matrices, Probability and Statistics

by Simeon M. Berman, *New York University*

The aim of this outstanding new textbook is to teach mathematical methods to the non-math, non-science major. Expressly written for the student in social sciences, business, economics, or life sciences, the book may also be used as a supplement in such courses as genetics, mathematical sociology, or operations research. Over 600 illustrations and numerous applications and exercises from all areas of interest aid in making abstract theory accessible to students of the humanistic professions. The four main areas covered in the text are analytic geometry, matrix algebra, calculus of one and two variables, and probability/statistics. Each section is followed by an initial set of exercises devoted to drill techniques and then by more specialized problems and applications. Answers to the odd numbered exercises may be found in the back of the book; others are located in the comprehensive Instructor's Manual.

Spring 1974/672 pages/$13.00 (tent.)

FINITE MATHEMATICS

by John M. Peterson, *Brigham Young University*

This textbook is designed for a one semester or a one quarter course for non-mathematics majors with introductory algebra as its only pre-requisite. The approach is intuitive and does not require or involve calculus. The chapters are written independently to allow for flexible arrangement and ordering of topics. Emphasis is on examples and applications to provide today's student with manipulative resources. Basic ideas, purposes, and uses of statistics are covered without excessive use of formulas and abstract theory. Concise chapter summaries and review exercises help make this new textbook a perfect guide for the professor and his student. A workbook and Solutions Manual accompany the text.

Spring 1974/350 pages/$12.00

ALGEBRA

by Thomas W. Hungerford, *University of Washington*

This is a basic text for an algebra course at the beginning graduate level. The major areas of algebra are treated in sufficient breadth and depth for this level, including sets and cardinal arithmetic; groups; rings; modules and vector spaces; fields and Galois theory; linear algebra; commutative algebra; structure of rings; categories and functors. There are over 800 exercises of varying difficulty.

December 1973/528 pages/$17.00

For complimentary copies please send your course title and approximate enrollment to:

Kaye A. Widmayer, Advertising Editor
HOLT, RINEHART AND WINSTON, INC.
383 Madison Avenue
New York, New York 10017

THE AMERICAN MATHEMATICAL MONTHLY

(FOUNDED IN 1894 BY BENJAMIN F. FINKEL)

THE OFFICIAL JOURNAL OF

THE MATHEMATICAL ASSOCIATION OF AMERICA

VOLUME 81 **NUMBER 7**

CODEN: AMMYAE

CONTENTS

(*Continued on inside cover*)

NOTICE TO AUTHORS

Research papers per se are unsuitable; see statement of policy (Vol. 81, p. 1).

Please follow the format in current issues of the MONTHLY. Manuscripts must be legibly typewritten or reproduced from typewritten copy, double spaced with wide margins. Three copies should be submitted to the appropriate editor and one kept by the author as protection against loss. The author's full address *must* appear at the end of the manuscript.

Backlog: Main Articles 14 months, Math. Notes 24 months, Research Problems 14 months, Classroom Notes 22 months, Math. Education 20 months.

EDITORIAL CORRESPONDENCE AND MAIN ARTICLES: to ALEX ROSENBERG, Department of Mathematics, Cornell University, Ithaca, N.Y. 14850; NOTES, etc.: to the corresponding Associate Editor; ADVERTISING CORRESPONDENCE: to RAOUL HAILPERN, Mathematical Association of America, SUNY at Buffalo, Buffalo, N. Y. 14214; CHANGE OF ADDRESS and SUBSCRIPTIONS: to A. B. WILLCOX, Mathematical Association of America, 1225 Connecticut Ave., N.W., Washington, D.C. 20036.

Annual dues for members of the Association (including a subscription to the American Mathematical Monthly) are $14.00 for each of the first two years of membership and $18.00 thereafter. Student Membership is available with annual dues of $9.00. For nonmembers the subscription price is $18.00.

PUBLISHED BY THE ASSOCIATION at Washington, D. C., and Menasha, Wisconsin, during the months of January, February, March, April, May, June–July, August–September, October, November, December.

Second-class postage paid at Washington, D. C., and additional mailing offices.

THE INDUSTRIAL MATHEMATICIAN VIEWS HIS PROFESSION: A REPORT OF THE COMMITTEE ON CORPORATE MEMBERS

R. E. GASKELL AND M. S. KLAMKIN

During 1972 the Committee on Corporate Members of the Mathematical Association of America sought the views of the Heads of non-academic Groups of mathematicians on several points that seem pertinent to the Association's relationship with a substantial fraction of its membership, viz., those in non-academic positions. The committee did not feel that it was ready for a survey, consequently a letter that we described as a "free form questionnaire" was sent to those listed as Heads of Mathematics Groups in the *Administrative Directory.*

We asked some rather general questions, but we requested also that they not merely answer, comment and suggest, but give us the benefit of any remarks that would bring to the Committee a better understanding of the nature and practice of industrial mathematics. We emphasized that our concern was not so much with numbers and data as it was with uncovering and defining the issues.

We asked the following three general questions, providing in each case some related questions or other comments to stimulate more extensive dialog.

A. *How do mathematicians fit into the industrial picture*? Do they carry out mathematical research and/or analysis, either self-directed or directed by mathematically oriented superiors? Do they frequently serve as mathematical specialists within a group of specialists? Are they retrained to work in other areas? Do they use mathematics only indirectly, benefiting principally from the discipline and logic that their mathematical training gives them?

B. *Are mathematicians adequately trained*? Do any inadequacies consist of entire areas of mathematics, or lacunae within otherwise adequate areas of preparation? Is more advanced training the answer? Is there an unbalanced choice of courses? Should there be substitution of courses from other fields? Do you find a lack of personal qualities, or of proper attitudes, to have an important bearing?

C. *Within your environment, what is the climate of opinion with respect to mathematics and mathematicians*? Does your company recognize mathematicians as professionals, with associated responsibilities and privileges? Does this recognition extend to encouragement?

Responses were received from 28 groups, all of them showing evidence of careful thought, and all quite frank. Many gave very extensive discussions of the issues we raised and others they brought out themselves. Some of them gave analyses that, with slight modification, might well deserve publication as separate articles or

notes in the MONTHLY. Those signing the responses, with their affiliations, are listed at the end of this report.

We are very greatly indebted to these individuals, and the organizations they represent, for the time they have taken, and for giving us the benefit of their very substantial cumulative experience in industry, industrial mathematics, and for many of them, their experience in the academic world as well. We extend to them our heartfelt thanks.

Since inclusion of all of the many responses would make up a document of prohibitive length, we shall present a selection of direct quotes, arranged according to the issues. Three dots (. . .) are supplied for any omission within a quote. So that the reader may re-assemble the bits and pieces of people that are scattered about in this process, each quote is identified by a code number, which is retained for that person throughout. We agree that this may be a dangerous procedure, but in nearly all cases the quotes are of sufficient length to minimize or eliminate the risk of distortion. We caution the reader against taking a preponderance of statements one way or another as a "vote," since all could not be quoted on each issue.

And now we turn to the comments themselves.

How do mathematicians fit into the industrial picture?

"I'm often asked to provide examples of applied problems. This is hard to do, since people expect something that can be described in a paragraph or so, and once this has been done the 'applied' part is mostly over and what remains is a textbook problem. The applied part consists mainly in hearing a client's description of his situation, plowing through his jargon, overcoming his tendency to start in the middle, digging out all the constraints that he has forgotten to mention, developing a model that takes account of enough detail to be realistic (but not so much as to become mathematically unwieldy), explaining it to the client in his language to determine if it is reasonable and relevant, solving whatever problems are suggested by the model, and explaining to the client, again in his language, how he can use the solution." **[18]**

"The Ph. D. mathematicians . . . normally carry out independent research within certain designated areas, which may be quite variable in breadth, and under the direction of an R & D manager of some kind. The manager may or may not have a lot of graduate training in mathematics as such, but he is almost always mathematically-oriented. The areas of research can encompass a wide variety of mathematical disciplines, including numerical analysis, the formulation and solution of systems of ordinary or partial differential equations of varying degrees of complexity, transform theory, time series analysis, the solution of large systems of linear or algebraic equations, operations research techniques, potential theory, statistics, etc. Although quite basic mathematical work may be done, it is carried out with the objective of application toward some industry problem. . . . In general, the problems tackled are very difficult and require pioneering research in the mathematical areas involved." **[19]**

"In order for a mathematician to have effective communications with management, it is his responsibility to translate management's problems into a technical statement and find a solution to that statement through the use of his own unique training. However, most of our graduates are not prepared to take the next step, that of translating the solution back into the language generally used and understood by management. Without this skill, the mathematician is not able to perform a useful function for his company." [8]

Can a more specific description be given?

"Often our mathematicians work on the development of mathematical models under the direction of an engineer, on the simulation of engineering problems or on data analyses. Ultimately this work nearly always involves computer programming." [5]

"I would describe our activity largely as problem-solving, although the word 'analysis' is rather appropriate. Normally I would not describe our work as falling under the word 'research'." [20]

"People with Bachelor's or Master's degrees are used in considerable numbers either as assistants to Ph. D. people in research, or independently as computer scientists in various computer organizations.... There may be instances where such individuals are not using their mathematics training to any great extent, but I believe this is the exception rather than the rule." [19]

"Most, but not all, of the mathematicians are associated with the use of the Company's computing facilities to solve technical problems. A much smaller number are involved in operations research and mathematical modeling activities." [23]

"Our far more numerous electrical engineers and physicists use a good deal of mathematics, relying on our mathematicians quite often only for their computer programming or computations." [5]

"Much of my work is non-mathematical, involving logical and legal interpretations, controls, measurements, systems work, graphics, computer programming, etc." [13]

Does the mathematician do research?

"Only a very small fraction of those trained as mathematicians with graduate degrees carry out mathematical research, and the remaining fraction is engaged in mathematical analysis, 'non-mathematical' work or management." [6]

"Actual mathematical research has practically no place in an industrial environment, except where specific techniques are being developed to solve a pressing engineering problem... the mathematician entering industry cannot expect to 'perform research.' He must be prepared to solve other people's problems, not those which he generates himself." [23]

"As far as I know, there are almost no mathematicians in industry doing research

in mathematics, at least not as the academic mathematician would see it. We do have plenty of mathematicians using their math as a tool to do research in other fields. Even more use a little math and a lot of the discipline and habit of thought in their performance as problem solvers." [21]

"Mathematicians are used in applications work; very little, if indeed any, basic mathematical research is performed." [2]

How is mathematics organized?

"Most mathematicians with graduate degrees ... are normally assigned to serve as mathematical consultants within teams of specialists (engineers, accountants, etc.) in charge of carrying out a particular project.... Only occasionally are the projects undertaken by our company purely mathematical, so that very seldom all the specialists on the team are mathematicians. Thus, most of the mathematicians with graduate training perform most of the time a support function as consultants." [6]

"Mathematicians work in groups doing primarily mathematical studies under a mathematically oriented supervisor. Some find themselves as individual mathematicians within groups of some other specialty which, in my experience, is most frequently engineering." [17]

"To a large extent our research programs in Mathematics are self-directed by the Principal Investigators, though in recent years emphasis has shifted to 'near term relevance' and away from basic research." [15]

"A large fraction of those trained as mathematicians do carry out mathematical research and/or analyses, either self-directed or directed by mathematically oriented superiors. One does not very often find a professional mathematician acting as a single consultant to a group of engineers or physicists." [1]

"(Of our approximately 110 people) about 1/3 work in computer sciences. Another 1/3 serve with groups of specialists, largely engineers and economists. Except for 5 or 6 who carry out mathematical research and analysis, the remainder benefit mostly from the discipline and logic that their training has given them." [27]

"We employ approximately 20 mathematicians. These math analysts are directed by non-mathematically oriented supervisors. Most of our mathematicians are part of a group of specialists usually working in a system analysis section. We use personnel with mathematical training in many other disciplines.... A significant number of mathematical trained personnel (more than 50%) do not use mathematics directly." [10]

Does he fit well, or must he be retrained?

"Frequently personnel with mathematical training, mainly those with undergraduate training, are retrained to work in another area. Probably the majority of those with undergraduate mathematical training hired by our company in the

last 10 years were retrained to work in computing and computer programming, and some were retrained to work in operations research and industrial engineering. Probably only a very small fraction of the pure mathematics training of our mathematically trained employees (undergraduate as well as graduate) is used directly in our work. Some benefit is derived, however, by these employees from the discipline and logic acquired through this training. . . . Certain areas in mathematics (mainly in pure mathematics) are not applicable in the activities of our Company. In the past we had to hire personnel with pure mathematics training and retrain them because it was impossible to find enough personnel with the right kind of training. . . . This process is obviously rather inefficient and it appears not necessary in the present and anticipated near future job market situation. . . . Several of our mathematically trained personnel, mostly at the graduate level, found it relatively easy to adapt to other non-mathematical assignments (e.g., in operations research) to which they had to be transferred as a result of these budget reductions. This perhaps implies that training in mathematics gives a person a fair amount of flexibility with respect to job assignments." **[6]**

"The bulk of the people in our group trained as mathematicians have entered the area of computing and computer programming. Mainly, these people hold an undergraduate degree, but many have had some graduate training up to and beyond a master's degree — a few hold a Ph. D. degree." **[16]**

"Newly graduated mathematicians joining (us) receive on-the-job training in the particular specialty they work in. In this respect, they are. . . retrained to work in 'applied mathematics.' We find our mathematically trained employees do use their mathematics directly in whatever area of applications they work in." **[2]**

"(Most mathematicians are retrained), except for those who majored in computer science or in probability and statistics." **[3]**

"Mathematicians, in my experience, have been retrained to work in some other area other than pure mathematics. These areas are Management Sciences, Operations Research, or Systems Analysis. They do not use mathematics directly but the discipline and logic which they have learned through their training is a valuable part of their usefulness to the Company." **[8]**

"I would say that in industry not most, but all mathematicians are also trained in other fields, although I hesitate to draw lines which say that some of these are 'outside' the domain of mathematics." **[21]**

"In my opinion mathematicians do not fit well into the industrial picture, especially the Ph. D. To be successful most people with mathematical training are retrained to work in another area, or have to become competent in another area. In our Department, which employs most of the graduate trained mathematicians in the Company, it is necessary to learn some engineering, especially if you are a Ph. D. Almost without exception Bachelor and Master Mathematicians become computer programmers." **[14]**

"I believe that most of those with mathematical training are not retrained to work in another area, although their training must be widened. I expect that those with only a bachelor's degree in mathematics are more frequently retrained to be something other than a mathematician than are those who have studied beyond that level." [17]

"I don't believe that industrial laboratories hire mathematically trained individuals merely because of the discipline and logic that their training gives them. When we hire a mathematician, we are interested in his or her analytic skills and the ability to recognize a mathematical problem in a real-world situation and to solve it in a useful way." [1]

"It has been my experience that unless a mathematician has (or acquires) a good background and understanding of the fields in which he applies his mathematical training he is not apt to be very successful in his efforts, and consequently, his contributions are not always well appreciated." [22]

"Mathematicians must be able to adapt themselves to any branch of the sciences of interest at the moment to their employer." [3]

"Whatever one would do in re-orienting curricula, this type of broad spectrum will continue to exist and it is all but impossible to predict during the years of study and formation of young men, be they mathematicians or other, to forecast the trends in their careers." [11]

Is there a transition problem?

"From the employer's point of view, a mathematician will be either a math major with a scattering of science or engineering courses, or a science or engineering major with extra math courses sufficient for him to be classed as a mathematician by his employer. It is our experience that individuals with either class of training will end up with the same type of job, but that the transition from university to industry is far easier for the latter." [3]

"Everyone seems to be introduced to the Diet Problem, with about five variables and five or ten equations. Students are taught the mechanics of solving such problems. Then the applied mathematician finds that real problems may have dozens or hundreds of variables, while the mechanics are done by a computer. I find that this turn of events proves bewildering." [21]

Are mathematicians adequately prepared?

". . . Most of the mathematicians that we hire have a BA degree from a relatively small school. The benefit of the mathematics training is primarily in the discipline and logic rather than in the actual mathematical courses. This group is used very heavily in computer programming. A smaller group of mathematicians, typically with an MS degree from larger schools, are used in operations research work. In

this area the mathematics training is used extensively, but for us the MS level is completely satisfactory.

For our purposes the mathematicians are adequately trained. We have on our staff a sizable number of engineers, operations research specialists, geophysicists, statisticians and economists who have and use a substantial amount of mathematical training. This fits our needs as an industrial concern much better than a mathematician with a sprinkling of science courses." **[4]**

"We have found them to be well qualified when they come to us from the universities. These employees have good training in the basics of mathematics and in the disciplines and logic their training gives them." **[2]**

"Mathematicians are probably as adequately trained as any other professional group. . . . Overqualification, in my opinion, can be just as detrimental to personal progress as underqualification. After all, the person performing a job in numerical analysis for one of my computer routines need not know very much about Lie Algebras on Riemann Surfaces! But he should have some real skills in the area to which he is to apply his expertise." **[7]**

"It's easy for any industrial mathematician to point to large bodies of knowledge that he has acquired but never used. All of them contribute somewhat to his mathematical maturity, of course, and often it isn't easy to say in advance which subjects will be used. Basically, however, I think that nearly all applied mathematics depends on these areas:

(1) probability
(2) statistics
(3) design of experiments
(4) operations research
(5) classical analysis
(6) numerical analysis.

It is important for an applied mathematician to be strong in one of these fields, and it is better if he is proficient in two or more, and he should be acquainted with the potentials of all of them. If all his training is in algebra or topology (as mine originally was) then he can eventually become useful, but only by taking the time to become good at one or more of these fields. I would not hire such a person unless others were unavailable." **[18]**

"The field of mathematics is so vast, and applications so varied, that one cannot possibly prepare in advance for all eventualities, even at the Ph. D. level. However, anyone with a broad, solid mathematical background can learn new topics as he encounters them." **[13]**

"Many mathematicians (perhaps the majority) are not adequately trained for industrial work, but we have been successful in recruiting all the individuals required to fulfill our needs from those who are adequately trained. In recent years, most Bachelor mathematicians are obtaining training in critical subjects such as numerical analysis and computer science along with required basic courses in real and complex analysis, transform theory, matrix theory, etc. Hence, a larger percentage of undergraduates are obtaining training which will permit them at least to begin work in

industry. . . . As for the Ph. D. mathematicians, since 70–80% receive their special training in abstract fields such as topology, algebra, number theory, etc., most of these people are completely unsuited for industrial positions commensurate with their training. . . . A concerted effort must be made to direct more of these students into applied fields." [**19**]

"How many students, when asked to evaluate a finite integral whose integrand is a constant raised to a power which is a quadratic function of the variable of integration, know how to proceed? Think of the amount of effort wasted in trying to find an indefinite integral, and then giving up and using a computer to give a close approximation!" [**21**]

"In my experience, mathematically trained individuals in industry have ample opportunity to use any applied mathematical training they had received. However, exposure to topics in pure mathematics, such as modern algebra, real-variable theory, number theory, topology, etc., does not, in general, turn out to be directly useful. . . . A curriculum in Applied Mathematics, in general, does provide the appropriate type of background, but such specialization is usually not given prime emphasis in many mathematics departments." [**23**]

"(More) emphasis on statistics and their practical applications to business problems." [**24**]

"The mathematicians we hire are fairly adequately trained, particularly those with master's and doctor's degrees. For many of our newly hired personnel it would have been nice if they had studied probability and mathematical statistics in their undergraduate and graduate course work in mathematics. A knowledge of computer programming is helpful but not necessary, and is readily acquired here on the job. I do not believe that a watering down of the mathematics curriculum by taking courses in other fields (other than engineering, computer science, statistics, operations research, and similar mathematically oriented sciences) would be of any practical use to us. In fact, the more mathematics courses that an individual has, the more useful he is to us, particularly if these couises comprise diverse fields of mathematics such as both abstract algebra and analysis." [**25**]

"The mathematical training of the mathematician entering industry with a bachelor's degree, and usually even with a master's degree, is not adequate for the kind of work he is expected to be able to do and should be able to do. The engineer, on the contrary, with the same level of training in his field, usually knows more mathematics than the mathematician in that he is aware of and can use a broader spectrum of mathematics." [**9**]

"The mathematical curriculum has not in general adequately prepared most mathematicians for entry into the fields of Management Sciences, Operations Research, and the Computer Sciences. The areas lacking are not technical but in general are those which equip mathematicians to deal with their Company's management." [**8**]

"So far as college mathematics and to a certain extent high school mathematics are concerned, the curricula are so dominated by pure mathematicians (set theorists)

that students are often discouraged from pursuing the applied fields or find difficulty in finding appropriate courses." [12]

"Clearly the rise of both digital and analog computers has greatly influenced the mathematics courses in colleges. The result may be that the applications of non-computer oriented mathematics have been neglected in many collegiate mathematics courses. We have noted weaknesses in even the fundamental analytic techniques of Complex Variables and Probability Theory." [3]

"The push toward 'near term relevance' in our research programs has made it virtually impossible for us to afford the luxury of a topologist, an algebraist, . . . we can no longer afford an 'impractical dreamer'. . . . Several hundred applications from individuals at the Ph. D. — M. S. level cross my desk in a year and I'm certain that hundreds more at the B. S. level apply to the Company. Of those fresh from the universities I would guess that 99% do not have training suitable for consideration (even assuming positions were open)." [15]

Is it more breadth that is needed?

"A greater breadth of training would have been desirable in some cases. That is, the substitution of courses in applied areas for some of those in pure mathematics would have provided better preparation for the work they are now doing. Not unrelated to this comment is the fact that many members of the group studied in the Mathematics Department (at —) where pure mathematics dominates strongly." [16]

"The engineer is well grounded in the physical sciences, whereas the mathematician often is not. This statement implies that the mathematician in industry should have been exposed to courses in the fundamentals of the physical (and biological and social) sciences. This does not mean just a 'first course in physics' or a 'first course in chemistry', or the like, but something more on the order of the Richard Phillips Feynman lectures of UCLA. . . . But — and this is where the training of the mathematician would differ from that of the engineer or physicist — for the mathematician the broad spectrum courses should be followed by follow-on in-depth courses in the same and other areas. These would be courses designed to formalize the heuristic arguments of the initial courses into more abstract and rigorous proofs, to increase the scope of the content, and to develop a greater appreciation and understanding of the interrelations of the many branches or areas of mathematics." [9]

Should other courses be substituted?

"As far as courses in other subjects are concerned, our people would benefit from knowledge of physics, electrical engineering, systems engineering, computer science, economics, biology (we have problems ranging from medical systems to ecological studies), psychology, and others. Obviously you have to stop the list somewhere, but the point is that in applying mathematics you have to know (or be able to learn) enough about the area of application to be able to talk about it with

the client. Training could be improved by teaching courses in the formation of mathematical models — probably by showing students how to describe mathematically and try to optimize some of the non-mathematical activities that they are familiar with in their everyday life. . ." [18]

"In order for a mathematician to be effective and therefore be a desirable acquisition to an industrial organization, he should be educated in areas outside of mathematics so that he can effectively apply the mathematical training to real-world problems." [23]

"Students who wish to increase their facility in applied mathematics while at a university have found it necessary, in my experience, to resort to special courses set up by the engineering school or graduate chemistry and physics departments in order to obtain their training. I believe that I am not describing uncommon circumstances. It is my opinion that two worthwhile developments would be the strengthening of university curricula in applied mathematics under the aegis of the mathematics departments and the raising of requirements in associated fields which make considerable use of mathematics.

"The issue, as I mentioned above, is the availability in college curricula of strong courses in the common disciplines, e. g., differential equations, vector and tensor calculus, advanced algebra, complex variable theory, Fourier theory, analysis, transform theory, linear and dynamic programming, group theory, probability theory and so forth, taught, for example, with closer attention to practical applications in the physical and biological sciences than to existence theorems." [12]

"If any improvement could be made, I believe it would be in the area of (a) making sure that each bachelor's degree in mathematics has a solid content in both pure and applied mathematics including an insistence on such prosaic things as good algebraic manipulation together with a deep understanding of the foundations of the subject and (b) that the curriculum should include at least one course, seething with examples, showing how modern mathematics interfaces with engineering, the physical sciences, the biological sciences and to the degree possible, the social sciences and humanities." [20]

"I would suggest that industrial mathematicians be consulted in regards to the preparation and counseling of math majors (and that more stress be placed) upon the obtaining of numerical answers to dirty physical problems together with techniques specifying and limiting the errors involved." [3]

Is there value in advanced training?

"I prefer Ph. D.-level people for these reasons:

(1) Many of our clients are research scientists with fairly good mathematical backgrounds. We can't help them very much unless we know something they don't know.

(2) We get very few entirely routine problems. We must be able to make changes

in known techniques in order to fit the constraints of individual problems, and this means that we can't use cook-book mathematicians or statisticians.

(3) Much of our work is problem formulation rather than problem solving. The person who has the intelligence and common sense to do this has usually (barring extreme economic difficulties) had the foresight to equip himself with a Ph.D. degree." [18]

"While our work involves too little genuine mathematical research to justify a Ph. D. in this field, the master's degree in mathematics, applied mathematics, statistics, or computer science is highly desirable and almost essential in order to progress satisfactorily. This degree is often more valuable if obtained during or after employment in industry." [5]

How do the fields of statistics, computer science, and numerical analysis enter the picture?

"An undergraduate emphasis on probability, statistics, computer science and applied mathematics, and on the applications of mathematics would be helpful in providing a useful orientation for mathematicians, ... and it would facilitate their utilization in industry, with advanced training coming at a later time.. . . . Contact between students and industrial mathematicians could be very salutary." [5]

"In general, more advanced training in pure mathematics is not the answer to inadequate preparation, but a better balanced choice of courses taken is. In particular, courses in applied mathematics, numerical analysis, should be substituted for many of the pure mathematics courses taken, if the graduates of the program wish to compete for jobs that our company may have to offer in the future." [6]

"There is imbalance, not so much in the choice of courses as in the emphasis within each course: insufficient attention to numerical methods, estimating accuracy of answers from that of the inputs, methods of minimizing roundoff error, approximations designed for solution by digital computers, etc. Touch base with the real world from time to time." [13]

"Probability theory and mathematical statistics should be included as part and parcel of mathematics — and not dismissed with an 'Oh, but that's just statistics'! — implying that, after all, that's not really mathematics. The computer is here to stay, and should, of course, be considered an integral part of mathematical know-how, method and equipment, and should therefore be well imbedded in the mathematics curricula." [9]

"I find classical analysis, numerical analysis, probability and statistics to be used extensively in industry. Also linear algebra is used extensively but abstract algebra is not. I am surprised how difficult it is to get a bachelor's degree in mathematics without studying abstract algebra but how easy it is in many schools to get through with little or no statistics. While the tendency to provide numerical analysis is increasing, as it certainly should, I see, if anything, the split between mathematics and statistics to

be getting larger on the academic side but certainly not so on the industrial side. What I'm saying is that I see overspecialization at the undergraduate level." [17]

"The mathematician often finds himself participating on a team with individuals of different technical background, usually in the role of numerical analyst and computer programmer, and on a few occasions, as a mathematical analyst. Within my sixteen years of experience managing mathematics and computing organizations, I have found that mathematicians who are not capable of being trained to be effective computer programmers or operations analysts do not have a place in the industrial environment." [23]

"Successful utilization of our mathematicians generally involves their learning computer programming, statistics, or a certain amount of engineering or physics. The willingness to learn something from these fields is very important to our mathematicians' usefulness here, but those with a degree in physics or engineering, as well as in mathematics, often find themselves employed for their knowledge of that field rather than for their mathematics, which they may use infrequently although their knowledge of its potentialities serves them well." [5]

The question of communication was raised.

"I am teaching a graduate seminar in computer systems at — in which I stress more than anything the ability of the candidates to communicate and interact. They all are chuckfull of facts and some knowledge, but they can rarely express themselves well orally, although many of them write reasonably well. Often they fail to extract maximal values from existing situations, lacking the most rudimentary management skills or ability to plan their personal lives." [7]

"Personal qualities and proper attitudes are vital in any position requiring communication with others; adequate training should emphasize that fact. Indeed, it should be understood that communication is as important as anything else one does, and it merits much effort in writing and rewriting, as well as in face-to-face communication." [5]

"(Training is generally adequate) so far as purely mathematical concepts and techniques are concerned. Sometimes not in the methods of applied mathematics, i.e., ways of getting approximate analytical or numerical answers to real problems. Frequently (mathematicians) do not understand the language of physicists, engineers, or economists well enough to communicate effectively with people in these other disciplines and to formulate the mathematical problems arising in these fields. The remedy would appear to be less mathematical specialization and more courses in the other sciences, including subjects like economics and psychology which are just now in the process of being mathematized." [1]

"Communication, written and oral, is very important, of course, and to the extent that they aid communication, English courses are helpful. I would like to see an English course in which the students learn to describe precisely some of the more

complex things that they understand but probably haven't analyzed, like tying a shoe, mowing a lawn." **[18]**

"Above all, teach people to use the English language. At some point in every project, the mathematician's work must be communicated in plain English to those who will use it." **[13]**

"Of course, a capability to write and speak coherently is important and training of such capabilities must naturally be included in any educational program." **[23]**

What is the climate of opinion with respect to mathematics and mathematicians?

"In the climate in which I have been engaged for many years, a truly competent mathematician who is capable of working with individuals of many different specialties is respected and appreciated, but he must be flexible, capable of listening, and willing to dig deeply into those fields in which he is attempting to use his talents."**[22]**

"Inasmuch as mathematics provides the basis for most of our work here, it is highly respected for its usefulness, and mathematicians find themselves at no disadvantage in our electrical engineering environment. In fact, a large percentage of our unadulterated Ph.D.'s in mathematics in the past (three of them come immediately to mind) have risen to the level of laboratory manager and laboratory director; although in most cases they found no application here for the specialty of their doctoral dissertations (partial differential equations or algebraic topology)."**[5]**

"I think the climate of opinion here has become quite favorable, though it has not always been entirely so. Mathematics in industry is not an end in itself, so that we are primarily a 'service department.' Our main problem has been to keep people aware of the fact that this doesn't have to imply that our work is routine or low-level." **[18]**.

"To be perfectly frank, this is an engineering dominated Department and non-engineers do not have quite the status of engineers." **[14]**

Attitudes seem to play an important role.

"Unfortunately, for many years the attitude in some quarters of the collegiate mathematical world toward mathematicians who worked in applied fields was not as healthy as it should have been. Some of this attitude still exists." **[22]**

"Proper attitudes are vital in industry. Many businessmen distrust ivory tower research people unless they can produce results of measurable net benefit to the company. The mathematician must be willing to drop his technical jargon long enough to explain his work in laymen's terms without conveying the impression that he regards businessmen as shallow or stupid. This requires careful thought, patience, and mutual respect, but is a vital factor in gaining acceptance of the mathematician's work and opinions." **[13]**

"The natural tendency for university programs at the Ph.D. level to emphasize the 'independent research ideal' tends to cause the development of a negative attitude toward 'problem solving' to the detriment of the student if he later turns to the industrial job market." **[23]**

"The first thing that universities must do, if they are to improve the situation in applied mathematics, is to stop training their students to believe that the only respectable mathematical activity is the proving of new, self-generated theorems. If a person has been taught to scorn problem solving and problem formulation, then he is useless in industry, regardless of specific training. Unfortunately the majority (I believe) of mathematicians produced in the last generation have been so taught. . . . . As long as most academic people divide all mathematicians into new-theorem-provers and plumbers, communication will be difficult." [18]

"As is well known, some new Ph.D.'s in pure mathematics, as well as some professors, have the idea that applied mathematics is 'dirty', and that industrial mathematics is in some way inferior to university mathematics. However, such ivory-towerism is not found only among mathematicians! People who suffer from it do not generally seek or find employment in industrial laboratories. I would not say that we have found such attitudes a problem." [1]

"A solution of practical problems seems to be a second class way of life." [14]

"The majority of mathematicians in employment interviews do not seem to be sufficiently aware of the practical importance of finding usable and practical answers to the various problems presented to them. . . . We have interviewed many young graduates who would have made a much better impression if some job counsellor had told them about the industrial facts of life before they came here." [3]

"I believe that the advent of computers has pulled a large number of mathematicians into the electronic camp (where I see them much of the time). Many of them 'just work at their job', often without perceiving research problems which practically dangle in front of their noses for years on end." [7]

"The answers the specialists seek are almost never controlled by purely mathematical considerations. In rare cases, a mathematician may be able to show that a proposed solution is mathematically impossible, but in many instances the physical scientists or engineers simply decide that their physical arguments (or intuition) should prevail." [3]

"Far too often I find recent graduates who refuse to recognize that the answer management must have, is the best possible that (can) be obtained before they make a decision. Many communications barriers exist between management and the mathematician because the mathematician has been guilty in telling management that the wrong decision has been made in the past rather than providing guidance for a decision which will be made at some future time. I find that the tendency to stress theoretically perfect (even though late) rather than practical answers becomes more prominent with advanced degrees, so much so that few Ph.D.'s are equipped to be useful in a corporate environment other than research labs." [8]

"I decided years ago that my work was at least as interesting and challenging (but I may be unusually lucky) as any puzzles or contests which I might consider—and I get paid better for the former." [21]

All of the members of the Committee on Corporate Members are in industry or have had considerable experience in industrial mathematics. Perhaps the best way to sum up the reaction of Committee members is merely to say that the responses contained very few surprises. We feel that the remarks from this wide-ranging group of practicing mathematicians contain a great deal that should be taken to heart by those who are teaching or administering the teaching of mathematics.

Both of the writers of this report have commented earlier in this MONTHLY, perhaps too mildly, on the divergence between mathematical education and mathematical practice. Our traditional educational procedures have provided only training in mathematical manipulation, with extremely little attention to problem recognition, formulation, and follow-up. Modern trends have even narrowed the field of vision, which may be a necessity for true research efforts but is a handicap for the vast majority of our students who must compete with broadly trained individuals from other areas.

Many of the items, in the extensive bibliography attached, show that we have not been alone in our recognition of this very disturbing phenomenon. It can hardly be described as a transient situation since it has existed for so long. We observe something akin to a latent disease that the economic swings of the past twenty years have caused to surface, certainly as an embarrassment, and quite possibly as a dangerous threat to organized mathematical education.

We cannot refrain from including just one more quote from an entirely different source: "Thomas Godfrey, a self-taught mathematician, great in his way, and afterward inventor of what is now called Hadley's Quadrant. But he knew little out of his way, and was not a pleasing companion, as, like most great mathematicians I have met with, he expected universal precision in everything said, or was forever denying or distinguishing upon trifles, to the disturbance of all conversation. He soon left us." So complained Benjamin Franklin in his Autobiography, as he listed the members of his mutual improvement club, called the "Junto" — two hundred fifty years ago!

Appendix A

Those responding to the questionnaire were:

Albrecht, Mr. Norman E., Manager, Mathematical services, Investors Diversified Services, Inc., Eighth and Marquette, Minneapolis, Minnesota 55402.

Bargellini, Dr. P. L., Senior Staff Scientist, Communications Satellite Corp., Box 115, Clarksburg, Maryland 20734.

Benedict, Dr. T. R., Head, Computer Mathematics Dept., Cornell Aeronautical Laboratory, Inc., P. O. Box 235, Buffalo, New York, 14221.

† Burington, Dr. Richard S., Naval Air Systems Command, Washington, D. C., 20360.

Concus, Dr. Paul, Mathematics and Computing, Lawrence Berkeley Laboratory, University of California, Berkeley, California 94720.

Donaldson, Mr. W. G., Technical Director, Federal Electric Corporation, 621 Industrial Avenue, Paramus, New Jersey 07652.

Einhorn, Dr. Sheldon J., Project Manager, Kappa Systems, Inc., 1015 N. York Road, Willow Grove, Pennsylvania 19090.

Hammer, Dr. Carl, Director, Computer Sciences, Sperry Rand Univac Division, 1221 Wisconsin Ave. N. W., Washington, D. C. 20007.

Hooke, Dr. Robert, Manager Mathematics Department, Research and Development Center, Westinghouse Electric Corporation, Pittsburgh, Pennsylvania 15235.

Jacoby, Dr. Samuel L. S., Manager Analysis Systems, Boeing Computer Services, Inc., P. O. Box 24346, Seattle, Washington 98124.

Jenkins, Dr. T. R., Manager, Mathematics and Systems Analysis Laboratory, Lockheed Palo Alto Research Laboratory, 3251 Hanover St., Palo Alto, California 94304.

Johnson, Mr. Myron O., Manager, Computing Department, Phillips Petroleum Co., Bartlesville, Oklahoma 74003.

Karle, Dr. Jerome, Chief Scientist, Laboratory for the Structure of Matter, Naval Research Laboratory, Washington, D. C. 20375.

Kennard, Dr. R. W., Manager, Systems Engineering, E. I. DuPont de Nemours and Co., Wilmington, Delaware 19898.

Lawwill, Dr. Stanley J., President, Analytic Services, Inc., 5613 Leesburg Pike, Falls Church, Virginia 22041.

McCready, Dr. R. R., Applied Mathematician, Engineering Technologies, Vought Systems Division, P. O. Box 5907, Dallas, Texas 75222.

McKenzie, Mr. T. H., Vice President and General Manager, Electronic Systems Group, Western Division, GTE Sylvania, Inc., P. O. Box 188, Mountain View, California 94040.

Morgan, Dr. Samuel P., Director, Computing Science Research Center, Bell Laboratories, 600 Mountain Avenue, Murray Hill, New Jersey 07974.

Newman, Mr. Robert W., Manager, Planning Research Development Staff, General Electric Co., 570 Lexington Avenue, New York, New York 10022.

Nunnikhoven, Mr. J. A., Group Manager, Special Systems Services, American Can Co., Greenwich, Connecticut 06830.

Quade, Dr. E. S., Research Mathematician, The Rand Corporation, 1700 Main St., Santa Monica, California 90406.

Rice, Dr. R. B., Manager, Physics and Mathematics Department, Denver Research Center, Marathon Oil Co., Littleton, Colorado 80120.

Rosser, Dr. J. Barkley, Director, Mathematics Research Center, University of Wisconsin, Madison, Wisconsin 53706.

Schlesinger, Dr. Stewart, General Manager, Information Processing Division, The Aerospace Corporation, P. O. Box 92957, Los Angeles, California 90009.

Stahly, Mr. Glenn F., Chief, Mathematical Research, National Security Agency, Fort George G. Meade, Maryland 20755.

Stuhlinger, Dr. Ernst, Associate Director for Science, National Aeronautics and Space Administration, George C. Marshall Space Flight Center, Alabama 35812.

Van Wyk, Mr. R., Manager Applied Mathematics and Research Administration, Winchester Group Research, Olin Corporation, 275 Winchester Ave., New Haven, Connecticut, 06504.

Young, Dr. John W., Scientific Programming Dept., Martin Marietta Corp., P. O. Box 5837, Orlando, Florida 32805.

† Deceased.

Appendix B

Members of the Committee on Corporate Members during 1972 were:

F. A. Brooks, Jr., Mutual Benefit Life Corp.
Jim Douglas, Jr., University of Chicago
R. L. Graham, Bell Telephone Laboratories
M. S. Klamkin, Ford Motor Co.
H. E. Pickett, Aerospace Corp.
Andreas Thuswaldner, Communications Canada
Leonard Tornheim, Chevron Research Lab.
A. B. Willcox, MAA
R. E. Gaskell, Naval Postgraduate School (Chairman)

References

1. E. H. Bareiss, The college preparation for a mathematician in industry, this MONTHLY, 79 (1972) 972–984.

2. R. Bellman, Some Vistas of Modern Mathematics, University of Kentucky Press.

3. R. Bellman, G. Borg, Mathematics, Systems and Society, Report No. 2, April 1972, Committee on Research, Swedish Natural Science Research Council, Stockholm, Sweden.

4. E. A. Bender, Teaching applicable mathematics, this MONTHLY, 80 (1973) 302–306.

5. G. Birkhoff, The impact of computers on undergraduate mathematical education in 1984, this MONTHLY, 79 (1972) 648–659.

6. R. P. Boas, Jr., Calculus as an experimental science, this MONTHLY, 78 (1971) 664–666.

7. K. E. Bullen, Applied mathematics in twentieth century, Spectrum of Mathematics, Ed., J. C. Butcher, Oxford University Press, 1971.

8. R. S. Burington, For what types of careers should we train mathematicians? J. Eng. Education, 46 (1955) 115–120.

9. ———, New frontiers, Science, 101 (1945) 313–320.

10. ———, On the nature of applied mathematics, this MONTHLY, 56 (1949) 221–241. (Also has 62 references.)

11. ———, The problem of formulating a problem, Proc. Amer. Philos. Soc., 104 (1960) 429–443.

12. J. Crank, Mathematics and Industry, Oxford University Press, London, 1962.

13. T. C. Fry, Industrial mathematics, this MONTHLY, No. 6, part II, 48 (1941).

14. S. Garfunkel, A laboratory and computer based approach to calculus, this MONTHLY, 79 (1972) 282–289.

15. R. E. Gaskell, The practice of mathematics, this MONTHLY, 64 (1957) 557–565.

16. ———, Professional mathematicians in government and industry, SIAM Rev., 2 (1960) 3–10.

17. M. Gerstenhaber, Undergraduate mathematics training in 1984 — some predictions, this MONTHLY, 79 (1972) 658–661.

18. H. P. Greenspan, Applied mathematics as a science, this MONTHLY, 68 (1961) 872–880.

19. ———, Applied mathematics at M. I. T., this MONTHLY 80 (1973) 67–72.

20. H. B. Griffiths, 1871: our state of mathematical ignorance, this MONTHLY, 78 (1971) 1067–1084.

21. P. C. Hammer, Mind Pollution, preprint 1973.

22. Peter Henrici, Reflections of a teacher of applied mathematics, Quart. Appl. Math., (1972) 31–39.

23. H. W. Hethcote and A. J. Schaeffer, A computer laboratory course for calculus and linear algebra, this MONTHLY, 79 (1972) 290–293.

24. R. Hooke, D. Shaffer, Math and Aftermath, Walker, New York, 1965.

25. ———, Modern applied mathematics and the engineer, Westinghouse Engineer, Jan. 1972, 9–13.

26. How EE's rate math courses, IEEE Spectrum, January (1972) 55–58.

27. S. Karlin, Some mathematical models of population genetics, this MONTHLY, 79 (1972) 699–739.

28. M. S. Klamkin, On the ideal role of an industrial mathematician and its educational implications, this MONTHLY, 78 (1971) 53–76. (Also has 89 references.)

29. D. W. Lick, New directions and commitments for mathematics, Notices Amer. Math. Soc., (1972) 272–275.

30. Richard Martin, Many Schools Discover Kids Using 'New Math' can't divide 100 by 10, Wall Street Journal, May 31, 1973.

31. J. E. McKenna, Computers and experimentation in mathematics, this MONTHLY, 79 (1972) 294–295.

32. E. P. Miles, Jr., Calculus and the computer, the Cricisam Calculus Project — past, present, and future, this MONTHLY, 78 (1971) 284–290.

33. C. O. Oakley, Engineering for mathematicians, J. Eng. Education, 47 (1956) 292.

34. R. Oldenburger, Mathematical Engineering Analysis, Macmillan, New York, 1950.

35. L. J. Paige, Public understanding of science and its implication for mathematics, this MONTHLY, 78 (1971) 130–141.

36. B. B. Peterson, Survival for mathematicians or mathematics? this MONTHLY, 79 (1972) 70–76.

37. Y. A. Rocard, Opening Address, pp. 4–11, J. P. Mathieu, the teaching of mathematics, pp. 141–148. International Education in Physics, Proc. Intern. Conf. of Physics Education, Paris, July 18 — August 4, 1960, M. I. T. Press, Cambridge.

38. T. L. Saaty, Operations research: Some contributions to mathematics, Science, Dec. (1972) 1061–1070.

39. E. Spanier, The undergraduate program in mathematics, this MONTHLY, 77 (1970) 752–755.

40. O. G. Sutton, Mathematics in Action, Bell, London, 1954.

41. Symposium on "The future of applied mathematics", Quart. Appl. Math., April (1972).

42. Rene Thom, "Modern" mathematics: An educational and philosophic error, American Scientist, 59 (1971) 695.

43. J. W. Tukey, Mathematical consultants, computational mathematics, and mathematical engineering, this MONTHLY, 62 (1955) 566.

44. ———, What can mathematicians do for the Federal Government? this MONTHLY, 74 (1967) 101–109.

45. D. W. Ver Plank, B. R. Teare, Engineering Analysis, Chapman and Hall, London, 1954.

46. P. Wegner , A view of computer science education, this MONTHLY, 79 (1972) 168–179.

47. H. H. Walbesser, T. Eisenberg, What research competencies for the mathematics educator? this MONTHLY, 78 (1971) 667–672.

48. A. B. Willcox, England was lost on the playing fields of Eton: A parable for mathematics, this MONTHLY, 80 (1973) 25–40.

49. Mary B. Williams, Letter to the Editor, Notices, Amer. Math. Soc., (1971) 502–503.

50. L. A. Zadeh, Mathematics — A call for reorientation, Newsletter, CBMS, May 1972, 1–3.

Additional References (added in proof)

S. Chapman, Applying science to public technology, Amer. J. Phys., 41 (1973) 188–192.

P. J. Davis, Visual geometry, computer graphics and theorems of perceived type, preprint 1973.

J. A. Dieudonné, Should we teach "Modern Mathematics?" Amer. Scientist, 61 (1973) 16–19.

J. G. Kemeny, What every college president should know about mathematics, this MONTHLY, 80 (1973) 889–901.

M. Kline, Why Johnny Can't Add: The Failure of the New Math, St. Martin's Press, New York, 1973.

———, Why teach mathematics?, N.Y.S. Math. Teach. J., 24 (1974) 55–70.

D. E. Knuth, Computer science and its relation to mathematics, this MONTHLY, 81 (1974) 323–342.

R. R. McLone, The training of mathematics, CBMS Newsletter, 9 (March 1974).

J. D. Murray, Approximate methods in mathematics, Math. Spectrum, 6 (1973) 19–24.

Oxford Study Groups with Industry, 1968–1971, Progress report on applications of differential equations, Mathematical Institute, Oxford.

G. C. Papanicolaou, Stochastic equations and their applications, this MONTHLY, 80 (1973) 526–545.

F. Reif, Educational challenges for the university, Science, 184 (1974) 537–542.

J. J. Seidel, The mathematical education of engineers, and the education of mathematical engineers, in the Netherlands, Inst. Math. & Appl., 9 (1973) 305–307.

T. Seidman, A proposal for a professional program in mathematics, this MONTHLY (to appear).

T. P. Speed, Statistics in school and society, Math. Spect., 6 (1973/74) 7–11.

DEPARTMENT OF MATHEMATICS, NAVAL POSTGRADUATE SCHOOL, MONTEREY, CA 93940.

SCIENTIFIC RESEARCH STAFF, FORD MOTOR COMPANY, P. O. BOX 2053, DEARBORN, MI 48121.

THE SHIFT OPERATOR

P. A. FILLMORE

1. Introduction. By a (simple, unilateral) **shift operator** we understand a bounded linear transformation S on a separable complex Hilbert space $\mathscr{H}$ for which there exists an orthonormal basis $e_0, e_1, \cdots$ of $\mathscr{H}$ such that $Se_n = e_{n+1}$ for all $n \geqq 0$. Any two shift operators $S: \mathscr{H} \to \mathscr{H}$ and $S': \mathscr{H}' \to \mathscr{H}'$ are unitarily equivalent. If $\{e_n\}$ and $\{e'_n\}$ are the corresponding orthonormal bases, the equations $We_n = e'_n$ for all $n \geqq 0$ determine an isomorphism $W: \mathscr{H} \to \mathscr{H}'$ such that $WS = S'W$. There are two realizations of the shift operator on concrete Hilbert spaces that are particularly useful. The first, on the space l^2 of square-summable sequences of complex numbers, is defined by

$$S(x_0, x_1, x_2, \cdots) = (0, x_0, x_1, x_2, \cdots).$$

The other is on the Hardy space H^2, consisting of all measurable complex functions f on the unit circle that are square-integrable with respect to normalized Lebesgue measure and whose Fourier coefficients of negative index all vanish:

$$\int_0^{2\pi} f(e^{i\theta})e^{in\theta}d\theta = 0, \qquad n \geqq 1.$$

Here the shift operator appears as

$$(Sf)\,(e^{i\theta}) = e^{i\theta}f(e^{i\theta}),$$

and the corresponding orthonormal basis consists of the functions e^n, $n \geqq 0$, where e is the identity function $e^{i\theta}$.

The shift operator has been known for many years, at first as an interesting example, but more recently as a fundamental building block in the structure theory of operators on Hilbert space. The purpose of the present note is to make more widely known the modern role of the shift operator.

2. Characterizations. The shift operator evidently has the following properties:
(i) it is an **isometry:** $\| Sf \| = \|f\|$ for all $f \in \mathscr{H}$,
(ii) it is **pure:** $\bigcap_{n=0}^{\infty} S^n\mathscr{H} = \{0\}$.

Is every pure isometry a shift? For example, consider the operator T determined by the mapping $Te_n = e_{2n}$ on an orthonormal basis $\{e_n \mid n \geqq 1\}$. This is a pure isometry, but not a shift. However, it is a **direct sum of shifts** in the following sense: for each odd integer $k \geqq 1$, let $\mathscr{M}_k$ be the subspace spanned by $e_k, e_{2k}, e_{4k}, \cdots$; then these subspaces are mutually orthogonal and span the whole space, and in each, T is a shift.

THEOREM 1. *Any pure isometry is a direct sum of simple shifts.*

Proof. Let V be a pure isometry on $\mathscr{H}$, and let $\mathscr{K} = (V\mathscr{H})^{\perp}$, the orthogonal

complement of the range of V. The subspaces $V\mathscr{K}$, $V^2\mathscr{K},\cdots$ are contained in the range of V, and thus are orthogonal to $\mathscr{K}$. Any isometry has the property

$$(Vf, Vg) = (f,g),\ f,\ g\in\mathscr{H}$$

and it follows that the subspaces $\mathscr{K}, V\mathscr{K}, V^2\mathscr{K},\cdots$ are mutually orthogonal. Moreover these subspaces span $\mathscr{H}$; indeed, $\mathscr{H}$ is spanned by the subspaces

$$\mathscr{K},\ V\mathscr{K},\cdots, V^{n-1}\mathscr{K} \text{ and } V^n\mathscr{H}$$

for every $n \geqq 1$, and therefore by the subspaces $\mathscr{K}$, $V\mathscr{K}$, $V^2\mathscr{K},\cdots$ and $\bigcap_{n=0}^{\infty} V^n\mathscr{H} = \{0\}$. Now let $\{e_\alpha \mid \alpha\in A\}$ be an orthonormal basis of $\mathscr{K}$, and for each $\alpha\in A$ let $\mathscr{M}_\alpha$ be the subspace spanned by the orthonormal set $\{e_\alpha, Ve_\alpha, V^2e_\alpha,\cdots\}$. As in the example, these subspaces are mutually orthogonal and span $\mathscr{H}$, and in each, V is a shift. □

On the other hand, any pure isometry may be regarded as a shift of a suitably general type. To make this precise, let $\mathscr{K}$ be any Hilbert space, and let $l^2(\mathscr{K})$ be the Hilbert space of norm-square-summable sequences of vectors from $\mathscr{K}$. The **shift operator** on $l^2(\mathscr{K})$ is defined in the same fashion as the shift on l^2.

THEOREM 2. *Any pure isometry V on a Hilbert space $\mathscr{H}$ is unitarily equivalent to the shift operator on $l^2(\mathscr{K})$, where $\mathscr{K} = (V\mathscr{H})^\perp$.*

Proof. It was shown above that the subspaces $\mathscr{K}$, $V\mathscr{K}$, $V^2\mathscr{K},\cdots$ are mutually orthogonal and span $\mathscr{H}$. It follows that the map

$$W\colon (k_0, k_1, k_2,\cdots) \to \sum_{n=0}^{\infty} V^n k_n$$

is an isomorphism of $l^2(\mathscr{K})$ with $\mathscr{H}$. Since

$$\begin{aligned} WS(k_0, k_1,\cdots) &= W(0, k_0, k_1,\cdots) \\ &= \sum_{n=0}^{\infty} V^{n+1}k_n = VW(k_0, k_1,\cdots) \end{aligned}$$

for all $(k_0, k_1,\cdots)\in l^2(\mathscr{K})$, we have $WS = VW$ as required. □

The next result describes the structure of arbitrary isometries. It was discovered by von Neumann [**7**] in the course of investigating extensions of symmetric operators (as will be explained in the next section). Recall that a **unitary operator** is an isometry of a Hilbert space onto itself. The structure of unitary operators is completely described by the spectral theorem [**1**, §62].

THEOREM 3. *Any isometry is uniquely the direct sum of a pure isometry and a unitary operator.*

Proof. Let V be an isometry on $\mathscr{H}$. It must be shown that there is a unique subspace $\mathscr{M}$ of $\mathscr{H}$ with the properties $V\mathscr{M} \subset \mathscr{M}$, $V(\mathscr{M}^\perp) \subset \mathscr{M}^\perp$, $V \mid \mathscr{M}$ is unitary, and

$V|\mathscr{M}^{\perp}$ is a pure isometry. It is easy to see that $\mathscr{M} = \bigcap_{n=0}^{\infty} V^n\mathscr{H}$ is the only possibility for such a subspace. To see that this one works, observe that

$$\mathscr{H} \supset V\mathscr{H} \supset V^2\mathscr{H} \supset \cdots$$

so that $V\mathscr{M} = \mathscr{M}$, and that from this the first three requirements follow. The last one results from

$$\bigcap_{n=0}^{\infty} V^n(\mathscr{M}^{\perp}) \subset \bigcap_{n=0}^{\infty} V^n\mathscr{H} = \mathscr{M}.$$

3. Symmetric operators. A **symmetric operator** on a Hilbert space $\mathscr{H}$ is a linear transformation A, defined on a dense linear man fold $\mathscr{D}_A$ in $\mathscr{H}$, such that $(Af, g) = (f, Ag)$ for all $f, g \in \mathscr{D}_A$. Such operators need not be bounded; the usual substitute is the requirement that A be closed (i.e., that the graph of A be closed in the Cartesian product $\mathscr{H} \times \mathscr{H}$). Symmetric operators arise naturally in the study of differential equations. A useful example is the following: let $\mathscr{H} = L^2(0, \infty)$, let $\mathscr{D}$ consist of those $f \in \mathscr{H}$ such that f is absolutely continuous, $f' \in L^2(0, \infty)$, and $f(0) = 0$, and let $Df = if'$ for all $f \in \mathscr{D}$. To see that D is symmetric, we need the fact that $f(t) \to 0$ as $t \to \infty$ for $f \in \mathscr{D}$. This follows from the formula

$$\int_0^t f\bar{f}' = |f(t)|^2 - \int_0^t f'\bar{f}$$

and the fact $|ff'|$ is integrable on $(0, \infty)$. Then

$$\begin{aligned}(Df, g) &= \lim_{t\to\infty} \int_0^t if'\bar{g}\\ &= \lim_{t\to\infty} \left(f(t)\overline{g(t)} - \int_0^t if\overline{g'}\right)\\ &= \lim_{t\to\infty} \int_0^t f(\overline{ig'}) = (f, Dg).\end{aligned}$$

It may also be shown that D is closed but not bounded.

In [7] von Neumann proved that any symmetric operator possesses maximal symmetric extensions, and described the structure of these extensions. We give a brief account of his reasoning. Let A be symmetric with domain $\mathscr{D}_A$. Then $A + iI$ is one-to-one on $\mathscr{D}_A$, so $V = (A - iI)(A + iI)^{-1}$ is a well-defined linear operator with domain $\mathscr{D}_V = (A + iI)\mathscr{D}_A$ and range $\mathscr{R}_V = (A - iI)\mathscr{D}_A$. This operator is called the **Cayley transform** of A. That this is an isometry follows from the easy relations:

$$\|(A + iI)f\|^2 = \|Af\|^2 + \|f\|^2 = \|(A - iI)f\|^2.$$

Not every (partially-defined) isometry arises in this way. In fact, with A and V as above we have

$$(I - V)\mathscr{D}_V = \mathscr{D}_A,$$

and so $(I - V)\mathscr{D}_V$ is dense. Conversely, if V is an isometry such that $(I - V)\mathscr{D}_V$ is dense, then $I - V$ is one-to-one on $\mathscr{D}_V$,

$$A = i(I + V)(I - V)^{-1}$$

defines a symmetric transformation on $\mathscr{D}_A = (I - V)\mathscr{D}_V$, and the Cayley transform of A is V.

The basis of von Neumann's argument is now clear: symmetric extensions of A correspond, via the Cayley transform, to isometric extensions of V. In particular, maximal symmetric operators correspond to maximal isometries. An isometry V is maximal if and only if either $\mathscr{D}_V = \mathscr{H}$ or $\mathscr{R}_V = \mathscr{H}$. Any isometry has such an extension, and therefore any symmetric operator has a maximal symmetric extension.

An interesting case occurs when both $\mathscr{D}_V$ and $\mathscr{R}_V$ are all of $\mathscr{H}$; i.e., V is unitary. In this case A is self-adjoint and is described by the spectral theorem [**1**, §66].

Now let A be maximal symmetric. Since $\mathscr{D}_V$ and $\mathscr{R}_V$ are interchanged when A is replaced by $-A$, it can be assumed that $\mathscr{D}_V = \mathscr{H}$, so that V is an isometry defined on all of $\mathscr{H}$. According to Theorems 1 and 3, V is a direct sum of a unitary operator and a number of copies of the simple shift. Hence A is a direct sum of a self-adjoint operator and a number of copies of the simple maximal symmetric operator (i.e., the operator with Cayley transform the simple shift).

To complete this discussion, we remark that the differential operator D introduced above is simple and maximal. In fact, if V is the Cayley transform of D and $h(t) = e^{-t}$, then $\{h, Vh, V^2h, \cdots\}$ is an orthonormal basis of $L^2(0, \infty)$, and consequently V is a simple shift [**1**, §82].

4. Models. A subspace $\mathscr{M}$ of a space $\mathscr{H}$ is **invariant** for a linear transformation T on $\mathscr{H}$ if $Tf \in \mathscr{M}$ for all $f \in \mathscr{M}$. One way to obtain new operators from old is by restricting to invariant subspaces. By a **part of an operator** T we shall mean a restriction of T to an invariant subspace.

A part of a pure isometry is itself a pure isometry, and therefore, by Theorem 2, a part of a shift is another shift. On the other hand, an astonishing variety of operators arise as parts of the adjoint of the shift. This situation, discovered by Rota [**9**], will now be described. Recall first that the **adjoint** of a bounded operator T on a Hilbert space $\mathscr{H}$ is the unique operator T^* satisfying

$$(Tf, g) = (f, T^*g), \; f, \; g \in \mathscr{H}.$$

The adjoint of the shift S on $l^2(\mathscr{H})$ is the **backward shift**, given by

$$S^*(f_0, f_1, f_2, \cdots) = (f_1, f_2, f_3, \cdots).$$

Let T be an operator on a Hilbert space $\mathscr{H}$, and consider the map

$$R: f \to (f, Tf, T^2f, \cdots), \qquad f \in \mathscr{H}.$$

We want this sequence to be in $l^2(\mathscr{H})$, and for this it is sufficient that T be a strict contraction (i.e., $\| T \| < 1$):

$$\sum_{n=0}^{\infty} \| T^n f \|^2 \leqq \sum_{n=0}^{\infty} \| T \|^{2n} \| f \|^2 = (1 - \| T \|^2)^{-1} \| f \|^2 < \infty.$$

Of course R is linear, and this computation also shows that it is bounded (by $(1 - \| T \|^2)^{-\frac{1}{2}}$). Moreover, it follows from the inequality

$$\| Rf \| \geqq \| f \|, \; f \in \mathscr{H}$$

that the range of R (call it $\mathscr{N}$) is a closed subspace of $l^2(\mathscr{H})$. Then the closed graph theorem implies that the inverse operator $R^{-1}: \mathscr{N} \to \mathscr{H}$ is bounded. Finally we have

$$\begin{aligned} RTf &= (Tf, T^2 f, T^3 f, \cdots) \\ &= S^*(f, Tf, T^2 f, \cdots) = S^* Rf \end{aligned}$$

for all f in $\mathscr{H}$. This says that R carries the action of T on $\mathscr{H}$ to that of S^* on $\mathscr{N}$. Two operators related in this fashion, by a bounded operator with a bounded inverse, are said to be **similar**. Thus we have shown that T and $S^* |\mathscr{N}$ are similar.

THEOREM 4. *Any strict contraction is similar to a part of a backward shift.*

This result has implications for the **invariant subspace problem**, which asks whether any bounded linear operator on a complex Hilbert space of dimension greater than 1 has a proper (different from $\{0\}$ and $\mathscr{H}$) invariant subspace, and which remains unsolved in spite of the efforts of many mathematicians. Since any bounded operator can be "scaled" so as to be a strict contraction, the theorem gives the following reformulation of the problem: are the minimal nonzero invariant subspaces of backward shifts one-dimensional? Invariant subspaces of shifts are considered in the next section.

Soon after Rota's result appeared, De Branges and Rovnyak [3] and Foias [4] noticed that a modification of his argument will produce a description, up to unitary equivalence, of all the parts of backward shifts. Backward shifts have the properties $\| S^* \| \leqq 1$ and $\| S^{*n} f \| \to 0$ as $n \to \infty$ for all f, as do all of their parts. These conditions are also sufficient.

THEOREM 5. *Any contraction with powers tending strongly to zero is unitarily equivalent to a part of a backward shift.*

Proof. Let T be such an operator. We want to duplicate the situation of the last proof, but with R replaced by an isometry. Now

$$\begin{aligned} ((I - T^*T)f, f) &= (f, f) - (f, T^*Tf) \\ &= \| f \|^2 - \| Tf \|^2 \geqq 0 \end{aligned}$$

since $\| T \| \leqq 1$. This means that the operator $I - T^*T$ is *positive*, and as such has a unique positive square root D [8]. Consider the map

$$W: f \to (Df, DTf, DT^2 f, \cdots), \qquad f \in \mathscr{H}.$$

Since

$$\begin{aligned}\|DT^nf\|^2 &= (T^nf, D^2T^nf)\\ &= (T^nf, T^nf) - (T^nf, (T^*T)T^nf)\\ &= \|T^nf\|^2 - \|T^{n+1}f\|^2,\end{aligned}$$

the series $\sum_{n=0}^{\infty} \|DT^nf\|^2$ is telescoping, and we have

$$\|Wf\|^2 = \|f\|^2 - \lim_{n\to\infty} \|T^nf\|^2 = \|f\|^2$$

since the powers of T tend to 0 strongly. Hence W is an isometry with range in $l^2(\mathscr{D})$, where $\mathscr{D}$ is the closure of the range of D. The rest of the argument is as before.

5. Invariant Subspaces. The result of the previous section makes the nature of the invariant subspaces of backward shifts a matter of great importance. For any operator T, the subspaces invariant for T^* are precisely the orthogonal complements of the subspaces invariant for T. Thus it will suffice to study shifts.

One of the few operators whose invariant subspace structure has been completely and satisfactorily described is the simple shift, in a fundamental paper of Beurling [2] (see also [5]). For this we use the realization of the shift as multiplication by e (the identity function $e^{i\theta}$) on the Hardy space H^2. To begin with, there are the obvious invariant subspaces e^nH^2, consisting of all e^nf, $f \in H^2$, and spanned by $\{e^n, e^{n+1}, \cdots\}$. More generally, if $\phi \in H^2$ is of unit modulus almost everywhere (such functions are said to be **inner**) then multiplication by ϕ is an isometry on H^2, so the range ϕH^2 is a closed subspace that is evidently invariant. Conversely:

THEOREM 6. *Any closed nonzero invariant subspace of the shift on H^2 is of the form ϕH^2 for a suitable inner function ϕ.*

Proof. Let $\mathscr{M}$ be invariant, and assume for the moment that the function 1 is not orthogonal to $\mathscr{M}$, so that the component ψ of 1 in $\mathscr{M}$ is not zero. Then $e^n\psi \in \mathscr{M}$ for all $n \geqq 1$, and therefore

$$\begin{aligned}0 = (e^n\psi, 1-\psi) &= \frac{1}{2\pi}\int_0^{2\pi} e^{in\theta}\psi(\theta)(1-\overline{\psi(\theta)})\,d\theta\\ &= \frac{1}{2\pi}\int_0^{2\pi} e^{in\theta}\psi(\theta)d\theta - \frac{1}{2\pi}\int_0^{2\pi} e^{in\theta}|\psi(\theta)|^2 d\theta\\ &= -\frac{1}{2\pi}\int_0^{2\pi} e^{in\theta}|\psi(\theta)|^2 d\theta\end{aligned}$$

since $\psi \in H^2$. By conjugation we obtain

$$\int_0^{2\pi} e^{in\theta}|\psi(\theta)|^2 d\theta = 0 \text{ for all } n \neq 0,$$

and so $|\psi(\theta)|$ is equal a.e. to a nonzero constant. Thus, a constant multiple of ψ is

inner, and it will suffice to show $\mathscr{M} = \psi H^2$. It is clear that $\mathscr{M}$ contains ψH^2 (since $\mathscr{M}$ contains $e^n\psi$, $n \geqq 0$), so suppose that f is in $\mathscr{M}$ and orthogonal to ψH^2. Since $e^n\psi \in \psi H^2$ for all $n \geqq 0$, we have

$$0 = (f, e^n\psi) = \frac{1}{2\pi}\int_0^{2\pi} e^{-in\theta} f(\theta)\overline{\psi(\theta)}d\theta, \qquad n \geqq 0.$$

On the other hand, arguing with $(e^n f, 1 - \psi)$, just as in the first calculation above, gives

$$\int_0^{2\pi} e^{in\theta} f(\theta)\overline{\psi(\theta)}d\theta = 0, \qquad n \geqq 1.$$

Hence $f\overline{\psi} = 0$ a.e., and, since ψ has constant nonzero modulus, $f = 0$. Thus $\mathscr{M} = \psi H^2$.

Finally, if 1 is orthogonal to $\mathscr{M}$, then all functions in $\mathscr{M}$ have vanishing Fourier coefficient of order zero, so $\mathscr{M} = e\mathscr{N}$ with $\mathscr{N}$ closed and invariant. Since $\mathscr{M} \neq \{0\}$ there is a largest integer n for which $\mathscr{M} = e^n\mathscr{M}_0$ with $\mathscr{M}_0$ closed and invariant. Then by the above $\mathscr{M}_0 = \phi H^2$ with ϕ inner, so $\mathscr{M} = e^n\phi H^2$ and $e^n\phi$ is inner. $\square$

This representation of invariant subspaces in terms of inner functions is not unique; however, it is easily seen that if $\phi_1 H^2 = \phi_2 H^2$ with ϕ_1 and ϕ_2 inner, then ϕ_1/ϕ_2 is constant almost everywhere. A great deal is known about inner functions [6], and therefore the theorem is a useful tool for answering questions about invariant subspaces of the simple shift.

In particular, the question of the previous section can be shown to have an affirmative answer in this case: the minimal nonzero invariant subspaces of the simple backward shift are one-dimensional. The same is true of the backward shift on $l^2(\mathscr{H})$ for $\mathscr{H}$ finite-dimensional (see [5]), but the general question remains open.

Based on a talk at the Indiana sectional meeting, Franklin College, May 13, 1972.

References

1. N. I. Akhiezer and I. M. Glazman, Theory of Linear Operators in Hilbert Space, Ungar, New York, 1963.

2. A. Beurling, On two problems concerning linear transformations in Hilbert space, Acta Math., 81 (1949) 239–255.

3. L. de Branges and J. Rovnyak, Perturbation Theory and Its Applications in Quantum Mechanics (Calvin H. Wilcox, ed.), Wiley, 1966.

4. C. Foias, A remark on the universal model for contractions of G.-C. Rota, Comm. Acad. R. P. Romîne, 13 (1963) 349–352.

5. H. Helson, Invariant Subspaces, Academic Press, New York, 1964.

6. K. Hoffman, Banach Spaces of Analytic Functions, Prentice-Hall, Englewood Cliffs, 1962.

7. J. von Neumann, Allgemeine Eigenwerttheorie Hermitescher Funktionaloperatoren, Math. Ann., 102 (1929) 49–131.

8. F. Riesz and B. Sz. Nagy, Functional Analysis, Ungar, New York, 1955.

9. G. -C. Rota, On models for linear operators, Comm. Pure Appl. Math., 13 (1960) 469–472.

DEPARTMENT OF MATHEMATICS, DALHOUSIE UNIVERSITY, HALIFAX, NOVA SCOTIA, CANADA.

RECURSIVE UNDECIDABILITY — AN EXPOSITION

JAMES P. JONES

1. Introduction. The discovery of proofs to the effect that certain mathematical problems are unsolvable has been one of the more remarkable achievements of mathematics of the nineteenth and twentieth centuries. Consider the ancient Greek geometry problems, doubling the cube and trisecting the angle with compass and straightedge. The search for solutions went on for over 2000 years. Then it was proved that what was sought did not exist. The history of attempts to solve the fifth degree polynomial equation is a similar story. Early in the nineteenth century Abel, Galois and Ruffini proved the impossibility of solving it by means of radicals. Somewhat later Lindemann proved the transcendence of π, thus disposing of the age-old problem of squaring the circle.

The classical unsolvability proofs were a rich source of new ideas for mathematics. The work of Galois would later inspire the entire development of modern algebra. But bold explorers did not wait long before taking the next step: proofs of *unprovability*.

In the first half of the nineteenth century Gauss, Bolyai and Lobachevsky obtained the independence of the parallel postulate. They proved that this postulate cannot be proved from Euclid's axioms for geometry. Since the parallel postulate is consistent (assuming the theory of real numbers consistent), its negation is also unprovable. Thus the parallel postulate is undecidable on the basis of Euclid's axioms.

In 1931 K. Gödel showed [**13**] that the theory of numbers also contains undecidable propositions. More importantly, he proved that it was impossible to alter the foundations of mathematics so as to exclude undecidable propositions. In 1963, P. Cohen [**7**] proved that the axiom of choice and the generalized continuum hypothesis were independent of the axioms of Zermelo-Fraenkel set theory. Gödel had already obtained the relative consistency of these two propositions in 1940 [**12**]. Thus these two famous problems were shown to be undecidable in axiomatic set theory.

It is interesting to note that Carl Frederick Gauss may have anticipated such developments in mathematics already in 1816. The Paris Academy had proposed the proof or disproof of Fermat's Last Theorem as its prize problem for the period. When attempts were made to persuade Gauss to compete, the Prince of Mathematicians replied: "Fermat's Last Theorem is an isolated proposition which has little interest for me, because I can easily lay down a multitude of such propositions, which one can neither prove nor disprove."

This expository article contains an explanation of recursive unsolvability, examples of non-computable functions and an example of a two-person game in which neither player has an algorithm to win. The intention is to provide a simple explanation of the subject, comprehensible to the non-logician. The results presented here are especially suited to this purpose because their proofs may be based entirely upon simple combinatorial arguments which avoid the formal arithmetization usually necessary in recursive function theory.

2. Recursive unsolvability. Dramatic as they may seem, the classical unsolvability proofs have nevertheless a distinctly relative character. The problem of doubling the cube can be solved if the use of a ruler in the form of a right angle is permitted.

Similarly, the proofs of unprovability have their relative nature. There is no concept of a proposition undecidable *per se*, only that of a proposition undecidable on the basis of given axioms. The continuum hypothesis becomes provable if we supplement the axioms of set theory with Gödel's axiom of constructibility [**12**]. Moreover, the presence of undecidable sentences in a theory is not particularly surprising unless we feel that the axioms in question ought to be categorical. Before the discovery that the parallel postulate was independent, it had been believed for centuries that Euclid's axioms were a categorical description of space.

It is natural to ask if there are problems in some sense absolutely unsolvable.

In the 1930's, proofs of a different sort of undecidability appeared. For the first time mathematical problems were proven unsolvable in the sense that there is no finite algorithm for dealing with them. The early explorers were the logicians A. Church, K. Gödel, S. C. Kleene, E. Post, J. B. Rosser and A. Turing. Problems of the type then considered have come to be known as *decision problems.*

Suppose we are given a countable set W and a particular subset A of W. By a *decision procedure* for A we mean an algorithm, in the form of a finite list of instructions, which permits us to decide effectively, for each element of W, whether or not it is an element of the set A. The **decision problem** for the set A is the problem of finding such a decision procedure, or proving that no such procedure exists. The set A is said to be *decidable* or *undecidable* according as a decision procedure exists or not. In the case that it does not exist, one also says that the decision problem is *recursively unsolvable.*

The original researchers, mentioned previously, defined the exact concept of an algorithm in the 1930's. This important work made recursive unsolvability proofs possible. Of course these researchers were, at that time, motivated primarily by decision problems in logic; questions of existence of mechanical procedures, like truth tables, for determining the truth or falsity of propositions. Today however, spurred by the advent of the large digital computer, questions of decidability have acquired much more than their former philosophical interest. The past quarter century has seen the solution of important decision problems in nearly every branch of mathematics. Let us consider some examples.

One of the most important decision problems is the very difficult *word problem for groups.* Finitely presented groups arise in a natural way in algebraic topology, for example in the problem of classification of knots. A group is presented as a finite system of generators and relations. The word problem is the problem of deciding, given a word on the generators, whether or not this word is equal to the identity, in consequence of the relations. (Equivalently, whether two words are equal.)

The word problem was one of the first decision problems to be formulated. It was first considered by M. Dehn and A. Thue in 1912–14. An important first step toward

its eventual solution (in the negative) was taken by E. Post and A. Markov in 1947. They (independently) proved the analogous *word problem for semi-groups* unsolvable (cf. [**8**, Chap. 6]). As seems to be the rule rather than the exception in this subject, the (negative) solution of the word problem for groups was also obtained nearly simultaneously in the East and West. During the period 1954–56, P.S. Novikov and W. W. Boone independently proved the word problem for groups undecidable. For groups the problem was much more difficult than for semi-groups. Novikov's original paper [**27**] is 143 pages long. Boone's proof, contained in a long series of six papers extending over four years, was later shortened by J. L. Britton. Britton's proof is now available in a textbook [**39**].

Not only is the word problem for groups unsolvable in the general sense; Novikov and Boone produced *particular* groups with unsolvable word problem. Concerning this it is interesting to note that in consequence of the Higman, Neumann, Neumann embedding theorem [**15**], such groups must exist with only 2 generators.

Not all results on the word problem have been negative. For many groups, for example finite groups and free groups, the word problem is solvable. K. Reidemeister proved that the word problem for an Abelian group is solvable [**33**]. Other positive results may be found in [**20**].

The word problem concerns the elements of a group. More important to algebraists are decision problems concerning algebraic properties of groups as a whole. M. Rabin [**31**] and S. I. Adjan [**1**] (independently) obtained important results on these decision problems. They showed that for a wide class of algebraic properties, including finiteness, commutativity, cyclicity, simplicity, solvability and many others, there does not exist an effective method of deciding from presentations, whether the defined group has the property in question. This theorem led Rabin and Adjan to the negative solution of the *isomorphism problem*: There is no algorithm to decide, given two presentations, whether or not the groups defined by them are isomorphic.

The Rabin-Adjan theorem also applies to the property of having a solvable word problem. Thus in general we cannot tell from a presentation, whether the word problem is solvable or not. And oddly enough, it turns out that there is no partial algorithm to solve the word problem even for just those groups with solvable word problem. This last result is proved in [**5**]. More information on decision problems in group theory may be found in [**25**].

Since the word problem was not on its face a problem in logic, its solution did much to convince mathematicians that the work of logicians could have significant consequences outside logic. This opinion was further reinforced in 1958 when an undecidability result appeared in topology. A. Markov proved [**21**] that the *homeomorphism problem for* 4-*manifolds* was undecidable. Roughly, this is the problem of deciding, given (suitable descriptions of) two 4-dimensional manifolds, whether or not they are homeomorphic (cf. [**4**]). The homeomorphism problem for 2-manifolds is solvable by well-known methods. For 3-manifolds the problem is still open.

Also in combinatorial geometry one finds decision problems: the so-called *domino problem* (proved undecidable by Berger [2]) and various other problems of *tiling the plane* (cf. [37]). John Conway has announced that his "Game of Life" is undecidable.

Decision problems arise naturally even in analysis. In calculus one is confronted with the *elementary integrability problem.* Recall that a function of a real variable is *elementary* if it can be built up from integers, the rational functions, nth roots, exponentials, logarithms, trigonometric functions and their inverses, by addition, multiplication and composition. The derivative of an elementary function is again elementary but some elementary functions do not have elementary integrals. For example, it is well known [35] that the following integrals are not elementary:

$$\int e^{-x^2}dx, \qquad \int \frac{\sin(x)}{x}\,dx, \qquad \int \sin(x^2)dx.$$

Thus it is natural to ask if there is an algorithm by means of which we may recognize whether a given elementary function is elementarily integrable or not. Interestingly enough, D. Richardson [34] has shown that no such algorithm exists.

Bound up with many results in logic is the important *Entscheidungsproblem*, the decision problem for provability of sentences of an axiomatic theory. Numerous important cases of this problem have now been solved, notably by A. Tarski [40], and not all of them in the negative. A celebrated theorem of Tarski [41] asserts that there is a decision procedure for deciding the truth or falsity of *all propositions of elementary algebra and geometry.* The propositions of elementary algebra here include all formulas involving integers, inequalities, addition and multiplication in which the variables range over all real numbers.

The tenth problem on David Hilbert's famous list [16] was a decision problem. Hilbert asked for *an algorithm to decide of a polynomial equation, in several variables, with integer coefficients, whether or not the equation had solutions in integers.* Ju. V. Matijasevič recently showed that this problem is unsolvable. Matijasevič proved that there is no such algorithm [22], [23].

Hilbert's tenth problem has had a long and interesting history. Hilbert first stated the problem at the International Congress of Mathematicians in Paris in 1900. At that time, one could imagine only a positive solution to the problem. The concept of algorithm had not yet been worked out. Indeed, largely for this reason, significant work was not done on the problem until the 1950's. At about this time, the early fifties, Julia Robinson, Martin Davis and Hilary Putnam began studying Diophantine sets. [A relation $R(x_1, x_2, \cdots, x_n)$ is said to be *Diophantine* if there exists a polynomial $P(x_1, x_2, \cdots, x_n, y_1, y_2, \cdots, y_m)$, with integer coefficients, such that $R(x_1, x_2, \cdots, x_n)$ holds if and only if integers $y_1, y_2, \cdots, y_m$ exist for which $P(x_1, x_2, \cdots, x_n, y_1, y_2, \cdots, y_m) = 0$.] In 1961 Robinson, Davis and Putnam came very close to the negative solution of Hilbert's tenth problem. They succeeded in showing that its unsolvability would follow from the existence of a single Diophantine predicate with exponential growth [10]. In 1970 Matijasevič was able to use this result to deliver the *coup de grâce* to

Hilbert's tenth problem. By means of clever number theoretic arguments, termed "elementary but ingenious" by J. W. S. Cassels, Matijasevič proved that the relation "v is the $2u$th Fibonacci number" was Diophantine.

The solution of this long outstanding problem is one of the most significant events to occur in pure mathematics in the last decade. Only one byproduct of the ingenious solution of Hilbert's tenth problem is the long sought polynomial formula for the prime numbers. From results of Putnam [**29**] and Matijasevič [**22**] [**23**] it follows that there exists a polynomial whose *positive* values are identical with the set of prime numbers. (Negative integers also appear in the range.) In reference [**24**] Matijasevič has shown that 21 variables are sufficient in this polynomial.

It is interesting to consider variants of Hilbert's tenth problem. We may enquire about existence of solutions in sets of numbers other than the integers. Of course for complex numbers the problem is decidable. A polynomial has solutions in complex numbers if and only if its degree is nonzero. Also for the set of real numbers, the problem is decidable. This fact is less obvious. The algorithm of Tarski for elementary algebra [**41**] provides a decision method in this case. For the integers, non-negative integers and positive integers, the problem is unsolvable [**23**], [**9**]. Oddly enough however, for rational numbers the problem is still open.

Unsolvability results have also appeared in *game theory*. M. Rabin proved in 1957 [**30**] that there exists a two person win-lose game, with perfect information and decidable rules, in which neither player has a computable winning strategy. The author has also discovered an example of such a game (somewhat simpler than Rabin's) [**17**]. We describe our *Turing machine game* in Section 7.

In considering recursive unsolvability results, it is important to keep in mind the very wide scope of the concept of *algorithm*. An algorithm, in the form of a finite list of instructions, is potentially a quite powerful problem-solving procedure. Today, computing machines, using algorithms, not only play games, but *learn* to play games. A computer can be programmed to play a better game of checkers than can be played by the person who wrote the program. Furthermore, the digital computer is the man-made analogue most closely resembling the brain. Thus an algorithmic unsolvability proof establishes unsolvability in a quite strong sense. Post has written [**28**]: Like the classic unsolvability proofs, these proofs are of unsolvability by means of given instruments. What is new is that in the present case these instruments, in effect, seem to be the only instruments at man's disposal.

3. Turing machines. An algorithmic unsolvability proof is supposed to establish the impossibility of an algorithm. But the concept of *algorithm* is vague. Before it can be proved that something does not exist it must be defined precisely. What is an *algorithm*?

If we analyze the notion we see that it is closely bound up with the concept of *computability*. Consider a function from the natural numbers to the natural numbers. Intuitively, the function should be computable if there exists an algorithm permitting

us to obtain its values for each argument. Conversely, the decision problem for a set reduces to the computation problem for a certain function; the characteristic function of the set.

Thus we will have made our ideas precise once we succeed in defining the class of computable functions (of a natural number variable). What is meant exactly when we say that a function is *computable*?

If an algorithm exists for computing a function then it should be possible to program a computer to carry out the algorithm and calculate the value of the function. At least this should be possible in principle, allowing the computer sufficient time (an arbitrary finite amount) and sufficient external storage for intermediate calculations (again an arbitrary finite amount).

Today, one might attempt to base a definition of computability upon physically existing computers. However, this would be unsatisfactory. It would leave unanswered two very important questions. What is to be the exact relationship between the computer and its external memory (the machine-tape coupling)? And, more importantly, what operations need a computer be capable of in order to perform "all possible computations"?

For the purpose of *defining* computability one is forced to abandon physically existing machines. One must instead describe an idealized mathematical model of a computer.

This is what was done by A. Turing in a classic paper published in 1936 [**42**]. In this paper, Turing defined the class of theoretical computing machines that now bear his name. Here is specified in exact detail a simple mechanical basis sufficient to carry out any computation whatsoever.

We shall not reproduce here Turing's original formulation in detail. Rather we present a simplification (due to Kleene [**18**] and Rado [**32**]). The essence of the conception is nevertheless Turing's.

One visualizes a Turing machine in the form of a finite reading and writing device operating upon a two-way infinite linear tape. The tape is ruled off into squares. Upon each square of the tape may be printed a symbol from a finite alphabet, including "0" (blank) and "1". At each moment of time, the machine is assumed to be in only one of a finite number of internal *states*, which we number $0, 1, 2, \cdots n$. The machine can *scan* only a single square of the tape at a time. Its behavior is completely determined by its current internal state and the contents of this scanned square. It erases the symbol on this square, prints a new one (possibly the same), shifts itself left or right on the tape (so as to scan an adjacent square), and changes itself into a new internal state (possibly the same state).

Turing believed that any computation could be reduced to a succession of such simple acts.

Turing machines operate sequentially, performing one act per unit of time They are assumed to begin all computations initially in state 1 (the starting state). State 0

is the stopping state. The machine is said to *halt* (cease operations) upon entering this inactive state.

It is more accurate to think of each individual Turing machine as a *program* rather than a machine. We represent this program with a finite set of "computer cards". Each instruction card corresponds to an active state of the machine. When the alphabet is restricted to the two symbols "0" and "1", this representation is particularly simple.

EXAMPLE 1. A TURING MACHINE WITH TWO STATES.

Card 1	
0	1,R,2
1	1,L,2

Card 2	
0	1,L,1
1	1,L,0

The ith card represents the ith state of the Turing machine. On each card, the first column gives the *scanned symbol*, the second column is the *overprint by* column, the third column is the *shift right or left* column and the fourth column gives the *next state to enter.*

Suppose the machine of example 1 is started in state 1 scanning the symbol 1, the remainder of the tape assumed blank. Then card 1 instructs it to leave this symbol unchanged, shift left on the tape and enter state 2. By assumption the square now scanned contains 0. Card 2 instructs the machine to overprint this 0 by 1, move left and (re)enter state 1. The scanned square is again 0 so card 1 instructs the machine to overprint this 0 by 1, move right and enter state 2. Now the machine sees 1, so card 2 instructs it to leave this 1 unchanged, shift left and stop. The result is three ones printed on the tape.

One could represent the machine's behavior with the sequence of successive tape conditions:

$$\begin{array}{ccccccccc} 001 & \rightarrow & 001 & \rightarrow & 011 & \rightarrow & 111 & \rightarrow & 111. \\ 1 & & 2 & & 1 & & 2 & & 0 \end{array}$$

Apparently, the effect of this machine is to "add 2". If this machine is started in state 1 scanning the leftmost of n consecutive ones, then it will be seen to halt after 4 shifts. And when it does, it will be scanning the leftmost of $n+2$ consecutive ones.

We represent the natural number x on the tape with a string of $x + 1$ consecutive ones (tallies). The number 0 is represented on the tape as a single 1 to distinguish it from a blank square. An ordered pair could be represented with two groups of tallies separated by a blank (0).

With these conventions we may now define computability for functions of a natural number.

DEFINITION. *A function f is* **Turing computable** *if there exists a Turing machine M such that whenever M is started in state* 1 *scanning the leftmost of* $x + 1$ *consecutive ones, M eventually halts scanning the leftmost of* $f(x) + 1$ *consecutive ones.*

In this sense the machine of example 1 computes the function $f(x) = x + 2$. The following Turing machine computes the function $f(x) = 2x$.

EXAMPLE 2. A TURING MACHINE WHICH COMPUTES $f(x) = 2x$.

Card 1		Card 2		Card 3		Card 4		Card 5	
0	0,R,4	0	1,R,3	0	1,R,1	0	0,L,4	0	0,R,0
1	0,L,2	1	1,L,2	1	1,R,3	1	0,L,5	1	1,L,5

It is important to note that although the amount of information stored at any one time by a Turing machine (on its tape) is always finite, we do not place any preassigned upper bound on this amount. To do so would be tantamount to making our definition dependent upon computer technology and therefore mathematically uninteresting.

4. Church's Thesis. Is every computable function computable by a Turing machine? The answer to this question appears to be "yes". This may be difficult for the reader to believe. Perhaps, since computation is normally done on sheets of paper, it might be thought that the two-dimensional nature of paper is essential. However, as we allow any finite number of symbols in the alphabet, there is no reason why each individual tape square cannot be thought to correspond to an entire sheet of paper. Also, our insistence that there be only a finite number of symbols in the alphabet is not a severe restriction. If more symbols are needed it is always possible to use sequences of symbols in place of individual symbols. (Indeed, for this reason, we may restrict our machines to the use of only two symbols "0" and "1". It is well known that the class of functions computed is not diminished by so doing. For a proof of this see [**26**, p. 129].

The assertion that *every computable function is Turing computable,* is known as the Church-Turing thesis. It is called a "thesis" rather than a "theorem" because it is not susceptible to proof. The thesis is rather in the nature of a definition. It is a proposal that we identify our intuitive concept of computable function with the exact mathematical substitute (Turing computability).

Turing believed that any process which could effectively be carried out, could actually be realized on one of his machines. And the years have tended to bear out Turing's view. Today, the Church-Turing thesis is generally accepted by virtually all workers in logic.

Convincing evidence for the Church-Turing thesis is provided by certain historical

events. Recall that Turing put forward his proposed characterization of computability in 1936 [**42**]. At about this same time, two other characterizations were also proposed: The *λ-definable functions* (Church, Kleene), and the *general recursive functions* (Herbrand, Gödel). Superficially, the three formulations appeared to be different. Between them they encompassed all methods of calculation known at the time, including the various definitions by recursion (induction). Yet the three different characterizations were proved equivalent. The same (evidently fundamental) class of functions is obtained in each case.

An additional argument in favor of the Church-Turing thesis is given by Kleene [**18**]: A great stock of intuitively computable functions (all that have been investigated in this connection) are known to be Turing computable. Likewise we have a great stock of methods or operations (for obtaining new intuitively computable functions from others) which are paralleled by operations for building new Turing machines from given Turing machines (or the analog in terms of recursiveness). If there were a function which is intuitively computable but not Turing computable, it would have to be "inaccessible" by any process of building up toward it from this stock of functions and operations already mastered.

So it was that the *recursive functions* were discovered. Concerning this class of functions Gödel remarks [**11**]: "... with this concept of recursiveness one has for the first time succeeded in giving an absolute definition of an interesting epistemological notion, i.e., one not depending upon the formalism chosen." Post has remarked: "Indeed, if general recursive function is the formal equivalent of effective calculability, its formulation may play a role in the history of combinatory mathematics second only to that of the formulation of the concept of natural number." [**28**].

5. A non-computable function. The machine presented in Example 1 is one of the highest scorers in a game invented by T. Rado [**32**]. Rado's "Busy Beaver n game" is the problem of programming an n-state (2-symbol) Turing machine to print the largest possible number of ones onto a (blank) tape and halt. If the machine of example 1 is started in state 1 on a blank tape it will be seen to halt after 6 shifts and when it does there will be 4 ones printed on its tape. 4 is the highest score obtainable in the Busy Beaver 2 game.

What is the highest score possible in the Busy Beaver n game? Formally, a 2-symbol Turing machine of n states is simply a mapping

$$f: \{0,1\} \times \{1,2,3,\cdots n\} \to \{0,1\} \times \{R,L\} \times \{0,1,2,3,\cdots n\}.$$

Thus there are exactly $(4n+4)^{2n}$ Turing machines with n-states, a finite number. Some of these will fail to halt if started on a blank tape; the machine of example 2 loops forever. However, among the machines which halt, there must be one or more which print the maximum possible number of ones. Let $\Sigma(n)$ denote this maximum score.

Evidently, Σ is a well defined function of n, because a finite set of numbers has

a greatest element. The function Σ has been calculated for a few small values of n. It is known that $\Sigma(1) = 1$ and $\Sigma(2) = 4$. S. Lin proved in [19] that $\Sigma(3) = 6$. Also, one sees immediately that for each n, $\Sigma(n) < \Sigma(n + 1)$. For, with an additional card, a machine may be instructed to move right (or left) across ones to overprint with 1 the first 0 encountered, and halt.

Card $n+1$	
0	1,R,0
1	1,R,$n+1$

Can a formula for $\Sigma(n)$ be found?

There are difficulties in determining $\Sigma(n)$ for large values of n. First of all, there is an enormous number of machines to consider. There are 25,600,000,000 different Turing machines with 4 states. Of course, many of these fail to halt, and may therefore be eliminated from consideration. However, it is very difficult to decide, given a Turing machine, whether or not it will eventually halt. This is the so called *halting problem*. We shall see that the function Σ is not computable (non-recursive).

Now, whether Σ were computable or not, one would nevertheless expect that it would be possible to give an upper bound for Σ. One would expect to be able to prove that $\Sigma(n) \leqq 2^n$ or perhaps $\Sigma(n) \leqq n^{(n^n)}$ or at least some such inequality. Surprisingly enough, no such inequalities hold! These two proposed bounds are computable functions. Not only is Σ non-computable, Rado discovered [32] that Σ has no computable upper bound.

THEOREM 1. *Let f be any increasing computable function. Then for n sufficiently large, $f(n) < \Sigma(n)$. In particular Σ is non-computable.*

The proof is almost trivial. Let M be a Turing machine with c states which computes f. Denote by T the Turing machine of 5 states which computes the function $2x$ (example 2).

For each x, let $M^{(x)}$ be a Turing machine with $x + 1$ states which will print $x + 1$ consecutive ones onto a blank tape and halt scanning the leftmost of these. For example, $M^{(2)}$ could be taken to be the following machine:

Card 1	
0	1,L,2
1	——

Card 2	
0	1,L,3
1	——

Card 3	
0	1,R,3
1	1,L,0

Now consider the composition of machines, which we denote by

$$M[T[M^{(x)}]].$$

It is obtained by relabeling states and is constructed so as to print $f(2x)+1$ ones onto a blank tape and halt. Since this composite machine has $x+6+c$ states it follows that

$$\text{(i)} \qquad f(2x) < \Sigma(x+6+c).$$

Now for $x \geqq 6+c$ we have $x+6+c \leqq 2x$. Therefore, since f is assumed to be increasing,

$$\text{(ii)} \qquad f(x+6+c) \leqq f(2x).$$

Let $n = x+6+c$. Then for $n \geqq 12+2c$ we have $f(n) < \Sigma(n)$ by (i) and (ii). Since Σ itself is increasing we have proved that Σ is non-computable.

COROLLARY 1. *Let f be any computable function (increasing or not). Then for n sufficiently large, $f(n) < \Sigma(n)$.*

Proof: For each computable function f we can find an increasing computable function $\hat{f}$ such that $f(n) \leqq \hat{f}(n)$. For example we may take

$$\hat{f}(n) = \sum_{i=0}^{n} f(i).$$

Intuitively, $\hat{f}$ is computable. (For a rigorous proof of this see [**18**], p. 224.)

The halting problem. Evidently, the only obstacle to the computation of Σ is the halting problem: *To decide, given a Turing machine M, whether or not M will eventually halt.* The halting problem was the first decision problem to be proved unsolvable. Its unsolvability was obtained by Church in 1936 [**6**].

An informal proof of the undecidability of the halting problem may be given, based on the non-computability of Σ. Assume for contradiction that an algorithm A existed for deciding the halting problem. Then Σ could be computed algorithmically as follows: Given n, list the $(4n+4)^{2n}$ n-state Turing machines. Using A, strike out from the list those which will not halt. Operate the others until they halt. Examine the number of ones on their tapes for the largest. Since Σ is not computable this is a contradiction. (For a rigorous proof see [**18**].)

For programming, the fact that the halting problem is recursively unsolvable means that there does not exist *one* program capable of determining of an *arbitrary* program whether or not that arbitrary program will eventually halt.

7. An unsolvable game. Consider the following two-person impartial win lose game with perfect information. We shall refer to it as the *Turing machine game.*

THE TURING MACHINE GAME. *There are two players, I and II. They take turns choosing positive integers:*

first	*I picks n,*
then	*II picks m (knowing n),*
finally	*I picks k (knowing m).*

Player I wins if some n-state Turing machine halts in exactly $m + k$ shifts when started on a blank tape. Otherwise, player II wins.

In contrast to many games considered in the theory of games, the Turing machine game may conceivably be played. *It is possible to decide, after the game is over, who has won.* After I and II have selected their integers, a referee need only list the $(4n + 4)^{2n}$ n-state Turing machines and then operate each of them through exactly $m + k$ shifts to determine the winner. It is *not* necessary for the referee to be able to solve the halting problem. A machine can be programmed to decide the winner.

Now it is well known that a game of finite length is *determined.* There exists a winning strategy for one of the players. In this case it is player II. However, we shall show that the Turing machine game is nevertheless not trivial to play. In this game neither player has an algorithm for winning.

THEOREM 2. *In the Turing machine game neither player has a computable winning strategy.*

Proof: Let $SH(n)$ be the maximum possible number of shifts which an n-state Turing machine can perform before halting (after being started in state 1 on a blank tape). Obviously $\Sigma(n) \leqq SH(n)$. A strategy for player II is a function $m = f(n)$. Plainly, f is a winning strategy for player II if and only if $SH(n) \leqq f(n)$. Thus player II has a winning strategy $SH(n)$. However, by Corollary 1, for each computable function f, eventually $f(n) < \Sigma(n) \leqq SH(n)$. Hence player II has no computable winning strategy.

8. Generalizations of the Busy Beaver problem. We have defined $SH(n)$ to be the maximum possible number of *shifts* which an n-state (2-symbol) Turing machine can perform when started in state 1 on a blank tape. Analogously define $SC(n)$ to be the maximum possible number of tape squares which an n-state Turing machine can *scan* in an active state before halting (when started in state 1 on a blank tape). Define $H(n)$ to be the *number* of different n-state Turing machines which halt when started in state 1 on a blank tape. Plainly

(1) $\Sigma(n) \leqq SC(n) \leqq SH(n)$ and (2) $H(n) < (4n + 4)^{2n}$.

From these inequalities it follows that the functions SC and SH are non-computable. The function H is also non-computable. For it is not difficult to show that if an algorithm existed for computing H, then the halting problem would be solvable. The

known values of these functions appear in the following table.

TABLE 1

n	1	2	3	4	5	6	7	8	10
$\Sigma(n)$	1	4	6	$13 \leqq$	$16 \leqq$	$35 \leqq$	$22{,}961 \leqq$	$9 \times 10^{41} \leqq$	$10^{2 \times 10^{44}} \leqq$
$SC(n)$	1	4	7	$14 \leqq$					
$SH(n)$	1	6	21	$107 \leqq$		$436 \leqq$	$10^{693} \leqq$		$10^{4 \times 10^{44}} \leqq$
$H(n)$	32	9,784	7,571,840						

Results in Table 1 are due to T. Rado, S. Lin, C. Y. Lee, P. Fischer, M. Green, D. Jefferson, J. Slater and the author.

As was already evident from Theorem 1, the rate of growth of the function Σ is astronomical. This is illustrated in row 1 by the very large scores in the Busy Beaver 8 and 10 games (achieved by Green [**14**]). Not only is it true that $\Sigma(n) < \Sigma(n+1)$ for each n. Indeed it is easy to see that for any computable function f

$$f(\Sigma(n)) < \Sigma(n+1)$$

for infinitely many values of n. Thus, for example, infinitely often we have

$$\Sigma(n)! < \Sigma(n+1).$$

The functions SC and SH are related to Σ. We can prove

$$(3) \quad SC(n) < \Sigma(3n) \quad \text{and} \quad (4) \quad SH(n) \leqq n \cdot SC(n) \cdot 2^{SC(n)}.$$

Inequality (3) is derived from the fact that each n-state Turing machine may be simulated (on every other square of the tape) by a $3n$-state Turing machine which records the number of squares scanned with ones. The inequality (4) is derived by counting tape-state configurations. These inequalities are due to T. Rado, S. Lin and the author. It is not difficult to show, using (3) and (4), that all four non-recursive functions have the same degree of recursive unsolvability ($0'$). The range of the function Σ is an example of a hyperimmune set. The range of H is retraceable and therefore immune but not hyperimmune. The set $\{(m, n): m \leqq H(n)\}$ is r.e. nonrecursive.

Consider a Turing machine programmed by a monkey. The probability of such a random Turing machine halting when started on a blank tape might be expressed by the limit of the frequencies:

$$\lim_{n \to \infty} \frac{H(n)}{(4n+4)^{2n}}.$$

From Table 1 we find that the first three values of this quotient are (approximately) .500, .471 and .451. However, the exact value of the limit is unknown.

Acknowledgment. The author wishes to thank Professors Verena H. Dyson and Roger C. Lyndon for many helpful suggestions regarding the preparation of this paper.

References

1. S. I. Adjan, The algorithmic unsolvability of problems concerning certain properties of groups (Russian), Dokl. Akad. Nauk SSSR, 103 (1955) 533–535.

2. R. Berger, The undecidability of the domino problem, Mem. Amer. Math. Soc., no. 66, (1966).

3. W. W. Boone, Certain simple unsolvable problems of group theory, Indag. Math., 16 (1954) 231–237, 492–497; 17 (1955) 252–256, 571–577; 19 (1957) 22–27, 227–232.

4. W. W. Boone, W. Haken and V. Poénaru, On recursively unsolvable problems in topology and their classification, Contributions to Mathematical Logic. Proceedings of the logic colloquium, Hannover, Germany 1966. Edited by H. A. Schmidt, K. Schütte and H. J. Thiele. North Holland, Amsterdam, 1968.

5. W. W. Boone and H. Rogers Jr., On a problem of J. H. C. Whitehead and a problem of Alonzo Church, Math. Scand., 19 (1966) 185–192.

6. A. Church, An unsolvable problem of elementary number theory, Amer. Jour. Math., 58 (1936) 345–363.

7. P. Cohen, The independence of the continuum hypothesis, Proc. National Acad. Sciences, U. S. A., 50 (1963) 1143–1148 and 51 (1964) 105–110.

8. M. Davis, Computability and Unsolvability, McGraw Hill, New York, 1958, xxv + 210 pp.

9. ———, Hilbert's tenth problem is unsolvable, this MONTHLY, 80 (1973) 233–269.

10. M. Davis, H. Putnam and J. Robinson, The decision problem for exponential Diophantine equations, Ann. Math., 74 (1961) 425–436.

11. K. Gödel, Remarks before the Princeton bicentennial conference on problems in mathematics 1946. Reprinted in Davis, The Undecidable, Raven Press, Hewlett, N. Y., 1965, pp. 84–88.

12. ———, The Consistency of the Axiom of Choice and of the Generalized Continuum-Hypothesis with the Axioms of Set Theory, Annals of Math. Studies, no. 3 (1940), Princeton University Press, Princeton, N. J., 66 pp. Second printing 1951, v + 69 pp.

13. ———, Über formal unentscheidbare Sätze der Principia Mathematica und verwandter Systeme I, Monatsh. Math. und Physik, 38 (1931) 173–189. English translations: (1) On formally undecidable propositions of Principia Mathematica and related systems, Martin Davis (editor) The Undecidable, Raven Press, 1965, pp. 5–38. (2) Jean van Heijenoort (editor), From Frege to Gödel, Harvard University Press, 1967, pp. 596–616.

14. M. Green, Proceedings of the Fifth Annual Symposium on Switching Circuit Theory and Logical Design, Princeton University Conference, Oct. 1964, pp. 91–94.

15. G. Higman, B. H. Neumann and H. Neumann, Embedding theorems for groups, J. London Math. Soc., 24 (1949) 247–254.

16. D. Hilbert, Mathematische Probleme, Vortrag, gehalten auf dem internationalen Mathematiker-Kongress zu Paris 1900. Nachrichten Akad. Wiss. Göttingen, Math.-Phys. Kl. (1900) 253–297. English translation: Bull. Amer. Math. Soc., 8 (1901–1902) 437–479.

17. J. P. Jones, A simple unsolvable game, Abstracts of the IV-th International Congress for Logic, Methodology and Philosophy of Science, Bucharest, Rumania, 1971.

18. S. C. Kleene, Introduction to Metamathematics, Van Nostrand, Princeton, N. J., 1952, x + 550 pp.

19. S. Lin and T. Rado, Computer studies of Turing machine problems, J. Assoc. Comp. Mach., 12 (1965) 196–212.

20. R. C. Lyndon, On Dehn's algorithm, Math. Ann., 166 (1966) 208–228.

21. A. Markov, Unsolvability of the problem of homeomorphy, Proc. Int. Cong. Math., Edinburgh 1958, pp. 300–306.

22. Ju. V. Matijasevič, Enumerable sets are Diophantine, Dokl. Acad. Nauk SSSR, 191 (1970) 279–282. English translation: Soviet Math. Doklady, 11 (1970) 354–357.

23. ———, Diophantine representation of enumerable predicates, Izvestija Akademii Nauk SSSR. Serija Matematičeskaja, 35 (1971) 3–30. English translation: Mathematics of the USSSR — Izvestija, 5 (1971) 1–28.

24. ———, Diophantine representation of the set of prime numbers, Dokl. Akad. Nauk SSSR, 196 (1971) 770–773. English translation: Soviet Math. Doklady, 12 (1971) 249–254.

25. C. F. Miller III, On group-theoretic decision problems and their classification, Annals of Math. Studies, 68 (1971), Princeton, N. J., vii + 106 pp.

26. M. Minsky, Computation: Finite and Infinite Machines, Prentice-Hall, Englewood Cliffs, N. J. 1967, xii + 317 pp.

27. P. S. Novikov, On the algorithmic unsolvability of the word problem in group theory, Akademiya Nauk SSSR, Mathematicheskii Institut Trudy no. 44 (1955), Moscow. Amer. Math. Soc. Translations, ser. 2, 9 (1958) 1–122.

28. E. Post, Recursively enumerable sets of positive integers and their decision problems, Bull. Amer. Math. Soc., 50 (1944) 284–316.

29. H. Putnam, An unsolvable problem in number theory, J. Symbolic Logic, 25 (1960) 220–232.

30. M. Rabin, Effective computability of winning strategies, Annals of Math. Studies, Princeton, 39 (1957) 147–157.

31. ———, Recursive unsolvability of group theoretic problems, Ann. of Math., 67 (1958) 172–194.

32. T. Rado, On non-computable functions, Bell System Technical Jour., 41 (1962) 877–884.

33. K. Reidemeister, Einführung in die Kombinatorische Topologie, Braunschweig, 1932. (Also Chelsea, New York, 1950.)

34. D. Richardson, Some undecidable problems involving elementary functions of a real variable, J. Symbolic Logic, 33 (1968) 514–520.

35. J. F. Ritt, Integration in Finite Terms, Columbia University Press, New York, 1948.

36. J. Robinson, Hilbert's tenth problem, AMS Proceedings of Symposia in Pure Mathematics, 20 (1971) 191–194.

37. R. M. Robinson, Undecidability and nonperiodicity for tilings of the plane, Invent. Math., 12 (1971) 117–209.

38. H. Rogers Jr., Theory of Recursive Functions and Effective Computability, McGraw Hill, New York, 1967, ix + 482 pp.

39. J. Rotman, The Theory of Groups: An Introduction, Allyn and Bacon, Boston, 2nd ed., 1973.

40. A. Tarski, A. Mostowski and R. M. Robinson, Undecidable Theories, North-Holland, Amsterdam, 1953, xi + 98 pp.

41. A. Tarski, A Decision Method for Elementary Algebra and Geometry, Univ. of California Press, Berkeley, California, 1951.

42. A. Turing, On computable numbers, with an application to the Entscheidungsproblem, Proc. London Math. Soc. ser. 2, 42 (1936–7) 230–265; corrections, Ibid., 43 (1937) 544–546.

DEPARTMENT OF MATHEMATICS, STATISTICS AND COMPUTING SCIENCE, UNIVERSITY OF CALGARY, CALGARY, ALBERTA, CANADA.

NEARNESS — A BETTER APPROACH TO CONTINUITY AND LIMITS

P. CAMERON, J. G. HOCKING, AND S. A. NAIMPALLY

This article advocates an unusual introduction to the theory of continuity and limits. Surely an ideal presentation would begin with a simple concept having strong intuitive appeal. That appeal should be retained along a direct line from the crude beginnings to complete rigor. Furthermore, a student should be able to "drop off" at any point along that line without losing all his feeling for the topic. A natural unification of related ideas should come easily as should the extensions to more advanced material. And any presentation should lend itself to class participation. We think our approach comes closer to such an ideal than any other we have seen.

In discussing continuity the calculus teacher frequently sets a subtle trap. There usually comes the moment when the teacher considers some fixed point c in the domain and says that points near c are to be carried to images near $f(c)$. Then by looking at such a function as $1/x$ with $c = 1/1000$, he may refine this by saying that points as near c as you like must be considered. He may speak of the function having no jumps, etc. All of this discussion has led the unsuspecting students to a "co-variant" view of continuity. Then the teacher switches direction, hitting the students with the "contravariant" ε-δ definition. The resulting confusion leaves the class foundering.

Another major difficulty with the standard approach is what could be called its structural problems. From the usual ε-δ definition it is not at all clear what really lies behind the basic concepts of limits and continuity. Students are apt to think that ordering, absolute values, etc., are involved with continuity. This makes the transition from Calculus to Analysis very difficult—a matter which was the subject of an MAA Panel Discussion (Winter Meeting, Denver, 1965).

Our approach to continuity and limits attacks the difficulties just outlined. The basic tool is the topological concept of "nearness" introduced by F. Riesz in 1908. This idea appeals strongly to the students' geometric intuition. It is very easily introduced, a very large majority of our students grasped its essence immediately and the passage to complete rigor is direct and readily accessible. The end result is equivalent to continuity in terms of the closure axioms but the entire theory remains closer to its intuitive foundations. In our experience students have more confidence in their own understanding even when that understanding is incomplete. We have found it considerably easier to lead a class to formulate definitions and to contribute to the proofs. The theory extends almost trivially to arbitrary topological spaces when necessary, and identical proofs hold in every setting. Finally, there is a striking unification of the theories of continuous functions, of various limits, uniform convergence, uniform continuity, etc. Thus the bridge to intermediate level courses is very easy to cross.

1. Nearness. Suppose $\varnothing \neq A \subset X \subset \mathbb{R}$ (or $\mathbb{R}^m$) and $x \in X$. The two notions

"x is near A" and its denial "x is far from A" are first suggested by means of discussion of special examples such as

(1) $A = \left\{\frac{1}{n} : n \in N\right\}$ with $x = 0, 2$, and N the set of natural numbers;

(2) $A = (0,1)$ with $x = -1, 0, 1, 2$.

Our goal here is to lead the class to a formulation of a precise definition.

1.1 DEFINITION OF NEARNESS. If $A \subset X \subset \mathbb{R}$ (or $\mathbb{R}^m$) and $x \in X$, then **x is near A in X** provided that for each $r > 0$ there exists $a \in A$ with $|x - a| < r$. We write $x \,\delta\, A$. The denial **x is far from A** means either $A = \varnothing$ or for some $r > 0$ we have $|x - a| \geqq r$ for every $a \in A$. The symbol is $x \not\delta A$.

A class usually comes up with the basic properties (a), (b), (c) and (e) below but rarely will find (d).

1.2 THEOREM.

(a) $x \,\delta\, A \Rightarrow A \neq \varnothing$

(b) $x \in A \Rightarrow x \,\delta\, A$

(c) $x \,\delta\, (A \cup B) \Leftrightarrow x \,\delta\, A$ *or* $x \,\delta\, B$

(d) $x \,\delta\, A$ *and* $\forall a \in A,\ a \,\delta\, B \Rightarrow x \,\delta\, B$

(e) $x \,\delta\, \{y\}$ *if and only if* $x = y$.

1.3 THEOREM. *If A is finite, then $x \,\delta\, A$ iff $x \in A$.*

1.4 THEOREM. $x \,\delta\, A$, $x \notin A \Rightarrow x \,\delta\, (A - F)$ *for any finite set F.*

1.5 DEFINITION. $x \subset \mathbb{R}$ (or $\mathbb{R}^m$) is **discrete** iff $x \not\delta (X - \{x\})\ \forall x \in X$.

1.6 THEOREM. *Any subset of the set of integers is discrete.*

We foreshadow important ideas by pointing out that the proofs of these results depend only on the five "axioms" in Theorem 1.2. Then we provide (not necessarily metric) examples of nearness satisfying the axioms. These include

1. $x \,\delta\, A \Leftrightarrow x \in A$ (the discrete topology).
2. $x \,\delta\, A \Leftrightarrow A \neq \varnothing$ (the indiscrete topology).
3. The nearness on $\mathbb{N}^* = \mathbb{N} \cup \{\infty\}$ from Definition 3.2 below.
4. Some example involving family membership. (Mothers-in-law always seem to be far away by such definitions.)

2. Continuous functions. A function is continuous if it preserves the nearness relation. This "covariant" definition is not only appealing to the intuition, it is precise.

2.1 DEFINITION. Let $c \in X \subset \mathbb{R}$ (or $\mathbb{R}^m$) and $f: X \to \mathbb{R}$ (or $\mathbb{R}^n$) be a function. Then f is **continuous at c** provided that $\forall A \subset X$, $c \,\delta\, A \Rightarrow f(c) \,\delta\, f(A)$. We say that f is **continuous** on X if f is continuous at every point of X and if f fails to be continuous at c, we say that f is **discontinuous** at c.

A little care in the discussion will lead a class to formulate this entire definition. Incidentally, the discussion always includes proofs of the first two statements in the next result.

2.2 THEOREM.
(a) *Constant functions are continuous.*
(b) *Identity functions are continuous.*
(c) *Restrictions of a continuous function are continuous.*
(d) *A composition of continuous functions is continuous.*

To illustrate the methods used to prove that the continuous functions on $X \subset \mathbb{R}^m$ to $\mathbb{R}$ form a ring, we include a proof of closure under addition. Note that we have written the (contrapositive) proof in very terse form. This is not the way we present it to a class. In fact, preparatory discussion results in the class constructing much of the proof. (We have experienced an unusually high degree of class participation throughout this entire presentation.)

2.3 THEOREM. *Let $c \in X \subset \mathbb{R}$ (or $\mathbb{R}^m$). If both $f: X \to \mathbb{R}$ and $g: X \to \mathbb{R}$ are continuous at c, so is $f + g$.*

Proof. Suppose to the contrary that $f + g$ were not continuous at c. Then there would exist $A \subset X$ such that $c \delta A$ but $(f+g)(c) \not\delta (f+g)(A)$. In other words, for some $r > 0$, $|(f+g)(c) - (f+g)(a)| \geqq r$ for each $a \in A$, or

$$(*) \qquad |f(c) - f(a) + g(c) - g(a)| \geqq r \text{ for each } a \in A.$$

Define sets

$$A_1 = \{a \in A : |f(c) - f(a)| < r/2\}$$
$$A_2 = \{a \in A : |f(c) - f(a)| \geqq r/2\} = A - A_1.$$

If $a \in A_1$, then we must have $|g(c) - g(a)| \geqq r/2$ or else (*) would be contradicted. Thus $g(c) \not\delta g(A_1)$. Since g is continuous at c we must have $c \not\delta A_1$. By identical reasoning, we obtain $c \not\delta A_2$. Theorem 1.2(c) now tells us that $c \not\delta (A_1 \cup A_2)$. Because $A_1 \cup A_2 = A$, this contradicts the original assumption and completes the proof.

We are careful to point out that such proofs depend only on the "axioms" in Theorem 1.2. This is in preparation for our treatment of limits to be given in Section 3. The final step in this presentation of continuity is to establish the usual ε-δ characterization and discuss its utility.

3. Limits. We have not used limits, in the usual sense, to define continuity. In order to develop the needed theories, however, we employ our knowledge of continuity. By defining a limit to be an "expected" value, the value causing a certain new function to be continuous, all of our theorems on continuity apply directly to provide the properties of limits.

3.1 DEFINITION OF LIMIT OF A FUNCTION. Let $X \subset \mathbb{R}$ (or $\mathbb{R}^m$) and let $c\,\delta(X-\{c\})$. Given $f\colon X \to \mathbb{R}$ (or $\mathbb{R}^n$) we write

$$\lim_{x\to c} f(x) = l \in \mathbb{R} \text{ (or } \mathbb{R}^n),$$

provided that the function f^* on $X \cup \{c\}$ defined by

$$f^*(x) = f(x) \text{ if } x \in X - \{c\},\ f^*(c) = l,$$

is continuous at c.
(At last we see the $\lim_{x\to c} f(x) = f(c)$ property of continuous functions.)

To study convergence of sequences we append an element ∞ to $\mathbb{N}$ and postulate that ∞ is near $A \subset \mathbb{N}$ if A is infinite. More precisely, on $\mathbb{N}^* = \mathbb{N} \cup \{\infty\}$ we provide a nearness δ as follows:

3.2 DEFINITION OF NEARNESS ON N^*. Let $x \in \mathbb{N}^*$ and $A \subset \mathbb{N}^*$. Then $x\,\delta\,A$ provided that either

(i) $x \in \mathbb{N}$ and $x \in A$, or

(ii) $x = \infty$ and $\infty \in A$ or A is infinite.

The students easily verify that this nearness satisfies the "axioms."

If $\phi\colon \mathbb{N} \to \mathbb{R}$ (or $\mathbb{R}^n$) is a sequence, we again define its limit as an "expected" value.

3.3 DEFINITION OF LIMIT OF A SEQUENCE. Let $\phi\colon \mathbb{N} \to \mathbb{R}$ (or $\mathbb{R}^n$) be a sequence and let $l \in \mathbb{R}$ (or $\mathbb{R}^n$). Then $\phi \to l$ or

$$\lim_{n\to\infty} \phi(n) = l$$

provided that the function $\phi^*\colon \mathbb{N}^* \to \mathbb{R}$ (or $\mathbb{R}^n$) defined by

$$\phi^*(n) = \phi(n) \text{ if } n \in \mathbb{N},\ \phi^*(\infty) = l$$

is continuous at ∞.

3.4 THEOREM. *Let $\varnothing \neq A \subset X \subset \mathbb{R}$ (or $\mathbb{R}^m$) and let $x \in X$. Then $x\delta A$ iff there exists $\phi\colon \mathbb{N} \to A$ with $\phi \to x$.*

3.5 COROLLARY. *Let $X \subset \mathbb{R}$ (or $\mathbb{R}^m$) and consider $f\colon X \to \mathbb{R}$ (or $\mathbb{R}^n$). f is continuous at $c \in X$ iff, for each $\phi\colon \mathbb{N} \to X$, $\phi \to c \Rightarrow f \circ \phi \to f(c)$.*

In the same way, infinite limits, $\lim_{x\to\infty} f(x)$ are defined by producing a nearness on $\mathbb{R}^* = \mathbb{R} \cup \{-\infty, \infty\}$ and then using obvious analogs of Definition 3.3. The same proofs hold in far more sophisticated settings, too. Let X be a set and denote by $B(X)$ the space of bounded functions $f\colon X \to \mathbb{R}$ with the metric

$$d(f,g) = \sup_X \left| f(x) - g(x) \right|.$$

This provides a nearness δ_d on $B(X)$ and if Definition 3.3 is used in this setting we have uniform convergence!

3.6 THEOREM. *Let $\phi: \mathbb{N} \to B(X)$ be a sequence of bounded real-valued functions on X and let $g \in B(X)$. Then $\phi \to g$ iff for each $\varepsilon > 0$ there exists $m_0 \in \mathbb{N}$ such that $\forall n > m_0$ $\forall x \in X$*

$$\left| \phi_n(x) - g(x) \right| < \varepsilon .$$

The point of this program is that the usual theorems about all of these variously defined limits are now no more than special cases of the corresponding results concerning continuous functions. We need leave nothing unproven because it is too difficult at an early stage nor need any proof be "left as an exercise" because its inclusion is too tedious.

4. Proximities. The program sketched above does not include uniform continuity and of course it need not. Even in an honors calculus course one usually evokes uniform continuity only to prove that a continuous function is integrable. That usage is unnecessary (see our offer of sets of notes later.) However, in an intermediate analysis course uniform continuity becomes important and an easily motivated generalization of nearness applies.

It is a natural step to pass from "nearness of a point to a set" to "nearness of a set to a set." The latter notion, called a **proximity**, also was introduced by F. Riesz in his 1908 paper.

4.1 DEFINITION. Let $A, B \subset X \subset \mathbb{R}$ (or $\mathbb{R}^m$). We say that ***A* is near *B* in *X***, and write $A \delta B$, provided that for each $r > 0$ there exist points $a \in A$ and $b \in B$ such that $|a - b| < r$. (Surely if $B = \{x\}$, this is precisely the definition of $x \delta A$).

By calling for properties analogous to those found in Theorem 1.2 the instructor can lead the class to formulate most of the next result.

4.2 THEOREM.
(a) $A \delta B \Rightarrow A \neq \varnothing, B \neq \varnothing$.
(b) $A \delta B \Rightarrow B \delta A$.
(c) $A \cap B \neq \varnothing \Rightarrow A \delta B$.
(d) $A \delta (B \cup C) \Leftrightarrow A \delta B$ or $A \delta C$.
(e) $A \delta B$ *and* $b \delta C$ $\forall b \in B \Rightarrow A \delta C$.
(f) *If* $\exists x \in X$ *with* $x \delta A$ *and* $x \delta B$, *then* $A \delta B$.

The converse of 4.2 (f) is not true. The standard counterexample is $A = \mathbb{N}$ and $B = \{n - 1/n : n \in \mathbb{N}\}$.

4.3 THEOREM. *If A and B are finite, then $A \delta B$ iff $A \cap B \neq \varnothing$.*

4.4 THEOREM. *$A \delta B$, $A \cap B = \varnothing \Rightarrow A \delta (B - F)$ for any finite set F.*

Examples of proximities satisfying the "axioms" (a)-(f) of Theorem 4.2 can easily be found.

1. $A\ \delta_1\ B$ iff $A \cap B \neq \varnothing$ (the discrete proximity).
2. $A\ \delta_2\ B$ iff $A \neq \varnothing \neq B$ (the indiscrete proximity).

3. $A \, \delta_3 \, B$ iff $\exists x$ with $x \, \delta \, A$ and $x \, \delta \, B$.

4. $A \, \delta_4 \, B$ iff either $A \, \delta_3 \, B$ or both A and B are infinite.

5. On $\mathbb{R}$, $A \, \delta_5 \, B$ iff either $A \, \delta_3 \, B$ or both A and B are unbounded above or both are unbounded below.

One of the motivations for Definition 4.1 is this: Starting with a topology, as given by a metric in our case, a proximity was defined so that whenever one of the sets involved is a singleton set, the definition reduces to that of the original nearness. One says then the proximity is compatible with the nearness. For instance, the proximities 3, 4 and 5 above are compatible with the usual nearness 1.1 on $\mathbb{R}$. In fact, there are 2^c different proximities compatible with 1.1 on $\mathbb{R}$. One sees that it is very significant to pass from nearness to proximity.

4.5 DEFINITION. Let $X \subset \mathbb{R}$ (or $\mathbb{R}^m$) and consider a function $f\colon X \to \mathbb{R}$ (or $\mathbb{R}^n$). We say f is **proximally** or **uniformly continuous** on X provided that for all sets $A, B \subset X$,

$$A \, \delta \, B \Rightarrow f(A) \, \delta \, f(B).$$

It is obvious that a uniformly continuous function is continuous. However, the converse is not true. For a counterexample, consider the function $f\colon \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2$. Let $A = \mathbb{N}$ and $B = \{n - (1/n) \colon n \in \mathbb{N}\}$. Then $A \, \delta \, B$ but $f(A) \, \not\delta \, f(B)$ because the distance $|f(a) - f(b)|$ is never less than 1.

4.6 THEOREM.

(a) *Constant functions are uniformly continuous.*

(b) *Identity functions are uniformly continuous.*

(c) *A restriction of a uniformly continuous function is uniformly continuous.*

(d) *A composition of uniformly continuous functions is uniformly continuous.*

The algebra of uniformly continuous functions is not so well-structured as that of continuous functions. For instance, note that $f(x) = x^2$ is a product of identity functions but is not uniformly continuous on $\mathbb{R}$.

Of course, both the nearness and the ε-δ definitions of uniform continuity should be known and used. However, the equivalence of the two is not a trivial theorem. It was proved first by Efremovič in 1952 (Geometry of Proximity, Math. Sbornik Vol. 31, pp. 189–200). Cleveland (this MONTHLY, 80 (1973) 64–66) has provided a readily accessible proof.

We conclude this article with an offer. A reader interested in the details of the material sketchily outlined here may obtain them from the author at his Thunder Bay address. There are three sets of notes available at cost. The first covers material on continuity and limits needed in an elementary calculus course. A second more extensive set of notes augments an introductory analysis or advanced calculus course. The third set of notes on elementary integration theory was mentioned earlier. The level of difficulty is suitable for an honors calculus course, for instance, and the approach provides a proof of the integrability of continuous functions without

using uniform continuity. In fact, this theorem comes out naturally in a program leading to the fundamental theorem of calculus.

The author S. A. Naimpally was partially supported by an operating grant from N. R. C. (Canada) and a research grant from the President of Lakehead University.

DEPARTMENT OF MATHEMATICS, LAKEHEAD UNIVERSITY, THUNDER BAY, ONTARIO, P7B 5E1, CANADA.
DEPARTMENT OF MATHEMATICS, MICHIGAN STATE UNIVERSITY, EAST LANSING, MICHIGAN, 48823, U. S. A.
DEPARTMENT OF MATHEMATICS, LAKEHEAD UNIVERSITY, THUNDER BAY, ONTARIO, P7B 5E1, CANADA.

THE PROFESSOR'S SONG

Words by Tom Lehrer — Tune: "If You Give Me Your Attention"
from *Princess Ida* (Gilbert and Sullivan)

If you give me your attention, I will tell you what I am.
I'm a brilliant math'matician — also something of a ham.
I have tried for numerous degrees, in fact I've one of each;
Of course that makes me eminently qualified to teach.
I understand the subject matter thoroughly, it's true,
And I can't see why it isn't all as obvious to *you*.
Each lecture is a masterpiece, meticulously planned,
Yet everybody tells me that I'm hard to understand,
And I can't think why.

My diagrams are models of true art, you must agree,
And my handwriting is famous for its legibility.
Take a word like "minimum" (to choose a random word), (*)
For anyone to say he cannot read that, is absurd.
The anecdotes I tell get more amusing every year,
Though frankly, what they go to prove is sometimes less than clear,
And all my explanations are quite lucid, I am sure,
Yet everybody tells me that my lectures are obscure,
And I can't think why.

Consider, for example, just the force of gravity:
It's inversely proportional to something — let me see —
It's r^3 — no, r^2 — no, it's just r, I'll bet —
The sign in front is plus — or is it minus, I forget —
Well, anyway, there *is* a force, of that there is no doubt.
All these formulas are trivial if you only think them out.
Yet students tell me, "I have memorized the whole year through
Ev'rything you've told us, but the problems I can't do."
And I can't think why!

(*) This was performed at a blackboard, and the professor wrote:

CORRECTIONS TO "CURRENT TRENDS IN ALGEBRA"

(This MONTHLY, 80 (1973) 760–782)

GARRETT BIRKHOFF, Harvard University

1. p. 763, last line. Change "Cayley (1878)" to "Cayley (1858)."
2. p. 765, lines –10, –9. Delete: "I shall... next lecture."
3. p. 771, Section 10, lines 6–7. Change: "*group... field... valuation*" to "*field... ideal... group.*"
4. p. 771, line –6. Change "students" to "younger colleagues." For a list of Emmy Noether's students, see Auguste Dick, "Emmy Noether," *Elem. Math., Beiheft* No. 13, Birkhäuser, Basel, 1970.

QUERIES

EDITED BY A. C. ZITRONENBAUM

This Department welcomes queries from readers about mathematics at the collegiate level, such as sources for exposition of a particular topic from a special point of view, references to vaguely remembered articles, descriptions of special kinds of courses or teaching methods, and methods for constructing illustrative examples for exercises of particular kinds (questions on research topics should, in general, be addressed to the "Queries Department" of the Notices of the American Mathematical Society). Replies will be forwarded to the questioner and may also be edited into a composite answer for publication in this Department. Consequently all items submitted for consideration for possible publication should include the name and complete mailing address of the person who is to receive the reply. Queries and answers should be sent to A. C. Zitronenbaum, Department of Mathematics, Cornell University, Ithaca, NY 14850.

Replies to Query 4. In this Query, Professor Pedrick asked for material for an interdisciplinary seminar in mathematical ecology. Many readers responded with source material. The following is a selection of some of the suggestions: E. C. Pielou, *An Introduction to Mathematical Ecology*, Wiley 1969; J. M. Emlen, *Ecology*, Addison-Wesley 1973; the series of monographs edited by R. MacArthur and published by Princeton University Press on Population Ecology; B. C. Patton, *Systems Analysis and Simulation in Ecology*, Academic Press. An example of the misuse of mathematics in ecology can be found in the note by Brussard et al, Science 22 (1971), 174, 435–436. Professor S. Levin, Department of Theoretical and Applied Mechanics, Cornell University, Ithaca, New York 14850, has a detailed bibliography available.

11. A. A. Mullin. I would appreciate receiving references to expository material on the use of lattice points to couple H. Minkowski's theory of convex bodies and E. L. Post's theory of recursively enumerable sets.

EDITED BY DAVID ROSELLE

Material for this Department should be sent to David Roselle, Department of Mathematics, Louisiana State University, Baton Rouge, LA 70803.

RECIPROCITY THEOREM FOR DEDEKIND SUMS

R. C. GRIMSON

For any real number x, let $[x]$ denote the largest integer which does not exceed x and let $((x))$ denote the function which is zero on the integers and $x - [x] - \frac{1}{2}$ elsewhere. The **Dedekind sum** $s(a,b)$ is defined by

$$s(a,b) = \sum_{\mu \bmod b} \left(\left(\frac{\mu}{b}\right)\right)\left(\left(\frac{a\mu}{b}\right)\right).$$

The sum is independent of the complete residue system (mod b) through which μ ranges but it is often convenient to let μ range through the simplest system $0, 1, \cdots, b-1$. In a recent article in this MONTHLY [**1**], Grosswald discusses the history of Dedekind sums and presents a reconstruction of one of Rademacher's proofs of the reciprocity theorem.

RECIPROCITY THEOREM. *If* $(a,b) = 1$, *then*

$$s(a,b) + s(b,a) = -\frac{1}{4} + \frac{1}{12}\left(\frac{a}{b} + \frac{1}{ab} + \frac{b}{a}\right). \tag{1}$$

Several other proofs have been given for this fundamental result; see [**1**] for references.

Using the familiar identity

$$\sum_{\mu=0}^{b-1} \left[\frac{a\mu}{b}\right] = \tfrac{1}{2}(a-1)(b-1), \tag{2}$$

equation (1) becomes

$$s(a,b) = \frac{b-1}{12b}(4ab - 2a - 3b) - \frac{1}{b}A(a,b), \tag{3}$$

where

$$A(a,b) = \sum_{\mu=1}^{b-1} \mu\left[\frac{a\mu}{b}\right]. \tag{4}$$

Rademacher [**2**] and [**3**] proved the following result which we shall regard as a Lemma.

LEMMA. *If* $(a,b) = 1$, *then*

$$aA(a,b) + bA(b,a) = \frac{1}{12}(a-1)(b-1)(8ab-a-b-1).$$

In this note we shall give a short elementary proof of the Lemma and, hence, of the Reciprocity Theorem.

Proof of the Lemma. From (4) we have

$$A(a,b) = \sum_{\mu=1}^{b-1} \sum_{k=1}^{[a\mu/b]} \mu.$$

Interchanging the order of summation and using (2), we find

$$\begin{aligned} A(a,b) &= \sum_{k=1}^{a-1} \sum_{\mu=[bk/a]+1}^{b-1} \mu \\ &= \frac{1}{2}\sum_{k=1}^{a-1}\left(b^2 - b - \left[\frac{bk}{a}\right]^2 - \left[\frac{bk}{a}\right]\right) \\ &= \frac{(a-1)(b-1)(2b-1)}{4} - \frac{1}{2}\sum_{k=1}^{a-1}\left[\frac{bk}{a}\right]^2. \end{aligned}$$

Therefore,

$$\sum_{k=1}^{a-1}\left[\frac{bk}{a}\right]^2 = \frac{(a-1)(b-1)(2b-1)}{2} - 2A(a,b). \tag{5}$$

Next put

$$bk = a\left[\frac{bk}{a}\right] + r_k. \tag{6}$$

Observe that $1 \leqq r_k \leqq a-1$ for $k = 1,2,\cdots,a-1$. Furthermore, using the assumption that $(a,b) = 1$ and letting j and k be unequal integers which do not exceed $a-1$, we see from (6) that $r_j \neq r_k$. Therefore, the sequence $r_1, r_2, \cdots, r_{a-1}$ is some permutation of $1,2,\cdots,a-1$. Accordingly,

$$\sum_{k=1}^{a-1} r_k^2 = \frac{a(a-1)(2a-1)}{6}. \tag{7}$$

Now, dividing both sides of (6) by a, then squaring both sides, and then replacing r_k/a by $bk/a - [bk/a]$ only in the term $2(r_k/a)[bk/a]$, we find that

$$\left(\frac{bk}{a}\right)^2 = -\left[\frac{bk}{a}\right]^2 + \frac{2bk}{a}\left[\frac{bk}{a}\right] + \left(\frac{r_k}{a}\right)^2.$$

Summing on k from 1 to $a-1$ in both sides of this expression, and using (7), we have

$$(8) \qquad -\sum_{k=1}^{a-1}\left[\frac{bk}{a}\right]^2 = \frac{(a-1)(2a-1)(b^2-1)}{6a} - \frac{2b}{a}A(b,a).$$

The proof of the Lemma is completed by adding (5) and (8).

Proof of the Reciprocity Theorem. Note from (3) that

$$abs(a,b) = \frac{(b-1)a}{12}(4ab-2a-3b) - aA(a,b)$$

and

$$abs(b,a) = \frac{(a-1)b}{12}(4ab-2b-3a) - bA(b,a).$$

The Reciprocity Theorem follows at once by first adding these two equations and then applying the Lemma.

This work was partially supported by a National Science Foundation summer research grant at Louisiana State University. The author wishes to thank the referee for some helpful suggestions.

References

1. E. Grosswald, Dedekind-Rademacher sums, this MONTHLY, 78 (1971) 639–644.
2. H. Rademacher, Eine arithmetische Summenformel, Monatshefte für Mathematik und Physik, 39 (1932) 221–228.
3. ———, Zur Theorie der Modulfunktionen, J. Reine Angew. Math., 167 (1932) 312–336.

DEPARTMENT OF MATHEMATICS, ELON COLLEGE, ELON COLLEGE, N.C. 27244.

ON CONVEX POLYGONS DETERMINED BY A FINITE PLANAR SET

W. E. BONNICE

This paper gives a short proof of the fact that if nine points are sprinkled on a plane, so that no three are collinear, then some five of these points are the vertices of a convex pentagon. This is the case $n = 5$ of the conjecture [**4**, p. 44; **1**] that in a plane, any collection of at least $2^{n-2}+1$ points such that no three are collinear contains the vertices of at least one convex n-gon. For $n = 5$, this has been proved before by Kalbfleisch, Kalbfleisch and Stanton [**2**], but the conjecture is still open for $n \geqq 6$.

Given a finite collection of points in a plane, the statement that X is $(k_1, k_2, \cdots, k_j)$ will mean that $\operatorname{card} X = k_1 + k_2 + \cdots + k_j$ and that the convex hull, $\operatorname{con} X$, of X is a k_1-gon; that, when the vertex set of $\operatorname{con} X$ is taken away from X, the convex hull of the points which remain is a k_2-gon; etc. Thus X is $(4,3,2)$ will mean that X has nine points, that $\operatorname{con} X$ is a quadrilateral, and that, when the vertices of the

quadrilateral are removed from X, the points of X that remain determine a triangle with two points of X in its interior. When a set Y contains the vertex set of at least one convex n-gon, we will say simply that *Y determines an n-gon.* Also ray v_1v_2 denotes the ray emanating from a point v_1 through a point v_2; v_1v_2 denotes the segment from v_1 to v_2; $v_1v_2 \cdots v_k$ $(k \geqq 3)$ designates a convex k-gon with the vertices ordered such that if the perimeter of this k-gon is traversed in a counterclockwise direction beginning at v_1, the k vertices are encountered in the order of increasing subscripts. If $abcd$ is such a convex quadrilateral with vertices ordered counterclockwise, beam $ab\colon cd$ denotes the section of the plane obtained by deleting $\mathrm{con}\{a,b,c,d\}$ from the convex section of the plane bounded by segment ab, ray ad, and ray bc. Similarly, if x, y, and z are not collinear, beam $x\colon yz$ will denote the infinite section of the plane obtained by deleting $\mathrm{con}\{x,y,z\}$ from the convex cone which has vertex x and is bounded by ray xy and ray xz.

LEMMA. *If Y is $(3,3,2)$, or $(4,3,1)$, or $(3,4,2)$, then Y determines a pentagon.*

Proof. First, assume that Y is $(3,3,2)$. Let y_1, y_2, and y_3 be the vertices of con Y; let triangle $v_1v_2v_3$ be the second triangle, $\mathrm{con}(Y - \{y_1,y_2,y_3\})$; and let z_1 and z_2 be the two points of Y interior to $v_1v_2v_3$. We may assume that line z_1z_2 intersects sides v_1v_2 and v_1v_3 of $v_1v_2v_3$ with the orientation such that ray z_1z_2 intersects v_1v_2.

The vertices y_1, y_2, and y_3 of the outside triangle are in the union of beam $z_1z_2\colon v_2v_3$, beam $z_1\colon v_1v_3$, and beam $z_2\colon v_1v_2$. If one of these vertices, say y_1, is in beam $z_1z_2\colon v_2v_3$, then $z_1z_2v_2y_1v_3$ is a convex pentagon. Thus we may assume that two of the three points y_1, y_2, and y_3, say y_1 and y_2, are in beam $z_1\colon v_1v_3$. Since $\mathrm{con}\{y_1,y_2,y_3\}$ contains $v_1v_2v_3$, triangle $v_1v_2v_3$ must lie in one of the open half-planes determined by line y_1y_2. Thus line y_1y_2 does not intersect $\mathrm{con}\{v_1,v_2,v_3\}$, and therefore $\mathrm{con}\{z_1,v_1,v_3,y_1,y_2\}$ is a convex pentagon.

Now assume that Y is $(4,3,1)$. Let y_1, y_2, y_3, and y_4 be the vertices of con Y; let $v_1v_2v_3$ be the inside triangle, $\mathrm{con}(Y - \{y_1,y_2,y_3,y_4\})$; and let z denote the point of Y inside it. Partitioning the plane outside $v_1v_2v_3$ into beam $z\colon v_1v_3$, beam $z\colon v_1v_2$, and beam $z\colon v_2v_3$, we may assume that two of the four points y_1,y_2,y_3,y_4, say y_1 and y_2, are in beam $z\colon v_1v_3$. Then, as above, $\mathrm{con}\{z,v_1,v_3,y_1,y_2\}$ is a pentagon.

Finally, assume that Y is $(3,4,2)$. The technique is the same: Let $\mathrm{con}\{y_1,y_2,y_3\}$, $v_1v_2v_3v_4$, and z_1z_2 be the triangle, quadrilateral, and line segment given by the fact that Y is $(3,4,2)$. If line z_1z_2 cuts off a vertex, say v_1, of $v_1v_2v_3v_4$, then $\mathrm{con}\{z_1,z_2,v_2,v_3,v_4\}$ is a pentagon. So assume that line z_1z_2 cuts sides v_1v_4 and v_2v_3 with the orientation such that ray z_1z_2 intersects v_2v_3. As above, if there is a point of $\{y_1,y_2,y_3\}$ in either beam $z_1z_2\colon v_3v_4$ or beam $z_2z_1\colon v_1v_2$, a convex pentagon is formed. Thus we may assume that $\{y_1,y_2,y_3\}$ is a subset of the union of beam $z_1\colon v_1v_4$ and beam $z_2\colon v_2v_3$. In particular, we may assume that both y_1 and y_2 are in beam $z_1\colon v_1v_4$ whence, as before, $\mathrm{con}\{z_1,v_1,v_4,y_1,y_2\}$ is a convex pentagon.

THEOREM. *If nine points in a plane have no three points collinear, then some five of the points are vertices of a convex pentagon.*

Proof. If the set of nine points determines no pentagons, then it must be one of the following: $(4,4,1)$, $(4,3,2)$, $(3,4,2)$, or $(3,3,3)$. Since no three points are collinear, each of the first two cases has a subset of eight points which is $(4,3,1)$, and the last case a subset which is $(3,3,2)$. Thus the lemma applies to all cases.

The question of whether 17 points always determine a hexagon seems considerably more difficult. One can appreciate this better if he considers that there are 12,376 six-element subsets of a seventeen-element set whereas there are "only" 126 five-element subsets of a nine-element set. Also, there are 70 distinct tuples $(k_1, k_2, \cdots, k_j)$ which represent the different ways that the successive convex hulls of a seventeen-element set might nest if it determines no hexagons.

In [1] Erdös and Szekeres claim to have constructed for each n an example of 2^{n-2} coplanar points which determine no convex n-gon. However, Kalbfleisch, Kalbfleisch, and Stanton [2] say that the proof in [1] is faulty, and they give a corrected version in [3] (unpublished).

It is interesting to note that the following related question appeared [5] in the Eleventh International Mathematical Olympiad (IMO) held in Bucharest in 1969: Given $n > 4$ points in a plane such that no three are collinear, prove that one can find at least $\binom{n-3}{2}$ convex quadrilaterals whose vertices are four of the given points. Both this and the case $n = 4$ of the conjecture stated in our first paragraph follow easily from the following result:

THEOREM. *Let X be any finite collection of at least five coplanar points with no three collinear, and let v_1, v_2, and v_3 be any three vertices of $\operatorname{con} X$. If a and b are any two points of $X - \{v_1, v_2, v_3\}$, then they, together with some two points of $\{v_1, v_2, v_3\}$, are the vertices of a convex quadrilateral.*

Proof. Line ab intersects exactly two or none of the three segments v_1v_2, v_2v_3, and v_1v_3. So we may assume that line ab does not intersect v_1v_2. Then a and b are both on the same side of line v_1v_2 (in which case $\operatorname{con}\{v_1, v_2, a, b\}$ is a convex quadrilateral). For, otherwise, let p be the point of intersection of line ab and line v_1v_2. It is notational to assume that v_2 is between v_1 and p. Then v_2 is in the interior of $\operatorname{con}\{a, b, v_1\}$, which contradicts the fact that v_2 is a vertex of $\operatorname{con} X$.

To see that this theorem proves the statement given in the Eleventh IMO, fix any three vertices of the convex hull. The other $n - 3$ points determine $\binom{n-3}{2}$ pairs, with each pair, associate the quadrilateral given in the theorem.

References

1. P. Erdös and G. Szekeres, On some extremum problems in elementary geometry, Annales Univ. Sci. Budapest, 3 (1960) 53–62.

2. J. D. Kalbfleisch, J. G. Kalbfleisch, and R. G. Stanton, A combinatorial problem on convex n-gons, Proc. Louisiana Conf. on Combinatorics, Graph Theory, and Computing, (1970) 180–188.

3. J. G. Kalbfleisch and R. G. Stanton, On the maximum number of coplanar points containing no convex n-gons, unpublished manuscript.

4. H. J. Ryser, Combinatorial Mathematics, MAA Carus Monograph no. 14, 1963.
5. F. R. Watson, XIth International Mathematical Olympiad, Bucharest, July 1969, Math. Gaz., 53 (1969) 395–396.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF NEW HAMPSHIRE, DURHAM, N. H. 03824.

A DIRECT INTERPRETATION OF GANDHI'S FORMULA

S. W. GOLOMB

Let $P_n = p_1 p_2 \cdots p_n$ be the product of the first n primes. Then the next prime, p_{n+1}, is the unique integer m such that

$$1 < 2^m \left(\sum_{d \mid P_n} \frac{\mu(d)}{2^d - 1} - \frac{1}{2} \right) < 2. \tag{1}$$

This formula, first obtained by J. M. Gandhi [1], was the subject of a recent note [2] in this MONTHLY. The purpose of the present article is to make Gandhi's mysterious formula totally transparent. As we shall see, Gandhi's formula is merely the Sieve of Eratosthenes performed on the binary expansion $1 = .11111111\ldots.$

To each positive integer n, assign the *probability* $p(n) = 2^{-n}$. Note that $\sum_{n=1}^{\infty} p(n) = 1$. In this distribution, the probability that a random integer is a multiple of a fixed integer d is given by

$$q(d) = \sum_{n=1}^{\infty} p(nd) = \sum_{n=1}^{\infty} 2^{-nd} = \frac{1}{2^d - 1},$$

and the probability that a random integer is relatively prime to a given integer m is then

$$1 - \sum_{p_1 \mid m} q(p_1) + \sum_{p_1 p_2 \mid m} q(p_1 p_2) - \sum_{p_1 p_2 p_3 \mid m} q(p_1 p_2 p_3) + \cdots$$
$$= \sum_{d \mid m} \mu(d) q(d) = \sum_{d \mid m} \frac{\mu(d)}{2^d - 1},$$

where $p_1, p_2, p_3, \cdots$ are distinct prime divisors of m and $\mu(d)$ is the Möbius function. In particular, the probability that a random integer is relatively prime to P_n (i.e., is divisible by none of the first n primes) is given on the one hand by

$$r(P_n) = \sum_{d \mid P_n} \frac{\mu(d)}{2^d - 1}, \tag{2}$$

but is given directly as

$$\sum_{(m, P_n) = 1} p(m) = \frac{1}{2^1} + \frac{1}{2^{p_{n+1}}} + \frac{1}{2^{p_{n+2}}} + \text{(reciprocals of some higher powers of 2)}. \tag{3}$$

Comparing (2) and (3), we see that

$$r(P_n) - \frac{1}{2} = \sum_{d|P_n} \frac{\mu(d)}{2^d - 1} - \frac{1}{2} = \frac{1}{2^{p_{n+1}}} + \frac{1}{2^{p_{n+2}}} + \text{(reciprocals of some higher powers of 2)},$$

so that

$$2^{p_{n+1}} \left(\sum_{d|P_n} \frac{\mu(d)}{2^d - 1} - \frac{1}{2} \right) = 1 + \theta_n, \tag{4}$$

where clearly $0 < \theta_n < 1$, which proves (1). In fact, for all $n \geqq 1$, we know that $p_{n+2} \geqq p_{n+1} + 2$, so that $0 < \theta_n < \frac{1}{2}$. Moreover, $\theta_n > \frac{1}{4}$, if and only if $p_{n+1} + 2 = p_{n+2}$ which gives a (highly impractical) test for twin primes.

All of this becomes even more obvious in binary notation. The probabilities of all the positive integers, written consecutively after the binary point, gives us

$$\sum_{n=1}^{\infty} p(n) = .111111111111111111\ldots = 1.$$

If we sieve out the even integers, we subtract

$$\sum_{n=1}^{\infty} p(2n) = .01010101010101\ldots = \frac{1}{2^2 - 1} = \frac{1}{3}.$$

Subtracting, we get

$$r(P_1) = .101010101010101\ldots = 1 - \frac{1}{3}.$$

Next we sieve out the multiples of 3, by subtracting

$$q(3) = .001001001001001\ldots = \frac{1}{2^3 - 1} = \frac{1}{7}$$

but adding back the twice-subtracted multiples of 6:

$$q(6) = .000001000001000001\ldots = \frac{1}{2^6 - 1} = \frac{1}{63}.$$

At this stage we are left with

$$r(P_2) = .1000101000101000101000101010\ldots = 1 - \frac{1}{3} - \frac{1}{7} + \frac{1}{63}.$$

After sieving out all primes through p_n, we are left with

$$r(P_n) = .100\ldots010\ldots10\ldots = 2^{-1} + 2^{-p_{n+1}} + 2^{-p_{n+2}} + \text{(reciprocals of some higher powers of 2)} \tag{5}$$

from which it is clear that $r(P_n) - \frac{1}{2} = .000\ldots010\cdots$, where the left-most non-zero bit is in position p_{n+1}.

Thus, Gandhi's formula is nothing more than the Sieve of Eratosthenes performed on the binary representation $1 = .11111111\ldots$.

The "higher powers of 2" referred to in (3) and (5) have as exponents all the remaining positive integers relatively prime to P_n. The *first* three positive integers relatively prime to P_n are always 1, p_{n+1}, and p_{n+2}, since the first composite number which survives the sieve is $(p_{n+1})^2$, which exceeds p_{n+2} for all $n \geqq 1$.

The actual use of formulas (1) or (4) merely to compute the next prime is not recommended. It is just as easy to perform the Sieve (using the binary notation, if desired) to determine *all* the primes on the interval $(p_n, (p_{n+1})^2)$, as indicated in (5).

This research was supported in part by the U.S. Army Research Office under Grant No. DA-ARO-D-31-124-173-G167.

References

1. J. M. Gandhi, Formulae for the Nth prime, Proc. Washington State University Conference on Number Theory, 1971, pp. 96–107.

2. Charles Vanden Eynden, A proof of Gandhi's formula for the nth prime, this MONTHLY, 79 (1972) 625.

DEPARTMENT OF MATHEMATICS AND DEPT. OF ELECTRICAL ENGINEERING, UNIVERSITY OF SOUTHERN CALIFORNIA, LOS ANGELES, CA 90007.

A THEOREM ABOUT ZIG-ZAGS BETWEEN TWO CIRCLES

W. L. BLACK, H. C. HOWLAND and B. HOWLAND

Let Γ and $\tilde{\Gamma}$ be two circles in euclidean 3-space, R^3. Suppose:

(1) *There exists a number x such that each point on either circle is distance x from exactly two points on the other circle.*

Then we can construct a zig-zag line between the two circles, each segment of length x, as follows:

$$\begin{array}{llll}
\text{Select a point } Z_1 & \text{on } \Gamma & \text{arbitrarily} & \\
\text{Select } \tilde{Z}_1 & \text{on } \tilde{\Gamma} & \text{with } |\tilde{Z}_1 - Z_1| & = x \\
\text{Select } Z_2 \neq Z_1 & \text{on } \Gamma & \text{with } |Z_2 - \tilde{Z}_1| & = x \\
\text{Select } \tilde{Z}_2 \neq \tilde{Z}_1 & \text{on } \tilde{\Gamma} & \text{with } |\tilde{Z}_2 - Z_2| & = x \\
\cdots & & & \\
\text{Select } Z_{n+1} \neq Z_n & \text{on } \Gamma & \text{with } |Z_{n+1} - \tilde{Z}_n| & = x \\
\text{Select } \tilde{Z}_{n+1} \neq \tilde{Z}_n & \text{on } \tilde{\Gamma} & \text{with } |\tilde{Z}_{n+1} - Z_{n+1}| & = x.
\end{array}$$

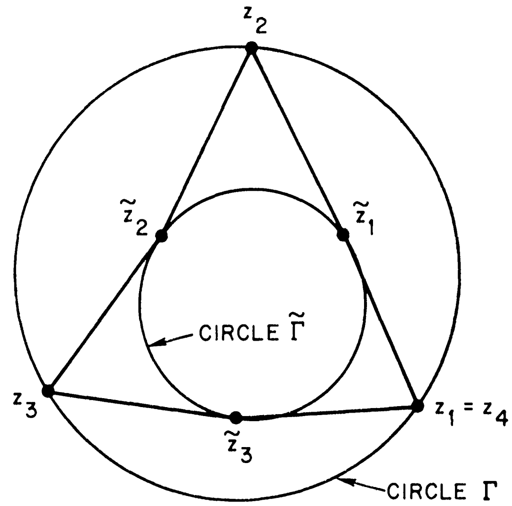


FIG. 1. Each straight line has length x, $n = 3$.

It may happen (illustrated for the case $n = 3$ in Fig. 1) that $Z_{n+1} = Z_1$; i.e., the zig-zag closes. We show that if this occurs, the zig-zag line can be started at *any* point on Γ and it will still close.

This remarkable fact was observed while performing certain calculations about ball bearings on a computer [**1**]. It bears a superficial resemblance to Steiner's Porism [**2**] and to Poncelet's Closure Theorem [**3**]. However, both of these are theorems about coplanar circles and have no obvious extension to 3-space.

Condition (1) can be simplified if the circles are *coplanar*. In this case one can easily show that an equivalent condition is:

(1′) *The smaller circle encloses the center of the larger circle.*

The proof requires the following:

LEMMA. *Let B, n, y_1 and y_2 be vectors in R^3 satisfying $|y_1| = |y_2|$, $|B + y_1| = |B + y_2|$, $n \cdot y_1 = n \cdot y_2$, and $y_1 \neq y_2$. Then $B \times n \cdot (y_1 + y_2) = 0$.*

Proof: The first three conditions imply that $y_1 - y_2 \neq 0$ is orthogonal to $y_1 + y_2$, B, and n. Thus these vectors lie in a plane and so are dependent.

The idea behind the proof is to compute how much Z_{n+1} moves if Z_1 is slightly altered. If $Z_{n+1} = Z_1$ we shall show $dZ_{n+1} = dZ_1$ and thus Z_{n+1} remains coincident with Z_1 as Z_1 is moved around Γ. More precisely, we shall select a point P on Γ and parametrize Γ by s, the directed arc length measured from P. Thus we can regard Z_1, and so Z_{n-1}, and ultimately t, the arc length from P to Z_{n+1}, as functions of s. We shall express the derivative dt/ds as a smooth function $f(s, t)$, where $f(s, s) = 1$ for all s. If for $Z_1 = P$, $Z_{n+1} = Z_1$, then $t(0) = 0$. Application of a well-known uniqueness theorem [**4**] assures us that the ordinary differential equation:

$$\frac{dt}{ds} = f(s,t); \quad t(0) = 0$$

has only one solution. Since $t(s) = s$ is a solution, we may conclude that $Z_{n+1} = Z_1$ for all s.

We will use the following notation:

n and $\tilde{n}$ are unit normals to Γ and $\tilde{\Gamma}$ respectively.

y_i is the vector from the center of Γ to Z_i.

$\tilde{y}_i$ is the vector from $\tilde{Z}_i$ to the center of $\tilde{\Gamma}$.

Δ is the vector from the center of $\tilde{\Gamma}$ to that at Γ.

Imagine a slight motion of y_1 along Γ. Since dy_1 is perpendicular to both n and y_1 we can write:

$$dy_1 = n \times \frac{y_1}{|y_1|} |dy_1|.$$

As y_1 moves, so must $\tilde{y}_1$, and as above:

$$d\tilde{y}_1 = \tilde{n} \times \frac{\tilde{y}_1}{|\tilde{y}_1|} |d\tilde{y}_1|.$$

The motion of $\tilde{y}_1$ must be such as to keep $|\Delta + y_1 + \tilde{y}_1| = x$, or squaring and differentiating:

$$(\Delta + y_1 + \tilde{y}_1)\cdot(dy_1 + d\tilde{y}_1) = 0.$$

Substituting into this last equation the previous expressions for dy_1 and $d\tilde{y}_1$ gives:

$$\frac{|d\tilde{y}_1|}{|dy_1|} = -\frac{|\tilde{y}_1|(\Delta + \tilde{y}_1)\cdot n \times y_1}{|y_1|(\Delta + y_1)\cdot \tilde{n} \times \tilde{y}_1}.$$

Similarly:

$$\frac{|dy_2|}{|d\tilde{y}_1|} = -\frac{|y_2|(\Delta + y_2)\cdot \tilde{n} \times \tilde{y}_1}{|\tilde{y}_1|(\Delta + \tilde{y}_1)\cdot n \times y_1}.$$

Multiplying these last two equations, using the lemma in the forms:

$$(\Delta + \tilde{y}_1)\cdot n \times y_1 = -(\Delta + \tilde{y}_1)\cdot n \times y_2,$$

$$(\Delta + y_2)\cdot \tilde{n} \times \tilde{y}_1 = -(\Delta + y_2)\cdot \tilde{n} \times \tilde{y}_2,$$

and noting that for all i, $|y_i| = |y_{i+1}|$ gives:

$$\frac{|dy_2|}{|dy_1|} = \frac{(\Delta + y_2) \cdot \tilde{n} \times \tilde{y}_2}{(\Delta + y_1) \cdot \tilde{n} \times \tilde{y}_1}.$$

Multiplying n similar expressions gives:

$$\frac{dt}{ds} = \frac{|dy_{n+1}|}{|dy_1|} = \frac{(\Delta + y_{n+1}) \cdot \tilde{n} \times \tilde{y}_{n+1}}{(\Delta + y_1) \cdot \tilde{n} \times \tilde{y}_1} = f(s,t).$$

Clearly if $Z_{n+1} = Z_1$, then $y_{n+1} = y_1$ and $\tilde{y}_{n+1} = \tilde{y}_1$, so indeed $f(s,s) = 1$. This completes the proof.

The above proof assumes that none of the various denominators vanish. Fortunately this is guaranteed by condition (1). Suppose for example, that $B \times n \cdot y_1 = 0$ where $B = \Delta + \tilde{y}_1$. From the lemma, $B \times n \cdot y_2 = 0$ also. Thus, $y_1 - y_2$ (which is non-zero) is normal to $B \times n$ as well as B and n. This implies that B and n are dependent, i.e., that $\tilde{Z}_1$ is on the symmetry axis of Γ in contradiction to condition (1).

The above proof is defective in the sense that it uses analytic techniques to prove what is presumably an algebraic theorem. It would be satisfying if one could obtain this theorem as a simple corollary of a general algebraic theorem, much as Poncelet's closure theorem can be proven by an application of Chasles' correspondence principle [**5**].

References

1. Quarterly Progress Report No. 101, Research Laboratory of Electronics, Massachusetts Institute of Technology, (15 April 1971) p. 268.

2. H. S. M. Coxeter and M. C. Greitzer, Geometry Revisited, Random House, New York, 1967, pp. 124, 125.

3. H. Dorrie, 100 Great Problems of Elementary Mathematics, Dover, New York, 1965, p. 193.

4. E. A. Coddington and N. Levinson, Theory of Ordinary Differential Equations, McGraw-Hill, New York, 1955, p. 10.

5. J. Dieudonné, The historical development of algebraic geometry, this MONTHLY, 8 (1972) 832.

W. L. BLACK: 14 PULSIFER STREET, NEWTONVILLE, MA 02160.

H. C. HOWLAND: DIVISION OF BIOLOGICAL SCIENCES, CORNELL UNIVERSITY, ITHACA, N. Y. 14850.

B. HOWLAND: MIT LINCOLN LABORATORY, LEXINGTON, MA 02173.

RESEARCH PROBLEMS

EDITED BY RICHARD GUY

In this Department the Monthly presents easily stated research problems dealing with notions ordinarily encountered in undergraduate mathematics. Each problem should be accompanied by relevant references (if any are known to the author) and by a brief description of known partial results. Manuscripts should be sent to Richard Guy, Department of Mathematics, Statistics, and Computing Science, The University of Calgary, Calgary, Alberta, Canada, T2N 1N4.

THE EXPONENTIAL DIOPHANTINE EQUATION $1 + a + a^2 + \cdots + a^{x-1} = p^y$

HUGH M. EDGAR

Throughout the article Latin letters denote positive integers, and p and q are odd primes. Various authors ([3], [7] and [8]) have shown, independently, that the title equation has at most one solution (x, y) if $a > 1$ and p are preassigned. Szymiczek [8] has proved the more general result for the equation $a^x - b^x = (a - b)c^y$ where a, b and c are preassigned. He also noted that another proof of his result could be obtained from the paper of Birkhoff and Vandiver [2].

THEOREM. *If $a > 1$ and p are given then the equation*

$$1 + a + a^2 + \cdots + a^{x-1} = p^y \tag{1}$$

has at most one solution (x, y). If (1) is solvable and if (x, y) is the solution then $x = \operatorname{ord}_p a$ is a prime and $y = u \operatorname{ord}_a p$ for some u. (Here $\operatorname{ord}_m a = t$ is defined to be the least t such that $a^t \equiv 1 \pmod m$).

The easiest and most natural way to increase the mathematical content of the above theorem is to obtain some information about u.

CONJECTURE. *If the equation*

$$1 + q + q^2 + \cdots + q^{x-1} = p^y \tag{2}$$

is solvable then we must have $y = \operatorname{ord}_q p$, i.e., $u = 1$ is forced.

There seems to be a reasonable amount of evidence to warrant the conjecture. Although $y = 1$ is not always satisfied, $y > 1$ seems to be a pretty rare phenomenon. The only examples of equation (1) with $y > 1$, of which I am aware, are $1 + 18 + 18^2 = 7^3$ and $1 + 3 + 3^2 + 3^3 + 3^4 = 11^2$, where $\operatorname{ord}_7 18 = 3$ and $\operatorname{ord}_{11} 3 = 5$ show u to be one. On the other hand, there is a substantial number of examples in which $y = 1$ so that $u = 1$ is forced. Bateman and Stemmler [1] found 776 examples of $1 + q + q^2 = p$ for $p < 1.275 \times 10^{10}$. With the same upper bound on the prime p they found sixteen examples of $(q^5 - 1)/(q - 1) = p$, six examples of $(q^7 - 1)/(q - 1) = p$, one example of $(q^{11} - 1)/(q - 1) = p$ and three examples of $(q^{13} - 1)/(q - 1) = p$.

Various conditions are known to be satisfied by u. Ljunggren [**5**] has shown that u is odd and also that $3 \nmid u$ if $\operatorname{ord}_p a \not\equiv -1 \pmod 6$. His results apply to the more general equation $(a^x - 1)/(a - 1) = c^y$. It is easily seen that we always must have $u \leqq \operatorname{ord}_p a - 1$.

The conjecture is also known to hold in certain special cases. The requirement $u \leqq \operatorname{ord}_p a - 1$ guarantees that $u = 1$ in the special case $\operatorname{ord}_p a = 3$. Recent unpublished work of Jamison and the writer shows that $u = 1$ is forced whenever x and y form a twin prime pair. In still other cases one can prove that the equation has no solution; an example is $1 + q + q^2 + q^3 + q^4 = p^3$ (see Inkeri [**4**]).

Marshall [**6**] has shown that the two sets of conditions

$$p^y > q^x \text{ and } p^y \mid \sigma(q^x) \tag{3}$$

and

$$p^y = \sigma(q^x) \tag{4}$$

are equivalent ($\sigma(n)$ is the usual sum of the positive integral divisors of n), so we can reformulate our conjecture.

CONJECTURE. *If* (2) *is solvable then we must have*

$$p^{\operatorname{ord}_q p} > q^{\operatorname{ord}_p q - 1}.$$

We conclude with two related questions.

QUESTION 1. (Erdös) *Does* $y > 1$ *occur infinitely often?*

QUESTION 2. *Can* $q > p$ *occur in* (2)? *The example* $1 + 18 + 18^2 = 7^3$ *shows that* $a > p$ *can occur in* (1).

References

1. P. T. Bateman and R. M. Stemmler, Waring's problem for algebraic number fields and primes of the form $(p^r - 1)/(p^d - 1)$, Illinois J. Math., 6 (1962) 142–156.

2. G. Birkhoff and H. S. Vandiver, On the integral divisors of $a^n - b^n$, Ann. of Math., 5 (1904) 173–180.

3. H. Edgar, On a Theorem of Suryanarayana, Proc. Washington State Univ. Conf. Number Theory, Pullman (1971) 52–54.

4. K. Inkeri, On the diophantine equation $a(x^n - 1)/(x - 1) = y^m$, Acta Arith., 21 (1972) 299–311.

5. W. Ljunggren, Some theorems on indeterminate equations of the form $(x^n - 1)/(x - 1) = y^q$, (Norwegian), Nordisk Mat. Tidskr., 25 (1943) 17–20.

6. A. Marshall, Problem E2376, this MONTHLY, 79 (1972) 906; solution, 80 (1973) 948.

7. D. Suryanarayana, Certain Diophantine equations, Math. Student, 35 (1967) 197–199.

8. K. Szymiczek, On the equation $a^x - b^x = (a - b)c^y$ (Polish), Wiadom. Mat., (2) 7 (1964) 233–236.

DEPARTMENT OF MATHEMATICS, SAN JOSE STATE UNIVERSITY, SAN JOSE, CALIFORNIA, 95192.

THIRD PARTY MOVEMENTS

R. E. PRATHER

Often a classroom presentation is most successful when it seems to impart a degree of relevance to an otherwise classical subject. Then it does not matter if the model is not entirely faithful; the effect is more important. And the model can then be made more realistic as a result of the ensuing classroom discussion. In this note, we present just such an application for material usually encountered in an elementary course in differential equations.

Assume, as in the United States, that the political arena is dominated by two major parties, but that we wish to examine the effect created by an emerging third party. At time t, let $\lambda(t)$ and $\rho(t)$ denote the per-unit (percentage/100) of the voting population belonging to the two established parties, on the left and right, respectively. If $v(t)$ similarly represents the per-unit figure attained by the new party, we have

$$v(t) + \lambda(t) + \rho(t) = 1$$

at all times. With the major parties ideologically close to the political center, the chances are that the third party will emerge at an extreme end of the spectrum. Suppose therefore, without much loss of generality, that our new party is a radical left-wing movement. There being virtually no tendency for conservative defections to the new party, we may view $\rho(t) = \rho$ as constant in this simplified treatment.

The rate $v'(t)$ of increase would seem to be influenced by several factors. We should expect a contribution proportional to $v(t)$ itself, reflecting the "ground-swell" aspect of the movement. Thus, the larger is $v(t)$, the more extremists would there be found in local elected office and in other increasingly influential positions, where they would be able to bring about a still larger percentage in their number.

Because their programs and aspirations are somewhat compatible, the liberals (i.e., those of the major party on the left) might be expected to cooperate with the radical left initially, at least to some degree. And defections to the new party would not be uncommon. But we cannot expect this trend toward collusion to continue, say for $v > \lambda$. The major parties seem to have an amazing resilience, and in fact, the earlier tendency toward cooperation may be reversed entirely at this point, and a real political struggle would very likely ensue. So let us suppose that an otherwise linear support-opposition balanced at $v = \lambda$ is reinforced to a degree that is inversely proportional to $\lambda(t)$ so as to reflect this resilience: the fewer the liberals, the more resourceful and defiant they become in their attempt to re-establish their relative strength.

We are thus led to the first order differential equation

$$v' = av - b\frac{v - \lambda}{\lambda},$$

where $a, b > 0$ are the "ground-swell" and (in this case) "radic-lib" coefficients,

respectively. Our equation involves only the variable v after the substitution

$$\lambda = (1-\rho) - v = c - v$$

and we have

$$v' = av + b\frac{2v - c}{v - c},$$

where c is the per-unit non-conservative figure "up for grabs."

A fairly realistic situation might result if we were to choose, for the sake of argument:

(i) $a = b = \frac{1}{2}$

(ii) $\rho = 3/8 \quad (c = 5/8)$

(iii) $v(0) = 1/8$

with a decade as the unit of time. One then obtains a corresponding initial value problem for $v(t)$. And the student may be asked to show that

(a) $$\lim_{t\to\infty} v(t) = \frac{-11 + \sqrt{281}}{16} \approx .36 \qquad (< 3/8 = \rho);$$

(b) $\lambda(t) < \rho(t)$ inside four years time;

(c) $\lambda(t) < v(t)$ within seven years time.

It follows that within the span of one quadrennial election, the conservatives would come into power; and by the next election, the liberals cease even to be a majority party. And yet, the new party will never dominate.

Of course, the problem is also well suited for illustrating the various numerical methods. And the preliminary results above give some idea as to what to expect in these investigations. The instructor wishing to pursue this aspect more carefully may appreciate a few tabulations. We include results for $v(t)$ with the Euler Method, An Improved Euler Method, The Three-Term Taylor Series Method, and the Runge-Kutta Method, for spacings of .01 and .001 respectively.

t	EULER	IMPROVED EULER	TAYLOR	RUNGE-KUTTA
0.0	0.12500000	0.12500000	0.12500000	0.12500000
0.1	0.16716628	0.16699091	0.16699334	0.16699223
0.2	0.20538056	0.20501940	0.20502412	0.20502220
0.3	0.23904472	0.23850628	0.23851291	0.23851063
0.4	0.26775577	0.26707067	0.26707854	0.26707647
0.5	0.29139296	0.29061136	0.29061966	0.29061830
0.6	0.31015722	0.30933976	0.30934769	0.30934737
0.7	0.32453751	0.32374268	0.32374967	0.32375042
0.8	0.33521202	0.33448533	0.33449109	0.33449271
0.9	0.34292387	0.34229259	0.34229708	0.34229924
1.0	0.34837548	0.34784946	0.34785283	0.34785519

.01 spacing

t	EULER	IMPROVED EULER	TAYLOR	RUNGE-KUTTA
0.0	0.12500000	0.12500000	0.12500000	0.12500000
0.1	0.16700961	0.16699222	0.16699224	0.16699223
0.2	0.20505794	0.20502217	0.20502222	0.20502220
0.3	0.23856383	0.23851059	0.23851066	0.23851063
0.4	0.26714405	0.26707641	0.26707649	0.26707647
0.5	0.29069532	0.29061823	0.29061832	0.29061830
0.6	0.30942787	0.30934729	0.30934737	0.30934737
0.7	0.32382871	0.32375035	0.32375042	0.32375042
0.8	0.33456434	0.33449264	0.33449269	0.33449271
0.9	0.34236156	0.34229917	0.34229922	0.34229924
1.0	0.34790722	0.34785513	0.34785516	0.34785519

.001 spacing

On the basis of all of these results, will the student conclude that third party movements are to be discouraged to one's own side of the political spectrum? Probably not. It is more likely that he will decide that the model needs revision. And it is here that the classroom discussion can become most interesting. One may ask why there is no accounting for the special powers of an incumbent party; or what provision should be made for treating the effects of a political scandal. Moreover, it would probably be observed that the model does not allow for the eventual demise of the third party, as seems so often to happen in actuality. Perhaps a disenchantment or apathy factor should have been included. But in any case, these and other ramifications should lead to an active student participation, and this is all to the good.

Reference

1. W. E. Boyce and R. C. DiPrima, Elementary Differential Equations, Wiley, New York, 1969.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF DENVER, DENVER, CO 80210.

A CONDITION FOR TWO MATRICES TO BE INVERSES OF EACH OTHER

C. M. BANG, Emory University

1. Introduction. It is not being presumptuous to say that we teach some forms of elementary matrix theory virtually at every college. Defining two matrices A and B to be inverses of each other if $BA = I = AB$, we would quickly prove the following fact:

(a) *If* $BA = I = AC$, *then* $B = C$.

The proof, $B = BI = BAC = IC = C$ is very simple. To show in examples that A and B are inverses of each other, we have to check both the conditions $BA = I$ and $AB = I$. At this point, we may guide the class to think through the following handy theorem and call for its proof or disproof.

THEOREM. *Two $n \times n$ matrices A and B are inverses of each other if (and, of course, only if) $BA = I$ or $AB = I$.*

Most students who just saw the short, simple proof of (a) may try to get $AB = I = BA$ from the given one condition in a similar manner by playing around with matrix identities. This is not possible, as we all know (see Epilog). Indeed, a perfect proof of this little theorem would involve some nontrivial results about linear transformations or determinants. The main purpose of this paper is to give a simple proof of the theorem by two different approaches.

2. Unique solution method. This method is for the course if a study of matrices is preceded by a study of systems of linear equations. An $n \times n$ system of linear equations

$$(\alpha) \quad \begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = s_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = s_2 \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdots \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = s_n \end{cases}$$

has a unique solution if, and only if, the *left side* of (α) can be *elementarily* transformed to a diagonal form

$$\cdot \quad \begin{cases} x_1 & & & & = [\ ] \\ & x_2 & & & = [\ ] \\ & & \cdot & & \cdot \\ & & & \cdot & \cdot \\ & & & \cdot & \cdot \\ & & & x_n & = [\ ]. \end{cases}$$

Since elementary transformations include additions and multiplications only and no divisions at all, we can carry out any elementary transformations on $s_1, \cdots, s_n$ whatever these constant terms may be. Thus, the following is true.

(b) *If an $n \times n$ system of linear equations once has a unique solution for one set of constant terms, then it has a unique solution for any replacement of the constant terms.*

Let us next prove the theorem. Assume $BA = I$. Then, for any $n \times 1$ matrix S, the system[1] $AX = AS$ obviously has the solution $X = S$ and, besides, it is unique since $T = BAT = BAS = S$ whenever T is another solution, i.e., $AT = AS$. Thus,

[1] Right after the introduction of matrix multiplication, we can write systems of linear equations in matrix forms. Here, $X = [x_1, \cdots, x_n]^t$ is the matrix of the unknowns and AS is the matrix of constant terms.

by (b), the system $AX = D_j$ has a unique solution for any $n \times 1$ matrix D_j. Put

$$D_j = [\delta_{ij}]^t_{i=1,\cdots n} = [0,\cdots,0,1,0,\cdots,0]^t,$$

with 1 at the jth position, and let C_j be the corresponding solution, that is, $AC_j = D_j$. If we construct an $n \times n$ matrix $C = [C_1, C_2, \cdots, C_n]$, then clearly $AC = I$. We now have $BA = I = AC$ and hence by (a), we are finished.

3. Vector method. This is a suitable method when we introduce the vector space of k-tuples and learn some basic results about bases:

(c) *Any k linearly independent k-tuples form a basis for the vector space of k-tuples.*

(d) *Every vector is a linear combination of the elements of a basis.*

Let us again prove the theorem. Assume $BA = I$. We view an $n \times n$ matrix as an n^2-tuple vector. Let E_{ij} be the $n \times n$ matrix having all zero entries except 1 at the (i,j)-th position. We claim that $\{AE_{ij}\}_{i,j}$ is a basis for the n^2-tuple vector space by (c), since

$$\sum m_{ij}AE_{ij} = 0 \text{ implies } [m_{ij}]_{i,j} = BA \sum m_{ij}E_{ij} = B \sum m_{ij}AE_{ij} = 0,$$

i.e., $m_{ij} = 0$. Therefore, by (d), I is a linear combination of the basis, say, $I = \sum n_{ij}AE_{ij} = AC$ with $C = \sum n_{ij}E_{ij} = [n_{ij}]_{i,j}$. We now have $BA = I = AC$ and hence we are finished by (a). The point we should note in this method is that we need to know nothing about linear transformations nor about the confusing theory of matrix representation of linear transformations.

4. Epilog. A related interesting question is to ask how to convince students that they cannot prove the theorem in the way we proved (a). Consider two infinite square matrices

$$B = \begin{bmatrix} 0 & 1 & 0 & 0 & . & . & . \\ 0 & 0 & 1 & 0 & . & . & . \\ 0 & 0 & 0 & 1 & . & . & . \\ . & . & . & . & . & & \\ . & . & . & . & & . & \\ . & . & . & . & & & . \end{bmatrix} \qquad A = \begin{bmatrix} 0 & 0 & 0 & . & . & . \\ 1 & 0 & 0 & . & . & . \\ 0 & 1 & 0 & . & . & . \\ 0 & 0 & 1 & . & . & . \\ . & . & . & . & & \\ . & . & . & & . & \end{bmatrix}$$

Then, clearly $BA = I$, but $AC \neq I$ for all matrices C since the first row of AC is always zero. This example, hopefully, should convince students that the proof of the theorem must use somewhere the finiteness of sizes of the involved matrices. Thus, we cannot give an (a)-type proof which is free of sizes. In fact, the finiteness is absolutely necessary to get our needed claims such as (b), (c), and (d). This paper

originally included a third proof using elementary matrices only and, as the referee pointed out, a similar one to it can be found in [1]. We feel that these proofs are reasonable for an elementary course, if not perfect.

Reference

1. B. Kolman, Elementary Linear Algebra, Macmillan, New York, 1970.

DEPARTMENT OF MATHEMATICS, EMORY UNIVERSITY, ATLANTA, GA 30322.

MATHEMATICAL EDUCATION

EDITED BY SHIRLEY HILL AND PAUL MIELKE

Material for this Department should be sent to Shirley Hill, Department of Mathematics, University of Missouri, Kansas City, MO 64110, or to Paul Mielke, Department of Mathematics, Wabash College, Crawfordsville, IN 47933.

Editorial Announcement. Experiments in various forms of self-paced modular instruction are now widespread. The editors continue to receive many articles dealing with this topic. A necessary, but not sufficient, condition for acceptance of such an article will be that it include a careful analysis of the results, or a serious statistical study, or an innovative idea that has not appeared in these pages before.

A PERSONALIZED SYSTEM OF INSTRUCTION IN AN UNDERGRADUATE MATHEMATICS SERVICE SEQUENCE

D. L. YOUNG, H. E. MCKEAN AND F. L. NEWMAN

Summary. The problem of teaching required mathematics courses to students in the behavioral sciences with widely varying backgrounds and interests is a major one and is growing with increased emphasis on quantitative methods. Dissatisfaction with the traditional methods of instruction is also multiplying with these increased demands. This report describes using the Personalized System of Instruction (PSI) for teaching a nine hour sequence of mathematics and statistics to students in the behavioral sciences at New Mexico State University. The PSI method embodies student self-pacing, aspects of mastery learning and use of student proctors for immediate feedback and reinforcement.

Introduction. Many mathematics departments have the responsibility of teaching one or more courses in mathematics and/or statistics to students in the behavioral sciences. In particular, the Department of Mathematical Sciences at New Mexico

State University is in charge of a nine credit sequence of three courses required of students in such areas as business, economics, agriculture, psychology, biology and police science. Briefly, these courses (Math 121, Math 122 and Stat 251) review algebra and cover various aspects of finite mathematics and elementary statistics. In the past these courses were taught in large lecture sections, meeting three times a week with an additional recitation section taught by graduate assistants. The courses were taught differently each semester with the content varying appreciably according to the inclinations of the instructors in charge. The students entering the courses had widely disparate mathematical backgrounds and interests. An alarming number feared mathematics, and because of the open-door policy adhered to by most public institutions, many were generally poor students. The "curve" required to get a reasonable number of students through the courses was often very low and the success rate was discouraging. The areas serviced by these courses frequently complained that their students retained little of the material to which they were exposed. These problems were further compounded by a doubling of the number of students taking these courses over the past five years to the point where their enrollment presently exceeds that of the traditional calculus sequence.

The problems of teaching the unwilling the unwanted are well known to those teaching such service courses. They have been discussed often and some solutions proposed; the reader is referred to Riner [7] and Stein [9] for recent accounts. The solution proposed at NMSU was the introduction of the Personalized System of Instruction (Keller Plan) with certain modifications and extensions which seemed appropriate for teaching mathematics.

Personalized System of Instruction. The Personalized System of Instruction (PSI) is primarily the product of the psychologist Fred S. Keller and is described in various publications [3], [4], [5]. It has been used with considerable success in many disciplines ranging from psychology to engineering. (See Dessler [1] for a description of various courses and experiences.) PSI as developed by Keller and his followers entails the following five essential ingredients:

(1) Dividing the content of the course into *small meaningful units of instruction.*

(2) Devising *clearly written behavioral objectives* and other study materials for each unit (usually in conjunction with a text).

(3) Allowing the individual student to proceed through the course at his *own pace*, thus using lectures or demonstrations only for motivation and not as sources of critical information.

(4) Establishing a *unit perfection requirement* in the form of a quiz to be taken on a when-ready basis by the student with no penalty for unsatisfactory performance regardless of how long it takes to meet the requirement.

(5) *Using student proctors* whose presence in acceptable numbers provides immediate grading of quizzes in the student's presence, immediate feedback and reinforcement and the personal aspect of the teaching process.

PSI views testing as a learning experience rather than a basis for deciding what percentage of material was learned in a given amount of time for the purpose of determining a grade; incorrect responses to quiz items are guides to misunderstood concepts. While reviewing a quiz with a proctor the student discovers what he has and has not learned, receives information on what needs to be studied again and where he can find it, and then is encouraged and required to repeat (as often as necessary) a quiz on the same material until it is mastered.

Before discussing how PSI can be used to teach mathematics and statistics, let us consider the manner in which an individual learns. PSI should be implemented in such a way as to aid the learning process.

A Learning Model. In considering how an individual learns the learning process described in the following model seems to hold:

(L1) An individual first learns definitions and basic ideas via audio or visual stimulation — this is called the *stimulus-response phase.*

(L2) These new definitions and ideas are put to work in the formulation of concepts and the utilization of them through problem solving — the *concept formulation phase.*

(L3) An individual repeats steps (L1) and (L2) and then is able to consolidate the information as an integrated whole — the *consolidation phase.*

This learning model seems to be quite appropriate in mathematics. A student first learns a few basic definitions and ideas; he then works problems involving the definitions and ideas; and then he consolidates the concepts with other notions in mathematics which have been presented via the first two phases (L1) and (L2).

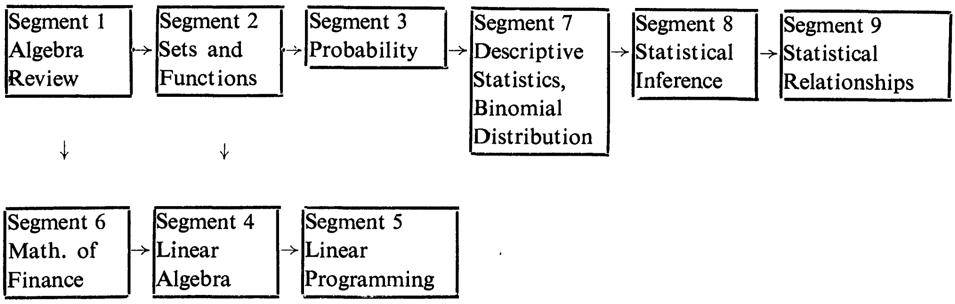
Basic Implementation of PSI. With a model of how mathematics is learned PSI should be applied to provide an efficient and effective way to carry out the various phases of the model. How this was done at NMSU will now be described.

SEGMENTATION. As a first approximation to dividing the material in the three courses into small, logically self-contained units of instruction and also to provide for a reasonable amount of material for the consolidation phase (L3) of the learning model, each of the three credit hour courses in the sequence was segmented into three one-hour courses, so that the nine-hour sequence consisted of nine separate one-hour courses. The logical flow of the segments, as well as a brief description of each, appears in Figure 1.

This segmentation allows a great deal of flexibility to the student for scheduling purposes (e. g., a student might be able to fit one or two segments into a crowded schedule but not an entire three hour course). Also provided for is the omitting of segments which the student has had elsewhere and which he would have had to repeat under the previous system of three three-hour courses.

FIG 1

Logical Flow of the Segments



THE LEARNING CYCLE. Each segment (or one-hour course) has been further subdivided into four units to yield the small meaningful units of instruction mentioned in the sections describing PSI. The unit is roughly equivalent to one week's work in a conventional course. Each unit in turn provides an execution of the first two phases of the learning model (L1) and (L2) as follows.

Step 1. The student receives a study guide which consists of two parts: (a) a tutorial-style narrative which steps the student through the reading assignment for the unit, and which, as needed, may explain, supplement or otherwise clarify the reference text; (b) a list of behavioral objectives which state what definitions, ideas and basic concepts the student should memorize and know in the unit.

When the student believes he has mastered the behavioral objectives at the required level he takes a brief closed-book quiz. There is no time limit imposed for taking the quiz and help may be asked of the proctors if difficulty is encountered in interpreting the questions. When the quiz is completed a proctor immediately goes over the quiz with the student and discusses the student's mastery of the material, pointing out strengths and weaknesses and offering suggestions for further study. The grade — provided it is above a prescribed level (we currently require 60% although higher requirements are under consideration) — is recorded on the student's progress sheet for the segment. Similar versions of the quiz must be repeated until the prescribed level is attained. The student may then elect to take the quiz once again in an effort to raise his grade or better master the material. Only the highest quiz grade is retained. Step 1 corresponds to the stimulus-response phase (L1) of the learning model.

Step 2. After satisfactory completion of the closed-book quiz and the first phase of the learning process, the student obtains an open-book quiz over the unit. The open-book quiz provides for the utilization phase of the learning model as the quiz is in essence a problem set. Again there is no time limit imposed for completion

and the student may consult his text and notes in working the problems and consult the proctors for points of clarification. It is ultimately required that the student present to a proctor a perfect set of well-organized solutions to the problems, and moreover, be fully prepared to discuss the questions and answers with the proctor. No penalty is exacted for incorrect solutions, but on the other hand, no credit is given until the solutions are completely correct. Thus the open-book quiz provides the unit perfection requirement of PSI. The proctor discusses the problems with the student and provides encouragement and hints for working the problems. The proctor does not solve the problems for the student.

After the open-book quiz is finished Steps 1 and 2 are repeated in the next unit and so on until all four units of the segment are completed. The next unit may not be started until the previous unit is finished.

Step 3. At the end of each segment a comprehensive examination is given to interrelate the four units. Again the exam is graded immediately and there is no time limit for completion. The student is allowed to take a second version of the examination if he is unhappy with his performance. This step incorporates the consolidation phase (L3) and is akin to a final examination in a conventional course.

GRADING SYSTEM. Each student's grades for a segment are recorded on a separate individual grade sheet. One can tell at a glance how a student is progressing on a particular segment by reviewing the grade sheet. The basis for assigning a grade for a segment is the simple average of the grades on the three parts of the learning cycle: the closed-book quiz average, the open-book quiz average, the comprehensive examination score. Because of the unit perfection requirement the open-book quiz average is automatically 100%. We have used the following basis for assigning grades: 95% and above — A, 87% to 94% — B, 78% to 86% — C, 69% to 77% — D. If a segment is unfinished at the end of the semester a grade of W (withdrawal) is assigned with the provision that the student may pick up where he left off the next time he enrolls for the segment. Because progress to the next unit is impossible until completion of the previous one, all open-books in a segment must be completed and all closed-book quizzes must be at a prescribed level so that grades below D are impossible. In our experience the preponderance of grades for those finishing a segment are in the A–B range.

Further Details of Implementation. The above section describes the basic meshing of the learning model with PSI. In dealing with the large number of students (approximately 1000 per semester) taking the nine hour sequence at NMSU the solution of several other problems became necessary for the effective and efficient handling of the sequence.

QUIZ GENERATION. As can readily be seen if students are to progress at their

own pace, if students are to be allowed to repeat quizzes and exams and if one desires that each student truly know the material and not just specific responses to questions on a few specific quizzes, it is required that a large number of different individual quizzes be available. This has been accomplished in two steps: (a) the development of a large data bank of quiz and examination questions and answers which were key-punched on cards and then transferred to a master tape; (b) development of a computer program for randomly selecting questions and generating and printing individualized quizzes and exams with answers appearing on a separate page. The program can subsample from strata of questions with a great deal of flexibility. The end effect is that each student receives a "custom-made" quiz.

SLIDE-TAPE SHOWS. Again since students may progress at their own pace conventional scheduled lectures become almost useless unless they are available on a when-ready basis. Another problem stems from the fact that all students do not learn in the same way and learning by reading alone may not provide the best results. As an attempt to overcome these difficulties a verbal-pictorial presentation of the material via "slide-tape shows" has been provided for each unit. A slide-tape show consists of a prerecorded cassette tape and a set of slides which provide a discussion of various aspects of the material. The apparatus for reviewing and listening to a show consists of a slide projector, cassette tape player and headphone which have controls enabling the student to pause for reflection or for checking calculations, or to repeat parts of the program that were not understood. The apparatus is housed in a specially built carrel. Again allowance has been made for student self-pacing.

LEARNING CENTER. Finding an adequate room in which to run the PSI program also presented problems. This problem was solved by the remodeling of the basement of the Mathematical Sciences Building and the construction of the Mathematics Learning Center. The Center is open approximately 40 hours a week and has a seating capacity of around 100. The students are required to take all quizzes and exams in the Learning Center. Every effort has been made to make the Learning Center a pleasant place for learning mathematics and statistics. It contains the carrels for viewing the slide-tape shows, a student lounge, several small conference rooms, several rooms for desk calculators, separate areas for taking the closed-book and open-book quizzes, partitioned study tables, a grading area for reviewing students' quizzes and an office for storage of quizzes and materials.

Advantages and Disadvantages of Using PSI. The advantages of PSI are readily apparent. The student is allowed great flexibility to learn on his own terms. He may take quizzes when he is ready and repeat them until he has mastered the material. He is not penalized for failure but rewarded for his final success. Course content must be explicitly defined and objectives stated very clearly. This aids in standardization of the courses and provides the student with a list of the things he must know.

Rather uniformly high standards are able to be maintained for student performance thus eliminating the problem of "lowering the curve" and helping to maximize the amount of material covered and learned. The responsibility for learning and how fast he learns is placed more squarely on the student's shoulders while he receives a great deal more individual one-on-one help than is possible in most traditional courses.

The problems and disadvantages in using PSI are not as apparent at first glance. As an instructor of a PSI course one must be prepared to relinquish the security and relative ease of lecturing, infrequent testing and little student contact. One must write study guides, state behavioral objectives (perhaps for the first time in one's career), prepare numerous quizzes and be ready for a great deal of student contact. Those who use PSI become course managers rather than course instructors and often feel that they are not really "teaching."

Many university administrators view PSI with horror because of the many administrative modifications necessary to run a PSI course efficiently. Often one's own colleagues are suspicious of PSI because of its newness and the large percentage of good grades obtained by students.

Student reaction to PSI is often mixed. As a rule students feel that they work harder in a PSI course than a conventional course. With more work and time required it is sometimes the case that a student doesn't progress as fast as he thinks he should and becomes a bit discouraged. It is also true that more students receive grades of W (withdrawal) in our PSI courses than in the traditional courses due to the increased work load and the fact that progression from one unit to the next is virtually impossible without mastery of material. Students are also highly suspicious of anything new and many adamantly maintain they cannot learn unless a teacher "teaches" the material to them. This attitude is difficult to overcome as is the fear of mathematics which students so often have. Because of these attitudes toward a new system and mathematics, students at times just don't get started in the segments. Thus far our success rate (grades of C or better) has been no better than the conventionally run course, but the percentage receiving D or F grades has dropped to almost zero. Our subjective feeling is that the students do learn more.

Another problem which has been encountered at NMSU is that of handling the almost 1000 students enrolled in the nine hour sequence each semester. It becomes difficult to keep track of such a large group — we presently have a secretary working on that full time — and the number of proctors required is quite large. The recommended ratio of students to proctors is around 10:1 and the problem of obtaining enough qualified proctors has been a major one. We have used various sources to obtain proctors, including students who have done well in the courses, undergraduate mathematics majors, graduate assistants and faculty members. We have provided remuneration for the undergraduate proctors by offering them academic credit in an undergraduate reading course when money was not available (as is usually the case). One of the great advantages to proctoring is that the proctor is the one who

learns the most. Training proctors in the material as well as the philosophy of PSI is not easy but has been greatly facilitated by a proctor manual written by Newman [**6**]. Proctors can make or break a PSI course. Properly trained and qualified proctors in adequate numbers are an absolute must.

Recommendations. Our experience has shown that special attention must be given to various points if a PSI course is being contemplated. We make the following recommendations:

(1) Start small, perhaps with a single section of your course.

(2) Work up study guides, behavioral objectives and quiz items well before the semester begins. You will have more stress than you anticipate during your first try at PSI, and you will not have time to handle everything in addition to writing materials.

(3) Try to generate enthusiasm among the students and recruit the best of them as proctors.

(4) Spend time with your proctors. Train them at the beginning of the semester or at the point in time when they will be used. Meet with them regularly.

(5) Consider segmenting your course into one credit courses. This gives enormous flexibility, and the students can choose the topics they may require.

(6) In no way compromise the intent of the PSI system. (See Sherman [**8**] for possible consequences of perverting the system.) If you must of necessity play with the system as your course grows larger, do so gingerly and only after you have had considerable experience with PSI in its pure form.

(7) Be sure PSI is right for you. See Green [**2**] and honestly assess whether your situation fits into any of those described there.

(8) If you possibly can, obtain the backing of a high administrator. Be frank with him and explain the pros and cons as experienced by others experimenting with PSI. If he is reluctant or wishes to impose constraints upon you, quietly drop the idea. (It's better to do this than wasting a year writing materials and then having to drop it!)

(9) If you do go the PSI route, be visible to the students, be outgoing and be ready to modify your execution of the model to accommodate the students, provided it does not affect the main thrust of PSI. Make every effort to get your students started in the course.

Conclusions. Is PSI really worth it? We believe the answer is yes if PSI is used properly. Although our views are only subjective (we hope to do a comparison study in the near future), students do seem to learn more and those who finish like PSI. PSI has enormous potential and could have tremendous impact on a university system which on the whole is outmoded and under fire.

Acknowledgements. The implementation of PSI at New Mexico State University was made possible by a National Science Foundation grant (GY-8304) and the

development of the Mathematics Learning Center was provided for by a grant from NMSU. Our sincere thanks go to Dr. Gerald Rogers, Dr. Eldon Boes, Dr. John Werth, and Dr. Robert Wisner for all their efforts in the development of the program.

Notes. The present address of Dr. H. E. McKean is Department of Statistics, University of Kentucky and the address of Dr. Frederick L. Newman is Department of Psychiatry, University of Pennsylvania.

References

1. A. J. Dessler, Proceedings of the Keller Method Workshop Conference, Rice University, Houston, Texas, 1972.

2. B. A. Green, Fifteen Reasons Not to Use PSI, Proceedings of the Keller Workshop Conference, A. J. Dessler, editor, Rice University, Houston, Texas, 1972.

3. F. S. Keller, "Good-bye Teacher...," J. of App. Beh. Anal., 1 (1966) 79–89.

4. F. S. Keller, Neglected Rewards in the Educational Process, Introductory Psychology Instructor's Manual, Vol. III, Validation Edition, Independent Learning Systems, Inc., San Rafael, Calif., 1970, 1–33.

5. F. S. Keller, PSI — Its History and Future, Proceedings of the Keller Method Workshop Conference, A. J. Dessler, editor, Rice University, Houston, Texas, 1972.

6. F. L. Newman, Proctor Manual for Math 121, 122 and Stat 251, Department of Mathematical Sciences, New Mexico State University, 1972.

7. J. Riner, Individualizing Mathematics Instruction, this MONTHLY, 79 (1972) 77–86.

8. G. Sherman, PSI — Some Notable Failures, Proceedings of the Keller Method Workshop Conference, A. J. Dessler, editor, Rice University, Houston, Texas, 1972.

9. S. K. Stein, Mathematics for the Captured Student, this MONTHLY, 79 (1972) 1023–1032.

DEPARTMENT OF MATH. SCIENCES, NEW MEXICO STATE UNIVERSITY, LAS CRUCES, N. M. 88003.

PROBLEMS AND SOLUTIONS

EDITED BY EMORY P. STARKE

All problems (both elementary and advanced) proposed for inclusion in this Department should be sent to E. P. Starke, 1000 Kensington Ave., Plainfield, NJ 07060. Proposers of problems are urged to enclose any solutions or information that will assist the editors. Ordinarily, problems in well-known textbooks and results in generally accessible sources are not appropriate for this Department. No solutions (except those accompanying proposals) should be sent to Professor Starke.

ELEMENTARY PROBLEMS

Solutions of Elementary Problems should be sent to Problems Group, Mathematics Department, University of Maine, Orono, ME 04473. To facilitate their consideration, solutions of Elementary Problems in this issue should be typed (with double spacing) and should be mailed before December 31, 1974.

An asterisk (*) means neither the proposer nor the editors supplied a solution.

E 2486. *Proposed by L. C. Washington, Princeton University*

Let p be an odd prime. Show that there exist perfect squares which when expressed in base p terminate in arbitrarily many repetitions of the digit "1". (This is in contrast to the result for base 10; see the 1970–1971 Putnam Examination, Problem A-3 [1971, 765–767].) Is there a similar result for higher powers?

E 2487. *Submitted by S. R. Conrad, Benjamin R. Cardozo High School, Bayside, New York (attributed to D. J. Newman)*

Let $S_k = x_1^k + x_2^k + \cdots + x_n^k$. If $S_k = k$ for $k = 1, 2, \cdots, n$, find S_{n+1}.

E 2488. *Proposed by Richard Stanley, University of California, Berkeley*

Let p be an odd prime. It has been conjectured that there exists a natural number $k \leqq p-1$ which is a primitive root modulo p and which is relatively prime to $p-1$. Prove this conjecture in the special case that $p \equiv 1 \pmod 4$ and $3\phi(p-1) > p-1$, where ϕ denotes Euler's totient function.

E 2489*. *Proposed by Bruce Reznick, California Institute of Technology*

Suppose that f is a continuously differentiable function which maps $[0, 1]$ onto itself and which satisfies $f(0) = 0$ and $f(1) = 1$. Can the arc length of f ever exceed the arc length of the composition $f \circ f$?

E 2490. *Proposed by James Brink, Pacific Lutheran University*

Let $A = (a_{ij})$ be an $n \times n$ matrix with the property that there exist constants $c_1, \cdots, c_n$ such that

$$\sum_{i=1}^{k} a_{ij} = c_k$$

for $1 \leqq j \leqq k \leqq n$. Find the eigenvalues of A.

E 2491. *Proposed by S. W. Golomb, University of Southern California*

Find all natural numbers n with the property that $[\sqrt{n}]$ is a divisor of n.

SOLUTIONS OF ELEMENTARY PROBLEMS

A Non-Generalization of an Identity of Euler

E 2403 [1973, 315]. *Proposed by W. A. Al-Salam, University of Alberta, and A. M. Chak, West Virginia University*

It is known that a generalization of the binomial theorem is Euler's identity

$$(1) \qquad (1+x)(1+qx)\cdots(1+q^{n-1}x) = \sum_{k=0}^{n} \frac{F(n,q)q^{k(k-1)/2}}{F(n-k,q)F(k,q)}x^k,$$

where q is a fixed complex number which is not a primitive kth root of unity for any $k \geqq 2$, where

$$F(k,q) = \sum_{j=1}^{k} (1+q+\cdots+q^{j-1}),$$

and where $F(0,q) = 1$. (Thus $F(k,1) = k!$.)

Show that the only solution of

$$(2) \qquad (1+x)(1+c_1x)\cdots(1+c_{n-1}x) = \sum_{k=0}^{n} \frac{\phi_n}{\alpha_{n-k}\beta_k}x^k$$

is this identity of Euler. That is, if given a sequence $c_1, c_2, \cdots$ of complex numbers there exist sequences $\phi_0, \phi_1, \cdots; \alpha_0, \alpha_1, \cdots; \beta_0, \beta_1, \cdots$ such that (2) holds identically for all values of n, then necessarily $c_1 = c_n/c_{n-1} = q$ is constant for $n = 2, 3, \cdots$ and thus (2) must have the form of (1).

Solution by Leonard Carlitz, Duke University. We note first that no $c_i = 0$, for if $c_n = 0$ and $c_i \neq 0$ for $i < n$, then

$$(1+x)(1+c_1x)\cdots(1+c_{n-1}x)(1) = \sum_{k=0}^{n+1} \frac{\phi_{n+1}}{\alpha_{n+1-k}\beta_k}x^k$$

and the left-hand side is a polynomial of degree n, whereas the right-hand side is of degree $n+1$. Now put $p_0 = p_1 = 1$ and $p_k = c_1c_2\cdots c_{k-1}$ for $k \geqq 2$ and put $\beta_k = \gamma_k/p_k$ for $k = 0, 1, \cdots$ so that (2) of the problem statement becomes

$$(3) \qquad (1+x)(1+c_1x)\cdots(1+c_{n-1}x) = \sum_{k=0}^{n} \frac{\phi_n}{\alpha_{n-k}\gamma_k}p_kx^k.$$

We can assume without loss of generality that $\alpha_0 = \gamma_0 = 1$ and thus $\phi_0 = 1$ also. It follows from (3) that $\phi_n/\alpha_n = \phi_n/\gamma_n = 1$ and so $\phi_n = \alpha_n = \gamma_n$ for $n = 0, 1, \cdots$. Thus (3) becomes

$$(4) \qquad (1+x)(1+c_1x)\cdots(1+c_{n-1}x) = \sum_{k=0}^{n} \frac{\alpha_n}{\alpha_{n-k}\alpha_k}p_kx^k.$$

We now show that $c_i = c_1^i$ for all i, using induction. Suppose first that c_1 is not a primitive nth root of unity for any $n \geqq 2$. Assume inductively that $c_i = c_1^i$ for $i = 1, 2, \cdots, n-1$ and consider

$$(1+x)(1+c_1x)\cdots(1+c_nx) = \sum_{k=0}^{n+1} \frac{\alpha_{n+1}}{\alpha_{n+1-k}\alpha_k} p_k x^k.$$

Comparing the coefficients of x and of x^n, we have

$$1 + c_1 + \cdots + c_n = \frac{\alpha_{n+1}}{\alpha_n\alpha_1},$$

$$c_1c_2\cdots c_n\{1 + c_1^{-1} + \cdots + c_n^{-1}\} = \frac{\alpha_{n+1}}{\alpha_1\alpha_n}p_n = \frac{\alpha_{n+1}}{\alpha_1\alpha_n}c_1c_2\cdots c_{n-1},$$

and thus

$$1 + c_1 + \cdots + c_n = c_n(1 + c_1^{-1} + \cdots + c_n^{-1}).$$

Then, by the inductive hypothesis,

$$1 + c_1 + c_1^2 + \cdots + c_1^{n-1} + c_n = c_n(1 + c_1^{-1} + c_1^{-2} + \cdots + c_1^{-n+1}) + 1,$$

$$c_1 + c_1^2 + \cdots + c_1^{n-1} = c_n(c_1^{-1} + c_1^{-2} + \cdots + c_1^{-n+1}),$$

$$c_1 + c_1^2 + \cdots + c_1^{n-1} = c_nc_1^{-n}(c_1^{n-1} + c_1^{n-2} + \cdots + c_1).$$

If $n = 2$, we can cancel the common factor of c_1 on both sides, showing that $c_n = c_1^n$. If $n > 2$, we note that by assumption, c_1 is not a primitive $(n-1)$st root of unity, so that $c_1 + c_1^2 + \cdots + c_1^{n-1} \neq 0$, and again $c_n = c_1^n$ follows.

It remains to be shown that c_1 cannot be a primitive nth root of unity for any $n \geqq 2$. Suppose to the contrary that it is. Using the above argument, we can show that $c_i = c_1^i$ for $i = 1, 2, \cdots, n$; the proof breaks down in trying to show $c_{n+1} = c_1^{n+1}$. But look at the coefficient of x in the left-hand side of (4); it is $1 + c_1 + \cdots + c_{n-1} = 1 + c_1 + c_1^2 + \cdots + c_1^{n-1} = 0$. The coefficient of x on the right-hand side of (4) is $\alpha_n/\alpha_{n-1}\alpha_1 \neq 0$, a contradiction. The result is shown.

Also solved by O. P. Lossers (Netherlands), and the proposers.

A Twin Prime Counting Function

E 2425 [1973, 692]. *Proposed by C. A. Nicol, University of South Carolina*

For each positive real number x let $\pi_2(x)$ denote the number of twin primes not exceeding x. Show that

$$\pi_2(x) = 2 + \sum_{7 \leqq n \leqq x} \sin\frac{\pi}{2}(n+2)\left[\frac{n!}{n+2}\right] \cdot \sin\frac{\pi}{2}n\left[\frac{(n-2)!}{n}\right],$$

where brackets denote the greatest integer function.

Solution by Allen Stenger, student, Pennsylvania State University. If $n > 5$ is composite, then $(n-2)!/n$ is an even integer so that

$$\sin\frac{\pi}{2}n\left[\frac{(n-2)!}{n}\right] = 0.$$

If p is prime, then by Wilson's theorem, $(p-2)! \equiv -(p-1)! \equiv 1 \pmod p$ implying

$$\left[\frac{(p-2)!}{p}\right] = \frac{(p-2)!-1}{p}.$$

Thus if $p > 5$, then $4 \mid (p-2)!$ and so

$$\sin\frac{\pi}{2}p\left[\frac{(p-2)!}{p}\right] = \sin\frac{\pi}{2}((p-2)!-1) = -1.$$

It follows that for $n > 5$, the term indexed by n in the sum is zero if either n or $n+2$ is composite and is $(-1)(-1) = 1$ if both n and $n+2$ are prime. The "$2+$" is necessary to count the twin prime pairs $(3,5)$ and $(5,7)$.

Also solved by Gwydion Anllw, M. G. Greening (Australia), D. Z. Kilhefner, E. S. Lander, Harry Lass, L. E. Mattics, M. R. Murty & V. K. Murty, Bob Prielipp, Kenneth Rosen, Michael Steiner (Sweden), E. W. Trost (Switzerland), and the proposer.

Editor's Note. As pointed out by Greening, Kilhefner, and Prielipp, the statement should read "...number of twin primes $(p, p+2)$ with $p \leqq x$."

Inscribing a Regular n-gon in a Regular $(n+1)$-gon

E 2426 [1973, 807]. *Proposed by C. A. Long, Bowling Green University*

It is easy to show that the equilateral triangle can be inscribed in a square, and that a square can be inscribed in a regular pentagon. Can a regular pentagon be inscribed in a regular hexagon?

Solution by N. G. Gunderson, University of Rochester. A regular $(n-1)$-gon cannot be inscribed in a regular n-gon for $n \geqq 6$. To show this, assume for some $n \geqq 6$ that the regular $(n-1)$-gon $A_1A_2\cdots A_{n-1}$ with unit side is inscribed in a regular n-gon $B_1B_2\cdots B_n$; we can assume that A_1 is on side B_1B_2, that $0 \leqq A_1B_1 \leqq A_iB_j$ for all i, j, and that A_2 is on side B_2B_3. Then A_3 is on B_3B_4, A_4 is on B_4B_5, and A_5 on B_5B_6. Let $\measuredangle A_{j+1}A_jB_{j+1} = \alpha_j$ and $\measuredangle A_jA_{j+1}B_{j+1} = \beta_j$, for $j = 1,2,3,4$. Compare angle sums to get for $j = 2,3,4$ that

$$\alpha_j = \alpha_{j-1} + 2\pi/n(n-1) \text{ and } \beta_j = \beta_{j-1} - 2\pi/n(n-1).$$

Since $\beta_j = (2\pi/n) - \alpha_j$, for $j = 1,2,3,4$, we have for $j = 2,3,4$ that

$$\alpha_{j-1} + \beta_j = 2\pi(n-2)/n(n-1)$$

and

$$\alpha_{j-1} - \beta_j = \alpha_1 - \beta_2 + 4\pi(j-2)/n(n-1).$$

Now apply the Law of Sines to triangles $A_iA_{i+1}B_{i+1}$ $(i = 1,2,3,4)$ to obtain for $j = 2,3,4$ that

$$\begin{aligned} B_jB_{j+1} \sin \measuredangle B_{j-1}B_jB_{j+1} &= (B_jA_j + A_jB_{j+1}) \sin(\pi - 2\pi/n) \\ &= \sin\alpha_{j-1} + \sin\beta_j \\ &= 2\sin\tfrac{1}{2}(\alpha_{j-1} + \beta_j)\cos\tfrac{1}{2}(\alpha_{j-1} - \beta_j) \\ &= 2\sin\frac{\pi(n-2)}{n(n-1)}\cos\left\{\frac{2\pi(j-2)}{n(n-1)} + \tfrac{1}{2}(\alpha_1 - \beta_2)\right\}. \end{aligned}$$

Since the n-gon is regular, the left-hand side of the above is independent of j and since the sine factor on the right is nonzero, we must have

$$\cos\theta = \cos\left\{\frac{2\pi}{n(n-1)} + \theta\right\} = \cos\left\{\frac{4\pi}{n(n-1)} + \theta\right\}$$

for $\theta = \frac{1}{2}(\alpha_1 - \beta_2)$. Since the cosine cannot assume the same value three times in an interval of length less than 2π, this is a contradiction.

Also solved by J. Alonso and his Bennett College Students, Anders Bager (Denmark), C. S. Gardner, Michael Goldberg, O. P. Lossers (Netherlands), L. E. Mattics, S. P. Morgan, D. K. Pickard, Primer for Research Group, Michael Shimshoni (Israel), and Leon Steinberg. Partial solution by Eric Chipman. Four correspondents claimed the inscription to be possible.

Alonso *et al.* state that a convex equilateral $(n-1)$-gon can be inscribed in a regular n-gon. In fact, any point of the contour of the n-gon can be chosen as a vertex of the equilateral $(n-1)$-gon. Also, if n is even, a convex equiangular $(n-1)$-gon can be inscribed in the regular n-gon in such a way that the two polygons have a common vertex. They ask the question: "What points on the contour of a regular n-gon are possible vertices of an inscribed convex equiangular $(n-1)$-gon?"

Bounds for Egyptian Fraction Partitions of Unity

E 2427 [1973, 807]. *Proposed by Harry Ruderman, Hunter College Campus School*

Suppose that 1 is written as a sum of n distinct Egyptian fractions. Find upper and lower bounds for the smallest fraction in the sum.

Comment by Paul Erdös, Mathematical Institute, Hungarian Academy of Sciences, Budapest. Write

$$1 = \frac{1}{x_1} + \cdots + \frac{1}{x_n}$$

with x_i integers and $x_1 \leqq x_2 \leqq \cdots \leqq x_n$ and set $y_n = \max x_n$, the max being over all such partitions. It is well known that

$$y_n = y_{n-1}(y_{n-1} + 1). \tag{1}$$

This was given by O. D. Kellogg, *On a Diophantine problem*, this MONTHLY 28 (1921), 300–303 and later by D. R. Curtiss, *On Kellogg's Diophantine problem*, this MONTHLY 29 (1922), 380–387. A generalization of (1) is proved in my paper, *On a Diophantine equation*, Mat. Lapok 1 (1950), 192–210 (in Hungarian). For further literature on Egyptian fractions see my paper *Quelques Problèmes de la Théorie des Nombres*, Monographie de L'Enseignement Math., No. 6, problèmes 72, 73, 74. (See also M. N. Bleicher, *A new algorithm for the expansion of Egyptian fractions*, J. Number Theory 4 (1972), 342–382, which has a good bibliography.—Ed.)

To determine $\min x_n$ is much more difficult; $x_1 < \cdots < x_n$ has to be assumed here, otherwise $x_1 = \cdots = x_n = n$ provides a trivial solution. Evidently

$$x_n > \frac{n}{1 - e^{-1}}$$

(for sufficiently large n) which follows from

$$\begin{aligned} 1 &= \frac{1}{x_1} + \cdots + \frac{1}{x_n} > \frac{1}{x_n} + \frac{1}{x_n - 1} + \cdots + \frac{1}{x_n - n + 1} \\ &= \log x_n - \log(x_n - n) - O(n^{-1}). \end{aligned}$$

Write $z_n = \min x_n$. I am reasonably sure that

$$z_n < (1 + o(1)) \frac{n}{1 - e^{-1}}, \tag{2}$$

but (2), if true, will not be easy to prove; I think even $z_n = O(n)$ will be very hard. (See the tables for small n given in the Editor's comment below—Ed.) The bound $z_n < n(\log n)^\alpha$ may follow by the methods of two papers of M. N. Bleicher and myself (*Denominators of Egyptian fractions I and II*, to appear in J. Number Theory).

One could also try to investigate $\max x_i$ and $\min x_i$ for $i < n$. Certainly $\min x_1 = 2$ is trivial and $\max x_1$ can be determined asymptotically as follows: assuming $x_1 < \cdots < x_n$ to avoid trivialities, we know

$$\begin{aligned} 1 &= \frac{1}{x_1} + \cdots + \frac{1}{x_n} < \frac{1}{x_1} + \frac{1}{x_1 + 1} + \cdots + \frac{1}{x_1 + n - 1} \\ &= \log(x_1 + n) - \log x_1 + O(n^{-1}), \end{aligned} \tag{3}$$

so that writing $y_1 = \max x_1$, we have for sufficiently large n

$$y_1 < \frac{n}{e - 1}.$$

In fact, I can show that for any $\varepsilon > 0$ there exist $x_1 < \cdots < x_n$ with

$$x_1 > (1 - \varepsilon) \frac{n}{e - 1}.$$

This follows from an argument similar to the one given by Straus and myself in our solution to Problem E 2232, this MONTHLY 78 (1971), 302–303. The exact determination of $\max x_1$ except for small values of n seems hopeless to me. For $k > 1$, $\max x_k$ can be estimated fairly accurately, too.

As for $\max x_{n-1}$, it follows fairly easily from my Hungarian paper that $\max x_{n-1} = 2y_{n-1}$—but only if $x_{n-1} = x_n$ is permitted. If we insist on $x_1 < \cdots < x_{n-1} < x_n$, then I think that $\max x_{n-1} = \frac{3}{2} y_{n-1}$ can be proved without too much trouble.

Editor's comment. Jane Evans, M. G. Greening (Australia), Helen Marston, and the proposer all discovered formula (1) above, although none had a convincing proof. The proposer noted that by induction, $y_n \leqq 2^t$ where $t = 2^{n-1} - 1$, and Arthur Marshall pointed out that if n is not of the form $m(m-1)$, then $1 = 1/2 + 1/6 + \cdots + 1/n(n-1) + 1/n$ so that $z_n \leqq n(n-1)$.

The following list of sums was submitted by E. P. Starke, who admitted that he has no general procedure and does not know if all of the sums are best possible.

Let z_n be the denominator of the desired least-valued fraction. Then

$$z_3 = 6,\ z_4 = 12, \text{ and}$$
$$z_5 \leqq 15,\ z_6 \leqq 15,\ z_7 \leqq 18,\ z_8 \leqq 20,\ z_9 \leqq 24,$$
$$z_{10} \leqq 24,\ z_{11} \leqq 35,\ z_{12} \leqq 36, \text{ etc.}$$

These figures correspond to the sets of unit fractions having the denominators shown:

$n = 5$:	2, 4, 10, 12, 15
$n = 6$:	3, 4, 6, 10, 12, 15
$n = 7$:	3, 4, 9, 10, 12, 15, 18
$n = 8$:	4, 5, 6, 9, 10, 15, 18, 20
$n = 9$:	4, 5, 8, 9, 10, 15, 18, 20, 24
$n = 10$:	5, 6, 8, 9, 10, 12, 15, 18, 20, 24
$n = 11$:	5, 6, 8, 9, 10, 14, 15, 18, 24, 30, 35
$n = 12$:	4, 6, 8, 12, 14, 15, 18, 20, 24, 30, 35, 36.

An Inequality of Statistical Interest

E 2428 [1973, 807]. *Proposed by M. S. Klamkin, Ford Motor Company*

If a_i $(i = 1, 2, \cdots, n)$ denote real numbers, show that

$$n \min(a_i) \leqq \Sigma a_i - S \leqq \Sigma a_i + S \leqq n \max(a_i),$$

where

$$(n-1)S^2 = \sum_{1 \leqq i < j \leqq n} (a_i - a_j)^2 \qquad (S \geqq 0)$$

and with equality if and only if $a_i = \text{constant}$.

I. *Solution by Ellen Hertz, Bronx Community College; Carolyn MacDonald, University of Missouri; Wolfe Snow, Brooklyn College; and Melvin Tews, University of California, Berkeley (independently).*

We can assume that $a_1 \leqq a_2 \leqq \cdots \leqq a_n$. Then

$$S^2 = \frac{1}{n-1} \sum_{i=2}^{n} \sum_{j=1}^{i-1} (a_i - a_j)^2 \leqq \frac{1}{n-1} \sum_{i=2}^{n} (i-1)(a_i - a_1)^2$$

$$\leqq \sum_{i=2}^{n} (a_i - a_1)^2 \leqq \left\{ \sum_{i=1}^{n} (a_i - a_1) \right\}^2.$$

Taking square roots we obtain

$$na_1 \leqq \sum_{i=1}^{n} a_i - S.$$

Similarly,

$$S^2 \leqq \frac{1}{n-1} \sum_{j=1}^{n-1} (n-j)(a_n - a_j)^2 \leqq \left\{ \sum_{j=1}^{n} (a_n - a_j) \right\}^2$$

from which follows that

$$\sum_{j=1}^{n} a_n + S \leqq na_n.$$

It is clear that equality holds anywhere if and only if it holds throughout and this is true if and only if $a_i =$ constant.

II. *Comment by O. P. Lossers, Technological University, Eindhoven, the Netherlands.* A statistical interpretation is possible. Let $a_1, a_2, \cdots, a_n$ $(n \geqq 2)$ be a random sample of a random variable A with mean μ and variance σ^2. As estimates for μ and σ^2 one usually takes

$$\bar{a} = \frac{1}{n} \sum_{i=1}^{n} a_i, \qquad s^2 = \frac{1}{n-1} \sum_{i=1}^{n} (a_i - \bar{a})^2.$$

The inequalities in the problem then take the form

$$(*) \qquad \min a_i \leqq \bar{a} - \frac{s}{\sqrt{n}} \leqq \bar{a} + \frac{s}{\sqrt{n}} \leqq \max a_i.$$

Note that if A_n is the random variable defined by averaging samples of size n from A, then the mean of A_n is also μ, but its variance is σ^2/n, so that quantities in (*) are related to the parameters of A_n.

Also solved by M. T. Bird, Brother Alfred Brousseau, Frederick Carty, Lawrence Dickson (Australia), Jane Evans, P. K. Garlick, Clark Givens, M. G. Greening (Australia), Ignace Kolodner, Robert Kopp, L. Kuipers, O. P. Lossers (Netherlands), L. E. Mattics, M. R. Modak (India), V. N. Murty (Indonesia), Radha Nath & Woon-Chung Lam, Robert Patenaude, Edward Rozema, Oto Strauch (Czechoslovakia), Temple University Problem Solving Group, George Tsintsifas (Greece), and the proposer.

The proposer notes that the case $n = 3$ is due to D. S. Mitrinović, *Analytic Inequalities*, Springer-Verlag, Heidelberg, 1970, p. 215.

A Generalization of the Euler-Fermat Theorem

E 2430 [1973, 808]. *Proposed by John Masley, University of Illinois at Chicago Circle*

Let a and m denote natural numbers, and let ϕ denote Euler's totient function. Euler's generalization of Fermat's "Little Theorem" asserts that if $(a, m) = 1$, then

$$(*) \qquad a^{\phi(m)+1} \equiv a \pmod{m}.$$

Show that (*) holds if and only if the following: If p is a prime that divides a, then $p^k \mid a$ whenever $p^k \mid m$.

Solution by G. A. Heuer, Mathematisches Institut der Universität zu Köln, Germany. First assume the proposed condition holds. Let p^e be any maximal prime power factor of m (i.e., $p^e \mid m$ but $p^{e+1} \nmid m$). If $p \mid a$ we have $p^e \mid a$. If $p \nmid a$, then $a^{\phi(p^e)} \equiv 1 \pmod{p^e}$, and since $\phi(p^e) \mid \phi(m)$ we have $a^{\phi(m)} \equiv 1 \pmod{p^e}$. Thus $p^e \mid a(a^{\phi(m)} - 1)$ in each case. Hence $m \mid a(a^{\phi(m)} - 1)$.

Conversely, if $p \mid a$ while some $p^k \mid m$ but $p^k \nmid a$, then $p^k \nmid a(a^{\phi(m)} - 1)$, since obviously $p \nmid (a^{\phi(m)} - 1)$. Thus $m \nmid a(a^{\phi(m)} - 1)$.

Also solved by D. M. Bloom, W. E. Bodden, S. C. Currier, Jr., L. J. Dickson (Australia), H. M. Edgar, E. W. Ewing, P. K. Garlick, R. A. Gibbs & Harold Stocker, Emil Grosswald, L. Kuipers, Eric Langford, P. W. Lindstrom, O. P. Lossers (Netherlands), Merry McDonald & Gary McDonald, M. R. Modak (India), Catherine Murphy, Alayne Parson, David Pickard, Bob Prielipp, W. J. Sánchez, Alan Stein, Temple University Problem Solving Group, E. T. H. Wang, J. A. Wehlen, Charles Wexler, and the proposer.

Editor's Comment: The interested reader is referred to Roger Osborne, *A good generalization of the Euler-Fermat Theorem*, Math. Mag. 47 (1974), 28–31; another generalization (with bibliography) is given by David Singmaster, *A maximal generalization of Fermat's Theorem*, Math. Mag. 39 (1966), 103–107.

ADVANCED PROBLEMS

All solutions of Advanced Problems should be sent to J. Barlaz, Rutgers — The State University, New Brunswick, N. J. 08903. Solutions of Advanced Problems in this issue should be typed (with double spacing) on separate, signed sheets and should be mailed before December 31, 1974.

An asterisk (*) means neither the proposer nor the editors supplied a solution.

5982. *Proposed by R. L. Bishop, University of Illinois*

Let A be a (not necessarily associative) finite dimensional algebra over a field K. If $x_1, \cdots, x_p$ are indeterminates and w is a nonassociative word in them, then for each $a_1, \cdots, a_p$ in A we get $w(a_1, \cdots, a_p) \in A$ by substituting $x_i = a_i$. Prove that there are words $w_1, \cdots, w_q$ such that for every $a_1, \cdots, a_p$ the algebra generated (using products and linear combinations over K) by $a_1, \cdots, a_p$ is the linear span of $w_i(a_1, \cdots, a_p)$, $i = 1, \cdots, q$.

5983*. *Proposed by H. D. Ruderman, Hunter College Campus School*

Let $\| x \|$ be the distance of x to the nearest integer, i.e.,

$$\| x \| = \min\{x - [x],\ 1 + [x] - x\}$$

for real x. For positive integers N find the greatest lower bound for $N \cdot \| N\pi \|$ and for $N \cdot \| N\sqrt{2} \|$.

5984*. *Proposed by Clark Kimberling, University of Evansville*

Suppose $\{x_i\}$ and $\{y_i\}$ are sequences of positive real numbers such that $\lim_{n\to\infty} \sum_{i=0}^{n} x_i x_{n-i} = 0$ and $\lim_{n\to\infty} \sum_{i=0}^{n} y_i y_{n-i} = 0$. Do we have $\lim_{n\to\infty} \sum_{i=0}^{n} x_i y_{n-i} = 0$? (The proposal is equivalent to a question about convolutions

$$f * g(x) = \int_0^x f(t) g(x-t)\, dt$$

of positive-valued functions f and g defined on $[0, \infty)$; namely, if $f * f(x) \to 0$ and $g * g(x) \to 0$, then must $f * g(x) \to 0$?)

5985*. *Proposed by Siemion Fajtlowicz, University of Houston*

Let f be a continuous function of the plane E into itself. Does there exist a closed proper subset X of E such that $f(X) \subseteq X$?

5986*. *Proposed by D. E. Daykin, Reading University, England*

Let E be the real Euclidean plane and $0 < \alpha < 1$. What can be said about maps $f: E \to E$ which send each triangle T into a triangle fT with area $(fT) \leqq \alpha$ area (T)?

5987. *Proposed by H. Kestelman, University College, London, England*

If $\{n_r\}$ is an increasing sequence of positive integers, prove that $\prod_{r=1}^{\infty} \cos n_r x = 0$ for all real x apart from a set of Lebesgue measure zero. Can the exceptional set be uncountable?

SOLUTIONS OF ADVANCED PROBLEMS

Finite Hausdorff Spaces

5912 [1973, 564]. *Proposed by R. B. Kirk, Southern Illinois University*

Let X be a compact Hausdorff space, and let C denote the space of continuous functions on X. Assume that C can be written as a countable union of equicontinuous sets. Prove that X is finite.

Solution by T. S. Bolis, State University College, Oneonta, New York. The space C with the uniform norm is a Banach space and hence a Baire space. Taking intersections with balls of integral radius if necessary, we may assume that C is the union of countably many equicontinuous bounded and closed sets (the closure of an equicontinuous set is equicontinuous). These sets are compact by Ascoli's Theorem.

Since C is a Baire space, one of these sets must contain a ball. Therefore C is finite dimensional and since X is Hausdorff, it must be finite.

Also solved by J. B. Conway, George Crofts, G. Cybenko, J. A. Guthrie, R. E. Hodel, M. J. Hoffman, P. D. Humke, A. A. Jagers (Netherlands), E. M. Klein, R. J. Loy (Australia), Arunava Mukherjea & J. R. Gard, S. A. Naimpally & P. L. Sharma, Lewis Pakula, M. Bhaskara Rao (England), R. F. Wheeler, J. B. Wilker, David Wong (Jamaica), and the proposer.

Editor's Note. Wheeler with his solution raises another question: When does C contain a countable family of equicontinuous sets whose union is dense in C (with the sup norm topology)? His answer: only when X is metrizable.

'Area' of a 'Triangle' in Complex Hilbert Space

5913 [1973, 564]. *Proposed by D. E. Daykin and J. K. Dugdale, University of Reading, England*

Let H be a real or complex Hilbert space and let x, y, z be points in H. We call $\langle x, y, z\rangle$ a triangle. As usual

$$L(y,z) = \{y + \alpha(y-z) : \alpha \text{ a scalar}\}$$

is the line through y and z; and the distance of x from $L(y,z)$ is

$$p(x, L(y,z)) = \inf\{\|x-w\| : w \in L(y,z)\}.$$

For convenience put $a = \|x-y\|$, $b = \|y-z\|$, $c = \|z-x\|$ and

$$s = \tfrac{1}{2}(a+b+c).$$

Euclidean geometry suggests two definitions

$$A_1 = \tfrac{1}{2}p(x, L(y,z))\,\|y-z\|$$

and

$$A_2 = \sqrt{s(s-a)(s-b)(s-c)}$$

for the area of triangle $\langle x,y,z\rangle$. Compare the values of A_1 and A_2 and determine when they are equal.

Solution by J. R. Kuttler, The Johns Hopkins Applied Physics Laboratory. Initially, let $z = 0$. Then straightforward computation shows

$$A_1 = \tfrac{1}{2}\sqrt{\|x\|^2\,\|y\|^2 - |(x,y)|^2}$$

$$A_2 = \tfrac{1}{2}\sqrt{\|x\|^2\,\|y\|^2 - [\mathrm{Re}(x,y)]^2}$$

where $(\ ,\ )$ is the inner product of H. Clearly

$$A_1 \leqq A_2$$

with equality if and only if (x, y) is real. The general case can be reduced to the special by a translation by $-z$, and the condition for equality becomes $(x-z,\ y-z)$ real.

Also solved by D. Ž. Djoković, O. P. Lossers (Netherlands), Oto Strauch (Czechoslovakia), Phil Tracy, J. B. Wilker, and the proposers.

An Analytic Inequality

5914 [1973, 564]. *Proposed by D. E. Daykin, C. E. Linderholm, and Albert Wilansky, University of Reading, England*

Show that if $A = \{z_1, z_2, \cdots, z_n\}$ is a finite set of n complex numbers, there is a subset B of A such that

$$\left| \sum_{z \in B} z \right| \geqq \pi^{-1} \sum_{i=1}^{n} |z_i|.$$

I. *Solution by H. E. Debrunner, University of Bern, Switzerland.* Reorder the $n+1$ numbers $z_1, z_2, \cdots, z_n, -(z_1 + z_2 + \cdots + z_n)$ to a sequence $w_1, \cdots, w_{n+1}$ in such a way that the corresponding sequence $\arg w_1, \cdots, \arg w_{n+1}$ (where we put $\arg w = 0$ if $w = 0$) is monotone. In the complex plane the partial sums $w_1,\ w_1 + w_2, \cdots, w_1 + \cdots + w_{n+1}$ then form the vertices of an oriented convex polygon P of which $w_1, \cdots, w_{n+1}$ are the side vectors. The diameter d of P is realized as the distance between two vertices of P and can be written as $d = \left| \sum_{i \in B} z_i \right|$ for some subset B of $\{1, \cdots, n\}$; for the perimeter p of P we get $p = \sum |w_i| = \sum_i^n |z_i| + \left| \sum_i^n z_i \right|$. The assertion results if we substitute these values in the well known inequality $\pi \cdot d(Q) \geqq p(Q)$ which is satisfied by the diameter $d(Q)$ and the perimeter $p(Q)$ of any compact convex figure Q of the plane (cf. Bonnesen-Fenchel, *Theorie der konvexen Körper*, p. 77).

If in place of $\pi \cdot d \geqq p$ we use Reinhardt's result (Jber. Deutsch. Math.-Ver. 31 (1922), 251–270) that the maximum of $p(Q)/d(Q)$ as Q varies through convex polygons with $2k+1$ vertices is realized by the regular $(2k+1)$-gons, we get the stronger assertion: for some $B \subset \{1, \cdots, n\}$ we have

$$\left| \sum_{i \in B} z_i \right| \geqq \frac{\sin (k\pi/(2k+1))}{(2k+1)\sin(\pi/(2k+1))} \left(\sum_{i=1}^{n} |z_i| + \left| \sum_{i=1}^{n} z_i \right| \right),$$

where $k = [\frac{1}{2}(n+1)]$.

II. *Solution by H. S. Witsenhausen, Bell Laboratories, Murray Hill, N. J.* The problem may be considered as finding the infimum over all $n > 0$ and all non-vanishing sequences $z_1, \cdots, z_n$ of vectors in E^2 of the maximum over $\theta_i \in \{0, 1\}$ (equivalently $[0, 1]$) of

$$\left\| \sum_{i=1}^{n} \theta_i z_i \right\| \Big/ \sum_{i=1}^{n} \| z_i \|.$$

This however is the definition of the Macphail number of the space E^2. (See M. S. Macphail, *Absolute and unconditional convergence*, Bull. Amer. Math. Soc., 53(1947), pp. 121–123). That the Macphail number of E^n is $2^{-1}\pi^{-\frac{1}{2}}\Gamma(n/2)/\Gamma((n+1)/2)$ is contained in the results of A. Mayer (*Grösste Polygone mit gegebenen Seitenvektoren*, Comm. Math. Helv., 10 (1938), pp. 288–301). For more general normed spaces see, e.g., Witsenhausen, *Some games on convex bodies and related inequalities*, J. of Math. Anal. and Appl., 40 (1972), pp. 79–106, where further references may be found.

Also solved by Ralph Boas, E. D. Bolker, Robert Breusch, R. G. Buschman, P. R. Chernoff, J. E. Connett, D. Ž. Djoković, M. S. Klamkin, E. M. Klein, Douglas Lind, O. P. Lossers (Netherlands), L. E. Mattics, D. S. Mitrinović (Yugoslavia), M. R. Murty & V. K. Murty, D. B. Price, Kenneth Rosen, K. A. Ross, and the proposers.

Editor's Note. Additional references provided for the problem are

(1) D. S. Mitrinović, Analytic Inequalities, p. 331.

(2) W. W. Bledsoe, An inequality about complex numbers, this MONTHLY, 77 (1970) 180–183.

(3) E. D. Bolker, A class of convex bodies, Trans. Amer. Math. Soc., 145 (1969) 323–345 (Section 4).

(4) R. P. Kaufman and N. W. Rickert, An inequality concerning measures, Bull. Amer. Math. Soc., 72 (1966) 672–676.

(5) Compare also Problem E 2373 [1973, 1139–1140].

Bound for $\Pi(1 - 1/p)$

5915 [1973, 595]. *Proposed by D. M. Battany, Oceanside, California*

Let p_n be the nth prime. Show that

$$\prod_{p \leqq p_1 \cdots p_n} \left(1 - \frac{1}{p}\right) \leqq \frac{1}{p_n}$$

for all prime p, or isolate the exceptional values.

Solution by P. T. Bateman, University of Illinois. Let $\zeta(x) = \sum_{p \leqq x} \log p$. By a theorem of Mertens and the prime number theorem we have

$$\prod_{p \leqq p_1 \cdots p_n} \left(1 - \frac{1}{p}\right) \sim \frac{e^{-\gamma}}{\log(p_1 \cdots p_n)} = \frac{e^{-\gamma}}{\zeta(p_n)} \sim \frac{e^{-\gamma}}{p_n},$$

where γ is Euler's constant. Thus the desired result holds for sufficiently large n. To obtain the result for all n we need some results from the paper, *Approximate formulas for some functions of prime numbers*, by J. B. Rosser and L. Schoenfeld, Illinois J. Math. 6(1962), 64–94. By Theorem 7 of that paper we have

$$\prod_{p \leqq x} \left(1 - \frac{1}{p}\right) < \frac{e^{-\gamma}}{\log x} \left(1 + \frac{1}{2\log^2 x}\right) < \frac{e^{-\gamma}}{\log x} \left(1 + \frac{1}{20}\right)$$

for all $x \geqq 30$. From Theorem 10 of the same paper we can get, after some calculation,

$$\zeta(p_n) > \frac{3}{5} p_n, \qquad n \geqq 3.$$

Thus

$$\prod_{p < p_1 \cdots p_n} \left(1 - \frac{1}{p}\right) < \frac{e^{-\gamma}}{\zeta(p_n)} \frac{21}{20} < \frac{7e^{-\gamma}}{4} \frac{1}{p_n} < \frac{1}{p_n}$$

for $n \geqq 3$. The proposed inequality is easily verified for $n = 1$ and $n = 2$ and therefore holds for all n.

Also solved by Robert Breusch, J. R. Kuttler, Arthur Marshall, L. E. Mattics, C. A. Nichol, David Seff, J. H. van Lint (Netherlands), S. S. Wagstaff, Jr., and the proposer.

Collections of Ordered Subsets

5918 [1973, 697]. *Proposed by S. W. Golomb, University of Southern California*

Prove or disprove: Let C be a collection of distinct subsets of the positive integers which are totally ordered by the inclusion relation. Clearly C must be either a finite or a countable collection.

Solution by Robert Gilmer, Florida State University. The collection C need not be finite or countable. For a proof, let f be a one-to-one correspondence between the set N of positive integers and the set of rational numbers. For each real number t, let $S_t = \{n \in N \mid f(n) < t\}$; $C = \{S_t\}_{t \in R}$ is an uncountable collection of distinct subsets of N totally ordered by inclusion.

Results of Reinhold Baer in *Dichte, Archimedezität und Starrheit geordneter Körper*, Math. Ann. 188(1970), 185–205, yield the following result.

If α is an infinite cardinal, then there is a totally ordered field F of cardinality α that can be imbedded as a dense (in the order topology) *ordered subfield in a totally ordered field F^* of cardinality 2^α.*

The preceding result, in turn, can be used to obtain the following theorem related to Problem 5918.

If X is an infinite set of cardinality α, then there is a collection C of distinct subsets of X such that C is totally ordered by inclusion and has cardinality 2^α.

Also solved by J. T. Annulis, James Bookey, R. W. Brooks, Jane Buddenhagen, P. W. Carlton, H. D. Carroll, D. Ž. Djoković, E. W. Ewing, James Fickett, Daniel Gallin, William Glassmire, F. D. Hammer, Melvin Henriksen, Ellen Hertz, G. A. Heuer (Germany), R. E. Hodel, J. P. Jones, J. E. Keesling & Nicholas Passell, K. O. Leland, P. W. Lindstrom, L. E. Mattics, Ka Menehune, J. G. Michaels, J. C. Morgan II, Masakazu Muro (Japan), Donald Plank, Jürg Rätz (Switzerland), P. L. Renz, Nina M. Roy, Eugene Sadowski, John H. Smith, S. K. Stein, Oto Strauch (Czechoslovakia), Herbert Taylor, Albert Wilansky, Thomas Wray, and the proposer.

(1) Hammer notes that the problem appears in Grätzer's *Lattice Theory*, # 28, p. 117. Henriksen notes that the problem is solved by Sierpinski in *Cardinal and Ordinal Numbers*, p. 82, and in an earlier paper. Another version is given on p. 37 (# 104) of Wilansky, *Topology for Analysis*. (2) Menehune notes that the problem and solution are similar to Problem 5622 [1969, 836], wherein a change from "simply ordered" to "well ordered" made the statement of 5622 true; likewise with the present 5918.

REVIEWS

EDITED BY J. ARTHUR SEEBACH, JR. AND LYNN A. STEEN

with the assistance of the mathematics departments of St. Olaf and Carleton Colleges

COLLABORATING EDITOR FOR FILMS: SEYMOUR SCHUSTER, CARLETON COLLEGE

We invite readers to submit reviews of significant recent college-level mathematics books. We especially encourage reviews based on classroom use, or comparative reviews of several related books. Reviews should ordinarily not exceed two pages (per book) typed double spaced. Manuscripts of reviews as well as books submitted for review should be sent to: Book Review Editor, American Mathematical Monthly, St. Olaf College, Northfield MN 55057.

A First Undergraduate Course in Abstract Algebra. By Abraham P. Hillman and Gerald L. Alexanderson. Wadsworth, Belmont, California, 1973. xiii + 413 pp. $12.95.

The prospect of teaching a first course in algebra to a mixed class of high school teachers on summer vacation and ambitious science students hoping to get done with their math credits a little faster is disheartening, to say the least. One expects a basic lack of enthusiasm, complicated by the summer doldrums, so I selected this text largely for its brief biographies of famous men and women of mathematics and its highly attractive appearance. It was a marvelous surprise to find that the many exercises are so interesting and so finely graded that when my students came to the problems labelled "Challenging and Supplementary," they not only did the ones I assigned them, but also tried many of the rest. The authors never condescend in their exposition, and have avoided the old-fashioned and irritating forms "he" and "himself" when referring to the reader, so the book is very pleasant to read. Moreover, the exposition is so clear and direct that several students read the material on extension fields by themselves, even though I had skipped it in favor of Euclidean constructions.

The authors list several possible one-semester goals for their book. One such is the Euclidean construction goal which covers the following material: basic properties of the integers such as the Euclidean algorithm; groups, which begins with permutation groups, proceeds to groups of symmetries and abstract groups, and covers cosets, the alternating group, and quotient groups; mappings and homomorphisms: only the bare essentials are recommended by the authors, but they include Cayley's theorem, direct product, and even the fundamental theorem of Abelian groups; rings and ideals, where the relationship between ideals and homomorphisms is brought out and integral domains and fields are introduced (the chapter contains much more); polynomials (over a commutative ring), through integral and rational roots, splitting, partial fractions, extension fields, and the insolvability of the quintic; and finally Euclidean constructions, which describes

closure under Euclidean construction, closure under square roots and finally constructible points, lines and circles.

Each chapter has a set of review problems and a set of supplementary and challenging problems which my students found very invigorating.

In sum, this text presents mathematics in such a way as to have convinced my students that mathematics is alive, vivid, and within their reach.

JUDITH Q. LONGYEAR, Dartmouth College

Modern Geometries. By James R. Smart. Brooks/Cole, Monterey, California, 1973. viii + 371 pp. $9.95. (Telegraphic Review, November 1973.)

Judging from the number of new geometry books published during the past five years, there appears to be a revival of interest in geometry at the undergraduate level. This interest is partly a response to the need for new and better ways of teaching geometry in the secondary schools, but it is also a recognition that better geometric insight will be beneficial to all persons who use mathematics. It has been said that geometry is the result of a particular way of thinking, rather than a well-defined body of knowledge, and a student of mathematics discovers that the concepts and methods of geometry permeate every branch of mathematics.

Smart's book is a survey of a number of facets of modern mathematics which can be designated as geometry. The table of contents indicates the main topics, which are: finite geometries, geometric transformations, convexity, advanced Euclidean geometry, constructions, inversion, projective geometry, geometric topology, and non-Euclidean geometry. The chapters are largely independent of each other, so that the instructor can organize the content to fit mathematical interests and the time available. For example, three different professors at Miami University have used this textbook in a one-quarter course, and each has presented a different subset of the topics. Seldom is this kind of flexibility found in a mathematics text, and it is especially welcome in an area whose subject matter is so diverse.

While the content and organization of this book are commendable, the text itself leaves something to be desired. For example, many definitions are stated informally in the course of an expository paragraph. The resulting lack of precision causes considerable confusion for some students. The proofs of most theorems are also given in an informal, conversational style. This is certainly acceptable, but some proofs appear before the statement of the theorem, some immediately after, some after an intervening paragraph, and others not at all. This inconsistency, coupled with some "hand-waving" and the author's failure to designate the beginning or ending of a proof, frequently leaves the reader uncertain as to just what has been proved.

There are a number of errors in the text, most of them of a typographical nature. The chapter on convexity seems to be especially "error prone." There is an occasional blunder, such as in the proof of Helly's theorem in 3-space. Here the author uses

mathematical induction on the number of sets, N, beginning with $N = 4$. In the proof, however, he assumes the truth of the statement for $N = 5$. That this follows from the truth of the statement for $N = 4$ requires a nontrivial lemma, which the author has overlooked.

The book contains a large number of easy exercises, which help the students to sharpen their understanding of definitions and the consequences of theorems. However, there is a need for more non-trivial exercises which will challenge the students at higher levels of conceptual understanding.

When students who had already taken calculus and linear algebra used *Modern Geometries* as a textbook, they rated it "good to average" as far as format, content, exercises, and usefulness for self-study were concerned. Expository style drew an "average" rating, and the examples were considered "average to poor." These students perceived the book's greatest strength to lie in the fact that it is easy to read and covers many interesting topics. Weaknesses noted were: vague or superficial explanations, unclear proofs, and not enough details or examples.

This book can still be recommended for its flexibility and breadth, but the instructor should be prepared to supplement it in those areas where it is deficient.

D. E. KULLMAN, Miami University

Fundamentals of Abstract Algebra. By Neal H. McCoy. Allyn and Bacon, Boston, Mass. 1972. x + 470 pp. $11.95. (Telegraphic Review, October 1972.)

First Course in Algebra and Number Theory. By Edwin Weiss. Academic Press, New York, 1971. xi + 547 pp. $12.95. (Telegraphic Review, November 1971.)

McCoy is an expansion of an earlier book by the same author to make it more suitable for use in a year course for average students at the junior-senior level. *First Course in Algebra and Number Theory* is designed to serve as a text in a year long introductory course for freshmen and sophomores. Each author succeeds in writing at the level of his intended readers.

Fundamentals of Abstract Algebra expands the author's 1968 book *Introduction to Modern Algebra.* Since almost all of the material in the previous book appears intact, *Fundamentals of Abstract Algebra* retains a classical approach. Rings, integral domains, and fields are considered first and are carefully tied to familiar number systems.

A major change is the addition of four new chapters on special topics. Two new chapters on the structure of finite groups appear after basic group theory. One presents the decomposition of a finite abelian group as a direct sum of cyclic subgroups of prime power order while a second gives rather elegant proofs of the Sylow theorems using double cosets. Another new chapter, this one on field extensions, follows the material on vector spaces and breaks up the four linear algebra chapters. The discussion here extends through uniqueness of splitting fields. The fourth new chapter is at the end of the book and provides a brief glimpse into such topics as quaternions, principal ideal domains, modules, and Zorn's lemma.

The added chapters are concise; main ideas are presented succinctly, few examples are given, and exercise lists are short. Students for whom the rest of the book is appropriate may well find this material quite difficult, and the instructor who decides to include it should be prepared to motivate, explain, and expand.

A minor criticism: the exercise sets throughout McCoy's book could be enriched, particularly with exercises of a more challenging nature. Again the instructor should be prepared to give extra problems to the best students.

The two editions of McCoy's *Introduction to Modern Algebra* have enjoyed rather wide adoption, particularly at smaller schools offering only one semester of abstract algebra during the junior or senior year. The additions make this new book much more useful for longer courses taught at the same level of sophistication. This level is intentionally below that of I.N. Herstein's *Topics in Algebra.* In summary, *Fundamentals of Abstract Algebra* warrants serious consideration for use in a course where Herstein's book would be too difficult.

In the preface to *First Course in Algebra and Number Theory*, Weiss argues that modern algebra should be offered at the freshman-sophomore level in place of linear algebra. He contends that topics in abstract algebra can be chosen and presented in a way more suited to the mathematical sophistication of such students. Weiss, remembering his proposed audience, takes great pains to write with care, precision, and thoroughness. Indeed, his book is one of the most carefully written books this reviewer has ever studied. Concepts are often rephrased several times for greater clarity, new material is carefully related to the preceding, and many theorems are proved in more than one way. The result is a slow-moving, detailed presentation with no ambiguities.

Again with an eye toward the intended reader, Weiss concentrates on a narrow range of topics with number theoretic origins. There are only four chapters in this book, each of considerable length. Chapter 1 contains standard topics in elementary number theory. Chapter 2, whose main goal is to present the integers as the only well-ordered domain, presents a very careful formal axiomatic approach to rings and integral domains. Chapter 3 considers polynomial rings in general and looks at the specific problem of solving polynomial equations with coefficients in Z_m. Chapter 4 is a study of groups, presented mostly as pure algebra with some remaining number theoretic influence.

Each chapter begins a new topic and progresses to more advanced material. Accordingly, I would expect many students to have difficulty with the sections on quadratic reciprocity and the group Z_m^* of units in Z_m, the last sections in Chapters 3 and 4, respectively.

A strong feature of the text is the extensive list of problems at the end of every section. Also, each chapter closes with a collection of challenging miscellaneous exercises (237 in all). Many of these problems introduce concepts not treated in the main body of the text and could serve as a vehicle for independent study by the best students.

While Weiss' book is very well suited to its intended audience, smaller schools do not generally offer an abstract algebra course at the freshman-sophomore level, their curricula being filled with courses either of a more elementary nature or with greater application to other disciplines. To use the text in an introductory course for juniors and seniors would involve a decision to treat a narrow range of subjects in some depth. Topics often included in such a course but omitted from this book include: structure of finite groups and the Sylow theorems, ideals (these are defined in the miscellaneous exercises), fields, vector spaces, homological algebra, and categories.

The material in Weiss is particularly applicable to classes of prospective or current elementary and secondary teachers. For example, the basic number theory in Chapter 1 should be thoroughly understood by anyone teaching at those levels.

To summarize, the quality of writing makes *First Course in Algebra and Number Theory* a book strongly recommended to any instructor teaching a course for which it is appropriate.

CHARLES JEPSEN, Grinnell College

The Fascination of Groups. By F. J. Budden. Cambridge University Press, New York, N.Y. 1972. xviii + 596 pp. $19.50. (Telegraphic Review, January 1973.)

This book was designed to provide background material for teachers and supplementary material for university students. It covers only as much actual group theory as would occupy about thirty pages of a standard text. The fact that it takes the author over 500 pages to reach the same point indicates the extreme leisureliness of the development. He devotes a whole chapter to each of the group axioms and does not define a group until page 73; he savors each individual group to the fullest, in one case spending 13 pages on a single group.

After defining a group he gives examples, both finite and infinite, and discusses subgroups, cosets, isomorphisms, normal subgroups, quotient groups, homomorphisms (though omitting even the *First Isomorphism Theorem*), direct products, generators and relations and automorphisms. There is a wealth of illustrative and motivational material backed up by an ample bibliography. The author shows how groups arise in many unexpected places. He includes chapters on groups in geometry, in music (with a separate chapter on campanology) and in the symmetry of polyhedra and patterns. There are innumerable questions and exercises with answers, largely dealing with specific finite or infinite groups.

In his preface the author states his intention "to interest, to enlighten and to transport the reader, rather than provide him with the strict discipline of a mathematical education." There is no doubt that he succeeds in being interesting or enlightening but unfortunately the strict mathematical discipline is lacking, in many places, to the extent that certain vital hypotheses are missing from the statements of theorems. Fermat's theorem is stated without the assumption of co-primeness. One exercise asks the reader to prove that "every Abelian group is the direct product of cyclic

groups" (unreasonably difficult without a lead-in, even in the finite case, for a book at this level), and another states that "the mapping $x \to x^2$ is an automorphism... if and only if the group is Abelian and contains no element of even period." The proof that A_5 is simple needs some adjustment since it assumes that the 24 5-cycles in A_5 form a single conjugacy class. The section which touches briefly on soluble groups and the solution of equations by radicals is somewhat confused with statements such as "the fact that A_2, A_3 and A_4 do allow themselves homomorphic images may be shown to account for the fact that quadratic, cubic and quartic equations may be solved...", "to solve the general equations of the nth degree it would be necessary to find a chain of subgroups $1 \lhd G_1 \lhd G_2 \cdots \lhd G_n$", and "... it is not even known whether the number of [non-Abelian simple] groups is finite." The convention that "subgroup" shall imply "non-trivial proper subgroup" is made but then frequently ignored.

Apart from such defects this book could be recommended not only for the beginner, but also for someone who wishes to widen rather than deepen his knowledge of group theory.

C. D. H. COOPER, Macquarie University, Australia

The Theoretical Side of Calculus. By Colin Clark. Wadsworth, Belmont, California, 1972. xv + 240 pp. $10.95. (Telegraphic Review, April 1973.)

This book solved my problem of choosing a text for a one-semester introduction to analysis course. Most existing books were over the heads of the students, too long to cover in a semester, or completely oriented towards applications. Clark's book seemed to avoid these pitfalls. The students were juniors and seniors, all of whom had had three semesters of primarily cookbook calculus, including basic multivariable calculus.

The preface indicates that Clark wrote his text to offer an "approach in which the theoretical part of elementary calculus is studied after the technical aspects of the subject have been mastered in an earlier course." The book begins with the real number system and proceeds to develop rigorously the classical theorems of basic calculus. The first chapter discusses sequences and the real number system. Attention is given to the growth properties of exponential sequences. Sequences are employed to discuss the Completeness Axiom and the uncountability of the reals, and a brief account of infinite series is given. Chapter 2 introduces limits, continuity, and differentiability, while Chapter 3 develops standard theorems about continuous and uniformly continuous functions. Chapter 4 covers the definition of the Riemann Integral, improper integrals, uniform convergence, and power series. Chapter 5 considers limits and continuity in n-space.

The book is very well written. Students found it to be quite readable, and I was able to concentrate on problems during class rather than constantly lecturing. Clark provides a great deal of numerical motivation (almost too much) for the main theorems. He finds space to insert some history of the development of the concepts

of calculus. I particularly like his inclusion of material that students are often assumed to know but have never had, such as his detailed study of suprema and infima.

The main fault I found with the book is the very hasty treatment of infinite series. Only a few of the standard convergence tests are given, and this necessitated much supplementation to the class. By way of atonement, however, an unusually clear treatment of Taylor's Theorem is presented.

The text is well suited for a one-semester introduction to analysis course, particularly if additional material on series can be given. It supplies the theoretical details customarily skipped in introductory calculus, and it provides enough new material to challenge the best students. I look forward to using the book again.

R. P. WEBBER, Longwood College

What Are Numbers. By Louis Auslander. Scott, Foresman and Co., Glenview, Illinois, 1969. 128 pp. $3.95.

This book is one that ought to be listed under the heading, *Caveat Studiosus.* According to the author, it is suitable for precalculus students as well as those in teacher-training programs or engineering courses. For any of these audiences the results well might be disastrous for four reasons, one of which is the source of the book's strength. Auslander presents one of the most beautiful constructive treatments of the real numbers that I have ever seen. The approach uses sequences of finite decimals to approximate real numbers (written as infinite decimals). This is the heart of the text and the subject of the middle chapter of this three chapter work.

The problem is that the treatment is not elementary. For example, using the approximating sequences α_n and β_n of positive finite decimals whose limits are the infinite decimals α and β, respectively, the author proves that if the distance between the sequences can be made arbitrarily small, then the real numbers α and β are the same (Theorem II. 1.14, p. 68). But to do this he first needs four lemmas and one theorem. The theorem and one lemma establish the result for those real numbers that have two different decimal forms. Theorem II. 1.14 is presented without proof, but with the suggestion that it should be done by induction using the argument of the previous lemma which is a case of the theorem. The other three lemmas establish the curious relationship of the distance between two finite decimals and the number of digits that the decimals have in common, i.e., if there are enough common digits, then the decimals are close to each other, but they may be close without having many common digits.

To prepare the student to cope, the first chapter should at least contain material dealing with induction and indexing sets. But neither topic appears. Instead, there is an abundance of subjects on a much lower level, enough to bore even the poor student. The sections on the geometry of inequalities, composition of mappings, the graph of a mapping and working with absolute values and inequalities (sections 1.4, 1.5, 1.6, 1.8), could be shortened considerably without taking anything away from the theory

of inequalities necessary to understand what happens when approximations are used instead of the "real" numbers.

In the final chapter, the author considers convergent sequences and series. The motivation is natural since a convergent sequence can be defined in terms of an approximating sequence. But from this point on leaps in reasoning abound. In about ten pages, Auslander covers the basic properties of convergent sequences and the evaluation of their limits, using methods which never really make clear the need to devise a suitable "ε-radius challenge, $N(\varepsilon)$ starting index response" routine. For example, in J. Chover's, *The Green Book of Calculus*, (Review this MONTHLY, October 1973), these ideas occupy the greater part of chapters 8 and 9, a total of about fifty pages.

Finally, the number of errors is astonishing. In the first hundred pages there are about two dozen errors, the majority typographical or incorrect citations of previous definitions and theorems. The errors in worked out examples and in the answer section only compound the problem.

This book was used with a class of sophomores most of whom had intentions of majoring in mathematics or computer science. I chose it because I thought the formal approach to real numbers would be a good introduction to the required abstract algebra courses of the upper division, while the discussions of how errors behave when finite decimals are used as approximations for real numbers are important for computer science courses. The better student emerged with a healthy skepticism for the printed word; the average student was confused and frustrated; the poor student simply turned off.

FRANCINE ÁBELES, Kean College of New Jersey

FILMS

Fundamental Mathematics: A Mixed-Media Program. By James Streeter and Gerald Alexander. Harper and Row, New York, N. Y., 1972. Unit 1 \$119.00, Unit 2 \$137.00, Unit 3 \$137.00, Unit 4 \$99.00. Study Guides \$2.35 — \$2.95 each.

Audio cassettes, 35 mm color film strips and multilithed study guides comprise this self-study program package covering standard pre-calculus algebra and trigonometry. The program is separated into four units: Fundamentals of Algebra, Intermediate Algebra, College Algebra and Trigonometry. The units are subdivided into modules each consisting of coordinated film-tape materials — about forty frames of film and about thirty minutes of unbroken running time — and a printed study guide.

New material and associated problems are introduced in the film-tape portions of the program with the study guides providing review, further exercises, practice tests (with solutions) and mastery tests (solutions separate). The latter measure how well objectives outlined at the beginning of each module are met. The objectives are realistic and include those topics calculus instructors wish their students to know.

Unit 1, Fundamentals of Algebra, in twelve modules, includes basic set terminology and notation, the natural numbers, arithmetic operations on the integers, linear equations, factoring, operating on fractions, the rectangular coordinate system, and solutions of linear equations.

Unit 2, Intermediate Algebra, in fourteen modules, covers the real numbers, polynomials and factoring, rational algebraic expressions, quadratics, relations and functions, inequalities, exponents and radicals.

Unit 3, College Algebra, in fourteen modules, reinforces the material of Units 1 and 2, and introduces new material on matrices, determinants, sequences, series, the binomial theorem, and the exponential and logarithmic functions.

Unit 4, Trigonometry, in ten modules, develops the circular functions via a wrapping of a number line around a unit circle and includes the usual graphs, identities and equations of trigonometry, some numerical solutions of triangles with applications to vectors, and the sine and cosine laws. The unit concludes with an introduction to complex numbers and the theorem of DeMoivre.

The film visuals are neatly done in two colors on a light green background. Problems appear on a yellow background with solutions on the following frame with a blue background. The first several and the last several frames of each module are color-matched to their plastic containers and corresponding study guides.

The advantage of the film strips over simply a profusely illustrated workbook eludes the reviewer. A large percentage of the visuals are merely displayed relations and equations. As an extreme example, the display on frame 17, Module 5, Unit IV is "Problem: Find the tangent of $25°22'$."

The narration on the tapes is smooth and clear. The pace is unhurried and is keyed to the film strip. The monotony of a single narrator is avoided by having three — one female and two males.

The study guides are well-produced summaries but not suitable as texts if used alone. Except for some minor inconsistencies in notation, no serious errors were detected.

This reviewer shares the authors' feeling that while the program is self-contained it is best used in conjunction with regular classroom sessions or conferences with instructors. Although the requirement of recorder and projector is judged an inconvenience, the film-tape combination permits the student to go back as often as necessary to points which are giving him difficulty.

The authors claim that student response to the program has been favorable and that classroom test results indicate improvement in the performance of students using the materials.

Institutions offering pre-calculus courses might consider augmenting them with this mixed-media program. The different units are available separately.

J. D. E. KONHAUSER, Macalester College

TELEGRAPHIC REVIEWS

Telegraphic reviews are designed to give prompt notice of new books with sufficient information to assist our readers in deciding whether to order an examination copy or to suggest library purchase. Possible uses are indicated as follows:

T = textbook P = professional reading
S = supplementary reading L = undergraduate library purchase
13 to 18 = freshman to second year graduate level usage
1 to 4 = appropriate time in semesters to cover text

Asterisks (*) or question marks(?) denote special positive or negative emphasis, respectively. Publishers are denoted by standard abbreviations; complete addresses may be found in *Books in Print*.

GENERAL, P*, L. *La Mathêmatique, Les Mathêmatiques, La Mathêmatique Moderne*. M.-L. Guerard des Lauriers. Doin, 1972, 184 pp, $8.30 (P). Discussion of the ends, essence and consequences for pedagogy of modern mathematics, especially the "new math." More philosophical in nature than Morris Kline's *Why Johnny Can't Add*. Despite strong criticism of aspects of *le bourbakisme* ("It reduces mathematics to an esoteric game"), the author concludes with the slogan *" A bas l'ensemblisme; et, alors, vive Bourbaki!"*. PJC

GENERAL, T(13: 1), S(13), L. *Number Systems*. Benjamin Bold, Alan Wayne. American Book, 1972, vi + 266 pp, $2.52 (P). A semi-formal, semi-axiomatic development of the real and complex number systems from the Peano postulates for the natural numbers, with frequent diversions into elementary number theory (e.g., congruences, primes) and abstract algebra (e.g., rings, matrices). Intended as a transition between intuitive high school and rigorous college mathematics. LCL

GENERAL, T(13-14), S*, P*, L. *Le Livre du Probleme*. IREM de Strasbourg. CEDIC, Paris, 1973. *Fascicule 1: Pedagogie De L'Exercices et du Probleme*, 100 pp, (P); *Fascicule 2: Exercices Elementaires de Geometrie Affine*, 93 pp, (P); *Fascicule 3: A Propos D'un Theme Mathematique: La Parite*, 53 pp, (P). Three pamphlets on teaching mathematics. No. 1 is an exposition of the underlying philosophy while Nos. 2 and 3 apply the principles of the first volume to affine geometry and the notion of parity. Written as a collective effort, this is an excellent series. Volume 1 should be required reading for any mathematics teacher; the other two volumes would be good for short (interim) courses. The level of the books is very wide-ranging. The French is easy, but a translation would be worthwhile. PJM

GENERAL, T*(13-15: 1), S*, P*, L. *Social Issues in Computing*. C.C. Gotlieb, A. Borodin. Acad Pr, 1973, xiv + 284 pp, $9.50. An excellent treatment of a wide spectrum of issues including, for instance, invasion of privacy, effects on unemployment, anti-trust suits, and various uses of computing to improve society. Balanced presentation of conflicting viewpoints on each of the topics. No technical knowledge of computers or computer programming assumed or provided. Intended for wide readership. Good references at end of each chapter. DAA

GENERAL, P. *Transactions of the Moscow Mathematical Society for the Year 1971, V. 25*. AMS, 1973, iii + 288 pp, $22.95.

BASIC, S(13). *Geometry for the Practical Man, Third Edition.* J.E. Thompson. Van N-Rein, 1962, xv + 256 pp, $3.95 (P). Essentially the content of a classical high school geometry course. Written for self study. Contains some historical perspectives. Probably not as effective as modern "programmed" texts. TAV

BASIC, S(13). *Arithmetic for the Practical Man, Third Edition.* J.E. Thompson. Van N-Rein, 1962, xiv + 266 pp, $3.95 (P). For home study to review arithmetic techniques. Includes a variety of topics including logs, progressions, measures, areas, graphs, interest, etc. A few exercises are included. TAV

BASIC, S(13). *Algebra for the Practical Man, Third Edition.* J.E. Thompson. Van N-Rein, 1962, xvii + 277 pp, $3.95 (P). For home study as a review of algebraic manipulations and techniques. Covers introductory high school algebra through cubics with introductions to probability, Boolean algebras and series. TAV

PRECALCULUS, S(13). *Trigonometry for the Practical Man, Third Edition.* J.E. Thompson. Van N-Rein, 1962, xv + 199 pp, $3.95 (P). Essentially the content of a 1940 high school trigonometry course taught from the point of view of "solving triangles" as opposed to the properties of the trigonometric functions as functions. Intended for self study.TAV

FINITE MATHEMATICS, T*(13-15: 1, 2), S, L. *Introduction to Finite Mathematics, Third Edition.* John G. Kemeny, J. Laurie Snell, Gerald L. Thompson. P-H, 1974, xi + 484 pp, $11.50. A thorough, refreshing face-lift, including mostly new or revised problems, an introduction to BASIC, less emphasis on logic and sets, completely revised treatment of linear programming and matrix game theory. An attractive, contemporary version of the canonical textbook of finite mathematics. LAS

FINITE MATHEMATICS, T(13: 1). *Finite Mathematics.* John M. Peterson. HR&W, 1974, viii + 307 pp, $12. One semester finite math text written at the pre-calculus level. Topics: logic, basic set theory, counting, discrete probability, descriptive statistics, matrix algebra, linear programming, and game theory. SG

FINITE MATHEMATICS, T(13: 1). *Fundamentals of Finite Mathematics.* R.W. Negus. Wiley, 1974, xiii + 402 pp, $10.95. The first 100 pages concern sets, deductive reasoning and symbolic logic; for the book's intended audience, this amount seems excessive. What follows: relations and functions, matrices and vectors, linear inequalities and linear programming, counting patterns, probability, Markov chains. Attractive style and format. Many exercises with an applied flavor. DFA

FINITE MATHEMATICS, T*(13: 1), S*, L*. *Graphs, Models, and Finite Mathematics.* Joseph Malkevitch, Walter Meyer. P-H, 1974, x + 515 pp, $10.95. Guiding philosophy: present mathematics that is "interesting, useful, and accessible." Mathematical modeling and graphs (networks) replace usual opening chapters on logic and sets; probability, matrices, game theory, linear programming, computers, a chapter on theory of elections. The level lies somewhere between Jacobs and Kemeny. The presentations are lively and fun to read. LCL

EDUCATION, P*. *Dialogues sur la Géomêtrie; Dessi, Mati, Logi.* Lucienne Félix. Blanchard, 1971, vii + 153 pp, (P). A geometer responds to the French algebraists. An attempt to bring to his audience (probably high school teachers) a modern French development of geometry, up to and including a little topology, based on discovery. The format is that of a dialogue, with interludes of recapitulation, among three "students", Dessi, Mati, and Logi, who make some pretty sophisticated discoveries "on their own." Definitely out of the ordinary. JAS

EDUCATION, S, P, L. *Teaching School Mathematics.* Ed: W. Servais, T. Varga. Penguin, 1971, 308 pp, $4.50 (P). A UNESCO source book designed to assist persons involved in curriculum development. In three parts, one on practice and theory of teaching mathematics, one on the new mathematics, with sample syllabuses, and the last on teacher training. Excellent for those wanting a better understanding of European mathematics education. Emphasis on secondary level. Should be available to all methods students. PSJ

EDUCATION, T(13-14: 2). *Mathematics for Elementary Teachers.* W.H. Spragens. Allyn, 1972, vii + 328 pp, $9.95. For a one-year mathematics course for elementary teachers. Develops number systems from natural numbers to rationals with intuitive consideration of reals; metric geometry. Deductive proof central; can be de-emphasized. Answers to selected exercises. PSJ

EDUCATION, T(14, 17: 1). *Modern Algebra for Elementary Teachers.* J. Eldon Whitesitt. A-W, 1972, ix + 290 pp, $10.50. For pre- and in-service elementary teachers. Meets CUPM recommendations for a course in algebra to follow number systems course. Completely solved examples precede exercises. Answers to selected exercises. PSJ

EDUCATION, S, P, L. *The Psychology of Learning Mathematics.* Richard R. Skemp. Penguin, 319 pp, $2.25 (P). Written by a mathematics teacher turned psychologist. The book is in two parts. The first attacks the question: what is understanding and how can it be brought about? The second part applies the knowledge of the first part to develop some basic mathematical notions such as number, problem solving, mapping and functions, and geometry. For teachers of mathematics and laymen. PSJ

EDUCATION, T(15-17: 1). *Mathematics and the Elementary Teacher, Second Edition.* Richard W. Copeland. Saunders, 1972, ix + 336 pp, $9.25. Methods text for pre- and in-service elementary teachers. Revision includes strong emphasis on implications of Piaget's research. Geometry plays significant role; introduced before arithmetic topics. Laboratory approach also emphasized. Solutions to selected exercises. PSJ

EDUCATION, S(15-16), P, L. *The Mathematics Laboratory: Theory to Practice.* Robert E. Reys, Thomas R. Post. Prindle, 1973, xvii + 297 pp, $9.95. Develops a rationale for laboratory method of instruction; also provides suggestions to illustrate applications of the theories proposed. Based on theories of Gagne, Piaget, Dienes, Bruner and others. Appendices include good bibliographies and sources of laboratory materials. An important book for use in methods courses for elementary and secondary teachers. PSJ

EDUCATION, T(13: 1, 2), L. *Topics in Elementary Mathematics.* Abe Mizrahi, Michael Sullivan. HR&W, 1971, xi + 394 pp, $10. Accurate, easily readable text for training students intending to teach at the elementary school level. Adheres to Level I recommendations of CUPM (development of real number system, applications). LCL

EDUCATION, P. *Enseignement Élémentaire 1: Somme de Naturels, Addition dans N.* Marguerite Robert. CEDIC, 1973, 96 pp, (P). Modern math for elementary school teachers, intended to clarify notions about addition and provide specific hints about teaching it. PJC

HISTORY, P. *Rapport historique sur les progres des sciences mathématiques depuis 1789.* J.B.J. Delambre. B.M. Israel, 1966, vii + 362 pp. Unaltered reprint of original Paris edition of 1810, tracing developments in the period of 1789-1808 in geometry, algebra, mechanics, astronomy, geography, physics, industry. LAS

HISTORY, P, L. *Coulomb and the Evolution of Physics and Engineering in Eighteenth-Century France.* C. Stewart Gillmor. Princeton U Pr, 1971, xvii + 328 pp, $13.50. First biography of the man, including a full account of his life and an assessment of his work in physics (torsion, electricity and magnetism) and engineering (strength of materials, soil mechanics, structural design, and friction). PJC

HISTORY, P, L. *Essai sur L'Application de L'Analyse à la Probabilité des Décisions.* M. le Marquis de Condorcet. Chelsea, 1972, cxci + 304 pp, $10.50. A photographic reprint of the 1785 work which played a major role in the beginnings of the theory of probability. JAS

FOUNDATIONS, S*(17-18), P**, L*. *Studies in Model Theory.* Ed: M.D. Morley. Stud. in Math., V. 8. MAA, 1973, 197 pp, $8. Six well-written expository essays: Barwise (infinitary logic), A.R. Bernstein (non-standard analysis), Chang (saturated models), Keisler (forcing and omitting types), A. Robinson (model theory and algebra), and J. Silver (large cardinals and constructibility). Also, short appendices on predicate calculus, ultrapowers, ZF axioms. A jewel of a book! Readers will need some sophistication, however. PJC

FOUNDATIONS, T(16-17), P. *Formal Semantics and Logic.* Bas C. van Fraassen. Macmillan, 1971, xi + 225 pp, $9.95. Intended for philosophy students with one previous course in formal logic. Follows current trend in emphasis on applications to non-classical logics and non-classical interpretations of classical logic. Formal languages, evaluation spaces, ultraproducts and compactness, classical quantification and identity theory, appraisal of logical systems, and non-classical logics. Argues that Tarski's theory of truth does not carry over unchanged to non-classical logics. PJC

FOUNDATIONS, T(14-17: 1, 2). *Symbolic Logic: An Introduction.* Richmond H. Thomason. Macmillan, 1970, xiii + 367 pp, $9.50. Two-valued sentential logic and predicate logic with identity, plus chapters on set theory and mathematical induction. Uses natural deduction modeled after Fitch, proves completeness theorem after fashion of Henkin. No non-classical logics. Open, informal style will help bridge the reader over the conceptual difficulties and interpretation of the symbols. Greater depth than similar texts; suitable for a mathematical logic course, although aimed at philosophy students with no specialized training in mathematics. PJC

FOUNDATIONS, T(17-18: 1, 2), S, P*, L. *Model Theory.* C.C. Chang, H.J. Keisler. Stud. in Logic and Found. of Math., V. 73. North-Holland, 1973, xii + 550 pp, $27.50. For the first time, a definitive and encyclopedic coverage of first-order model theory and many of its applications to algebra and set theory. A significant and massive undertaking--ten years in the making. Suitable for a wide variety of model theory courses. LCL

FOUNDATIONS, S. *La Théorie des Ensembles, Deuxième Edition.* Alain Bouvier. Pr U France, 1972, 124 pp, (P). Popular exposition of elements of naive set theory in the series *Que sais-je.* Proceeds through elementary notions, Cartesian products, relations, functions, and order and equivalence relations. A quick chapter on cardinals, and another on the history of set theory and its paradoxes, close the book. Starts slow but accelerates. PJC

FOUNDATIONS, P, L. *Philosophical Problems of Space and Time, Second Enlarged Edition.* Adolf Grünbaum. Boston Stud. in Philo. of Sci., V. XII. Reidel, 1973, xxiii + 884 pp, $17.90 (P). Corrected reprint of the influential 1963 original edition published by Alfred Knopf together with seven new chapters reprinted or revised from the author's recent papers, modified by an extensive appendix giving recent interpretation, reactions to critics, or major corrections to each of the preceding 22 chapters. A definitive contemporary treatise on a problem of fundamental interest to mathematics, physics and philosophy, marred only by the patchwork nature of the enlarged edition. LAS

FOUNDATIONS, P. *A General Interpreted Modal Calculus.* Aldo Bressan. Yale U Pr, 1972, xxviii + 327 pp, $15. Introduces quantifiers into modal logic and features a new analysis of predication in modal logic which does not compel all predicates to be extensional. Heavy on notation. Author is a physicist and he uses the modal calculus presented to make sense out of Mach's definition of mass in classical mechanics. Includes development of proof-theory and proof of relative completeness for the language. PJC

COMBINATORICS, T(17: 2), S, P. *Graphs, Groups and Surfaces.* Arthur T. White. Math. Stud., V. 8. North-Holland, 1973, x + 142 pp, $7.50 (P). Introduction of graphs is followed by study of automorphism groups of graphs. Group presentations lead to Cayley color graphs. Finally, these concepts are related to the study of surfaces. Among the topics encountered after the basics are: imbedding problems, genus of a group, map coloring (including the author's interesting work on K-degenerate graphs), quotient graphs and quotient manifolds. A careful exposition, with exercises, an extensive bibliography and index. SS

COMBINATORICS, P. *Graphes et Questionnaires.* Claude-Francois Picard. Gauthier-Villars, 1972. *Tome 1: Graphes,* xiv + 145 pp, (P); *Tome 2: Questionnaires,* xv + 211 pp, (P). "Questionnaires" are valued graphs which serve as models of choice or decision involving some information (in the technical sense). Standard graph theory, plus details of graphs appropriate to questionnaires, is developed in the first volume, with an entire chapter devoted to theory of information through discrete sources. Second volume sets out the elements of information theory and investigates optimality of various kinds of questionnaires. Extensive bibliography, by chapters. PJC

COMBINATORICS, T*(17-18; 2), S, P*. *Graphical Enumeration.* Frank Harary, Edgar M. Palmer. Acad Pr, 1973, xiv + 271 pp, $14.50. Long-awaited definitive treatise on enumeration of graphs, classes of graphs and related structural configurations. Contains complete and detailed treatment of basic techniques, numerous exercises ranging from routine to extremely difficult. Extensive bibliography, index of symbols and definitions. SS

NUMBER THEORY, S(13), L. *Perfect Numbers.* Richard W. Shoemaker. NCTM, 1973, iii + 28 pp, $1 (P). Focused on the fascinating properties of perfect numbers--all generally well-known with the exception of the section on perfect numbers in base negative two. Written especially for secondary teachers. Scattered exercises. LCL

NUMBER THEORY, P, L. *Empirical Study of Aliquot Series.* Jack Alanen. Math Centrum, 1972, 121 pp, DFL. 12,50 (P). The author's dissertation. History and analysis of aliquot series (e.g., n, $s(n)$, $s^2(n)$,... where $s(n)$ is the sum of all positive divisors of n except n itself.) Considerations include recurrence relations, asymptotic formulae, upper and lower bounds of various sorts, a related oriented graph, numerical investigations of certain problems. SG

LINEAR ALGEBRA, T(16-17), S, P*, L. *Non-Negative Matrices: An Introduction to Theory and Applications.* E. Seneta. Halsted Pr, 1973, x + 214 pp, $19.50. A valuable and much needed presentation; aimed at breadth with a view toward relating various aspects of the theory. First four chapters on finite matrices, final two on infinite analogues. Particularly useful to workers in applied fields such as probability, numerical analysis, demography, mathematical economics, dynamic programming; presumes knowledge of matrix theory, real variable and some complex variable. Extensive bibliographic remarks; exercises. LCL

ALGEBRA, S(14-15). *Topics in Pure Mathematics, 16 Volumes.* Open University. Har-Row, 1973, 1041 pp, $56.15 set (P). Modular workbooks for a multi-media self-instructional second level course for the Open University of Great Britain designed to supplement four regular texts on set theory, algebra, topology and machine theory. Syllabus covers sets, groups, metric spaces, rings and ideals, recursive functions, categories, topological spaces, compactness, proof theory, fundamental groups, fixed point theorems, and Galois theory. Each volume is available separately for $1.75-$5.50, as are 33 associated films ($150@) and eight tapes $10@); see Har-Row catalog for details. LAS

ALGEBRA, P*. *Cohomology Theories for Compact Abelian Groups.* Karl H. Hofmann, Paul S. Mostert. Springer-Verlag, 1973, 236 pp, $18.50. A compact abelian group combines two of the nicest structures in mathematics. In developing a cohomology theory, one would want to take advantage of the combination. That is the approach and motivation of the authors in this excellent book. After studying cohomology of abelian groups, and finding it lacking for most compact abelian groups, the authors look at the Cech cohomology--not of the space of the group, for that would be ignoring the multiplicative structure--of the classifying space. An appendix by Eric Nummela extends results to compact monoids (=groups without inverses). PJM

ALGEBRA, T(16-18: 1), S, P. *Algebraic Theory of Lattices.* Peter Crawley, Robert P. Dilworth. P-H, 1973, vi + 201 pp, $11.95. Self-contained but advanced. Focuses on structure: modularity, decompositions, representations, dimension theory. Routine proofs left to reader. Includes open problems and conjectures. Takes reader to level of current research. LH

ALGEBRA, P. *Lecture Notes in Mathematics-359: Homology in Group Theory.* Urs Stammbach. Springer-Verlag, 1973, viii + 183 pp, $7 (P). Uses homological methods to obtain results about extensions with abelian kernel in an arbitrary variety of groups, lower central series, central extensions, and localization and rationalization of nilpotent groups. The main tools are the functors $\hat{V}$, V of an arbitrary variety $\underline{\underline{V}}$. DFA

ALGEBRA, T(15-16: 1, 2), S, L. *Introduction to Group Theory.* W. Ledermann. B&N, 1973, vii + 176 pp, $5 (P). Starts from scratch and treats finitely generated Abelian groups, the Jordan-Hölder Theorem and the Sylow Theorems, as well as the usual more elementary topics. About 80 exercises, with solutions to most of them. Very clearly written; should make an excellent text. A bargain for the price, even though it is a paperback. JD-B

ALGEBRA, T(18), P. *Infinite Linear Groups.* B.A.F. Wehrfritz. Ergebnisse der Math., B. 76. Springer-Verlag, 1973, xiv + 229 pp, $21.90. An account of the group-theoretic properties of infinite groups of matrices written for graduate students of group theory wishing to make a serious study of the subject. LCL

CALCULUS, T*(13), S*. *Problems for A Computer-Oriented Calculus Course with an Appendix on Elementary Fortran Programming.* Richard C. Allen, Jr., G. Milton Wing. P-H, 1973, xiii + 206 pp, $3.50 (P). One-third problems and exposition, one-half (!) selected solutions (computer printouts--lots of white space), one-sixth FORTRAN introduction. De-emphasizes computer itself (used strictly as a tool), stresses demonstrating basic concepts of calculus: function evaluation, limits, differentiation, integration, sequences, series, Taylor polynomials. Numerical analysis avoided, so no trapezoid or Simpson's rule. Smoothly written, carefully designed. Would go better without including so many solutions (complete solutions available in separate teachers manual)--they could be posted. Using a time-sharing system would seem more in tune with the authors' intentions not to let the computer get in the way. PJC

CALCULUS, T(13: 1, 2). *Calculus for Business, Biology, and the Social Sciences.* David G. Crowdis, *et al.* Glencoe Pr, 1972, xii + 547 pp, $10.95. Suitable for one or two semester courses. Approach is intuitive, with probability as model for integration, marginal considerations for differentiation. Examples and exercises taken primarily from economics and biology. Contains partial derivatives, max-min in several variables (no Lagrange multipliers), multiple integrals and 32 pages bound upside down. TAV

CALCULUS, S(15), L. *Laplace Transforms: Programmes and Problems.* K. A. Stroud. Wiley, 1973, x + 275 pp, $5.75 (P). Programmed text which assumes some familiarity with differential equations. Definitions, solution of scalar linear equations and 2-dimensional systems, transforms of the Heaviside unit step function, periodic functions, the

Dirac delta function. Discussion of the convolution (and other) theorems. "If you read the frames carefully and follow the directions exactly, you are bound to learn", the author tells the reader; he's right. DFA

CALCULUS, T(15: 2), L. *Applicable Mathematics: A Course for Scientists and Engineers.* R.J. Goult, *et al.* Crane, Russak, 1973, viii + 491 pp, $12.75. The usual fare for this genre of text, but with these exceptions: a high level of rigor prevails in much of the book, and presentation of numerical techniques takes place throughout. For its intended user, the text makes challenging reading; unfortunately, the exercises are all routine. DFA

CALCULUS, T?(13), S. *Calculus for the Practical Man, Third Edition.* J.E. Thompson. Van N-Rein, 1962, xvi + 280 pp, $3.95 (P). For home study as a review of calculus. The content is quite standard for a 1950's course. The style is informal. There are very few exercises, a major flaw. One would probably do better with a Schaum Outline or similar work. TAV

REAL ANALYSIS, T(14-15: 1). *An Introduction to Real Analysis.* Derek G. Ball. Pergamon Pr, 1973, xvii + 305 pp, $7.25 (P). For an intermediate level course. An introduction motivates subsequent topics: sets, relations and functions; numbers; sequences; series; functions of a real variable; the derivative; some important functions and expansions; the Riemann integral. Written for students in a British college of education. RBK

REAL ANALYSIS, T(14-15: 1). *Real Analysis: An Introductory Course.* J.R. Giles. Wiley, 1972, ix + 171 pp, $13.50. Elementary advanced calculus--real numbers, sequences, series, continuity, differentiation, integration. Neighborhood concept de-emphasized in favor of sequential approach to limits. No open or closed sets, cluster points, topological or metric spaces. Very short sections. Extensive solutions to selected exercises. PJC

COMPLEX ANALYSIS, T***(17: 2), S, P*, L. *Introduction to the Theory of Entire Functions.* A.S.B. Holland. Acad Pr, 1973, x + 221 pp, $18. A beautifully written introduction to entire functions, readable to any with a first course in complex analysis. The treatment through elementary Nevanlinna theory is orderly and compelling. An excellent preparation for reading R. Boas' recent *Entire Functions* and W. Hayman's *Meromorphic Functions.* Contains an excellent bibliography as well. TAV

COMPLEX ANALYSIS, P*. *Quasiconformal Mappings in the Plane, Second Edition.* O. Lehto, K.I, Virtanen. Transl: K.W. Lucas. Grund. math. Wissenschaften, B. 126. Springer-Verlag, 1973, ix + 258 pp, $27.50. A translation of the 1965 German edition, the definitive treatment of QC mappings. The geometric treatment of Grötzsch is employed to give new insights into the subject. A welcome English version of a significant work. TAV

COMPLEX ANALYSIS, P. *Value-Distribution Theory, Part A.* Ed: Robert O. Kujala, Albert L. Vitter III. Dekker, 1974, xi + 269 pp, $15.50; *Part B: Deficit and Bezout Estimates,* Wilhelm Stoll, 1973, xi + 271 pp, $15.50. Papers from a special semester-long program (spring, 1973) at Tulane on value distribution theory in complex analysis and

differential geometry. Lecture notes of a course given by Stoll comprise *Part B;* various short papers from other participants comprise *Part A.* LAS

COMPLEX ANALYSIS, T(18: 1), P. *An Introduction to Complex Analysis in Several Variables.* Lars Hörmander. Math. Lib., V. 7. North-Holland, 1973, x + 213 pp, $14.50. Uses the $\bar{\partial}$ partial differential operator to develop the properties of analytic functions of several variables (in analogy to the Cauchy-Riemann equations for one variable). Differs from the earlier edition (TR, January 1967) in expansion of results on L^2 existence theorems and updated bibliography. TAV

DIFFERENTIAL EQUATIONS, P. *Linear Partial Differential Equations.* Francois Trèves. Gordon, 1970, x + 120 pp, $17.50. A brief graduate level introduction. Begins with a review of some functional analysis, esp. approximation theorems; other topics: L^2 inequalities (e.g., Hörmander, Garding, and hyperbolic inequalities), existence and uniqueness of solutions, L^2 estimates and pseudo-convexity. SG

DIFFERENTIAL EQUATIONS, P. *Linear Differential Transformations of the Second Order.* Otakar Borůvka. Transl: F.M. Arscott. English U Pr, 1971, xvi + 254 pp, £5.45. Linear homogeneous second-order ordinary differential equations in Jacobian form. Provides classical qualitative theory, then studies dispersions and general transformations, often using an algebraic approach. Adds material on the abstract algebraic model of the transformation theory--and its realization in the analytical case--to the 1967 German edition. DFA

DIFFERENTIAL EQUATIONS, P. *Non-Homogeneous Boundary Value Problems and Applications, V. II.* J.L. Lions, E. Magenes. Transl: P. Kenneth. Grund. math. Wissenschaften, B. 182. Springer-Verlag, 1972, xi + 242 pp, $18.40. Joins volumes I (TR, August 1972) and III (TR, February 1974). Studies regularity, transposition, and interpolation for parabolic and (Petrowski and Schroedinger) hyperbolic evolution operators. Applications to optimal control problems. Notes, open problems, 400 references. DFA

DIFFERENTIAL EQUATIONS, T(14-15: 1), S. *Applied Differential Equations.* N. Curle. New Math. Lib., V. 1. Van N-Rein, 1972, viii + 108 pp, $5.95. A brief introduction to differential equations stressing applications. Covers standard first and second order equations only. No power series, no systems, no existence and uniqueness. A modest number of exercises. SG

DIFFERENTIAL EQUATIONS, P. *Nonlinear and Random Vibrations.* Florea Dinca, Cristian Teodosiu. Transl: Cristian Teodosiu. Editura Academiei Romania, 1973, 413 pp, $29.50. Concerns mechanical single-degree-of-freedom conservative and dissipative systems. Studies free vibrations of nonlinear ones and forced vibrations of nonlinear ones with deterministic excitations and of linear and nonlinear ones with random excitations. Qualitative and quantitative; valuable to engineers. Updates original. DFA

DIFFERENTIAL EQUATIONS, P. *Qualitative Theory of Second-Order Dynamic Systems.* A.A. Andronov, *et al.* Transl: D. Louvish. Wiley, 1973, xxiii + 524 pp, $44. Translation of the first volume of Andronov's

two-volume treatise, the second of which appeared in English *(Theory of Bifurcations of Dynamic Systems on a Plane*, Israel Prog. for Scientific Transl.) in 1971. Classical theory together with results of Leontovich, Maier and Gubar'. The exposition and a handy appendix make the book usable for an introductory course. DFA

NUMERICAL ANALYSIS, T(15-16: 1). *Numerical Computation*. P.W. Williams. B&N, 1972, viii + 191 pp, $6.95 (P); $13.75. Numerical methods for science and engineering students. Errors are exemplified and discussed but not analyzed. Topics include nonlinear equations, linear systems, ODEs, curve fitting, numerical integration and matrix eigenvalue problems. Good numerical examples. RWN

NUMERICAL ANALYSIS, T(17-18: 1), P. *Spline Analysis*. Martin H. Schultz. P-H, 1973, xiii + 156 pp, $10.50. "A unified and mathematically rigorous approach to the finite element method." Theoretical basis using variational formulations for applications to interpolation, least squares, integral equations, eigenvalue problems, elliptic and parabolic differential equations and optimal control. Uses easily-followed model problems. Although not complete, this book provides a solid basis for the rest of the literature. Exercises. RWN

NUMERICAL ANALYSIS, P. *Cardinal Spline Interpolation*. I.J. Schoenberg. CBMS Reg. Conf. Ser. in Appl. Math., No. 12. SIAM, 1973, vi + 125 pp, $7.40 (P). An excellent summary of the mathematical properties of cardinal splines, including B-splines, exponential splines, semi-cardinal splines, the relationship with finite splines, interpolation, quadrature, extremal problems and applications. RWN

NUMERICAL ANALYSIS, T(17-18: 1, 2), S, P. *Solution of Equations in Euclidean and Banach Spaces. Third Edition of Solution of Equations and Systems of Equations*. A.M. Ostrowski. Acad Pr, 1973, xx + 412 pp, $34. Smaller print and about 25% more material than the second edition. As the change in title indicates, much of the addition uses recent works (including several by the author) applying functional analysis. Includes preliminary definitions and essential theory, general Newton-Raphson methods, convergence tests, acceleration, and applications to polynomials and finite systems. This book will continue to be a major reference. RWN

NUMERICAL ANALYSIS, P. *Lecture Notes in Mathematics-333: Numerische, insbesondere approximationstheoretische Behandlung von Funktionalgleichungen*. R. Ansorge, W. Tornig. Springer-Verlag, 1973, vi + 296 pp, $10.80 (P). Papers from a meeting in Oberwolfach in December 1972. JAS

FUNCTIONAL ANALYSIS, P. *Algebraic and Analytic Aspects of Operator Algebras*. Irving Kaplansky. CBMS Reg. Conf. in Math., No. 1. AMS, 1970, iv + 20 pp, $2.20 (P). From lectures at a Regional Conference at the University of Hawaii in June 1969. Concerns C*-algebras and special classes of them (W*, AW*, CCR, GCR). Descriptive in nature, with 98 citations and few proofs. DFA

FUNCTIONAL ANALYSIS, P. *Compact Non-Self-Adjoint Operators*. John R. Ringrose. Van N-Rein, 1971, vi + 238 pp, $9.95 (P). Hilbert space setting. Von Neumann-Schatten classes of operators, Fredholm theory for trace class operators, superdiagonal representation of compact linear operators. From a graduate course at Pennsylvania in 1964-65. Presupposes acquaintance with elementary functional analysis. DFA

OPTIMIZATION, T(15-18; 1), S. *Nonlinear Optimisation*. L.C.W. Dixon. Crane, Russak, 1972, ix + 214 pp, $11.95. Basic introduction to several optimization techniques. Static and dynamic problems. Extensive references, adequate examples, but no exercises for students. Explicit description of many algorithms using block diagrams. LH

OPTIMIZATION, S(16-17), P. *Lecture Notes in Economics and Mathematical Systems-63: Monte Carlo Simulation im Operations Research*. J. Kohlas. Springer-Verlag, 1972, vi+162 pp, $6.20 (P). An account (in German) of Monte Carlo simulation methods written for students of economics and operations research. A long chapter on the generation of random numbers is followed by a discussion of the use of simulation in two concrete examples and in queueing theory. JD-B

OPTIMIZATION, P, L. *Fortran Codes for Mathematical Programming: Linear, Quadratic and Discrete*. A.H. Land, S. Powell. Wiley, 1973, 249 pp, $14.95. A well-documented set of programs for linear, quadratic and discrete programming. Intended as a reliable package for running test cases, comparisons and modification; not particularly efficient. RWN

OPTIMIZATION, P. *Topics in Differential Games*. Ed: Austin Blaquière. North-Holland, 1973, ix + 450 pp, $25.50. A number of papers from all over the world dealing with recent advances in both zero-sum and non-zero-sum games. JAS

ANALYSIS, P. *Introducere in Teoria Punctelor Fixe*. Vasile I. Istratescu. Editura Academiei Romania, 1973, 426 pp, (P). Fixed point theory with applications to functional analysis; in Rumanian.JAS

ANALYSIS, P. *Harmonic Analysis on Homogeneous Spaces*. Ed: Calvin C. Moore. Proc. of Symp. in Pure Math., V. XXVI. AMS, 1973, x + 467 pp, $43.50. Invited lectures and seminar papers from a 1972 summer symposium at Williams College. LAS

ALGEBRAIC GEOMETRY, P. *Algebraic Geometry, Oslo 1970*. Ed: F. Oort. Wolters-Noordhoff, 1972, viii + 332 pp, $31.50. The proceedings of the 5th Nordic Summer School in Mathematics, Oslo, August 5-25, 1970. Part of an attempt to provide access to modern methods of algebraic geometry more efficiently than through the more than 1900 pages of Grothendieck and Dieudonné. JAS

DIFFERENTIAL GEOMETRY, T(16-17; 1, 2), P. *Géométrie différentielle intrinsèque*. Paul Malliavin. Hermann, 1972, 307 pp, 60F. A moderately abstract approach to differential geometry. Four parts: differentiable manifolds, differential calculus on manifolds, local theory of Lie groups, and calculus of variations. The French is intermediate in difficulty. PJM

GEOMETRY, S*(14-16), L*. *Euclid and His Modern Rivals*. Lewis Carroll. Dover, 1973, xxxi + 275 pp, $3.50 (P). "Oh, give it full marks! What have we to do with logic, or truth, or falsehood, or right, or wrong? 'We are but markers of a larger growth'..." A stunningly witty dramatic satire of the many nineteenth century British geometry textbooks which sought to improve on Euclid's presentation. A fun way to learn some classical geometry as well as an invitation to imagine twentieth century parallels. LAS

GEOMETRY, T(18), S, P. *Harmonic Analysis on Homogeneous Spaces.* Nolan R. Wallach. Dekker, 1973, xv + 361 pp, $25.75. A text for, or a supplement to, a one-year course in Lie groups or Lie algebras. It does contain a number of problems but has a rather brief index for use as a reference-text. An attempt to give access to some of the newer and more complex (pun intended) areas of geometry. JAS

GEOMETRY, T(13: 1), S. *Conceptions of Space: Beginning Geometries for College.* William Hemmer. Canfield Pr, 1973, 112 pp, $5.95 (P). Intended for students who have not previously studied geometry, this delightful, but expensive, "workbook" maintains an elementary character while strongly emphasizing axiomatics in its investigation of Euclidean, hyperbolic and elliptic geometries. Unfortunately, each geometry is carried only to the determination of the angle sum of a triangle. JNC

GEOMETRY, T(14-16: 1, 2), L. *Géométrie pour l'elève-professeur.* Jean Frenkel. Hermann, 1973, 353 pp. 52F (P). From the author's preface: "Without geometry, algebra is blind; without algebra, geometry is paralyzed." The book covers affine, Euclidean, and finite dimensional projective geometries, primarily from an affine viewpoint, but always with the idea in mind that geometry is a form of linear algebra. The French is intermediate in difficulty. A good book, if not as a text (because of the language) surely as a secondary reference. PJM

PROBABILITY, T(13-17: 1, 2), P, L*. *An Introduction to Bayesian Inference and Decision.* Robert L. Winkler. HR&W, 1972, xi + 563 pp, $12.50. Elementary probability, Bayesian inference for discrete and continuous probability models, decision theory, the value of information, Bayesian and other approaches to inference and decision compared, plus 80 pages of tables. Requires no previous knowledge of probability or statistics, only college algebra. Excellent chapter references, classified by level and emphasis, based on 20-page bibliography. Examples, exercises are "realistic"; none require use of computer. Solutions manual available. PJC

PROBABILITY, P. *Proceedings of the Fourth Conference on Probability Theory.* Ed: Bernard Bereanu, *et al.* Editura Academiei Romania, 1971, 644 pp. Invited papers, communications and abstracts from a September, 1971 conference in Brasov, Romania. LAS

PROBABILITY, P. *Modele Matematice ale Asteptării.* Gh. Mihoc, *et al.* Editura Academiei Romania, 1973, 464 pp. Exposition of basic queueing theory in Romanian. JAS

PROBABILITY, T(15-16: 1). *Introduction to Probability.* James E. Huneycutt, Jr. Merrill, 1973, vi + 185 pp, $8.95. A somewhat abstract approach using only the Riemann integral. Does not consider stochastic processes or the central limit theorem. FLW

PROBABILITY, S(16), P. *Collection of Problems in Probability Theory.* L.D. Meshalkin. Transl: Leo F. Boron, Bryan A. Haworth. Noordhoff Intern, 1973, x + 148 pp, $19.50. More than 450 problems (with answers). Covers topics from basic concepts to infinitely divisible distributions and Markov chains. Tables for normal, t, χ^2, Poisson distributions appended. TAV

STATISTICS, T(13-14: 1, 2), S*. *Basic Statistics in Business and Economics.* George W. Summers, William S. Peters. Wadsworth, 1973, 445 pp, $11.95; *Self-Correcting Exercises*, 184 pp, $3.95 (P); *Self Instructional Supplement*, Freeman F. Elzey, Charles P. Armstrong, 294 pp, $3.95 (P). A well-developed package of instructional materials. The text is composed of four major sections: the first is descriptive; the second is on probability and contains an introduction to Bayesian decision procedures; the third covers classical statistical inference through analysis of variance and regression and correlation; and the last is a collection of specialized topics including nonparametric statistics, sample survey methods, time series and index numbers, and Bayesian statistics. The approach tends to be descriptive, and requires no mathematics beyond algebra. *Self-Correcting Exercises* is a collection of problems related to the text, together with complete solutions. *Self-Instructional Supplement* is a set of programmed materials designed to provide supplemental instruction on the first three sections of the text. An *Instructor's Manual* is available which gives solutions for the exercises in the text, and suggestions for course organization and utilization of supplementary materials. Also available, for $10 upon adoption of the text, is a *Test Item Card File*, which contains 168 multiple-choice tests items with student performance ratings for each item. RSK

STATISTICS, S, L. *The Art and Science of Decision Making.* M. Tainiter. Timetable Pr, 1971, 79 pp, $1.85 (P). A well written, informal introduction to probability and the basic ideas of decision theory in business. FLW

STATISTICS, T(16-17: 1, 2), S, L. *Estimation Theory with Applications to Communications and Control.* Andrew P. Sage, James L. Melsa. McGraw, 1971, xi + 529 pp, $17.50. Stochastic processes, stochastic differential equations, decision theory, estimation theory, the optimum linear filter, and nonlinear estimation. "Questions of true mathematical rigor have been largely ignored." "Written for engineers." FLW

STATISTICS, T(14-17: 1). *Modern Methods for Statistical Analysis.* Harold L. Pazer, Lloyd A. Swanson. Intext, 1972, ix + 483 pp, $12. Intended for a second level course in statistics for business and social science students. Uses some calculus, but assumes no previous exposure. Advanced topics include chi-square tests, analysis of variance, regression and correlation, analysis of covariance, and nonparametric and distribution-free tests. Employs both Bayesian and classical approaches. Appendices describe the use of canned FORTRAN programs which are available to institutions using the text. RSK

STATISTICS, T(18), P. *Random Data: Analysis and Measurement Procedures.* Julius S. Bendat, Allan G. Piersol. Wiley, 1971, xv + 407 pp, $21.50. Extensive revision of the authors' 1966 book *Measurement and Analysis of Random Data* (TR, May 1967). Assumes background in probability, statistics, stochastic processes and transform methods of applied mathematics, although each is reviewed briefly. Primarily concerned with the measurement and analysis, including both analog and digital procedures, of stationary data representing one dimensional (time series) random processes. Concludes with a chapter on nonstationary, transient, and multidimensional data. RSK

STATISTICS, P. *Families of Frequency Distributions.* J.K. Ord. Griffin's Stat. Mono., No. 30. Hafner, 1972, viii + 231 pp, $14.95 (P). Discusses the main properties of the major systems of probability distributions, both continuous and discrete, beginning with the well-known Pearson family of curves. Also discusses the problems involved in selecting and fitting an appropriate model. Good list of references and tables. RSK

STATISTICS, T(17: 1, 2), S, P, *Statistics of Directional Data.* K.V. Mardia. Acad Pr, 1972, xx + 357 pp, $19.50. Presupposes undergraduate statistics. Extensive bibliography. FLW

STATISTICS, T(13-17: 2), *Probability and Statistics for Decision Making.* Ya-lun Chou. HR&W, 1972, xv + 623 pp, $12. Non-calculus course in probability and statistics for business and economics students. Abundance of "realistic" examples and exercises. All the standard topics, plus chapters on Bayesian inference, trending, and short-run fluctuations. Good references at end to where topics treated can be pursued further. PJC

STATISTICS, T(14-17: 1), *Fundamentals of Experimental Design, Second Edition.* Jerome L. Myers. Allyn, 1972, xii + 465 pp, $12.95. Assumes a background of college algebra and elementary statistics, but no previous analysis of variance. In addition to the usual material on various designs, it contains short sections on such topics as analysis of covariance, multiple comparison procedures, analysis of interaction, and trend analysis by means of orthogonal polynomials. Emphasizes understanding why as well as how. RSK

STATISTICS, P**, L*, *Multivariate-Analysis: A Selected and Abstracted Bibliography, 1957-1972.* Kocherlakota and Kathleen Subrahmaniam. Dekker, 1973, xi + 265 pp, $19.75. Gives abstracts of 1189 papers (no books) dealing with the multivariable normal and related distributions, and then classifies them by subtopic. Begins with 1957 since T.W. Anderson's 1958 book, *An Introduction to Multivariate Statistical Analysis*, has an excellent bibliography of all previous work. More descriptive and up-to-date, because of its specialized nature, than Anderson, Gupta and Styan's *A Bibliography of Multivariate Statistical Analysis* (TR, October 1973). RSK

STATISTICS, S, P*, *L'Analyse Des Données.* J.-P. Benzécri, *et al.* Dunod, 1973. *I, La Taxinomie*, viii + 615 pp; *II, L'Analyse des Correspondances*, viii + 619 pp. Structured collection by authors from the statistical laboratory of the University of Paris VI. First volume concentrates on factor analysis, pattern recognition, numerical taxonomy, and theory of classification; the second on use of the χ^2 metric, automatic classification, and analysis of questionnaires. Both volumes emphasize algorithms; some computer programs are reproduced in the appendices. Most valuable of all are the research studies in botany, zoology, economics and psychology where the techniques are shown in use; these articles comprise 40% of the book. PJC

STATISTICS, S. *Business Control Through Multiple Regression Analysis: A Technique for the Numerate Manager.* James H. Heward, Peter M. Steele. Halsted Pr, 1973, 116 pp, $8.50. An informal discussion with case histories and suggestions concerning computer use. FLW

STATISTICS, S(13). *Statistical Thinking; A Structural Approach.* John L. Phillips, Jr. Freeman, 1973, xv + 124 pp, $5.95; $2.50 (P). Designed to be used for a unit on statistics as part of another course, or to provide a framework for further study. Its purpose is to introduce quantitative concepts in a logical manner, emphasizing relationships while de-emphasizing computations. Treatment is somewhat uneven--very good in some sections, but possibly confusing in others. RSK

STATISTICS, S(18), P. *Time-Series.* M.G. Kendall. Hafner, 1973, ix + 197 pp, $11.95. A basic treatment, presenting both theory and practical applications. Contains many examples and references. RSK

STATISTICS, T(13-14: 1, 2). *General Statistics, Second Edition.* Audrey Haber, Richard P. Runyon. A-W, 1973, xiii + 401 pp, $9.95. Presupposes only high school mathematics. Same basic structure as the first edition reviewed here in October 1969. FLW

STATISTICS, T(16-17: 1, 2), S, P, L. *Bayesian Inference in Statistical Analysis.* George E.P. Box, George C. Tiao. A-W, 1973, xviii + 588 pp, $16.95. Considers the Bayesian approach to standard problems of inference and to many problems where the classical approach is awkward. FLW

STATISTICS, P*. *Cluster Analysis for Applications.* Michael R. Anderberg. Acad Pr, 1973, xiii + 359 pp, $25. In their Probability and Mathematical Statistics Series. Designed to present a unified treatment of the diverse material in the literature on the subject, which deals with techniques for finding the "natural groups" or clusters in sets of data. Lengthy appendices contain listings of computer programs of all the major methods developed. Good set of references. RSK

STATISTICS, P. *Regression Estimation from Grouped Observations.* Yoel Haitovsky. Griffin's Stat. Mono., No. 33. Hafner, 1973, x + 94 pp, $8.50 (P). Treats the problems arising from trying to treat grouped rather than raw data. The topics covered include regression from cross classified, partially cross classified and one way tables as well as the treatment of bias introduced. A useful book. TAV

STATISTICS, T(13: 1). *Principles of Statistics and Probability.* Robert A. Crovelli. Prindle, 1973, xi + 307 pp, $9.95. Elementary introduction, emphasizing concepts rather than techniques. First half is devoted to descriptive statistics; last half to probability and statistical inference, concluding with a very short chapter on hypothesis testing. RSK

COMPUTER SCIENCE, P, L. *Cybernetic Machines.* T. Nemes. Transl: W.A. Ainsworth. Gordon, 1970, 260 pp, $18. Translated from the Hungarian and German editions. The history, design and construction of machines which attempt to simulate limited human intellectual activities, e.g., game-playing, theorem-proving, translating, decision-making, perception. Includes many of the author's inventions. Somewhat dated. Definitely of historical value. RWN

COMPUTER SCIENCE, P. *Theory of Machines and Computations.* Ed: Zvi Kohavi, Azaria Paz. Acad Pr, 1971, xiii + 416 pp, $12.50. Papers presented at an international symposium at Haifa, Israel, in August, 1971, touching on computability theory, formal languages, automata theory and switching theory. LAS

COMPUTER SCIENCE, T?(15-17; 1), S, P, L, *A Programmer's Introduction to Computability and Formal Languages*. Reino Kurki-Suonio. Auerbach, 1971, 140 pp, $7.95. Aims to introduce computer science students to relevance and limitations of applying computability theory and formal languages to computer programming. Turing machines, Markov algorithms, Chomsky's hierarchy of formal languages, finite automata and regular languages, pushdown automata and context-free languages and relevance of syntax analysis to programming of compilers. Short for a full term; also, few exercises. Neatly reproduced from typescript. PJC

COMPUTER SCIENCE, P, *On Programming: An Interim Report on the SETL Project*. Jacob T. Schwartz. Courant Inst, 1973. *Installment I: Generalities*, viii + 160 pp, $3.75 (P); *Installment II: The SETL Language and Examples of Its Use*, viii + 520 pp, $13 (P); *A SETLB Primer*, Henry Mullish, Max Goldstein, v + 201 pp, $5.25 (P). SETL is a set-oriented language developed at NYU under the direction of Jacob Schwartz. The set operations include union, intersection, symmetric difference, etc. The language also contains facilities for operating on more standard data types, e.g., numbers and strings. Program logic resembles that of ALGOL and PL/I. The current implementation, described in the primer, is based on BALM (which appears to be a compromise between LISP and ALGOL 68). The usefulness and generality of the language is demonstrated by a wide variety of sample programs. At the moment, the main difficulty is inefficiency. Part of this is due to basing the implementation on a high-level language. Another reason is the inherent conflict between unordered objects and ordered machines. *Installment III: Extensions and Optimization* may discuss ways to control this problem. RWN

COMPUTER SCIENCE, T(16-17: 1), *Computer Semantics: Studies of Algorithms, Processors and Languages*. John A.N. Lee. Van N-Rein, 1972, xvi + 397 pp, $17.95. Uses an extension of the Vienna Definition Language to describe and analyze data structures, algorithms (arithmetic and sorting), a processor (PDP-8) and a language (BASIC). After the system is informally described, a formal definition machine is defined, studied and applied. No exercises. RWN

COMPUTER SCIENCE, T(15-16: 2), S, P, L, *A Practical Guide to Minicomputer Applications*. Ed: Fred F. Coury. IEEE Pr, 1972, vi + 211 pp, $9.95. A collection of 28 reprints from various sources, all devoted to minicomputers. Organized into four sections: Peripheral and Software Considerations, Selecting a Minicomputer, General Applications and Specific Applications. Aimed primarily at the user or potential user with some computing expertise. Two papers giving methods for ranking the various models are of particular interest.TAV

COMPUTER SCIENCE, T(15), S, *Operating System for Multiprogramming with a Variable Number of Tasks*. Ivan Flores. Allyn, 1973, xiv + 431 pp, $17.50. Useful for 360 users. Based on some of the author's previous books. Management of memory, time, interrupts and tasks. Use of IOS, LINKPACT, etc. RWN

COMPUTER SCIENCE, S(13), L, *Data Processing*. B.H. Blakeley. SMP Computing in Math. Cambridge U Pr, 1973, xi + 68 pp, $3.75. Although very short, this book highlights many important aspects of data processing. The author has taken the teacher's work register to illustrate file storage, updating, usage, sorting and merging. No problems or exercises. RB

COMPUTER SCIENCE, T(13-15: 1), S. *Computing and Computers, 15 Volumes.* Open University. Har-Row, 1973, 1059 pp, $68.45 set (P). Eight volumes (really, chapters) of text material accompanied by seven modular workbooks for a "post-experience" course of the Open University in Great Britain. Each module is available separately ($2.25-$6.50@), as are 12 related films ($150@) and four tapes $10@); see Har-Row catalog for details. Syllabus covers systems analysis and design, people in the electronic age, computability, computer structure, computers in action and three case studies. LAS

COMPUTER SCIENCE, S(14-15). *An Algorithmic Approach to Computing, 7 Volumes.* Open University. Har-Row, 1973, 756 pp, $36 set (P). Modular workbooks to accompany two standard computer science texts (by Forsythe, Keenan, Organick and Stenberg) for the Open University of Great Britain. Each workbook is available separately for $3.75-$5.75, as are 10 related films ($150@) and three tapes ($10@). Syllabus covers algorithms, BASIC, computer hardware, data structures, strings, compiling, other programming languages. LAS

COMPUTER SCIENCE, S(13). *BASIC.* Robert L. Albrecht, LeRoy Finkel, Jerry Brown. Wiley, 1973, ix + 325 pp, $3.95 (P). Starts on a very elementary level (perhaps too much so) and ends with an advanced topic on files. Each chapter ends with a self test section with answers. A good book for self-instruction, but may not be complete enough for a full classroom course. RB

APPLICATIONS, P, L*. *A Treatise on Time and Space.* J.R. Lucas. Methuen, 1973, xi + 321 pp, $25. A broad-ranging, elegant philosophical treatment using just enough mathematics and physics to make subtle concepts precise. Essentially neo-Kantian, Lucas often argues that the space-time continuum is the way it ought to be. An impressive amalgam of humanistic and scientific scholarship. LAS

APPLICATIONS, P. *The Study of Time.* Ed: J.T. Fraser, F.C. Haber, G. H. Müller. Springer-Verlag, 1972, viii + 550 pp, $20.30. Proceedings of a 1969 interdisciplinary conference at Oberwolfach on the nature of time including physical, biological, philosophical, sociological, psychological and religious issues. Each paper has since appeared in the journal *Studium Generale.* LAS

APPLICATIONS, P. *La Mathêmatisation des Doctrines Informes.* Georges Canguilhem. Hermann, 1972, 237 pp, 46F (P). Papers and discussions from a colloquy held at the Institute of the History of Science, University of Paris, June 1970. *Doctrines informes* in this context refers to "informal ideas" or "unstructured theories"--examples in this collection being drawn from mechanics, physics, biology, psychology, economics, and social science. *Mathêmatisation* is meant to include quantification, axiomatization, and reduction to simpler or better-described theories and structures. Several essays are historical in nature, including J.R. Ravetz's "Galileo and the mathematisation of 'speed'", the only one in English. Worthy of note are F. Bresson's survey of mathematization in psychology and F. Perroux's description of the limits of modelling in economics. PJC

APPLICATIONS (ARCHITECTURE), P. *Computers in Architectural Practice.* Bryan Guttridge, Jonathan R. Wainwright. Halsted Pr, 1973, xiii + 121 pp, $7.75. Analysis of current use in Britain of computers in pre-contract stage of architectural activities, with some comparisons

to the U.S. Uses mentioned: networking, scheduling, site analysis, cost control, system building, design and production information, brief information, schedules of accommodations. Includes 18 pages of sample printouts. Greatest potential appears to be in data retrieval at conclusion of design stage. High cost deters British firms from more widespread computer use. PJC

APPLICATIONS (ARCHITECTURE), S, *Dome Cookbook of Geodesic Geometry.* David Kruschke, 2135 W. Juneau Ave., Milwaukee, Wisc., 53233, 1972, 46 pp, $2 (P). For Fuller freaks of a mathematical bent. Author gives a correct construction of chord factors for a 3-frequency triacon dome all of whose ground-level vertices are coplanar; the proof is not provided. This construction puts to shame the authors of various books on domes (except Fuller), especially *Domebook Two*, which claimed such a dome was impossible. The booklet is hand-lettered with slightly sloppy illustrations. It also includes calculations (to 10 decimal places, of which perhaps 5 are significant) of chord factors for the 4-frequency dome, which agree closely with Fuller's unpublished results. This booklet would have been unnecessary if Fuller had published his derivations. PJC

APPLICATIONS (URBAN PLANNING), S**, P*, L*, *Compact City: A Plan for a Liveable Urban Environment.* George B. Dantzig, Thomas L. Saaty. Freeman, 1973, xi + 244 pp, $4.50 (P); $9. A specific bold proposal and feasibility study by operations researchers for a Soleri-like new city, designed to exploit the vertical (thru layering) and time (thru 24-hour scheduling) dimensions. Includes chapters on layout, transportation, cost ("bargain"), advantages, how to get it built. Lots of quick-and-dirty estimates. Parts II and III discuss the role of operations research in urban planning and the social implications of Compact City. Many illustrations and diagrams, attractive format. An intellectually convincing study, but the question lingers: who would like to live in Compact City? PJC

APPLICATIONS (PHYSICS), P, *Lecture Notes in Mathematics-313: Spectral Properties of Hamiltonian Operators.* Konrad Jörgens, Joachim Weidmann. Springer-Verlag, 1973, 140 pp, $6 (P). N-particle Hamiltonians in quantum mechanics: summary of preceeding work is followed by new results on the essential spectrum. JAS

APPLICATIONS (PHYSICS), P, *Tables of Branching Rules for Representations of Simple Lie Algebras.* Jiri Patera, David Sankoff. Pr U Montreal, 1973, 99 pp, $6.25 (P). Computer generated tables designed to assist physicists in determining possible inclusions among semi-simple Lie groups and corresponding branching rules. LAS

Reviewers Whose Initials Appear Above

Donald A. Alton, U. of Iowa; David F. Appleyard, Carleton; Ralph Bjork, St. Olaf; Paul J. Campbell, St. Olaf; Judith N. Cederberg, St. Olaf; John Dyer-Bennet, Carleton; Steven Galovich, Carleton; Loren Haskins, Carleton; Paul S. Jorgensen, Carleton; Roger B. Kirchner, Carleton; Richard S. Kleber, St. Olaf; Loren C. Larson, St. Olaf; Pierre J. Malraison, Carleton; R.W. Nau, Carleton; Seymour Schuster, Carleton; J. Arthur Seebach, Jr., St. Olaf; Lynn A. Steen, St. Olaf; T.A. Vessey, St. Olaf; Frank L. Wolf, Carleton.

NEWS AND NOTICES

EDITED BY RAOUL HAILPERN, SUNY at Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to Mathematical Association of America, 1225 Connecticut Avenue, NW, Washington, D. C. 20036. Items must be submitted at least two months before publication can take place.

PERSONAL ITEMS

Ithaca College: Assistant Professors S. R. Hilbert, C. F. Keleman, and J. W. Rosenthal have been promoted to Associate Professors.

State University of New York College at Oswego: Assistant Professors Fredrick Barber and Spyros Magliveras have been promoted to Associate Professors.

Assistant Professor R. W. De Gray, Utica College, has been elected Coordinator, Science and Mathematics Division.

Assistant Professor Robert Johnson, Ohio State University, has been appointed Assistant Professor at Morningside College.

Assistant Professor R. A. Johnson, Washington State University, has been promoted to Associate Professor.

Associate Professor S. G. Leelamma, State University of New York College at Geneseo, has been promoted to Professor.

Assistant Professor J. L. Murphy, California State College, San Bernardino, has been promoted to Associate Professor.

Assistant Professor W. M. Patterson, U. S. Air Force Academy, has been promoted to Associate Professor.

Assistant Professor P. J. Ryan, University of Notre Dame, has been appointed Associate Professor at Indiana University, South Bend.

Associate Professor Hari Shankar, Ohio University, was the recipient of the "Outstanding Teacher" of Mathematics award in recognition of his outstanding teaching abilities and effective insight into the educational process. This honor bestows on him the title of University Professor and an honorarium of $1,000.

Assistant Professor Jesse Williams, Cheyney State College, has been promoted to Associate Professor.

Dr. Susan Williamson, Regis College, has been appointed Academic Dean.

Dr. Harold W. Bailey, Richmond, Virginia, died on January 24, 1974, at the age of 72. He was a member of the Association for fifty-two years.

Dr. Julius G. Baron, Rye, New York, died on July 7, 1973. He was a member of the Association for twenty years.

Professor Emeritus Herbert E. Buchanan, Tulane University, died on January 17, 1974, at the age of 92. He was a Charter Member of the Association.

Dr. Richard S. Burington, Consultant, Naval Air Systems Command, U. S. Navy, died on December 24, 1973, at the age of 72. He was a member of the Association for forty-seven years.

Professor William Fitch Cheney, Jr., University of Hartford, died on February 7, 1974, at the age of 79. He was a member of the Association for forty-seven years.

Professor Emeritus Walter H. Durfee, Hobart and William Smith Colleges, died on December 22, 1973, at the age of 84. He was a member of the Association for fifty years.

Assistant Professor Harold Thomas Fitzpatrick, Mohawk Valley Community College, died on June 30, 1973, at the age of 30. He was a member of the Association for five years.

Professor Emeritus Aubrey J. Kempner, University of Colorado, died on November 18, 1973, at the age of 93. He was a Charter Member of the Association.

Mr. Lawrence L. Mitchell, Eugene, Oregon, died on December 12, 1973. He was a member of the Association for seven years.

Dr. L. L. Silverman, University of Houston, died on October 18, 1973, at the age of 89. He was a Charter Member of the Association.

Professor Joseph L. Walsh, University of Maryland, died on December 10, 1973, at the age of 78. He was a member of the Association for fifty-three years.

A RENEWED APPEAL FOR THE PRESERVATION OF ARCHIVAL MATERIALS BY THE CBMS ADVISORY COMMITTEE ON HISTORY

Mathematics as a specialty and mathematical research are nearing their centenary in the United States — if we take as their beginning such symbolic bench-marks as the founding of the Johns Hopkins University (opened to students in 1876) and the publication of *The American Journal of Mathematics* (founded by Sylvester in 1878). Our major mathematical organizations (the A.M.S., M.A.A., and N.C.T.M.) have all celebrated their semi-centennials relatively recently. The beginnings of American mathematics are rapidly passing out of the realm of personal knowledge and oral anecdote into written history. It is now more than proper, it is urgent that original sources and archival materials which are the basis of this history be preserved. This committee made an appeal for such preservation through notes in the mathematical journals a few years ago [see the MONTHLY, Vol. 77 (Jan. 1970) pp. 110–112]. A renewed and expanded appeal seems appropriate today with the passage of time and of many funded projects and committees. Our previous appeal suggested that:

1. Individual mathematicians and their families should preserve such items as unpublished and semi-published materials, course notes and photographs of important colleagues, data on the organization and early days of societies. College archives, local museums, academies, and historical societies are potential depositories. For example, the Archive of Contemporary History at the University of Wyoming has been given materials by the Rudolph Langer family and is interested in acquiring materials relating to mathematics and science; the Harvard University Archives have papers of Benjamin Osgood Peirce and George D. Birkhoff; while the Niels Bohr Library of the Centre for the History of Physics has copies of mathematical notebooks of Herman Minkowski. [There is a *Guide to Archives and Manuscripts in the United States* by P. M. Hamer (Yale University Press, 1961), the Library of Congress maintains *The National Union Catalogue of Manuscript Collections*, and the Center for the History of Physics maintains a National Catalog of Sources for the History of Physics and Astronomy.]

2. *Organizations* should be diligent in preserving their records.

3. *Mathematics Departments* of colleges and universities were also urged to take appropriate action. We would like to extend this latter suggestion as follows:

(a) have a *department history written and deposited* in the department office or in the college's library and archives. The committee is aware of such materials at Dartmouth, Harvard, and Michigan, but has found that most traces and memories of an important American mathematician, including a diary of studies abroad, have largely disappeared at another university which formerly went so far as to establish a fellowship in his name.

(b) collect, date, and label *photographs* of staff members, visiting lecturers, participants in colloquia and conferences held on campus.

(c) collect and preserve the unpublished minutes and proceedings of committees, conferences, projects, and perhaps even staff meetings.

(d) urge staff members to prepare a skeleton key to their own publications.

The Committee suggests that persons with historical interests would do the mathematical community a service by undertaking such tasks as are listed above and by pursuing studies of periods, people, or influences upon American mathematics such as the influx of foreign mathematicians in the period centered about World War II.

CARL B. BOYER
CHURCHILL EISENHART
PHILLIP S. JONES, *Chairman*
KENNETH O. MAY
DAVID ROSENBLATT
CHARLES WEINER

1974-75 SABBATICAL EXCHANGE INFORMATION SERVICE

In 1973–74 the MAA initiated a new service for its members, called the MAA Sabbatical Exchange Information Service (SEIS), to assist faculty members in universities and colleges (both two- and four-year) in arranging what might be called "No-Cost Sabbaticals." We are happy to announce that SEIS was successful and will be repeated in 1974–75. For the benefit of readers who missed the 1973 announcement of SEIS we repeat here the complete original description, with an additional suggestion for an innovative use of this service.

Many MAA members are in institutions not offering faculty members a program of Sabbatical leaves. Even in institutions with sabbatical leave programs individual faculty members often find themselves ineligible for such a leave at a time when the desire for one is strongest. We all recognize the rejuvenating effect of an occasional change of scene, even if it does not involve release from teaching duties. The Association therefore suggests that an occasional exchange between two faculty members of similar interests, training, and experience at different institutions could be of great benefit to the individuals and also to their institutions. The individuals and institutions all stand to gain from the refreshment of exchanged ideas and insights.

It is often possible for two such faculty members to trade identities, so to speak, for a year. Such an exchange might involve trading teaching responsibilities, living quarters, and some departmental responsibilities. The extent of the exchange would depend on the individual circumstances. It is suggested, however, that salaries should not be exchanged or even discussed. Each faculty member would remain on the payroll of his permanent institution and receive all of his normal fringe benefits. Financially, his institution would not recognize the exchange at all.

A type of exchange that might be attractive in view of today's employment market is the following: Occasionally, a university with a research-oriented graduate program employs one of its own new Ph.D.'s in a postdoctoral position for a fixed term, usually one or two years. The young mathematician may be attracted to such a temporary position because of the opportunity for one or two more years in a research center. However, having received his Ph.D. he may be more concerned about his credentials and preparation for a more permanent position, which may not be primarily in research. Even though he anticipates a career in a smaller teaching-oriented institution, the new Ph.D. may remain in such a postdoctoral position while waiting for the right opportunity to materialize. Under these circumstances, the young mathematician might be happier at a smaller teaching-oriented institution for this transition year. At the same time an established mathematics faculty member at such a teaching-oriented institution frequently feels the desire to return to a research center for a year of mathematical refreshment.

These two individuals can both benefit from an exchange with each other. The new Ph.D. obtains valuable teaching experience enhancing his credentials for a permanent position in a similar institution. The teaching-oriented institution benefits from the stimulating enthusiasm of a young mathematician fresh from several years in a research center. The older colleague benefits from a return to the well, and he undoubtedly contributes much to the teaching staff of the university from his years of classroom experience.

For these reasons we urge young mathematicians who anticipate occupying such post-doctoral positions in 1975, and who might be interested in an exchange of the type described above, to list themselves in SEIS. We also urge mathematicians in research centers who counsel prospective and recent Ph.D.'s to have on hand copies of the SEIS list in case such an exchange should be suggested. Finally, we urge faculty members in smaller colleges who would be interested in an exchange involving a return to a research center to list themselves in SEIS.

The MAA proposes to become involved only to the extent of assisting in bringing together like-minded mathematics faculty members who are interested in an exchange. The information exchange will be accomplished by the annual publication by the Association in December of a list containing the names, addresses, and other pertinent information about members of the Association interested in arranging a "Sabbatical Exchange" with a colleague in another institution. This list will be sent free of charge to all those on the list and to any other MAA member who request it.

Members interested in being listed in December 1974 should write to "SEIS, The Mathematical Association of America, 1225 Connecticut Avenue, N. W., Washington, D. C. 20036," enclosing the following information about themselves:

1. Name
2. Institution
3. Department
4. Address
5. Rank
6. Major field of interest
7. Highest earned degree
8. Names of from one to five courses recently taught
9. Normal teaching load
10. Section of country preferred for visit: Northeast, Southeast, Northcentral, Southcentral, Northwest, Southwest.
11. Period for which exchange is desired, e.g., all of the academic year 1975–76, or the first two quarters of 1975–76, or the second semester of 1975–76, etc.

Communications must reach the Washington office by November 18, 1974, for inclusion in the December 1974 list.

FACULTY EXCHANGE CENTER

The Faculty Exchange Center aims to make it possible for a faculty member to exchange positions for a year with a colleague from another institution either here or overseas where instruction is in English. The F. E. C. will publish a catalog containing details on those who register. For more information and registration forms write to the Faculty Exchange Center, P. O. Box 1866, Lancaster, Pennsylvania 17604.

MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

THE 1974 WILLIAM LOWELL PUTNAM MATHEMATICAL COMPETITION

The thirty-fifth annual William Lowell Putnam Mathematical Competition will be held at participating institutions on Saturday, December 7, 1974. This competition is supported by the William Lowell Putnam Prize Fund for the Promotion of Scholarship and is under the sponsorship of the Mathematical Association of America. Colleges and universities in the United States and Canada may register eligible undergraduates in the competition. Application forms will be mailed to institutions on the competition's mailing lists by September 23, 1974. Forms may also be secured by writing the director, A. P. Hillman; William Lowell Putnam Mathematical Competition; University of New Mexico, P. O. Box, 10; Albuquerque, New Mexico 87131. Completed applications must be received by October 28, 1974; this deadline will be enforced strictly.

Further details are provided in the Announcement Brochure which is mailed with the registration forms. Reports of previous competitions, including examination questions and outlines of solutions, may be found in past issues of the MONTHLY. The most recent of these reports were in the November 1973, February 1973, August-September 1971, and August-September 1970 issues.

NEW SECRETARY OF THE ASSOCIATION

Professor David P. Roselle of Virginia Polytechnic Institute and State University has been elected by the Board of Governors as Secretary of the Association for the five-year term 1975–79. He will assume his responsibilities after the Washington, D. C. meeting of the Association in January 1975. Effective January 28, 1975, all correspondence for the Secretary of the Association should, therefore, be addressed to him.

Professor Roselle has been a member of the Association since 1971. He is well known to the readers of this MONTHLY as the Associate Editor for Mathematical Notes and Classroom Notes. He has also served the Association in many other capacities including membership on the Committee on Institutes since January 1973 and the CUPM Panel on Applied Mathematics since 1973. He was a panelist at the MAA Conference on Special Problems of Minority Groups in Atlanta, Georgia, in February 1972. He is presently serving on the ad hoc Committee to Review Financial Management and Control and on the Nominating Committee for an Associate Executive Director of the MAA. He has been an invited speaker at conferences and meetings held throughout the United States and Canada. His mathematical interests are in number theory and combinatorics, and have resulted in numerous publications in various professional journals.

In order to assure a smooth transfer of responsibilities of the Secretary, Professor Roselle has, over the period of the last year and a half, acquainted himself thoroughly with his new duties by visiting the Secretary's office in Davis, by attending all meetings of the Board of Governors, the Executive Committee, and the Finance Committee, and other committees since August 1973, and by reviewing all recent history of the Association. As a result of my close work with him over the past year and a half, I have become convinced that the Board has elected an unusually well-qualified member as its new Secretary. I am very pleased to be able to turn over my duties to Professor Roselle.

HENRY L. ALDER, *Secretary*

NOVEMBER MEETING OF THE NORTHEASTERN SECTION

The nineteenth annual meeting of the Northeastern section of the MAA was held at Boston University, Boston, Massachusetts, on November 24, 1973; there were eighty-eight people in attendance. The section chairman, John Fraleigh, presided at the morning session at which the following talks were given:

Matrix Automaton Transformations of Continued Fractions, By G. N. Raney, University of Connecticut.

Some of the Algebra in Algebraic Topology, by F. P. Peterson, Massachusetts Institute of Technology.

At the afternoon business meeting Ernest Schlesinger, Chairman of the Nominating Committee, proposed the following slate of officers for the coming year: Chairman, L. Aileen Hostinsky, Connecticut College; Vice Chairman, Anne F. O'Neill, Wheaton College; Secretary-Treasurer, G. W. Best, Phillips Academy. The slate was elected unanimously. The nominating committee also recommended that the section consider amending the by-laws so that the immediate past chairman of the section join the current officers and sectional governor as a member of the executive committee. The business meeting concluded with Donald Small giving a report on the recent meeting of the section officers.

The following talks completed the program:

Irregular Primes: Their History and Some Current Problems, by R. W. Johnson, Bowdoin College.

Applications of Interactive Computer Graphics to Classical Number Theory (with computer generated illustrations), by C. M. Strauss, Brown University.

G. W. BEST, *Secretary-Treasurer*

NOVEMBER MEETING OF THE PHILADELPHIA SECTION

The forty-eighth annual meeting of the Philadelphia Section of the MAA was held at Drexel University, Philadelphia, Pennsylvania, on November 17, 1973. The Section Chairman, Professor W. J. Pervin, presided at the meeting. The meeting was attended by 179 persons, including 154 members of the Association.

The following section officers were elected: Chairman, Professor J. W. P. Mayer, Lebanon Valley College; Vice-Chairman, Professor E. A. Klotz, Swarthmore College; Secretary-Treasurer, Professor P. E. Bedient, Franklin and Marshall College; Executive Committee Members-at-Large, Professor J. P. King, Lehigh University, and Professor S. H. Plotkin, Montgomery County Community College.

The top performer from the Section in the 1972 Putnam Competition, D. S. Hough, Swarthmore College, was recognized and awarded a one year membership in the Association. Honorable mention citations were also presented to: D. S. Shucker, Swarthmore College, and Kim Schroeder, Bucknell University.

The following papers were presented:

Mathematics and the behavioral sciences, by David Rosen, Swarthmore College.

Ghosts of departed quantities, by M. D. Davis, Courant Institute, New York University.

The use of computers in undergraduate mathematics teaching, by Marialuisa McAllister, Moravian College.

Some mathematical operations research in government, by A. J. Goldman, Chief of Operations Research, National Bureau of Standards.

A. E. FILANO, *Secretary-Treasurer*

MARCH MEETING OF THE SOUTHEASTERN SECTION

The fifty-third annual meeting of the Southeastern Section of the MAA was held at the University of Tennessee, Knoxville, Tennessee, on March 29–30, 1974. A total of 324 persons attended the meeting, including 234 members of the Association. The local arrangements committee was headed by J. H. Carruth.

Three invited addresses were given: Professor J. W. Neuberger (Section Lecturer) of Emory University on "Derivatives and multilinear algebra"; Professor R. J. Plemmons of the University of Tennessee on "Modern algorithms for solving linear systems," and Professor Ernst Snapper of Dartmouth College on "Algebraic foundations of geometry". There was a Symposium on Mathematics in the Two-Year Colleges, featuring a talk by A. B. Hartung on "What are we looking for in a two-year college mathematics teacher?" and a panel discussion with Professors Hartung, Don England and Morris Marx.

There were nine sessions for contributed papers, including one session of papers by students. Presiders for the general sessions were H. V. Park (Chairman of the Section), Lida K. Barrett, D. L. Hunter, W. R. Wilson and J. H. Wahab and for the special sessions were J. C. Halsey, M. F. Neff, T. J. Pignani, F. C. Toney, V. R. R. Uppuluri, J. C. Wilson, T. H. Blackburn, H. E. Taylor and S. M. Lukawecki. Two MAA films were shown on Friday night.

Officers elected for 1974–5 were: Chairman, J. H. Wahab, University of South Carolina; Chairman-Elect, J. R. Wesson, Vanderbilt University; Vice-Chairman, D. L. Hunter, Central Piedmont Community College; Section Lecturer, W. R. Mann, University of North Carolina at Chapel Hill.

At the business meeting, Mr. A. S. Kyle of Davidson College was presented as the winner of the $25 given to the student in a Section school who scores highest in the Putnam Competition.

The following papers were presented:

1. *Characterizations of some standard concepts in nonstandard topology*, by J. W. Hall, Clayton Junior College.

2. *Pointwise limits of Darboux functions*, by M. H. Miller, Jr., University of Alabama.

3. *Convex topology*, by Douglas Moreman, Emory University.

4. *Gamma spaces and gammafications of topological spaces*, by J. A. Bond, Jr., Macon Junior College.

5. *Note on one-dimensional planar clans*, by Jewell McMorris, University of Dallas.

6. *A polynomial technique for factoring graphs*, by J. T. B. Beard and Ann D. Dorriss, The University of Texas at Arlington.

7. *F-reflexive modules*, by C. B. Myers, Austin Peay State University.

8. *An application of pairwise balanced designs to constructing nonisomorphic quasigroups satisfying two variable identities*, by C. C. Lindner, Auburn University.

9. *Generalizations of the spectral theorem for matrices*, by R. E. Hartwig, North Carolina State University.

10. *Necessary and sufficient conditions for a matrix to be completely positive*, by D. M. Jordan, University of South Carolina.

11. *Obrechkoff formulas and summation*, by C. H. Frick, White Rock, South Carolina.

12. *Matrix theory versus the theory of the Fredholm integral equation*, by F. Virginia Rohde, Mississippi State University.

13. *A proof of Stoilow's theorem (for analytic functions) suitable for a first course in complex analysis*, by H. T. Mathews, The University of Tennessee, Knoxville.

14. *Piecing functions together*, by K. E. Whipple, Georgia State University.

15. *A series that represents any number*, by F. L. Celauro, George Peabody College for Teachers.

16. *Mathematical modelling — pro and con*, by Robert Fennell and John Luedeman, Clemson University.

17. *Galton's quincunx revisited*, by J. D. Austin, Emory University.

18. *Some changes in the verbal behavior patterns of mathematics teachers during training in interaction analysis*, by J. W. Daniels, East Carolina University.

19. *Teaching mathematics by the Keller plan*, by Suzanne McGill, University of South Alabama.

20. *Are axioms assumed to be true? Are theorems proved to be true?*, by Fredrick Binford, Tennessee State University.

21. *A time-sharing computer system demonstration of Fourier series approximation of functions*, by L. L. Long, Tennessee Technological University.

22. *Numerical differentiation for calculus students*, by D. A. Smith, Duke University.

23. *The use of computer programming in the teaching of a service course in statistics*, by C. F. Kossack, The University of Georgia.

24. *Computer assisted instruction in statistics*, by T. F. Higginbotham, Auburn University.

25. *Computer assisted linear algebra*, by R. D. Fray, Furman University.

26. *Mathematical serendipity from a computer program*, by F. R. Norris and Thad Dankel, Jr., University of North Carolina at Wilmington.

27. *Describing puzzles to a computer*, by R. W. Gibson, Auburn University.

28. *Monthly problem E* 2446, by Nanetta B. Lowe, Bennett College.

29. *Monthly problem E* 2435, by Gloria J. Phillips, Bennett College.

30. *Monthly problem E* 2461, by Reba M. Turner, Bennett College.

31. *Two combinatorial problems*, by D. L. Webb, The University of Tennessee, Knoxville.

32. *Monthly problem E* 2450, by Cynthia Hardy, Bennett College.

33. *Construction of a strictly increasing continuous singular function in* $[0,1]$, by J. W. Crawley, Jr., The University of Tennessee, Knoxville.

34. *The extremal length problem*, by Jeffrey Wiener, Emory University.

35. *Weierstrass theorems in p-adic fields*, by C. G. Wagner, The University of Tennessee, Knoxville.

36. *On the correspondence between semigroups of operators and transition functions*, by Thad Dankel, Jr., University of North Carolina at Wilmington.

37. *Uniqueness criteria for solutions of initial value problems for ordinary differential equations*, by T. C. Gard, The University of Tennessee, Knoxville.

38. *Some oscillation criteria for delay-differential equations of even order*, by R. D. Terry, Georgia Institute of Technology.

39. *A generalization of the basis concept in Banach spaces*, by D. M. Dorris, Lawton, Oklahoma.

40. *Expository pitfalls in elementary algebra*, by S. A. Stricklen, Southern Technical Institute.

41. *An investigation and evaluation of goals of mathematics education for prospective elementary teachers*, by Iris Mack Dayoub, Atlanta, Georgia.

42. *The small college mathematics program in the South*, by W. M. Mitchell, Peabody College.

43. *Transformational geometry for teacher training*, by J. W. Lott, Georgia State University.

44. *One on one mathematics*, by D. M. Jordan, University of South Carolina.

45. *A billiard table in* E^n *space*, by H. S. Hahn, West Georgia College.

46. *Explicit solutions of a class of countable linear systems of ordinary differential equations*, by W. P. McKibben, Georgia Institute of Technology.

47. *Ambushing random walks I. Finite models*, by W. H. Ruckle, R. Fennell, C. Fennemore and P. Holmes, Clemson University.

48. *A disaggregated optimal control model of private incentives to housing maintenance*, by L. L. Dildine and F. A. Massey, Georgia State University.

49. *The protein vector — armament for nutritional self-defense*, by B. A. Fusaro, Queens College.

50. *A gradient method for long-term optimal scheduling of a reservoir system*, by E. C. Anderson and Ming Chen Shiao, Tennessee Valley Authority.

J. D. NEFF, *Secretary-Treasurer*

NEW SECTIONAL GOVERNORS OF THE ASSOCIATION

The following have been elected Governors of the Association representing the Sections indicated:

FLORIDA	Robert C. Meacham, Eckerd College
ILLINOIS	John A. Schumaker, Rockford College
IOWA	Elsie C. Muller, Morningside College
LOUISIANA-MISSISSIPPI	Russell A. Stokes, University of Mississippi
MARYLAND-DC-VIRGINIA	Robert H. Owens, University of Virginia
MICHIGAN	Ruel V. Churchill, University of Michigan
NORTH CENTRAL	C. Murray Braden, Macalester College
PHILADELPHIA	David Rosen, Swarthmore College
SOUTHERN CALIFORNIA	Thomas N. Robertson, Occidental College
TEXAS	Charles R. Deeter, Texas Christian University

The highest percentage of voters was 43%, occurring in the Iowa Section.
The Louisiana-Mississippi Section was the runner-up with 38%.

A. B. WILLCOX, Executive Director

CHANGE OF ADDRESS

AMERICAN MATHEMATICAL MONTHLY

OLD ADDRESS

Please affix address label from a recent issue here. The codes it contains are essential for prompt and accurate processing of your requested change.

NEW ADDRESS

NAME

ADDRESS

CITY STATE ZIP

Please return this form to

The Mathematical Association of America
1225 Connecticut Ave., NW
Washington, D.C. 20036

CALENDAR OF FUTURE MEETINGS

Summer Meeting 1974: There will be no joint summer meeting in 1974, in order that mathematicians may attend the International Congress of Mathematicians to be held in Vancouver, British Columbia, August 21–29, 1974.

Fifty-eighth Annual Meeting, Washington, D. C., January 25–27, 1975.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Editorial Director.

ALLEGHENY MOUNTAIN, Duquesne University, Pittsburgh, Pennsylvania, May 9–10, 1975.

FLORIDA, Manatee Junior College, Bradenton, March 7–8, 1975.

ILLINOIS, Rockford College, Rockford, May 9–10, 1975.

INDIANA, Indiana University — Purdue University at Indianapolis, Indianapolis, November 30, 1974.

IOWA, Iowa State University, Ames, April 18–19, 1975.

KANSAS

KENTUCKY

LOUISIANA-MISSISSIPPI, Centenary College, Shreveport, Louisiana, February 1975.

MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA

METROPOLITAN NEW YORK

MICHIGAN

MISSOURI, Missouri Western College, St. Joseph, Spring, 1975.

NEBRASKA, Nebraska Wesleyan University, Lincoln, April 18–19, 1975.

NEW JERSEY, Princeton University, Princeton, October 12, 1974.

NORTH CENTRAL, Moorhead State College, Moorhead, Minnesota, October 26, 1974.

NORTHEASTERN, Lowell Technological Institute, Lowell, Massachusetts, November 30, 1974.

NORTHERN CALIFORNIA, Menlo College, Menlo Park, February 8, 1975.

OHIO, University of Cincinnati, Cincinnati, November 1–2, 1974.

OKLAHOMA-ARKANSAS, Central State University, Edmond, Oklahoma, April 4–5, 1975.

PACIFIC NORTHWEST

PHILADELPHIA, Swarthmore College, Swarthmore, Pennsylvania, November 23, 1974.

ROCKY MOUNTAIN, Mesa College, Grand Junction, Colorado, April 11–12, 1975.

SEAWAY, St. John Fisher College, Rochester, New York, November 1–2, 1974.

SOUTHEASTERN, University of South Alabama, Mobile, March 21–22, 1975.

SOUTHERN CALIFORNIA

SOUTHWESTERN

TEXAS, Angelo State University, San Angelo, April 1975.

WISCONSIN, University of Wisconsin — Superior, Superior, April or May 1975.

FUTURE MEETINGS OF OTHER ORGANIZATIONS

AMERICAN ASSOCIATION FOR THE ADVANCEMENT OF SCIENCE

AMERICAN MATHEMATICAL SOCIETY, Washington, D. C., January 23–26, 1975.

AMERICAN SOCIETY FOR ENGINEERING EDUCATION, Colorado State University, Fort Collins, June 16–19, 1975.

ASSOCIATION FOR COMPUTING MACHINERY, San Diego, California, November 11–13, 1974.

ASSOCIATION FOR SYMBOLIC LOGIC, Shoreham Hotel, Washington, D. C., January 23–24, 1975.

FIBONACCI ASSOCIATION

INSTITUTE OF MATHEMATICAL STATISTICS

MU ALPHA THETA

NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS, Washington, D. C., January 25–26, 1975 (joint meeting with MAA).

OPERATIONS RESEARCH SOCIETY OF AMERICA, San Juan, Puerto Rico, October 16–18, 1974.

PI MU EPSILON, Western Michigan University, Kalamazoo, August 19–20, 1975.

SCHOOL SCIENCE AND MATHEMATICS ASSOCIATION, Sheraton-Gibson Hotel, Cincinnati, Ohio, November 7–9, 1974.

SOCIETY FOR INDUSTRIAL AND APPLIED MATHEMATICS, Shoreham-Americana Hotel, Washington, D. C., October 23-25, 1974.

THE AMERICAN MATHEMATICAL MONTHLY

(FOUNDED IN 1894 BY BENJAMIN F. FINKEL)

THE OFFICIAL JOURNAL OF

THE MATHEMATICAL ASSOCIATION OF AMERICA

VOLUME 81 NUMBER 8

CODEN: AMMYAE

CONTENTS

(*Continued on inside cover*)

OCTOBER **1974**

NOTICE TO AUTHORS

Research papers per se are unsuitable; see statement of policy (Vol. 81, p. 1).

Please follow the format in current issues of the Monthly. Manuscripts must be legibly typewritten or reproduced from typewritten copy, double spaced with wide margins. Three copies should be submitted to the appropriate editor and one kept by the author as protection against loss. The author's full address *must* appear at the end of the manuscript.

Backlog: Main Articles 12 months, Math. Notes 24 months, Research Problems 12 months, Classroom Notes 23 months, Math. Education 20 months.

EDITORIAL CORRESPONDENCE AND MAIN ARTICLES: to Alex Rosenberg, Department of Mathematics, Cornell University, Ithaca, N.Y. 14853; NOTES, etc.: to the corresponding Associate Editor; REPRINT PERMISSION: to Leonard Gillman, Mathematical Association of America, University of Texas, Austin, Texas 78712 (see also the copyright notice below); ADVERTISING CORRESPONDENCE: to Raoul Hailpern, Mathematical Association of America, SUNY at Buffalo, Buffalo, N. Y. 14214; CHANGE OF ADDRESS and SUBSCRIPTIONS: to A. B. Willcox, Mathematical Association of America, 1225 Connecticut Ave., N.W., Washington, D.C. 20036.

Annual dues for members of the Association (including a subscription to the American Mathematical Monthly) are $14.00 for each of the first two years of membership and $18.00 thereafter. Student Membership is available with annual dues of $9.00. For nonmembers the subscription price is $18.00.

Published by the Association at Washington, D. C., and Menasha, Wisconsin, during the months of January, February, March, April, May, June–July, August–September, October, November, December.

Second-class postage paid at Washington, D. C., and additional mailing offices.

PRINTED IN THE UNITED STATES OF AMERICA

MATHEMATICAL REFLECTIONS

SALOMON BOCHNER

Part I. Purely Mathematical Americana

My first awareness of American mathematics came soon after my enrolling as a student at the University of Berlin, in the late fall of 1918. In the early part of 1919 there began to appear Leonard Dickson's three-volume work *History of the Theory of Numbers* (1919-23), the scholarship of which made a deep impression. For instance, it was a rumor among students that the proof-reading of the galleys took Dickson two whole years.

It is a curious feature of this work, which I discovered only much later, that it apparently nowhere states, — certainly not with due emphasis, — the "fundamental theorem" on prime numbers, namely that every natural number is a product of primes, and uniquely so. The report does have statements, in abundance, which correlate properties of a natural number to properties of its prime number factors, but it is shy of stating — emphatically at any rate — that every natural number is a product of prime numbers and uniquely so. This reluctance to present the fundamental theorem in an otherwise very conscientious historical account may be due to the fact that, while this theorem is sometimes vaguely attributed to Euclid, it had apparently not been stated expressly before 1801, when Gauss featured it in his *Disquisitiones arithmeticae*, with some emphasis, and rather early in the work. The nearest that Euclid himself came to this theorem was his proposition (vii, 24): "if two numbers be prime to any number, their product also will be prime to the same."

G. H. Hardy and E. M. Wright, in their *An Introduction to the Theory of Numbers*, first edition 1938, make the following statement, on page 182, which, as they acknowledge on page 188, was triggered by remarks made to G. H. Hardy by myself, and I well remember that the remarks were made, sometime in 1933, in Hardy's large living room in Trinity College, in Cambridge, England:

> "It might seem strange at first that Euclid, having gone so far, could not prove the fundamental theorem itself; but this view would rest on a misconception. Euclid had no formal calculus of multiplication and exponentiation, and it would have been most difficult for him even to state the theorem. He had not even a *term* for the product of more than three factors. The omission of the fundamental theorem is in no way casual or accidental; Euclid knew very well that the theory of numbers turned upon his algorithm, and drew from it all the return he could." (Hardy and Wright, *loc. cit.*).

This is a 20th century insight, whereas Dickson's book is still rooted in Victorianism, and the difference is a telling one. Altogether, the work of Hardy and Wright, however technical, is a rich source book for the history of mathematical ideas.

Of course, what had discomfited Dickson was not just the fact that the fundamental theorem does not occur in Euclid, but, undoubtedly much more so, that none of the great number-theorists in the 17th and 18th centuries, like Fermat, Wallis, Euler, Lagrange, etc., had chanced upon it either. This may have been a general malaise among Victorian number theorists from which Dickson was unable to free himself. In fact, some textbooks in the 19th century did present and analyze the fundamental theorem in a forthright and satisfactory manner. But even when they expressly credited the fundamental theorem to Gauss, they did so in a very circumspect manner, as if to leave for themselves an escape tunnel in case somebody should suddenly discover that the theorem had been explicitly known in the 18th century after all.

But their circumspection was not called for. It was a very natural development of mathematical ideas that the fundamental theorem reached the stage of explicit articulation in the 19th century only. In my book *The Role of Mathematics in the Rise of Science*, (1966) on p. 16, I put it thus:

> "Arithmeticians have been showing embarrassment over the fact that the express formulation of the [fundamental] theorem came so late, and they have been trying to "pre-date" it. But there is no substance to assertions that the fundamental theorem had been consciously known to mathematicians before Gauss, but that they had neglected to make the fact known. We think that the 17th, and even the 18th, century was not yet ready for the peculiar kind of mathematical abstraction which the "fundamental theorem" involves; just as only the 19th century was comfortably prepared to conceptualize satisfactorily the notions of real number, limit, derivative, convergence, etc."

I am stating this here because, while Leonard Dickson was not yet receptive to this kind of explanation, a Charles Sanders Peirce, who was even almost a generation older than Dickson, already had been. I could not off-hand quote a reference in C. S. Peirce to this kind of comprehension, mainly because Peirce never wrote any kind of comprehensive treatise himself and because his purely mathematical writings have not yet been published at all. But they are being edited, by Carolyn Eisele, in 4 or 5 volumes, and some day I hope to trace this sort of thinking to Peirce, who was very advanced in many things of this kind.

Altogether, something about Dickson's *History* makes it a very "old-fashioned" number theory, and if one wants to have a brief concise summary of this old-fashioned number theory, then one may take to hand an article, from around 1928, by E. Bessel-Hagen, in the so-called Pascal's *Repertorium der höheren Mathematik*. But if one wants a good whiff of modernity, of genuine modernity, not just of surface "radicalism," then one may take to hand the 1923 book on number theory by E. Hecke (*Vorlesungen über die Theorie der algebraischen Zahlen*), which, I think, has never been translated. One of the first statements in the book is a formulation and proof of the then "abstract" theorem that a finite abelian group is a product of cyclic

groups. Before Hecke's book, this beginner's theorem was only presented in the midst of applications, usually in the theory of the cyclotomic equation for primitive roots of unity. Also, around 1923, there was still something "ancillary" about "abstract" theorems of this kind. Thus, among my fellow students in Berlin there was some brow knitting over the fact that Heinz Prüfer, who was several years older, dared to initiate his academic career in mathematics, for which he was obviously heading, with a doctoral thesis in 1921, — entitled *Unendliche Abelsche Gruppen von Elementen endlicher Ordnung* — on nothing other than the structure of abelian groups, albeit his groups were infinite ones. Some of us students took it even for granted — I don't know why — that the great algebraist Issai Schur, who was Prüfer's thesis advisor, was rather lukewarm towards this work.

Returning to Dickson it must be said that he himself was also quite "advanced" in some specialized directions. Thus, Issai Schur in a middling advanced classroom course, referred emphatically to his book: *Linear groups with an exposition of the Galois field theory*, which appeared, in English, in the prestigious Teubner Collection in 1901, as *the* source book for Galois Fields, i.e., finite fields. Around that time only very few books of this Collection were in a language other that German. However, several years after my student days van der Waerden's *Algebra* came out, in which the Galois Field theory occupies only several pages altogether; and the book of Dickson was soon eclipsed and forgotten.

Another book by an American, of which I became aware very early, was the *Elementary Principles in Statistical Mechanics* by Josiah Willard Gibbs, which appeared in 1902, and was almost instantly translated into German by somebody who later became a big name, namely by Ernst Zermelo, creator of the *Axiom of Choice*. And Zermelo performed the translation very admiringly even though Gibbs did not even quote Zermelo for an important theoretical objection — the so-called *Wiederkehreinwand* — to the assertion that the entropy is steadily increasing. This objection, raised by Zermelo in 1896, argues that "the usual kinetic model of a completely and permanently isolated gas behaves quasiperiodically" in time, and thus should not be endowed with a quantity, like the entropy, whose growth in time is in no wise periodic. An earlier objection, the so-called *Umkehreinwand* raised by Joseph Loschmidt already in 1876, is the "crude" argument that a mechanical system, and thus also any physical process described by a mechanical system, is reversible with regard to time, whereas the growth of the entropy is not. For references and details see Paul and Tatiana Ehrenfest, *The Conceptual Foundations of the Statistical Approach in Mechanics* (1912), Cornell University Press 1959, especially pages 14 and 15.

I do not remember whether I ran up against the name of Gibbs in a lecture course or by myself; but I do remember most distinctly that it was in a lecture course, again of Issai Schur, that I was introduced to the name of Joseph Henry Maclagan Wedderburn, which Schur wrote out, fully and solemnly, on the blackboard, as that of the "true" expert on "*Schiefkörper*," that is "non-commutative fields." Schur also mentioned that Wedderburn was at Princeton, in the U.S.A., and this was the

first time that I was introduced to the name of the place with which I would become associated for so long.

I do not think that Schur at the time introduced the word "hypercomplex system," or some such expression, and I am most certain that he did not speak of AN algebra, as a mathematical object with certain properties. Algebra was a part of mathematics, namely the part dealing with symbols, and AN algebra was, around 1920, still totally unfamiliar to a student on the Continent, like myself. In fact, AN algebra, as a term and object, is in my present-day view, the most outstanding 19th century "native" American contribution to present-day mathematics, and I must briefly report on how I found out about it. I heard the term for the first time sometime in 1925, in Oxford, at a small seminar of G. H. Hardy, in which there were, among others, myself, Besicovitch, Oppenheim, perhaps Titchmarsh and Echo Dolores Pepper, a young American girl on a fellowship. She came from Chicago, and she gave at this seminar a lecture on algebras, which left all of us totally uncomprehending what it was all about. She spoke in a well-prepared, well-articulated self-confident manner, but none of us had remotedly heard before the terms she used, and we were lost.

Still, to my own astonishment and education, only ten years later, in 1935, there appeared a German book by a young algebraist — no more than 28 years of age — one Max Deuring, with the title: *Algebren*, (*Algebras*). The word *Algebren* in German sounded to me harsh and unappealing, and yet when I turned over the pages in the book, the references were to mathematicians all over the world, with American mathematicians perhaps not even predominating.

This did not still in me the quest for knowing when the concept of AN algebra had really come into being. However, this quest was dormant in me for many years, and only about 10 years ago did I make an earnest attempt to answer the quest. I had to go back surprisingly far, until I finally landed in Benjamin Peirce's famous essay, *Linear associative algebra*, first published, quite informally, in 1870-71, based on lectures given in 1866–70 to the National Academy of Sciences, and then published formally in volume 4 of the American Journal in 1882.

I now wish to make some external observations on this article. It not only speaks of *an* algebra, but it also uses the plural "algebras," and this probably for the very first time anywhere in the English language. It also created the terms *idempotent* and *nilpotent* in our present-day sense. It tried to introduce a *nilfactor* which was used sporadically, but this is nowadays a *zero divisor*. Correspondingly, it also tried to introduce an *idemfactor*, to indicate a solution of $ab = a$ for $b \neq 1$, but this has apparently not caught on at all.

Furthermore, and very importantly, the edition of the essay in the *American Journal* is annotated by the son of Benjamin Peirce, Charles Sanders Peirce, by far the greater intellect of the two. C. S. Peirce created American pragmatism, which I do not understand at all, and which is, objectively speaking, not easy to define or characterize. For the latest about "Pragmatism," see the article under this title, by Philip P.

Wiener, in the *Dictionary of the History of Ideas*, 1973. In a different direction, C. S. Peirce created the algebra of relations, or algebra of "relatives," as he called it. This is probably his most specific achievement of a mathematical turn, and probably his most durable. But he also tried to create something specifically philosophical of a mathematical hue, which, if successful, would have become his greatest feat. In this he failed, even decisively so, but to this I want to devote the entire second part of this paper, because it was his most deeply mathematical attempt, although in a very non-technical, purely philosophical sense. In continuing, I want to report on three Americana, which I have chanced upon, one from the 20th century, and two from the 19th century.

The one from the 20th century is a very peculiar one, one that is hardly realized. Carathéodory published three major works: firstly his large book on real variables which appeared in 1918; afterwards, a book on the Calculus of Variations which appeared in the early 1930's, and finally a two-volumed work on functions of one complex variables (*Funktionentheorie*) which was put through the press posthumously in 1950. Actually Carathéodory was planning the book on complex variables ahead of the book on the calculus of variations, but what was holding up the book on complex variables was a widespread presumption among analysts in the 1920's that any book on complex variables, if to be complete, and to deserve the name, has to contain a complete and rigorous proof, without any prerequisites, for the theorem of C. Jordan that a simple closed curve in the plane decomposes the plane into two disjoint parts, an interior and an exterior. The analysts in the 1920's knew full well that this was a "topological" theorem, which already in 1911 had been extended by L. E. J. Brouwer from a simple closed curve in two dimensions to an $(n-1)$-dimensional topological sphere in n dimensions. Also, in 1923, K. Kerekjartó published an extensive work on two dimensional topology in which the Jordan curve theorem, and all other two dimensional prerequisites of complex function theory, as then needed, were treated in great rigorous detail. But, for all that, the complex variable analysts of the time were still vying with each other in the quest for producing, for the Theorem of Jordan, a proof to end all such proofs, and Carathéodory was experimenting for many years with a proof of his own.

What finally did persuade Carathéodory, however reluctantly, to dispense with such a proof of his own, was the following. About 1930 he made a grand tour of the United States, in which he was received almost regally in all great places on the East Coast, on the West Coast, and in the Mid-West. Now, at a stopover at Princeton, he met James Alexander, who had recently not only given a very elegant topological proof of Jordan's theorem for $(n-1)$-dimensional spheres in E_n, but had even succeeded in giving a meaningful extension to lower dimensional spheres in Euclidean E_n, something European mathematicians did not have in their vision at all. This finally gave Carathéodory the excuse for abandoning efforts of his own, and the published version of Carathéodory's work accepts two-dimensional topology as known.

Another American mathematical event in the 20th century, also a topological one, that had a sobering effect on Carathéodory was the great work of Marston Morse around 1928. Until then, Carathéodory mastered everything and anything ever produced in the Calculus of Variations, but I think the Morse Theory was something he no longer understood. I remember an evening, sometime in 1931 or 1932, in the home of Carathéodory, with S. Lefschetz, who was passing through Munich, also present, at which a student of Carathéodory tried to present the main paper of Morse. It was obvious to me that nobody present understood much. I was looking into the face of Lefschetz trying to gauge how much he understood at least, but he was very good at not giving himself away.

But now, for two very separate Americana from the 19th century. For the first I must return to my earliest student days. One of the first mathematics books I ever bought, perhaps literally THE first one, was Hermann Weyl's *Die Idee der Riemannschen Fläche*, (1913), or, as the translation has it, *The Concept of a Riemann Surface.* I studied this book for years on end — on and off, of course —, and many a piece of work of mine originated this way. But there was one thing in it that I have never been able to assimilate, to my own way of thinking, and this is the so-called Riemann-Roch theorem. On page 137 it reads as follows:

On a surface of genus p, if d is a divisor of order m ($m = m_1 + \cdots + m_r$; $d = p_1^{m_1} \cdots p_r^{m_r}$, $m_q \gtreqless 0$), *if B is the number of linearly independent* (in the complex sense) *differentials which are multiples of d; and if A is the number of linearly independent meromorphic functions which are multiples of* $1/d$, *then*

$$A + (p - 1) = B + m.$$

This is undoubtedly an aspect of a very comprehensive duality theorem, which lies athwart all of mathematics. Schematically, as I see it, the most important typical aspects of the comprehensive duality theorem are the following ones. If both sides of the duality are continuous sums, then the duality is a functional equation of the Riemann-Hecke type for zeta functions in number theory, and the functional equation is somehow a key to the Riemann hypothesis. If in the duality, one side is a continuous sum and the other is discrete, then in the simplest classical case this is the relation between a periodic function and its Fourier series, or, almost equivalently, it is the Cauchy residue formula; and in a very modern recondite version it seems to be the general Selberg trace formula. (The inversion formula for Fourier integrals, in which the sums on both sides are continuous, is a "limiting" case of this, and does *not* belong to the Riemann-Hecke functional equation.) The role of this formula is clear to me, I think. It is, in classical physics, the duality between an electromagnetic field on the one hand, and electric and magnetic "discrete" charges on the other hand. And in quantum physics it is the much more trenchant Janus-like de Broglie duality between the undulatory and corpuscular aspects of the same elementary particle valid simultaneously.

But in the Riemann-Roch theorem both sides are discrete sums, and both finite,

and the relation to the other aspects of the duality theorem is not easy to state. (The reciprocity formula for Gaussian sums, in which both sums are finite, does not belong here, but is a degenerate case of a Jacobi modular relation for theta functions which underlies the Riemann-Hecke functional equation; see my paper: Remarks on Gaussian sums and Tauberian theorems, *Journal of the Indian Mathematical Society*, vol. 15 (1951), 97–104.) Undoubtedly the Riemann-Roch duality is linked to the duality between gradience and co-gradience, and homology and cohomology, both of which have first been stated by Poncelet in projective geometry and by Grassmann in his nascent tensor calculus; but these links are somehow too obvious, and not too enlightening to me. Now, in my search for an understanding of the Riemann-Roch theorem, I have been looking at the references in the footnotes of Weyl's book. There, on pages 136 and 137 (of the English edition) there is a reference to one E. Ritter who in 1894 was one of the first who, except for one relatively minor flaw, gave a proof of the Riemann-Roch theorem even for so-called fractional divisors. And later on in Weyl's book, in a footnote to page 147, this E. Ritter is even credited with formulating an extension of the entire Riemann-Roch theorem from system of differentials and functions that are univalent on the closed Riemann surfaces to such differentials and functions that are defined on the covering space of the Riemann surface, but are reproduced by the elements of the Poincaré monodromy group by a fixed multiplicative character of the group. And Weyl adds that in a later paper in 1896 (Math. Annalen, vol. 47), E. Ritter even further generalizes this to solutions of certain types of differential equations on the Riemann surface.

This sounded to me like the work of a very good mathematician, and I wondered why I had not heard from Ritter otherwise. So, one day I decided to look up the last quoted paper, and I had a surprise. The very last sentence of the paper announces a continuation, but there is a sad footnote by the editor attached, stating that, alas, there will be no such continuation. The author, Dr. Ernst Ritter, had accepted a call to Cornell University at Ithaca, but on the way to Ithaca, in New York, he succumbed to a typhus attack, on September 23, 1895. Also, the footnote announced a detailed obituary in the *Jahresbericht der Deutschen Mathematiker Vereinigung*.

This obituary was done by F. Klein, and is, oddly enough, dated September 25, 1895. It made me doubly unhappy for Cornell, and the U.S.A. in general, for having missed out on such a gifted mathematician. Ritter had been born in January 9, 1867, and thus was only $28\frac{1}{2}$ years old when he died. He had become a *Privatdozent* at Göttingen only two terms before receiving the call from Cornell. It was meant to be a big, purely scientific position by which Cornell wanted to become aligned with other mathematically forward-striding great American universities. Klein added that at sailing time Ritter did not feel well, but having a robust constitution ignored it. Immediately on arrival he was taken to the Government hospital at Ellis Island, where he passed away, surrounded with all the care which friendly American mathematicians could show him. I have been unhappy for American mathematics since.

Our last item of Americana is about a mathematician at the other end of the 19th

century, at the beginning of it. This one *did* arrive in this country, he *was* active here, very much so, in fact, he was even hyperactive in a sense, and he even managed to carve out for himself a niche, an extremely tiny one, but a niche in the Hall of Immortal Mathematical Fame, so that he was something of a pioneer of American mathematics, beyond question; and yet it is extremely difficult to assess what it is that American mathematics really owes to him, durably, that is.

His name was Robert Adrain, and the best and most accessible source of information about him is an article by the somewhat redoubtable Harvard mathematician Julian Lowell Coolidge, entitled "Robert Adrain, and the beginning of American Mathematics," in the *American Mathematical Monthly*, volume 33, 1926, p. 61–67, in which earlier biographical information is listed. Of interest is also a subsequent brief appraisal, in the *Dictionary of American Biography*, volume 1, 109–110, by the able historian of mathematics, David Eugene Smith, who taught at Columbia University.

Adrain was born on September 30, 1775, at Carrickfergus, Ireland, came to the U.S. around 1800, and lived here until his death on August 10, 1843, on his farm in New Brunswick, New Jersey. He must have been a tempestuous person, because he was frequently changing jobs, not always for better ones. He was a master in the Academy of Princeton, New Jersey; then principal in similar schools in York, Pennsylvania, and in Reading, Pennsylvania; then Professor at Rutgers College (then called Queen's College) in New Brunswick, New Jersey; then Professor at Columbia College, New York City; then again at Rutgers; then at the University of Pennsylvania where he even became vice-provost; then, teacher at the Columbia College grammar school, yes grammar school, after which he retired, or was retired to his farm at New Brunswick. Far the most part he published only middling-interesting mathematics, mainly through books and in a short-lived journal of his own, called the *Analyst, or Mathematical Companion* (1808); but even this middling-interesting activity produced, in a journal called the *Mathematical Correspondent* (founded 1804 by George Baron), a curve called by Adrain the *catenaria volvens*, which half a century later was rediscovered, quite independently and very systematically, by the leading algebraic geometer Alfred Clebsch.

But Adrain's actual claim to fame was the presentation of two versions of what the 19th century called "the exponential law of (accidental) errors (of observation)," one of which even appeared in print a year before the publication of the version of Gauss, by whom the law was subsequently known and after whom it is usually named. The 19th century produced a large number of versions of the exponential law of errors (see, Emanuel Czuber, *Theorie der Beobachtungsfehler*, Leipzig 1891), and Adrain's versions are perhaps among the least satisfactory ones (see the article of Coolidge). But Adrain did achieve a measure of firstness in the matter, although adumbrations of the law reach back into the 18th century, to the work of De Moivre (see also I. Todhunter, *A history of the mathematical theory of probability from the time of Pascal to that of Laplace*, 1865); so that D. E. Smith may by forgiven for the

extravagant statement: "If, instead of being a self-made mathematician, he had come under the influence of men like Laplace, Legendre, and Gauss, [Smith might have here added the name of Poisson], he (Adrain) might have been a great leader. As chance had it, he was in a mathematical desert."

It is rash to assert, as D. E. Smith seems to do, that Adrain received no mathematical stimuli from the "ecological" setting in this country. The fact is that throughout the 19th and into the first decades of the 20th century there was in this country a *broad* interest in, and cultivation of, the kind of mathematical probability and statistics into which the 19th century law of errors naturally falls. The best evidence for this is the fact that even the above-named gentleman-mathematician J. L. Coolidge wrote a book on this subject. By mathematical avocation, Coolidge was a geometer, as evidenced by a string of textbooks, all published by the Clarendon Press, Oxford, and they were as follows: (1) *The elements of Non-Euclidean geometry*, 1909; (2) *A treatise on the circle and the sphere*, 1916; (3) *The geometry of the complex domain*, 1924; (4) *Algebraic plane curves*, 1931; (5) *A history of geometrical methods*, 1940; and (6) *A history of the conic sections and quadratic surfaces*, 1945. And he even ended his book-writing career with, — also for the Clarendon Press — (7) *The mathematics of great amateurs*, 1949. Now, amidst all such books of a rather uniform trend, he also had a very different book in 1925, likewise at the Clarendon Press, under the title: *An introduction to mathematical probability*, which also had chapters on "errors of observation," "errors in many variables," "indirect observations," "the statistical theory of gases," and, *note*, on "the principles of Life Insurance." And in the introduction the author states that "the present work is based upon the lectures which I have delivered, usually in alternate years, at Harvard University." This book, however hodge-podgey it may appear to be to a mathematician of today, was still very much in the style of 1925, and, in fact, the book of Coolidge was quickly translated into German, and I myself so read it.

But it is a curious fact that only a few years afterwards, this kind of book simply went out of style; and it is not the fault of J. L. Coolidge, if very soon after the appearance of his book, the work, original and translation, began to collect dust on the shelf which it has not shaken off since. Suddenly, around 1930, different conceptions of the cognitive nature and of the role of probability and of the role of statistics, and of their mutual relations came into being, and a different kind of textbook became necessary. All of a sudden, "exponential law of errors," which in the 19th century statistical activity was a household word, a state of mind, as it were, almost disappeared from textbooks and their subject indexes; and what — to a discerning antiquarian — was left of it were simply some versions of the so-called law of large numbers, more-or-less, and of some other asymptotic laws, perhaps. And it should be added that very soon after the change-over entirely different American crews joined in the "New Directions," both in probability and in statistics, and have been on top of developments since.

In some peculiar manner, the American folks-genius must have had some kind of

awareness in the late 1920's that an important phase of mathematical statistics had come to an end, because an American professional statistician even wrote a valedictory summary on this entire period. It was called *Studies in the History of Statistical Method, with Special Reference to Certain Educational Problems* (1931), and the author was Helen M. Walker, then Assistant Professor of Education, Teachers College, Columbia University. It is also a remarkable feature of this book, — and a comment on the still very nineteen-centuryish American outlook of it on Life and Education — that the educational aspect in the title of the book is taken very seriously indeed in the body of the book. Thus Chapter VII has the title: *Statistics as a subject of Instruction in American Universities*, and the sections of the chapter are as follows: (1) Purpose and method of the chapter, (2) Instruction in Statistics in 1890, (3) First College Courses in Statistics, (4) Pioneer Courses in Various Departments, (5) Outline of the Development of the Teaching of Educational Statistics in America, (6) The present situation, and, MARK!!, (7) Remarks by Florence Nightingale concerning Instruction in Statistics.

To come back to Adrain, I would like to repeat that it is not at all certain that by being in this country he was lost in a mathematical desert. It is even more hazardous to speculate that by staying in Europe, and in the right places in the vicinity of the right people, he might have come under the influence of men like Laplace, Legendre and Gauss, and thus become a great leader. Adrain's true activity falls into the period between 1790 and, say, 1830, a so-called *Age of Upheaval* in the terminology of the *New Cambridge Modern History* (I myself called it, for my purposes, *The Age of Eclosion*, and extended it from 1776–1825); and this era is, on record, one of the most difficult and recalcitrant to analyze and comprehend with regard to intellectual developments and spontaneities, and with regard to the influence of intellectuals upon each other. By what is known, but not comprehended about this era, it is equally valid to speculate — or at least could not be gainsaid — that it was precisely the relative isolation in the United States which brought out whatever originality there was in Adrain, and that, if he had stayed in Europe, whatever originality he was endowed with might have been nipped in the bud altogether, or smothered and choked off entirely. Because, if it is true that in the United States this era was in some respect a desert, then in Europe it was, in the same respect, a suffocating jungle, and it would be rash to say that Adrain would have fared better in a jungle than in an alleged desert.

What was reminiscent of a jungle in Europe at that time was an overwhelming outburst and uncontrollable proliferation of all kinds of knowledge, on all kinds of levels, all over the landscape. Leaving aside at *first* (and I mean at *first*) mathematics, and mechanics in all its parts (mechanics of point systems, and of continuous and of hydrodynamic and elastic media), it can be asserted that a whole spectrum of sciences, natural social, and even humanistic, came then into being; and if the sciences did not all actually come into being only then, it can nevertheless be stated that they then organized themselves in such a way, as if this were their true beginning, finally.

During that period there was the true beginning of physics *proper* (of thermodynamics, electricity, magnetism, even of optics, ancient as this discipline was), of chemistry (beginning with Dalton and Avogadro), of geology, biology (Lamarck), psychology (Herbart); of systematic sociology (Auguste Comte); even of economics, certainly of mathematical economics, or, at least of economics built on models; of linguistics; of the so-called "higher criticism" in all kinds of ancient literature, the Old Testament and Homer among others; of various specialized histories, like history of astronomy, of medicine, of Roman law, etc. (For all this, compare my book *Eclosion and Synthesis*, 1970.)

Even in mathematics the developments during that period were much more radical, and much more difficult to account for rationally than commonly realized, even if, externally, the developments did not appear to be as tumultuous as in other areas of knowledge, scientific or other. During this era mathematics and mechanics separated from each other; the conception of an existence theorem, and of a uniqueness theorem, and of a necessary and sufficient condition, was finally born, and only then. One need only compare the spirit of, and approach to analysis in Lagrange and Cauchy, — who after all were in a common French tradition — to see how radically analysis was reoriented. Lagrange and Cauchy simply do not speak the same mathematical language, whatever the similarity of vocabulary and grammar. For instance, to single out only one point, no matter how much Lagrange may assert and insist that a function is for him an "abstract" mathematical object, in his thought patterns it somehow is residually a mechanical orbit or perhaps a physical function of state; whereas in Cauchy, orbits and forces and pressures are always functions, as they are for us today. The history of this transition in mathematics must of course be pursued, to the limit of possibility, as a problem of the genetic unfolding of mathematics as mathematics. But it is also a fact that this transition in mathematics has parallels to transitions in many other areas of academic pursuits; and in order to round out the comprehension of the transition in mathematics such other transitions must be taken into account too.

The leading transition mathematicians were Laplace, Gauss, and Poisson. The provenance of Laplace is easy to see; he was in a continuing French tradition, from Lagrange to Cauchy. Poisson, although also a French mathematician, is much harder to understand and to place. And it is a curious fact that he is the only French mathematician of consequence whose *Oeuvres Complètes* have never been edited, and about whom there is no decent scientific biography. And Gauss, finally, is totally inexplicable as to his "ethnic" origin. His only mathematical "forbear" would be Leibniz. But it is very farfetched to attach him to Leibniz, and it is simply impossible to say where Gauss came from, mathematically, that is. He was suddenly there, just as some of the greatest prophets of the Old Testament were suddenly there. For instance, the first such great prophet in the Old Testament was Elijah, and the Bible introduces him suddenly, without any warning, as "Elijah the Tishbite, who was of the inhabitants of Gilead." That is all; no father's or grandfather's name, no name of a clan, nothing.

Also, our assertion that Gauss, for all his greatness, was the figure of a transition period, is borne out by the fact that while he already felt the mathematical need for giving a rigorous proof, and in fact more than only one proof, for the fact that every polynomial with complex coefficients has a zero, he was not yet in an intellectual frame to define the convergence of a sequence, or of a series of numbers, satisfactorily, that is. This was done only by Cauchy, who had the unusual distinction of being the first to give a reasonably satisfactory definition of continuity as it occurs in mathematics.

Part II. Charles Sanders Peirce

The statement just made about Cauchy is my cue for presenting my last and most significant item of Americana. In a very peculiar sense, which I shall try briefly to sketch, the conception of continuity was in the center of a dominant theme of 19th century intellectuality, in all its compartments, in and out of mathematics. And it is a remarkable fact that Charles Sanders Peirce was most peculiarly preoccupied with this theme, overtly and covertly, directly and by contrast. It was an intellectual obsession with him. At any rate, it is possible to view much, or even most, of his philosophical work, in all its complexity, as variations, however diverse, on the theme of continuity. If his work is now so viewed, it is admissible to say, that the preoccupation with this all pervading 19th century theme of continuity made Peirce into the greatest philosopher America has produced thus far, but that it also un-made him, as evidenced by the fact that Peirce was never able to compose a key treatise through which to express himself and in which to summarize himself, as philosophers are wont to do. His was a real American Tragedy. A great philosopher, and an even greater failure.

This harsh judgement was not originally mine. I encountered it in the book of Murray G. Murphey, *The Development of Peirce's Philosophy*, Harvard University Press, 1961, but found it very appropriate and just, however harsh it be. But Murphey did not substantiate it sufficiently, and I undertook to rationalize it from my own approach in the article: "Continuity and Discontinuity in Nature and Knowledge," which has appeared in the *Dictionary of the History of Ideas*, 1973.

Before briefly expounding my own view on the matter, I would first like to quote the decisive statement in Murphey, which is the last paragraph in his book. This is its text.

> "As one reads through the thousands of pages of manuscript which are all that remain from Peirce's life's labor, one cannot escape the feeling that these are the ruins of a once great structure. Every paragraph and every doctrine seem to be fragmentary parts of some larger whole. As Morris Cohen has remarked: 'In his [Peirce's] early papers, in the *Journal of Speculative Philosophy*, and in his later papers, in the *Monist*, we get indeed glimpses of a vast philosophic system on which he was working with an unusual wealth of material and apparatus.' (Charles S. Peirce and a Tentative Bibliography of His Published Writings,

Journal of Philosophy, Psychology and Scientific Methods, XIII, 727 December 21, 1916.) But this is an illusion — Peirce's illusion: the grand design was never fulfilled. The reason is that Peirce was never able to find a way to utilize the continuum concept effectively. The magnificent synthesis which the theory of continuity seemed to promise, somehow always eluded him, and the shining vision of the great system always remained a castle in the air."

We ought to observe though, that Murphey was in no wise implacable in his criticism of C. S. Peirce. Thus, after the book, Murphey also contributed the article on Peirce to *The Encyclopedia of Philosophy*, and there this kind of criticism does not occur at all.

But now we are turning to our own analysis, in which we will repeat much from our above-mentioned article in the *Dictionary of the History of Ideas*.

Continuity is a perplexing concept. The mere name of it poses a puzzle. Our common familiar name of it comes from the latin verb *con-tinere* which means "to hang or hold together," and forms of the verb *continere* are attested since the 3rd century B.C., from which the earliest large-scale Latin texts have come down. But long before that, a Greek equivalent of the entire word-complex had been in use. In fact, the Greek word for our "abstract" noun "continuity," as standardized by Aristotle, is the adverbial form *to syn-echés* (τὸ συνεχές), and the cognate verb *syn-echéin* is likewise translated in dictionaries by "to hang or hold together." To a layman this would suggest that the two words *con-tinere* and *syn-echéin* ought to be close linguistic relatives, especially since the prepositions *con* and *syn* have such similar sounds. But, surprisingly, the works on Indo-European linguistics which have been consulted in my layman's way, do not, in general, "identify" *con* and *syn*, and, in particular, they do not assert that the Greek and Latin stem words for *continuity* had a common root in Sanscrit.

Now, in Greek, this *synechés* has a venerable ancestry, meaning that it already occurs in Homer, than which there are no earlier texts. In fact, both the verb *synechéin* and the adverb *synechés* occur in Homer, and it is most remarkable that already in Homer they occur on different levels of abstraction, and that this difference has not worn off entirely ever since. The verb occurs on a lower level of abstraction. It occurs in *Iliad*, 4, 133, in the expression "the golden clasps of the belt were *held together*," in which its meaning is quite concrete. But the adverb, which is used twice, is both times used in a semi-abstract meaning, namely in the meaning *continually* (in time). The first occurrence is in *Iliad* 12,15–26, thus: "Zeus made it rain *continually*"; and the second occurrence is in *Odyssey* 9, 74, where Ulysses, in the narration of an adventure, uses it thus: "There for two nights and two days we lay *continually*." This peculiar intellectual tendency to have a heightened awareness of the abstractness and intricacy of continuity when dealing with continuity of Time has been with Philosophy, in all its compartments, ever since. For instance, a good deal of the philosophy of Henri Bergson can be so envisaged. But, of importance to us, is a section in Peirce,

6.86, entitled Time, in which the statements are rather intricate and even obscure, even by standards of the writings of C. S. Peirce, in which obscurity is a recurrent phenomenon.

Returning to Homer, to the passage in the *Odyssey* in which *synechés* means *continually* (in time), it is remarkable that the adverb *synechés* is there reinforced by the adverb *aiei* (αιεί) which means *always, ever, eternally*. The *Odyssey* thus adumbrated a tripartite bond between continuity, time, eternity; and this bond — by the inclusion of eternity — has been variously contemplated and exalted in theologically oriented (general) philosophy ever since. This bond is nowhere stronger than in the Old Testament, which, in a sense, is a vast dissertation on the Lord's *continuous* (because unceasing) concern for his "Chosen People," but the extant canon of the Old Testament does not have at all a word whose functions would correspond to those of *synechés*. It is further remarkable that both this bond and the intellectual preference for temporal continuity are verified by the order in which the cognates of our English word *continuity* have come into use. According to the entries in the *Oxford English Dictionary*, our word *continual* (in time) was the first to emerge. It occurs as early as 1340 A.D., in the phrase: "great exercise of body and continual travail of the spirit," in one of the so-called *English Prose Treatises* of the hermit Richard Rolle of Hampole (1290–1349). But for all other cognates of *continuity* the same dictionary quotes only from Chaucer, who wrote about half a century after Hampole, or from later sources, even much later ones. Thus, according to this dictionary, our adjective *continuous* gained currency in the 17th century only.

In mathematics, continuity is *nowadays* defined precisely and unequivocally, but only *nowadays*, since Cauchy, that is. There is no reason for being overly surprised at the fact that continuity has become a precise mathematical concept so relatively recently only, and one must not judge, as sometimes done, that mathematicians before that were "confused," or in some sense "defective" in their cognition. Continuity is nowadays as vital a mathematical conception as can be, but, for all that, it is not at all a mathematical conception to begin with, very far from it. This cannot be sufficiently emphasized. It is one of the greatest triumphs of mathematics that it has somehow managed to turn continuity into a genuinely precise and sharply defined concept as no other academic area of cognition has, although to begin with, continuity is not a peculiarly mathematical category of cognition at all. Rather, to begin with, it is a sprawling all-pervading notion, occurring, in all precincts of perception and intellection, ideation and abstraction; and outside of mathematics, it is ambiguously conceived and loosely applied, and it merges and fuses with neighboring concepts, like uniformity, steadiness, constancy, etc., all of which have, in mathematical contexts of today, definitions of their own.

That continuity has many meanings, or, at any rate, many shadings of meaning, can be recognized by the fact that it has many antonyms. The leading antonym to "continuous" seems to be not "discontinuous" but "discrete." But others are: saltatory, sudden, intermittent, indivisible, atomic, particulate, even: monadic.

A Monad however, being a kind of synonym to Unity and One-ness, may suggest both continuity and discreteness, self-dually. The monad of Leibniz, as presented in his *Monadology*, is apparently of such a kind; that is, it also suggests continuity, even if it is an irreducible ultimate unit, not only of physical structure but also of consciousness, cognition, and metaphysical coherence. Charles Sanders Peirce has a much more sophisticated version of duality between continuity and discontinuity, or at least an anticipation of such a duality, a version of duality which reached into the 20th century and even became a hallmark of it — but, of this, later on.

In one matter, though, Leibniz' insight may have been deeper than Peirce's. It concerns the relation between "function" and "continuity." A young mathematician nowadays is taught, even in high school, that the concept of a "function" $y = f(x)$ is a primary datum, whereas the property of "continuity" of a function is a secondary datum. A function, he is taught, is, to begin with, an "arbitrary" association of some y with each x, whereas continuity is a restrictive mode of association, which can only be brought about if the spaces of the objects x and of the objects y carry "topological structures," so called. Now, in the actual genesis of the two conceptions in the mathematics since 1600 there was no such hierarchical subordination of "continuity" under "function." It was almost the other way around; it was the urge to come to grips mathematically with the concept of continuity that was greatly responsible for the gradual emergence of the conception of a mathematical function too. Now, from my reading in Leibniz and in Peirce, such as it has been, I have the impression that Leibniz somehow knew better than Peirce what, cognitively, the true hierarchical relation between continuity and function really is.

Returning now to continuity *per se* we note that a most challenging problem in the History of Ideas and Knowledge, which has not yet been studied at all, are the manner and stages by which the sharp mathematical strand of continuity of today has arisen and evolved within the sprawling, weaving, undulating texture of continuity in the broad spread of cognition and knowledge in our total mental awareness. Also, most surprisingly, the study of Greek Rationality does not help at all. On the contrary, the Greek situation makes the problem ever so more perplexing, from our retrospect at any rate; the brutal fact being, that Greek mathematics, as known to us from the many documents that have come down to us, has absolutely never and in no wise come to grips with the conception of continuity at all, in a plain, honest, direct confrontation, that is.

In extant Greek mathematical texts, the word *synechés* occurs extremely rarely, and never, but never, in a technical sense; nor does any kind of recognizable express circumlocution appear in a technical sence. From knowing *all* extant Greek mathematical texts, but only those, and no philosophical texts, it would be impossible to surmise, that *synechés* was a potent philosophical concept that appeared in very meaningful philosophical contexts, meaningful and intricate ones. The most articulated such occurrence is in Aristotle's *Physics*, especially in his analysis of Zeno's puzzles (the flying arrow, Achilles [and the tortoise], the midpoint, the race-course).

But already well over a century before, *synechés* occurs in the great ontological poem of Parmenides; and the following two passages in which it occurs must be read slowly and thoughtfully, if the meaningfulness of the occurrence is to be savored. (The passages are lines 3–6 and 22–26 from fragment 8 in Diels-Krantz; all our translations of Parmenides are those of Leonardo Tarán.)

> "Being is ungenerated and imperishable, whole, unique, immovable and complete. It was not once [upon a time only], nor will it be [only in the future], since it is now altogether, one, continuous (*synechés*)."

> "Nor is it divisible, since it is all alike. Nor is there somewhat more here and somewhat less there that could prevent it from holding together (*synechéin*); but all is full of Being. Therefore it is all continuous (*synechés*), for being is in contact with Being."

Now, at first hearing this may sound like just another philosophical gibberish. Try however to replace here "Being," by "(Dedekind-Cantor) linear continuum," be it one-dimensional, two-dimensional, three-dimensional, etc., and then the passages immediately sound much more meaningful. The second passage especially has a thrilling ring, thus!

> "Nor is it divisible, since it is all alike. Nor is there somewhat more here and somewhat less there, that could prevent it from holding together. But all is full of continuum. Therefore it is all continuous (*synechés*), for continuum is in contact with continuum."

After this take the first passage, but in the sense of natural philosophy, and in a suitable paraphrase, thus:

> "Continuum is ungenerated, and imperishable, whole, unique, immovable and complete. It was not just sometimes in the past, nor will it be only in the future, since it is now altogether, one continuous (*synechés*)."

Before continuing with the subject matter proper, we wish to comment briefly on a famous passage in Parmenides in which he suggests that the universe has the shape of a homogeneous sphere: (lines 42–49 of fragment 8). The passage becomes rather luminous if one not only replaces "Being" by "(Cantorian) continuum," but also "non-Being" by "a gap (as in a Cantor ternary set)" in the following text.

> "But since there is a furthest limit, it is in every direction complete (*tetelesmenon*); like the body of a well-rounded sphere, from the middle everywhere of equal strength; for it need not be somewhat more here and somewhat less there, for neither is there non-Being to prevent it from reaching its like, nor is there Being so that it could be more than Being here and less than Being there, since it is all inviolable; for from every point it is equal to itself, staying uniformly in the limits."

That — from our retrospect — it is justified to impute to Parmenides a measure of preoccupation with the nature of the linear continuum is corroborated by developments engendered by Parmenides, not so much in the work of Melissus of Samos, as in the famed work of Zeno of Elea, both of whom were prominent successor-disciples of Parmenides. Melissus introduced some features of infinitude into the universe of Parmenides, and thus contributed, however slightly, to its eventual mathematization. And Zeno's puzzles contributed much more so, however slowly and even circuitously.

The work of Zeno, as known today, had two major aspects, which for our purposes are separable, however much a philosophical analysis in depth may find them overlapping, or even coextensive. One aspect was the general problem of One-and-Many, with which we need not concern ourselves as such; and another aspect was the more special, and more renowned, problem of the "puzzles" or difficulties relating to motion, with which any study of the notion of continuity in its metaphysical entirety must concern itself, in one form or another. The manner in which we are concerning ourselves with the puzzles is a purely negative one. We are not going to examine the puzzles at all on their own Zenonian standing and merit, or on the many philosophical expostulations and critiques of them, but we are pointing out, emphatically, the following closely interlocking facts.

(1) Whatever, nowadays, the philosophical interest in the puzzles continues to be, our *working* science and working mathematics is totally unaffected by them. There is not a trace of a hint of the puzzles in Newton's *Principia*, or in Lagrange's *Mécanique Analytique*, and a physicist of today is not less of a physicist if he does not know anything about the puzzles at all. In fact, there is not a trace of a hint in all of Greek scientific documentation that Archimedes was in the least interested in the puzzles, or that he had ever heard of them at all. I am not asserting or even suggesting that Archimedes never heard of them, but I am only pointing out that in the extant corpus of Greek writing there is not a trace of a hint of an allusion to his having known anything about them, and I do maintain that it is very pertinent and very important so to point out.

(2) The puzzles are known only because Aristotle reports on them, in Book VI of the Physics, and his purpose in reporting them is not "to save them for posterity," but to refute them. Now, from our retrospect, Aristotle's refutation consists, in his own thought patterns, in the following interconnecting attempts. (a) He attempts to establish the prerequisite continuity properties of the (Cantor-Dedekind) linear continuum. (b) He attempts to represent a *uniform* motion on the line by a linear function $x = ct$, and this is especially difficult for him, because the Greeks simply had no kind of analytical anticipation of a function in our sense. (c) And he finally attempts to argue that in this would-be linear function, the mathematical structural paradigm of the two quantities x and t is the same, namely that of the linear continuum so that, indeed, as we know it today, the puzzles simply do not arise.

(3) Now, a most remarkable commentary on these interlocking facts is this, that

these heroic attempts of Aristotle to establish such very basic and indispensable verities of analysis are not reflected in the least in the mathematics of the Greek professionals at all. While Aristotle is "experimenting" not only with the notion of "continuity" (*synechés*), but also with neighboring notions of the linear continuum, like: "together," "continuous," "overlapping," "everywhere dense," etc., professional mathematics is deathly silent about any such notions. One is forced to conclude that Greek professional mathematics was simply too cowardly to face the conception of continuity head-on, front-face, in a direct confrontation, as Aristotle, and perhaps also some other philosophers, had so urgently been prodding it to do. And the consequences of this "cowardice" were immeasurable. If nothing else, it took the Evolution of Ideas almost 2000 years fully and unequivocally to recognize that the puzzles of Zeno are, if properly viewed, problems of mathematics. At any rate, it appears that the first such recognition can be documented by a book of 1647 only, namely by the *Opus geometricum quadratura circuli et sectionum coni* of Gregory St. Vincent. Before that, the puzzles belonged to *Physics*, more-or-less, as they had so belonged in Aristotle, whatever Physics may have been understood to be, in Antiquity and in the Middle Ages.

Of course, it would be wrong, even preposterous, to say, that Greek mathematicians did not do anything about continuity at all. Of course they did, and even some of their most glorious achievements pertained to that part of our present-day analysis in which continuity is domiciled. Such achievements were: their theory of proportions, as presented in Book V of Euclid's *Elements*, and their theory of exhaustion that pervades the work of Archimedes. But, as far as continuity is concerned, these "substitute" achievements, when viewed coldly and unsentimentally, were maneuvers of evasions, instead of battles of confrontation; and Greek mathematics, and mathematics long after the Greeks, paid a high price for putting up with such evasions, the price being some unbelievably crude and obvious gaps in syllogistic reasoning — gaps which only the 19th century identified and filled.

Such a crude gap occurs in the very first theorem of Eculid's *Elements*. In it Euclid tacitly assumes that if a circle is given in the plane, and if a point inside the circle and a point outside the circle are connected by an arc (or a segment, or any simple curve), then the arc must meet the circumference of the cirle in at least one point. This gap is a gap in the knowledge of the structure of the linear continuum. And since Greek mathematics never faced up, directly, to the problem of the structure of the continuum — as the work of Aristotle should have urged it to do — it managed not to become aware of the gap at all. Not only did Archimedes accept this kind of gap unquestioningly, but even the great Gauss still did, when, in one of the proofs of the theorem that a polynomial with complex coefficients has a complex root, he takes it for granted that a polynomial with real coefficients of odd degree meets the real axis somewhere. And it must be stated again unsentimentally, that this gap in Gauss is of the same size and shape as the gap in the first theorem in Euclid, 21 centuries before. Only after Gauss, even soon after Gauss, did analysis finally deal

with the problem frontally, and even so it took mathematics the best part of the 19th century to master the problem.

About a century before Cauchy and others began to make the conception of continuity mathematically rigorous and operationally productive, Leibniz began to make it philosophically tangible, articulate, and useful. Thus he, finally, overcame the Aristotelian (meta-) physical contrariety between motion (*kinesis*) and rest (*eremía*) by pointing out, in terms of today, that rest is the limiting case of motion when velocities reduce to the value 0. But this was only a particular statement of Leibniz, which, in his view, was only a particular instance of a general Law of Continuity (*lex continui*) which runs through his entire metaphysics and science. Leibniz, from prudence or incapacity, did not present this law — important as it was to his thinking — in a systematically thought-out study of its own, but he frequently reverted to it in divers contexts, presenting some of its aspect each time. Perhaps the most representative single statement of the Law of Continuity is the maxim of cognition that "when the essential determinations of one being approximate those of another, as a consequence, all the properties of the former should also gradually approximate those of the latter." (*Leibniz selection*, edited by Philip P. Wiener, 1959, p. 187; see also pertinent passages in Hermann Weyl, *Philosophy of Mathematics and Natural Science.*)

This leading statement may be interpreted to assert that functions, and functional dependencies in nature are usually continuous. We know nowadays that this is an over-assertion, since chemical processes and nuclear processes can be discontinuous, even violently so. And Leibniz' emphasis on continuity was to an extent a part of his general attitude of optimism — which is a *leitmotif* of the 18th century althogether — that the world is rational, progressive, self-ameliorating, and perhaps even the best of all possible worlds. Nevertheless, the *lex continui* was an important event indeed, in that it revealed an aspect of (mathematical) continuity that had been hidden from view until then. Whatever insight into continuity Aristotle and other ancient and medieval philosophers may have had, this insight always referred only to the structure of the linear continuum, that is, to the topological structure associable with linear ordering; whereas Leibniz discovered, even if in a blurred way, the possibility of continuity of a *function*, that is, of a mapping from one topological space into another, and he even advanced towards a conception of a general function in the process. Professional mathematics and philosophy of mathematics pride themselves on the insight — and they spent the entire length of the 19th century on arriving at the insight — that a function, whatever that be, need not be continuous. But, as already stated before, it is a most significant fact that, in relatively recent Western Thought, the conceptions of function and of continuity have evolved simultaneously and in close intellectual interpenetration with and in dependence on each other, however much they were later disengaged from each other, in fact, or perhaps only in intent. The lasting residual dependence of these two conceptions upon each other comes about in the following way. It is all very well to say that a function is an "arbi-

trary" correspondence between two sets of objects, the domain and the range. It can be argued that however "arbitrary" a correspondence may appear to be, it does follow some kind of pattern nonetheless, and if one philosophically extends and generalizes the concept of continuity to subsume the presence of "any" kind of pattern, regularity, orderliness, etc., then, of course, most functions in Thought and Cognition are continuous indeed. The possibility of such an extension has been recognized and attempted by Charles Sanders Peirce, and it was this attempt in which he failed, or rather in which he never succeeded. Of course, his attempt was not quite as simplistic as our first statement, just made, would suggest, and we will try to describe, to an extent, how un-simplistic it was.

Returning to Leibniz, we have to say that it is not easy to assess the direct effect, if any, of the Law of Continuity on the growth of professional mathematics and physics or even on philosophy itself. There seems to be no manifest reference to the *lex continui* in Immanuel Kant's analysis of his *a priori* Space and Time, however much continuity of one kind or another is involved in their structure and mystique. Nor is there a manifest reference to the *lex continui* in the unusually slow unfolding of the conception of continuity in the working mathematics of the 18th and 19th centuries.

Of course, the "Law" of Leibniz may have been burrowing deep inside the texture of our intellectual history, thus affecting the course of Knowledge and Cognition. But the only outright corroboration for this is the fact that about 150 years after Leibniz a *lex continui* came into being in the thinking of C. S. Peirce and permeated it to a considerable extent. Except that Peirce called it not *lex continui* or *Law of Continuity*, or something similarly synonymous, but *synechism*, thus going back to the Greek form of the word continuity instead of the Latin. And may I state in parentheses that it was this "accidental," or perhaps not so accidental, trait of nomenclature that attracted my attention to Peirce, the general philosopher, at first. Peirce obviously did not want to admit, to himself or to others, his indebtedness to Leibniz in this matter, and he has been very successful in this respect, inasmuch as nobody states it outright that *synechism* is a *lex continui* by another name, however much Peirce may intend to differ from Leibniz in particular or general attitudes. It is altogether remarkable that in the vast collection of Peirce's papers the name of Leibniz appears extremely rarely, although Peirce apparently had Leibniz very much on his brain (see Max H. Fisch, Peirce and Leibniz, *Journal of the History of Ideas*, 33 (1972), 485–496). And, finally, Peirce absolutely never refers to the fact, which must have been known to him, that the verbal switch, in philosophical jargon, from "continuity" to "*synechés*" had been made earlier in the 19th century by the philosophers Johann Friedrich Herbart (1776–1841) and Gustav Theodor Fechner (1801–87), the first famous for psychology and pedagogy, and the second for the Webner-Fechner Law of quantitative psychology (intensity of sensation varies as the logarithm of the stimulus).

It must be stated though that Herbart uses *synechés* only in a *de*scriptive manner,

while Peirce uses it very *pre*scriptively, whereas Fechner uses it somewhere in-between. Thus Herbart has in his *Metaphysics* a long section on what he terms "Synechology," in which he *de*scribes, from the approach of a peculiarly Herbartian compound of psychology and realism, various basic features of continuity that are inherent to space, time, and matter; Peirce however maintains very *pre*scriptively, that continuity — of a Peircean hallmark, of course — pervades the decisive categories of cognition. Whereas Fechner, in a position somewhere in between, — which cannot be illuminated because Fechner is there impenetrably obscure — enlarges on the "synechological outlook versus the monadological outlook" (*Die Tagesansicht gegenüber der Nachtansicht*, 2nd edition, 1904, p. 204). Also, when Herbart and Fechner turned to the Greek root word *synechés* for the formation of "synechology" then this was much less of a linguistic affectation than when Peirce coined his "synechism." Because Herbart and Fechner wanted the adjective "synechological," in the sense of "pertaining to, or descriptive of continuity," which cannot be readily formed from "continere" whereas Peirce's coinage of "synechism," to take the place of Law of Continuity, had no such justification.

Now, Peirce's failure to establish his Law of Continuity as he envisaged and wanted it, had the following overall reason, stated schematically. Peirce was aiming at a conception of continuity that would be metaphysically at least as all-pervasive as Leibniz had envisioned it, be cognitively even vastly more general than the notion of Leibniz, and yet be as rigorous, unequivocal and precise as the mathematics and mathematical logic of Peirce's era could make it. But with this ambition Peirce was striving after an impossibility. Carrying out this program would have amounted to fusing mathematics with metaphysics and general cognition in their entirety, which, however, seems impossible. Mathematics is not as large as are philosophy and cognition in their full scope. To formulate this precisely, "if a conception from general philosophy has been made mathematically rigorous, then it can wear the vestments of mathematical rigor to advantage only when moving about in areas of mathematics proper, or, at best, in border areas which mathematics is in the process of penetrating, but certainly not when moving about in areas which are well outside of mathematics' sphere of influence" (*Dictionary of the History of Ideas*, vol. 1, p. 502), and outside of the philosophy of mathematics also.

It is not clear from the statements of Peirce, and it may have never become clear to himself, whether synechism is intended to be present effectively outside of areas of philosophy of mathematics, or whether, conversely, the philosophy of mathematics is meant to extend into every precinct of metaphysics in which the presence of synechism is detectable. Peirce was one of the first of a species of philosophers who by trend, intent, or circumstances had been blurring the several demarcations between mathematics, mathematical logic, philosophy of mathematics, and general philosophy. Outside of Peirce, very prominent representatives of this new species of philosopher were A. N. Whitehead and Bertrand Russell; and it is worth observing that they frequently mused that it ought to be possible to trespass on philosophy

proper with conceptions from mathematics, and they even made attempts in this direction. But they were prudent enough, especially Whitehead, not to become entangled in difficulties into which Peirce was stepping only too boldly. Philosophers like Whitehead and Russell were saved from the kind of impasse into which Peirce maneuvered himself, because they disciplined themselves to write up their findings in full-scale texts and monographs from time to time. Whatever did not fit into a monograph or become a monograph, simply had to be dropped or at least de-emphasized. Peirce — from whatever reason, perhaps from reasons of personal hardships — did not compose a single basic philosophical treatise of this kind, and it is probably idle to muse, whether he did not write a treatise because he was unable to think through synechism, or whether, conversely, he was not forced into making an intellectual decision with regard to synechism because he did not compose a systematic treatise about it.

After leveling these criticisms against Peirce, it must be stated that although his efforts to "synechize" our Universe, in its external and internal manifestations, did not succeed — and in fact, from our retrospect, were bound to fail — he did go about this self-appointed task in a very knowledgeable and interesting way. His outlooks and presumptions were much less "naive" than those of Leibniz; not only were they attuned to some of the scientific and other cognitive insights-in-depth of his late Victorian times, but they even foreshadowed some of the radical intellectual change-overs of the early decades of the 20th century too.

First of all, Peirce knew as sharply as anybody before him, that one cannot have continuity all by itself, only and exclusively, without any kind of discontinuity also present, however much the discontinuity be subordinate to continuity, existentially. This had been known long before Peirce. For instance, in the thinking of Leibniz, his *lex continui*, however universally valid, was inseparable from his *monadology*, which posited that anything there is anywhere, in nature and life, physics and metaphysics, logic and being, reality or mysticism, thought and impressions, *per*ception and *con*ception, is somehow built up of some ultimate units, which Leibniz called *monads*, and which, by their One-ness, somehow represented continuity and discreteness, both.

A very crude, low-leveled, but effective way of summarizing the monadology of Leibniz is to point out that even a physicist who abhors mechanics of discrete mass systems, but delights in mechanics of continuous media — and professes a scientific creed that only a mechanics of continuous media is metaphysically and ontologically meaningful — nevertheless knows that, in everyday life, all matter is somehow granulated, that there could be no chemistry without the assumption, however fictitious it be, that there are molecules, and that, however beautiful, perfect and meaningful Maxwell's purely field theoretic system of electrodynamics may be, it somehow is impossible not to perceive, at least in fictitious thought, that something like electrons are conceivable too. Now, this kind of physicist is not just a strawman of ours, there had indeed been such a one, long before Leibniz, and it was none other

than Aristotle. Aristotle was strongly opposed to the atomism of Leucippus and Democritus, and for very meaningful scientific reasons of his; but he understood atomism masterfully, and greatly respected it as an intellectual achievement. Furthermore, a relatively recent study of the history of "philosophical" atomism by Andrew G. van Melsen, in his illuminating book *From Atomos to Atom* (1952), has shown that, nevertheless, Aristotle had a theory of smallest particles, which the middle ages called *minima*, and that actually all along, after Aristotle, there had been, however ambivalently, two kinds of theories, or rather two kinds of irreducible elements, one being *atoms* and the other being *minima*.

Now, Peirce has also something corresponding to Leibniz' monadology as a companion to the *lex continui;* but, going far beyond that, Peirce has something which, incipiently, corresponds to the 20th century de Broglie duality between (continuous) waves and (discrete) corpuscles; and he furthermore has an insight into something entirely different that goes far beyond what Leibniz perceived, namely an insight into the fact that *evolution*, in its post-Darwinian explication, straddles and fuses continuity and discreteness both. Now, for a few condensed details about this.

Peirce's counterpart to the monadological element in the *lex continui* of Leibniz is imbedded in Peirce's recondite doctrine of a basic tripartite category of cognition, the parts of which Peirce denotes, in hierarchical ascent, simply by Firstness, Secondness, and Thirdness. For our purposes the role of this entire category of cognition can be exemplified by the following occurrence, which, however simplistic in outward appearance, is actually representative of Peirce's thought pattern, in its outward scheme, at any rate. Suppose that a working mathematican has to consider a certain general "space," that is, a general Cantorian point-set with some kind of structure, whether the "structure" be topological, algebraic, or relational. Now, in Peirce's scheme of cognition the steps by which the mathematician becomes fruitfully cognizant of this set are as follows. He first of all becomes aware of the set in a first tentative over-all perception, and this amounts to the logical intervention of the category of Firstness. Then, he "verifies" in his mathematical mind, that it is indeed a Cantor set, consisting monadologically of "separable" elements — so that from Peirce's approach, a feature of discontinuity is involved in all existence. But, finally, and decisively, the mathematician becomes aware of, and verifies the cogency of the actual structure of the space, that is of the manner by which the various elements of it are combined and linked with each other; and this he does by the exercise of the Category of Thirdness, which is the decisive one. However simplistic this above example perhaps is, it does legitimately explain how, from this entire approach, both Firstness and Thirdness involve continuity (Firstness does so tentatively and Thirdness decisively) and Secondness contributes a feature of discontinuity, simply by envisaging existence in its most elemental form. A trenchant statement of the synechological link between Firstness and Thirdness and of the generality of the

Peirce notion of Continuity is contained in the following passage from a small but helpful book about Peirce.

> "Thirdness is mediation, generality, order, interpretation, meaning, purpose. The third is the medium or bond which connects the absolute first and last, and brings them into relationship. Every process involves Continuity, and continuity represents Thirdness to perfection."

(See Eugene Freeman, *The categories of Charles S. Peirce*, 1931, p. 19.)

Next, Peirce's anticipation of the 20th century duality principle of quantum theory appears in the following insights and assertions of his.

The 19th century took it for granted that not only a mechanical system but also any other physical system is subject to a law, usually called determinism, that initial conditions, especially initial temporal conditions, *when precisely known*, fully determine the entire future course of events and developments within the system. This "law" was expressly stated by Laplace, and is attributed to him, but it actually was already contained in the *lex continui* more-or-less. But, seemingly inconsistently with this, the 19th century also arrived at the insight that it is possible to arrive at conclusions that are well-determined, even if initial conditions are not known precisely but only "statistically," that is, probabilistically; and the 19th century erected the mighty kinetic theory of gases and of other matter in this way. In this theory, position, direction, and velocity of molecules are assumed to be known by probabilistic averages only, however much, by deterministic hypothesis, they might be precisely determinable at every point. Furthermore, the 19th century took it for granted that this statistical approach was directed by "intellectual" and "practical" convenience, as it were, namely by the difficulties of truly precise observations, and by the mathematical inconvenience of dealing rigorously with a mechanical system of very many particles.

Now, Peirce liked the statistical approach but not its prevalent rationale, and he radically modified the approach by introducing the concept of "absolute chance," by which he meant the existence of a genuine indeterminacy as opposed to an indeterminacy arising merely from ignorance. Thus he says (6.46): "Try to verify any law of nature, and you will find that the more precise your observations, the more certain they will be to show irregular departures from the law. We are accustomed to ascribe these, and I do not say wrongly, to errors of observation; yet we cannot usually account for such errors in any antecedently probable way. Trace their causes far enough and you will be forced to admit they are always due to arbitrary determination, or chance." Peirce even introduced a "law" which maintains the existence of "absolute chance." and he called it "Tychism," from the Greek word *Tyche*, which was the name of the goddess of chance.

A very systematic and influential version of Tychism was created — or perhaps one should say that Peirce's own Tychism was vindicated — in the 1920's, around

1926, in quantum theory, and it came into being in several stages. In 1925 Heisenberg introduced his so-called law of uncertainty. Soon afterwards, Born and Heisenberg were led by this to the statistical interpretation of the Schrödinger Quantum theoretic wave function, which became known about the same time. And this statistical interpretation soon led to a deepening of the understanding of the de Broglie duality between wave and corpuscle which had been proclaimed by him in 1924, even before Heisenberg and Schrödinger made their discoveries.

It is no discredit to Peirce to say that he did not specifically anticipate any of these wonderful achievements of quantum theory. But this is a fitting occasion for "scolding" him, more sternly than ever, for not having collected his philosophemes into a systematic treatise. Doing so might have led him to confront and compare Synechism with Tychism in such a way as to make him a harbinger of the pronouncement of de Broglie that wave and corpuscle, that is continuity and discreteness, are theoretically and phenomenologically compatible and inseparable. What is even worse, although Peirce knew Aristotle extremely well, he did not comment on the observation of Aristotle that there are cases of genuine chance (*tyche*) which are disturbingly "un-random." For example, if someone goes "by chance (*tyche*)" to the market-place where he unexpectedly meets someone he has been wishing to meet, then this is, after all, a consequence of his "determined" decision to go to the market (*Physica* 196 a 4), and not of some "random" desire to meet the person whom he did meet.

And yet, Peirce had still one more insight to his credit. This one was in a way the profoundest of all, but it was also the least worked-out of all. Biological Evolution, Darwinian, or even Lamarckian, has very much the appearance of a phenomenon of continuity, and its emergence and development in the 19th century is, undoubtedly, one more corroboration, and a very powerful one, of the overall fact that the 19th century was very much continuity-oriented, that is, overwhelmingly *synechistic*, as Peirce might have put it. And yet the Late 19th century became very restive over the fact, that however synechistic a specific evolutionary change-over may appear to be, it still does represent steps of discontinuity, however small ones, and that it was not even able to account to itself how these steps of discontinuity really come about. In the early 20th century the uneasiness was compounded by the fact that *Mendelism* and *mutationism* entered upon the scene of genetics, and were, at first sight, even at cross-purposes with natural selection. The biologists finally resolved these difficulties, and peace now reigns on this particular battlefield.

Now, it is most remarkable that Peirce had had some kind of inkling of the intricacy of this biological situation, and that he even introduced a special law by which "intellectually" to control the situation. He should have called this law "evolutionism" or by some such name, but instead he injected "Love" into the context, and called it the Law of Evolutionary Love. This "Love" was not even sexual love, as would be quite befitting a context of Evolution of Species, but christological Love, the Love of the Gospel of John, which in Greek is called *agapé*. And so, Peirce

renamed the Law of Evolutionary Love *Agapism*, and as a name for some kind of semi-continuity this is one of the oddest coinages conceivable.

Altogether, Peirce created two triads. One was Firstness, Secondness, Thirdness — and the second was Synechism, Tychism, Agapism, and there seems to be some parallelism between these two triads. But, having never composed a treatise, Peirce cannot be called to account to explain how these two triads correspond to each other, and let alone how they also connect with the philosophical doctrine of Pragmatism of which C. S. Peirce was an architect. There is a rising flood of studies on the problems and puzzles which are the legacy of Peirce's philosophical efforts, and we want to terminate with referring to one relatively brief article that perhaps comes closest to justifiying our own critique of Peirce. It is Charles Hartshorne, "Charles Peirce's One contribution to philosophy and his most serious mistake", in Edward C. Moore and Richard S. Robin (editors), *Studies in the philosophy of Charles Sanders Peirce, Second Series*, Amherst 1964, pages 454–474.

Mathematics is the oldest organized knowledge there is. Yet "what makes mathematics so effective when it enters science is a mystery of mysteries" (*The Role of Mathematics in the Rise of Science*, p. v); and any philosopher bent on solving the mystery once for all, as C. S. Peirce tried, may come to grief.

(A comprehensive collection of Peirce's writings appears in *The Collected Papers of Charles Sanders Peirce*, Vols. I-VI, edited by Charles Hartshorne and Paul Weiss, Cambridge, Mass., 1931–1935.)

This is a paper which was presented at the Conference on History of American Mathematics, sponsored by Texas Tech University, May 1973. The article will also appear in "Graduate Studies of Texas Tech University" at a later date. The Association is grateful for permission to use material from the article "Continuity and Discontinuity in Nature and Knowledge," *Dictionary of the History of Ideas*, copyright 1973, by Charles Scribner's Sons.

DEPARTMENT OF MATHEMATICS, RICE UNIVERSITY, HOUSTON, TEXAS 77001.

THE NOISY CHANNEL CODING THEOREM FOR ERASURE CHANNELS

PETER ELIAS

1. Introduction. A noiseless binary channel accepts a binary input symbol, 0 or 1, and reproduces the same symbol as its output. Such a channel is said to have a **capacity** C of one bit per channel use, since it is clearly not possible to receive new binary input symbols over the channel at any rate R greater than one bit per use. The channel may of course be used at a rate $R < C$, for example by a transmitter who sends two copies of each input symbol so that a new input bit is received only once per pair of channel uses, giving $R = 1/2$.

A basic result in information theory is that even a noisy binary channel (like that shown in Figure 1a) also has a capacity C. If the error probability p is greater than 0 so that the channel is in fact unreliable, C is < 1, but C will be positive unless $p = 1/2$. And for any $\varepsilon > 0$, $\delta > 0$, it is possible by suitable encoding and decoding to communicate over the channel at rate $R = C - \delta$ with probability of error or ambiguity $< \varepsilon$ at the receiver. A more picturesque statement is that lack of reliability in a channel does not limit the *reliability* of communication through the channel, but only the *rate* at which reliable communication can occur.

Shannon's original proof of this theorem is an existence proof, which does not construct particular coding and decoding schemes with the desired reliable performance [**1**]. More general and precise results of the same existence-proof character have been developed since, and are given in Gallager [**2**]. However, no one has yet constructed a completely explicit coding and decoding scheme which permits arbitrarily reliable communication over any particular noisy channel at rates arbitrarily close to capacity.

This state of affairs has not prevented application of the theory. Schemes are available which require some random numbers for their construction but which can be shown to perform well for almost all choices of the random numbers. Also explicit schemes which communicate reliably at rates well below channel capacity have been found for particular noisy channels. Both kinds of schemes are now in use.

The development of algebraic coding theory, recently discussed in this MONTHLY by Levinson [**3**], arose from the search for explicit schemes to communicate reliably over a binary symmetric channel (BSC). Such a channel is shown in Figure 1a. At each use it accepts an input symbol, either a 0 or a 1, and with probability q delivers the same symbol as an output. With probability $p = 1 - q$ however, the output symbol received does not agree in value with the input symbol: the channel has introduced an error. The probability of an error in the channel is independent of whether the input is a 0 or 1, and successive uses of the channel introduce errors with statistical independence.

Most of the effort in the construction of explicit codes has been devoted to codes

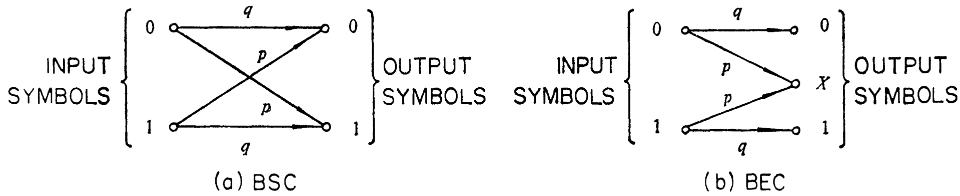


FIG. 1

The binary symmetric channel (BSC) and the binary erasure channel (BEC).

for the BSC, and its symmetric generalization to a larger alphabet of input and output symbols. Berlekamp [4] and van Lint [5] present this work. We choose instead a different channel to illustrate these ideas, the binary erasure channel (BEC) shown in Figure 1b.

Like the BSC, the BEC accepts a binary input symbol, 0 or 1, and with probability q reproduces its input symbol as its output. With probability $p = 1 - q$, however, the output symbol is an X, which marks the spot where the input symbol should be but gives no clue as to its value: the channel has erased the input symbol. On successive uses of the channel the erasures of the successive input symbols are statistically independent events.

The BEC is the pedagogical noisy channel *par excellence.* Results which are subtle and difficult to prove for the BSC have analogs for the BEC which are obvious and which have transparent proofs. We give a selection of such results.

The first is the demonstration that when a noiseless feedback channel is available, communication at a rate equal to channel capacity is possible with no residual error or ambiguity at the receiver. The second is the construction of a code which corrects single erasures in a block of N transmitted symbols. The third is the iterated use of such correction of single erasures to give communication at a positive rate with arbitrarily small probability of ambiguity at the receiver. The rate attained is less than capacity, but is positive for any channel with erasure probability $p < 1$. The fourth is the existence proof that there are codes which permit communication at rates arbitrarily near to channel capacity with arbitrarily small probability or error or ambiguity.

2. Noiseless feedback. Suppose that a noiseless feedback channel is available, so that after the transmitter has sent a symbol to the receiver over the BEC he can look over the receiver's shoulder and observe whether or not his symbol has been erased by the channel. Suppose further that the transmitter has a binary sequence of message symbols which he wants to communicate, in order, to the receiver. Then by sending each message symbol as many times as is required to get exactly one unerased copy through the channel to the receiver, he can send his sequence with no probability of ambiguous reception.

The probability that the transmitter must send a kth copy of a particular message symbol is just the probability p^{k-1} that the first $k - 1$ transmissions of that symbol

were all erased. Thus the average number of channel uses needed to get one unerased copy of a term in his sequence through the channel is just $1/q$ channel uses per message symbol since

$$1 + p + p^2 + p^3 + \cdots = 1/(1-p) = 1/q.$$

The transmitter is therefore introducing new message symbols at an average rate,

$$R = q \text{ new message symbols/channel use.}$$

We call this rate the capacity $C = q$ of the BEC, since it is clearly not possible to receive more than C bits per second over the channel. The receiver decodes his received sequence by simply discarding the X's. He has received each message symbol in its proper order just once, at an average rate of C message symbols/channel use, and has a residual probability P_a of ambiguity about the value of any message symbol which is zero as soon as that symbol has been received.

Without feedback the transmitter may supply the channel with a new message symbol at each successive use. Then he is providing input at a rate $R = 1$ message symbol/use, greater than the capacity $C = q$ of the channel. The channel will erase each symbol with probability p, and the receiver will receive unerased message symbols at an average rate C per channel use. A fraction p of the received symbols will be ambiguous (erasure symbols), so the average probability of ambiguous reception will be $P_a = p$.

If the transmitter wants to send each of his message symbols to the receiver with a probability of ambiguous reception $P_a < p$, the obvious strategy is to send N copies of each message symbol, encoding each 0 into a sequence of N transmitted 0's and each 1 into N transmitted 1's. Then the transmitter is sending at rate $R = 1/N$ message symbols per channel use, and the probability that a message symbol is ambiguous at the receiver is just the probability

$$P_a = p^N = p^{1/R}$$

that all N copies are erased by the channel.

P_a can be made arbitrarily small for any $p < 1$, but only by letting the rate R approach 0. Such simple repetition therefore cannot give reliable communication at a positive rate.

3. Correction of single erasures. A more fruitful scheme starts with the correction of single erasures in a block of N transmissions. To each sequence of $N-1$ message symbols, each 0 or 1, the transmitter adds an Nth (check) symbol, whose value is selected so as to make the total number of 1's in the block of N symbols an even number. Any single erasure in the block of N received symbols can be corrected by the receiver, who fills in the value 0 or 1 as required to make the total number of 1's in the block even again. The scheme fails when two or more erasures occur within

a block of N received symbols, but is still useful in reducing the average number of uncorrected erasures.

Let $p = p_0$, $q = q_0$. If the transmitter uses single erasure correction on blocks of N_1 transmitted digits, the channel introduces an average of $N_1 p_0$ erasures per block. With probability

$$N_1 p_0 q_0^{N_1 - 1}$$

the channel introduces one of the N_1 single erasures only, and the receiver corrects that erasure. Let $N_1 p_1$ be the average number of erasures which remain after correction of the single erasures in each block. Since each single erasure contributes 1 to the average number of erasures per block,

$$\begin{aligned} N_1 p_1 &= N_1 p_0 - N_1 p_0 q_0^{N_1 - 1} \\ &< N_1 p_0 (1 - q_0^{N_1}) = N_1 p_0 (1 - (1 - p_0)^{N_1}) \\ &< (N_1 p_0)^2, \end{aligned} \tag{1}$$

where the first inequality uses $q_0 < 1$ and the second uses the inequality

$$\prod_{n=1}^{N} (1 - \varepsilon_n) \geqq 1 - \sum_{n=1}^{N} \varepsilon_n \tag{2}$$

which holds if $0 < \varepsilon_n < 1$ for $1 \leqq n \leqq N$, with equality only at $N = 1$. (A proof by induction starting at $N = 2$ is easy.)

If $N_1 p_0 \geqq 1$, the inequality

$$N_1 p_1 < (N_1 p_0)^2 \tag{3}$$

does not guarantee an improvement (although direct evaluation of (1) will always give one). But if $N_1 p_0 \leqq 1/2$, (3) gives $N_1 p_1 \leqq 1/4$.

4. Iteration. An iteration of the correction of single erasures gives arbitrarily small values of the probability of ambiguity P_a while keeping the rate R positive.

Given a BEC with erasure probability p_0 and independence between erasures in successive uses of the channel, choose N_1 sufficiently small so that

$$N_1 p_0 \leqq 1/2.$$

Add one check digit to each sequence of $N_1 - 1$ message digits to make a single erasure correction block. Set $N_2 = 2N_1$ and take $N_2 - 1$ such blocks of N_1 digits each, align them one below another, and add an N_2th block of N_1 digits at the bottom which consists only of check digits. Each digit in this bottom block is selected to make the number of 1's in its column even. The digit in the lower right-hand corner is selected to make an even number of 1's in its column and an even number of 1's in

its row. There is no conflict between these two requirements, the proof of which is left as an exercise for the reader. Figure 2 shows an example with $N_1 = 3$, $N_2 = 6$.

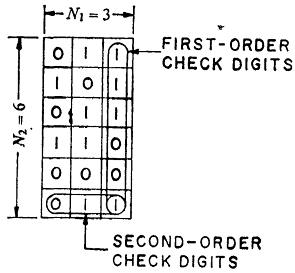


FIG. 2.

The iterative error correction scheme for $i = 2$.

Correcting the single erasures in rows gives by (3)

$$N_1 p_1 < (N_1 p_0)^2 \leqq (1/2)^2 = 1/4$$

as a bound on the average number of erasures left per row after correcting single erasures. Since $N_2 = 2N_1$, this gives

$$N_2 p_1 \leqq 1/2$$

remaining erasures per column, on the average, after correcting rows but before correcting columns. After correcting columns, the average number of erasures remaining per column is

$$N_2 p_2 < (N_2 p_1)^2 \leqq (1/2)^2 = 1/4$$

since, although the erasures remaining in any row are no longer statistically independent after correction of single erasures per row, the erasures remaining in any column *are* statistically independent after correction by rows. Therefore the inequality

$$N_i p_i < (N_i p_{i-1})^2 \tag{4}$$

holds for $i = 2$ as well as for $i = 1$.

To continue the pattern, stack $N_3 = 2N_2$ such layers of $N_1 \times N_2$ rectangles up to make a box, the N_3th being a check layer, $N_4 = 2N_3$ rows of such boxes, etc. In each case the N_i symbols checked by an ith order check digit have never been together in a lower order block, so the erasures in the ith order block remain statistically independent after all single erasures in lower order blocks have been corrected, and (4) applies. The result is that for all i,

$$N_i p_i < (N_i p_{i-1})^2 \leqq 1/4,$$
$$p_i < 1/4N_i.$$

Since $N_i = 2^{i-1}N_1$, this gives

$$p_i < 1/4 \times 2^{i-1}N_1 = 1/2^{i+1}N_1,$$

so the probability that an erasure has not yet been corrected after i stages goes to 0 as i gets large.

It is not clear that the rate stays positive as i gets large, since the fraction of check digits keeps increasing. To explore this question, observe that at the ith stage the number of message digits which have been sent is $(N_1 - 1)(N_2 - 1) \cdots (N_i - 1)$, out of a total of $N_1 \times N_2 \times \cdots \times N_i$. Thus the rate after i stages is

$$\begin{aligned} R_i &= \prod_{j=1}^{i} \left(\frac{N_i - 1}{N_j}\right) = \prod_{j=1}^{i} (1 - 1/N_j) \\ &> 1 - \sum_{j=1}^{i} 1/N_j = 1 - (1/N_1) \sum_{j=0}^{i=1} 2^{-j} \\ &> 1 - 2/N_1 \text{ for all } i. \end{aligned}$$

Therefore, even if i increases without bound, $R_i \geqq 1 - 2/N_1$ is the limit. If $N_1 p_0 = 1/2$, this gives

$$R_i \geqq R = 1 - 4p_0$$

which is positive for $p_0 < 1/4$, while

$$P_a = p_i \leqq 1/2^{i+1}N_1 = 2^{-i}p_0$$

decreases exponentially in i.

5. Comments on iteration. A few comments on the iterative scheme are in order before proceeding to the next topic.

(i) If the channel has an initial erasure probability $p > 1/4$, the inequalities do not guarantee a positive rate. In that case start by repeating each message digit N_0 times as a first step. Choose N_0 large enough so that

$$p_0 = p^{N_0} < 1/4$$

(which can be done for any $p < 1$). Proceeding as before after this first stage gives arbitrarily reliable communication, with

$$P_a = p_i \leqq 2^{-i}p_0 = 2^{-i}p^{N_0}$$

approaching 0 as i increases, at a positive rate for any channel with capacity $C = q = 1 - p > 0$.

(ii) For $N_1 p_0 < 1/2$, the inequalities guarantee that the average number of erasures $N_i p_i$ remaining in an ith order check set after the ith order correction is bounded above by 1/4. Since there are

$$B_{i-1} = N_1 \times N_2 \times \cdots \times N_{i-1} = N_1^{i-1} \cdot 2^1 \cdot 2^2 \cdot \cdots 2^{i-2}$$
$$= N_1^{i-1} \, 2^{(i-1)(i-2)/2}$$

such check sets, the average *total* number p_iB_i of erasures remaining among all $B_i = N_iB_{i-1}$ received digits after all i corrections could increase rapidly with i, even though the probability that any particular digit has not yet been corrected decreases with i. But the scheme in fact does better than that.

Let $N_1p_0 = c/2$, where $0 < c < 1$. Then

$$N_1p_0 < (N_1p_0)^2 = c^2/4,$$

and

$$N_iP_i < (N_ip_{i-1})^2 = (2N_{i-1}p_{i-1})^2 = c^{2^i}/4$$

by induction. So the average total number of erasures remaining after i corrections is bounded by

$$p_iB_i = p_iN_iB_{i-1} < c^{2^i}N_1^{i-1}\, 2^{(i-1)(i-2)/2}/4.$$

The quantity on the right may increase with i for small i and c near 1, but for any $c < 1$ it ultimately approaches 0 as i grows. In the limit $i \to \infty$, therefore, the average number of erasures remaining after all corrections is 0. So is the probability that any erasure remains, since

$$\text{(5)} \qquad \text{Prob}\{1 \text{ or more erasures}\} = \sum_{k=1}^{\infty} \text{Prob}\{k \text{ erasures}\}$$
$$\leqq \sum_{k=1}^{\infty} k\,\text{Prob}\{k \text{ erasures}\} = \text{average number of erasures.}$$

(iii) For large i, i stages of iterated correction of single erasures must completely correct most erasure patterns which have an average number p_0B_i of erasures when $N_1p_0 < 1/2$, for if it failed to deal with most such typical cases the scheme could not reduce the probability of any residual erasures to zero in the limit. Nonetheless there are some patterns containing many fewer erasures which cannot be corrected by this scheme.

If there are erasures in the first two positions in the first block of N_1, and in the same two positions in the second block, the square of 2^2 erasures cannot be corrected by either row or column checks. If the same square is erased in the next plane, the resulting cube of 2^3 erasures cannot be corrected by the first three orders of check digit. And if all 2^i corners of an i-dimensional cube are erased, the erasures cannot be corrected by all the check digits of order i or less. Since 2^i grows much more slowly than B_i, the smallest number of erasures which the scheme cannot correct is a rapidly decreasing function of i.

6. Random coding. Explicit schemes for communication at a rate R near the channel capacity $C = q$ are not known (not even for the BEC!) but it is easy to show the existence of such schemes for the BEC. We show that for any $\delta > 0$ and $\varepsilon > 0$ there are block codes of rate $R \geqq C - 3\delta$ and probability of ambiguity $P_a < 2\varepsilon$.

A block code of length N with M codewords is specified by an array of M rows and N columns, each entry filled in with a binary digit. The array is called a codebook and each row of N entries is called a codeword in the code. The rate of the code is defined by $R = (1/N) \log_2 M$, and there are just

$$G = 2^{NM} = 2^{N2^{NR}}$$

different possible codebooks, some of which are very bad, for example the codebook filled with NM 0's.

Given $C = q$ and ε and δ, choose an N so large that there is a rate R with NR an integer and $C - 2\delta \geqq R > C - 3\delta$. To send a sequence of NR message digits, the transmitter reads the message sequence as a binary number, goes to the corresponding row in his codebook and sends the N symbols in that row in sequence. The receiver, receiving $N - k$ unerased symbols and k erasures, scans his copy of the codebook row by row and marks each row which agrees with the received sequence in all of the $N - k$ unerased positions. The receiver will always mark the row which was actually transmitted. If he marks no other rows, the transmission is unambiguous. For the all-0 codebook all transmissions are always ambiguous, even when $k = 0$. For other codebooks whether a transmission is ambiguous or not depends on which row was sent, how large k is and where the k erasures occur.

Now take G identical erasure channels, in each one give the transmitter and receiver copies of a different one of the G codebooks, have all of the transmitters send message 1, have all of the channels erase the same k of the N digits, and have each receiver compare his received sequence with his codebook row by row and make a check mark after each row which agrees with all of the $N - k$ unerased binary digits in his received sequence.

All G receivers will make a check mark after the first row of their respective codebooks, since in each case that will be the pattern that was actually sent. We ask how many additional check marks will be made in each row by all of the receivers as they proceed down the $M - 1$ rows of their codebooks after the first.

The answer is that the total number of check marks after the mth row of the codebooks will be just $2^{-(N-k)}G$. For a receiver will make a check mark after row m if and only if the $N - k$ unerased digits in row m agree with the $N - k$ unerased digits in row 1. The number of codebooks in which this happens is the number of ways of filling in all of the $M \times N - (N - k)$ positions in the codebook excluding those $N - k$ positions in row m, which are then copied from the corresponding positions in row 1. And this number is just

$$2^{MN} \cdot 2^{-(N-k)} = G \cdot 2^{-(N-k)}.$$

Since there are the same total number of check marks after each row, the total number of check marks in all codebooks after all $M - 1$ rows following the first is

$$G \cdot 2^{-(N-k)} \cdot (M-1) < G \cdot 2^{-(N-k)} \cdot M = G \cdot 2^{-(N-k)+NR},$$

and the average number of incorrect checkmarks per codebook is obtained by dividing by G The probability $P_a(k)$ that a codebook chosen from the G possibilities with equal probability is ambiguous is the probability that it has one or more incorrect checkmarks, which (by (5) above) is less than the average number of incorrect checkmarks per codebook, so

$$P_a(k) < 2^{-(N-k)+NR} \tag{6}$$

The result does not depend on *which* k digits are erased by the channel, but only on how many And the same argument would clearly lead to the same answer if all transmitters sent some other message m rather than message 1. Different codebooks will be ambiguous for different patterns of k erasures and different transmitted messages, but the size of the ambiguous fraction does not change Finally, if the channel erases any number $j < k$ of the N digits the bound still holds, since then

$$P_a(k) \leqq 2^{-N} \cdot 2^{j+NR} \leqq 2^{-N+k+NR}$$

So (6) bounds the average probability of ambiguity (averaged over all G codebooks with equal weight) when *any message is transmitted and the channel introduces any set of k or fewer erasures* If the channel introduces more than k erasures, the bound

$$P_a(j) \leqq 1, \quad j > k \tag{7}$$

is always available.

Now we can bound the average probability of ambiguity P_a as the sum of the probabilities of two disjoint events. Either the channel introduces more than k erasures, which it does with some probability $Q(k)$, and we then use the bound (7) which assumes all codebooks to be ambiguous, or the channel introduces k or fewer erasures, with probability $1 - Q(k)$, in which case the probability of an ambiguous codebook is bounded by (8). Thus

$$P_a \leqq Q(k) \cdot 1 + (1 - Q(k)) \cdot 2^{-N+k+NR}.$$

The argument is completed by using the weak law of large numbers. Given any $\varepsilon > 0$ and $\delta > 0$, the weak law guarantees that for $k = N(\rho + \delta)$ and sufficiently large N, the probability $Q(k)$ that the channel introduces more than k erasures is less than ε. Thus for all sufficiently large N the average probability of ambiguity is bounded by

$$P_a \leqq \varepsilon + 2^{-N+N(\rho+\delta)+NR} = \varepsilon + 2^{-N(C-R-\delta)}$$

and $R \leqq C - 2\delta$, so

$$P_a \leqq \varepsilon + 2^{-N\delta} \tag{8}$$

which can be made $< 2\varepsilon$ by making N sufficiently large.

Enumerate the G codebooks and let $P_a(g)$, $1 \leqq g \leqq G$, be the ambiguity probability of the gth codebook. Then

$$P_a = (1/G) \sum_{g=1}^{G} P_a(g) \geqq (1/G)G \min P_a(g)$$

is the quantity bounded in (8). It follows that the best codebook has $P_a(g) \leqq P_a$, which completes the proof.

7. Comments on random coding. Two comments on the random coding scheme complete the presentation.

(i) Independence of successive erasures is used only to show that the weak law applies. Any other rules governing the occurrence of erasures which make the fraction of erasures in a long sequence of channel uses approach a constant except in a set of cases of probability $< \varepsilon$ will do. Erasures may, for example, be periodic, or occur in independent blocks of twelve, or be governed by a Markov process. The random coding argument still holds.

(ii) Let g be the number of the best code and $P_a(g)$ its probability of ambiguity, averaged over the M different rows in its codebook and the erasure patterns in the channel. Let $P_a(g, m)$ be the probability of ambiguity when row m is sent. Then

$$P_a(g) = (1/M) \sum_{m=1}^{M} P_a(g, m).$$

$P_a(g)$ may be small even though the codebook has two identical rows m_1 and m_2, so that $P_a(g, m_1) = P_a(g, m_2) = 1$. But at least half of the M terms $P_a(g, m)$ must have values $\leqq 2P_a(g)$. Construct a new codebook with only the $M/2$ best rows. Then that code has ambiguity probability

$$P'(m) \leqq 2P_a(g)$$

for every m, $1 \leqq m \leqq M/2$. Since $M = 2^{NR}$, cutting M in half reduces the rate R by only $1/N$, an amount which is negligible in the limit of large N. Thus not only is there a good code for any $R < C$, there is a uniformly good code.

References

1. C. E. Shannon and W. Weaver, The Mathematical Theory of Communication, Univ. of Illinois Press, Urbana, 1964.

2. R. G. Gallager, Information Theory and Reliable Communication, Wiley, New York, 1968.

3. N. Levinson, Coding theory: A counterexample to G. H. Hardy's conception of applied mathematics, this MONTHLY, 77 (1970) 249–258.

4. E. Berlekamp, Algebraic Coding Theory, McGraw-Hill, New York, 1968.

5. J. H. van Lint, Coding Theory, Lecture Notes in Mathematics 201, Springer-Verlag, Berlin, 1971.

DEPARTMENT OF ELECTRICAL ENGINEERING AND RES. LAB. OF ELECTRONICS, MIT, CAMBRIDGE, MA 02139.

HILBERT SPACE OPERATORS AND QUANTUM MECHANICS

E. W. PACKEL

1. Introduction. The beautiful theory of Hilbert space can be motivated in several ways. In one view, the Hilbert space axioms are a most natural generalization of properties of finite dimensional Euclidean space. From a historical standpoint, much of the theory of orthonormal bases and spectral decomposition grows directly out of Hilbert's work with quadratic forms and integral equations. Thirdly, and perhaps most significantly, Hilbert space provides a natural, elegant, and effective setting for one formulation of quantum mechanics. It is the purpose of this paper to elucidate in an elementary and necessarily simplified fashion this important physical role of Hilbert space.

Clearly we cannot hope to deal very generally or thoroughly with the mathematical and philosophical subtleties of the formalism of quantum mechanics without requiring more space and knowledge of physics than would be reasonable in this undertaking. For more detailed and advanced treatments see Mackey [**5**] and Jauch [**3**]. Our aim here is, rather, to use the machinery of Hilbert space theory to convey the spirit of the intellectual triumphs of the 1920's which resulted in quantum mechanics; and to do so without assuming any formal knowledge of quantum mechanics. The author hastens to point out his commitment to and qualification for this task. His very modest knowledge of quantum mechanics has resulted primarily from efforts to communicate with the intrepid and inquisitive physics students who invariably populate undergraduate courses in functional analysis. The author believes that any mathematics undergraduate or graduate who studies Hilbert space theory should have some idea of its important role in quantum mechanics. Two texts which support this belief are Packel [**6**] and Reed and Simon [**8**].

Bounded and unbounded operators on Hilbert space. With the space l^2 of square summable sequences and the space L^2 of square integrable functions as models, von Neumann proposed axioms for Hilbert space in 1929. We present below the definition, some properties, and an important example.

A **Hilbert space** H is a vector space over $\boldsymbol{C}$ on which is defined an **inner product** $\langle\ ,\ \rangle\colon H \times H \to \boldsymbol{C}$ satisfying for all f, g, and h in H and α, β in $\boldsymbol{C}$:

(i) $\langle f,g\rangle = \langle g,f\rangle^*$ (* denotes the complex conjugate),

(ii) $\langle \alpha f + \beta g, h\rangle = \alpha\langle f,h\rangle + \beta\langle g,h\rangle$,

(iii) $f \neq 0 \Rightarrow \langle f,f\rangle > 0$.

Furthermore, H must be complete with respect to the norm defined by

$$\|f\| = \langle f,f\rangle^{\frac{1}{2}}. \tag{1}$$

We shall make use of the well-known **Schwarz inequality**

$$|\langle f,g\rangle| \leqq \|f\|\ \|g\|,$$

which is used to show that

$$\|f+g\| \leqq \|f\| + \|g\|$$

so that (1) in fact does define a norm on H.

As a fundamental example of Hilbert space, and one which we shall use throughout, consider the Lebesgue integral on $\boldsymbol{R}$ and let the role of H be played by

$$L^2(\boldsymbol{R}) = \left\{f: \boldsymbol{R} \to \boldsymbol{C}: \int_{\boldsymbol{R}} |f|^2 < \infty\right\}.$$

If we define

$$\langle f, g \rangle = \int_{\boldsymbol{R}} fg^*$$

and if we play the traditional game of identifying functions which differ only on sets of Lebesgue measure zero (thereby regarding $L^2(\boldsymbol{R})$ as a space of equivalence classes), then $\langle\ ,\ \rangle$ becomes an inner product. Completeness of $L^2(\boldsymbol{R})$ is fundamentally related to the use of (and singlehandedly testifies to the importance of) the Lebesgue integral.

As with so many mathematical constructs (categories), our interest will be not so much in the Hilbert spaces themselves (the objects) as in the structure preserving mappings between spaces (the morphisms). Here, unless otherwise stated, we consider morphisms from a Hilbert space H into itself—these are the **operators** on H. Mathematicians have little trouble deciding what properties these operators should have. We define a (linear) **bounded operator** T on H as follows:

(i) $T: H \to H$ (T is defined on all of H),

(ii) $T(\alpha f + \beta g) = \alpha Tf + \beta Tg$ (T is linear),

(iii) $\|T\| \equiv \sup_{\|f\| \leqq 1} \|Tf\| < \infty$ (T is bounded).

It is readily shown that the operator norm defined in (iii) also satisfies for bounded operators S and T in H:

$$\|S+T\| \leqq \|S\| + \|T\| \tag{2}$$

and

$$\|ST\| \leqq \|S\| \, \|T\|. \tag{3}$$

The boundedness property ($\|T\| < \infty$) turns out to be equivalent to the continuity of T and hence seems a very natural condition to impose.

As some very special examples consider the operators I, U, and V on $L^2(\boldsymbol{R})$ defined by

$If = f$ (identity operator),

$(Uf)(x) = f(x+1)$ (shift one unit to the left),

$(Vf)(x) = e^{ix}f(x)$ (multiplication by e^{ix}).

It is clear that each of these operators is bounded and in fact has norm 1. It may also

be helpful to observe that, relative to a choice of basis, an n by n matrix with complex entries can be regarded as a bounded operator on the Hilbert space $\boldsymbol{C}^n$.

Despite the soundest of mathematical reasons for studying these bounded operators, we shall see that many operators required in the Hilbert space formulation of quantum mechanics (and hence of interest to physicists) are of necessity **unbounded operators.** We define T to be such an operator if:

(i) $T:\Omega \to H$ where Ω is a dense subspace of H,

(ii) T is linear on Ω,

(iii) $\| T \| \equiv \sup_{\substack{\|f\| \leqq 1 \\ f \in \Omega}} \| Tf \| = \infty.$

For the sake of accuracy (though we do not invoke it in what follows) we also require that T be **closed,** by which we mean that

$$\mathrm{graph}(T) = \{(f, Tf) \in H \times H : f \in \Omega\}$$

is a closed subspace of $H \times H$. In contrast to the bounded situation, it turns out that an unbounded operator is continuous at no point of its domain. For a thorough study of unbounded operators see Goldberg [**2**].

As examples of unbounded operators, consider p and q defined by

$$(pf)(x) = -if'(x) \qquad \text{(the ``momentum'' operator)},$$

$$(qf)(x) = xf(x) \qquad \text{(the ``position'' operator)},$$

on suitable domains in $L^2(\boldsymbol{R})$. Both p and q can readily be shown to have infinite norms (for q, look at $\| q(f_k) \|$ with $k = 1, 2, \cdots$, where

$$f_k(x) = \begin{cases} 1 & x \in [k-1, k] \\ 0 & \text{otherwise} \end{cases}$$

and note that $\|f_k\| = 1$ for all k). It is more demanding to show that these operators are closed and that their domains are dense. Indeed it requires more development than is appropriate here even to make precise what these domains are, though they are nicely related by the Fourier transform. The operators p and q play a crucial role in our glimpse of quantum mechanics.

We conclude this section by defining two additional subclasses of operators, the first of which we generalize to allow mappings between distinct Hilbert spaces.

Given Hilbert spaces H and K, a linear transformation $U: H \to K$ is called **unitary** if U is surjective and $\| Uf \| = \|f\|$ for all f in H. It is a standard result that U must also preserve inner products ($\langle Uf, Ug \rangle = \langle f, g \rangle$) and that U, preserving all existing structure, provides the notion of the "essential sameness" of H and K. In this case we call H and K **unitarily equivalent.** It is easily checked that each of our examples I, U, and V is a unitary operator on $L^2(\boldsymbol{R})$.

A (bounded or unbounded) operator T on H is called **self-adjoint** if

(i) $\langle Tf, g\rangle = \langle f, Tg\rangle$ for all $f, g \in \text{domain}(T)$,

(ii) range $(T+iI) = \text{range}(T-iI) = H$.

In the bounded case, (i) alone provides a complete definition of self-adjoint. In the unbounded case the above is not the standard definition, but is equivalent to it. The operators p and q serve as examples of unbounded self-adjoint operators on $L^2(\boldsymbol{R})$. Property (i) can be verified directly (using integration by parts for p), while (ii) requires more work.

3. Interplay with quantum mechanics. Physics in the early 20th century was in a state of great excitement and confusion. Einstein's theories of special and general relativity had initiated a break with classical Newtonian physics and had "proven" themselves by resolving numerous previously unexplained results and by successfully predicting new results. The determinism of Newtonian physics still remained as did many unresolved and seemingly contradictory experimental facts. A fundamental tenet of the emerging quantum theory was that a deterministic view of the universe must give way to a description of particle behavior by means of a probability distribution. Despite Einstein's belief that "God does not play dice with the world," this view of a universe behaving and evolving according to the laws of chance continues to provide the most satisfactory model to date of "the way things are." In what follows we outline (in a one-dimensional nonrelativistic setting) the way in which Hilbert space theory helps to formalize this view. We begin with a few basic notions from probability theory.

Given a variable quantity which takes on real values in some probabilistic fashion (more formally, a **random variable**), its **probability density function** is a function $g: \boldsymbol{R} \to \boldsymbol{R}$ such that for any subset S of $\boldsymbol{R}$

$$\int_S g = \text{the probability that the variable lies in } S.$$

With such a density function g arising from a random variable one associates an **expectation** E_g defined by

$$E_g = \int_{\boldsymbol{R}} xg(x)\,dx,$$

which can be thought of as an average value of the variable. The **variance** D_g of g (or of its generating random variable) is defined by

$$D_g = \int_{\boldsymbol{R}} (x - E_g)^2 g(x)\,dx.$$

The variance D_g provides a measure of how much the random variable deviates from its expectation value, and $(D_g)^{\frac{1}{2}}$ is called the **standard deviation** of the random variable. All these formulas are natural continuous analogues of the more familiar summation formulas arising from discrete probabilistic situations.

We now consider a simplified but fundamental situation with which quantum mechanics deals — that of a particle constrained to one dimension (i.e., $\boldsymbol{R}$) in a physical system. Whereas the instantaneous state of the system is described in classical mechanics by specifying the position and velocity of the particle, the instantaneous state of the system is described in quantum mechanics by specifying a unit vector ψ in $L^2(\boldsymbol{R})$. The most direct interpretation of this vector is that

$$\int_S |\psi|^2$$

represents the probability that the particle in state ψ is in the region S. Thus $|\psi|^2$ provides a probability density function for the random variable defined by the position of the particle in state ψ. Since the particle must be somewhere on $\boldsymbol{R}$, we are compelled to require that

$$\|\psi\|^2 = \langle \psi, \psi \rangle = \int_{\boldsymbol{R}} \psi\psi^* = \int_{\boldsymbol{R}} |\psi|^2 = 1.$$

This approach results in an identification between states of the system and rays (one-dimensional subspaces) in $L^2(\boldsymbol{R})$; and any norm 1 representative from a ray provides a probability density function for the state to which it corresponds.

If the use of Hilbert space stopped here, we should have achieved very little for all our effort. Fortunately there is very much more to be said. In studying a physical system, one generally considers various **observables** of the system such as position, momentum, and energy. In classical mechanics an observable is represented mathematically by a (real-valued) function of position and momentum, and in quantum mechanics the mathematical entity corresponding to the notion of observable for our system is a self-adjoint operator on $L^2(\boldsymbol{R})$. If T is an operator corresponding to some observable, then

$$E_\psi(T) \equiv \langle T\psi, \psi \rangle$$

represents the expectation value of the observable, given an initial state ψ. Since the expectation $\langle T\psi, \psi \rangle$ must be real for any state ψ, which can be shown to imply (i) in the definition of self-adjoint, we have considerable motivation for associating an observable with a *self-adjoint* operator. As expected, two most important examples of this observable-operator correspondence are the position and momentum operators.

It follows from the definition that the expectation value of the position of a particle in state ψ is

$$\int_{\boldsymbol{R}} x|\psi(x)|^2 dx = \int_{\boldsymbol{R}} x\psi(x)\psi^*(x)\, dx.$$

If we let q denote the position observable, then

$$E_\psi(q) = \langle q\psi, \psi \rangle = \int_{\boldsymbol{R}} (q\psi)\,(x)\psi^*(x)\, dx,$$

and we see a compelling reason for defining the position operator q on $L^2(\mathbf{R})$ by $(q\psi)(x) = x\psi(x)$. A more involved motivational argument (using Fourier transforms) which we omit suggests defining the momentum operator p on $L^2(\mathbf{R})$ as we have defined it; i.e., $p\psi = -i\psi'$. We have deliberately suppressed a factor $h/(2\pi)$ (h = Planck's constant) in the definition of p and we shall reinstate it after we have done some computation.

Having defined the expectation $E_\psi(T)$ for T self-adjoint and any state ψ, we approach a notion of variance by defining

$$D_\psi(T) \equiv \|(T - E_\psi(T)I)\psi\|^2 = \langle (T - E_\psi(T)I)\psi, (T - E_\psi(T)I)\psi\rangle.$$

We note that the idea of a state vector and the definitions of $E_\psi(T)$ and $D_\psi(T)$ are applicable in any Hilbert space H. If, in particular, T is self-adjoint on $L^2(\mathbf{R})$, then we may write

$$D_\psi(T) = \int_{\mathbf{R}} (T - E_\psi(T)I)^2\psi(x) \cdot \psi^*(x)\,dx. \tag{4}$$

A comparison of (4) with the definition of variance D_g suggests the idea that $D_\psi(T)$ is a variance in the sense that it provides a measure of how much the observable deviates from its expectation value.

It is possible to establish further connections between physical ideas and Hilbert space operators. In particular, measuring an observable for a particle in an initial state ψ can be related to operating on ψ (perhaps changing the state) with the corresponding self-adjoint operator. Also, allowable values of an observable (energy levels for example) correspond (in the prophetic terminology of Hilbert) to the **spectrum** or generalized eigenvalues of the operator. For an elementary exposition of these additional details see Gillespie [**1**].

We now prove a result in a general Hilbert space H from which the famous uncertainty relation of Heisenberg will swiftly emerge.

THEOREM 1. *If A and B are self-adjoint operators on a Hilbert space H and if ψ is in domain $(AB) \cap$ domain (BA), then*

$$D_\psi(A)D_\psi(B) \geqq \tfrac{1}{4}|E_\psi(AB - BA)|^2.$$

Proof. Using properties of inner products and of self-adjointness, we have

$$\begin{aligned} |E_\psi(AB - BA)|^2 &= |\langle (AB - BA)\psi, \psi\rangle|^2 = |\langle AB\psi, \psi\rangle - \langle BA\psi, \psi\rangle|^2 \\ &= |\langle AB\psi, \psi\rangle - \langle \psi, AB\psi\rangle|^2 = |\langle AB\psi, \psi\rangle - \langle AB\psi, \psi\rangle^*|^2 \\ &= (2\,\mathrm{Im}\langle AB\psi, \psi\rangle)^2 \quad \text{(Im denotes imaginary part).} \end{aligned}$$

Noting that for any a and b in $\mathbf{R}$

$$AB - BA = (A - aI)(B - bI) - (B - bI)(A - aI),$$

letting $E_\psi(A) = a$ and $E_\psi(B) = b$, and applying the identity developed at the beginning

of the proof to the self-adjoint operators $A - aI$ and $B - bI$, we obtain

$$\begin{aligned}\tfrac{1}{4}|E_\psi(AB - BA)|^2 &= \tfrac{1}{4}|E_\psi[(A - aI)(B - bI) - (B - bI)(A - aI)]|^2 \\ &= (\mathrm{Im}\langle(A - aI)(B - bI)\psi, \psi\rangle)^2 \\ &= (\mathrm{Im}\,\langle(A - aI)\psi, (B - bI)\psi\rangle)^2 \\ &\leqq \|(A - aI)\psi\|^2 \, \|(B - bI)\psi\|^2 \quad \text{(Schwarz)} \\ &= D_\psi(A)D_\psi(B).\end{aligned}$$

This completes the proof.

The quantity $AB - BA$ arising above is called the commutator of A and B, and is of considerable importance in quantum mechanics. A simple calculation using the differentiation product rule shows that the commutator for p and q on $L^2(\boldsymbol{R})$ is given by

$$pq - qp = -iI.$$

If we apply Theorem 1 to the self-adjoint operators p (momentum) and q (position), then we obtain for any appropriate state ψ (recall $\|\psi\| = 1$)

$$D_\psi(p)D_\psi(q) \geqq \tfrac{1}{4}E_\psi(I) = \tfrac{1}{4}. \tag{5}$$

In physics $D_\psi(p)$ and $D_\psi(q)$ are frequently denoted by $(\Delta p)^2$ and $(\Delta q)^2$, so Δp and Δq can be thought of as standard deviations. Making this replacement and taking square roots in (5), we obtain $\Delta p \cdot \Delta q \geqq \frac{1}{2}$. We now confess that in traditional units the momentum operator includes a factor $h/(2\pi)$, where $h = 6.625 \cdot 10^{-34}$ joule·sec is Planck's constant. If we allow for this, the inequality becomes

$$\Delta p \cdot \Delta q \geqq \frac{h}{4\pi},$$

and we have quantified the famous **Heisenberg uncertainty principle** that the position and complementary momentum of a particle cannot simultaneously be determined with complete precision. Specifically, the product of "uncertainties" associated with determining momentum and position must always exceed a magnitude on the order of Planck's constant, which fortunately for us all is rather small.

The above development can be carried out in a general Hilbert space H for any pair P and Q of self-adjoint operators whose commutator is a nonzero multiple of the identity. If (as indicated by the importance of p and q) we accept the importance to physicists of operators whose commutator has this property, the following problem suggests itself as one of fundamental significance.

General problem. Find all pairs of self-adjoint operators P and Q on H which satisfy

$$PQ - QP = -iI \tag{c}$$

on some "sufficiently large" domain Ω.

Any tendency on the part of mathematicians to confine their attention to the bounded (continuous) operators on H would appear to be challenged by the following elegant result.

THEOREM 2 (Wielandt, 1949). *There do not exist bounded operators P and Q on H satisfying* (c) *on all of H.*

Proof. There is no loss of generality if we replace (c) by

$$I = PQ - QP, \tag{6}$$

since this could be arranged by using iP instead of P. Suppose, as the basis of an indirect proof, that (6) holds everywhere on H for bounded operators P and Q. Then an induction argument shows that for every $n = 1, 2, \cdots$

$$nQ^{n-1} = PQ^n - Q^nP. \tag{7}$$

Indeed the $n = 1$ case is simply the assumed result (6); and, assuming (7) holds for a general n, we have

$$(n+1)Q^n = nQ^{n-1}Q + Q^nI = (PQ^n - Q^nP)Q + Q^n(PQ - QP) = PQ^{n+1} - Q^{n+1}P.$$

This establishes the induction step. Applying (2) and (3) to (7), we obtain for $n = 1, 2, \cdots$

$$n\| Q^{n-1} \| \leqq 2\| P \| \; \| Q \| \; \| Q^{n-1} \|.$$

This result requires that $\| Q^n \| = 0$ for *some* n since otherwise we would have $n \leqq 2\| P \| \; \| Q \|$ for every n, which is impossible under the assumption that P and Q are bounded. Finally, using (7) repeatedly, we obtain

$$\| Q^n \| = 0 \Rightarrow Q^n = 0 \Rightarrow Q^{n-1} = 0 \Rightarrow \cdots \Rightarrow Q = 0 \Rightarrow I = 0,$$

which is untenable (except in 0 dimensional Hilbert space). This establishes our contradiction and shows that such bounded operators cannot exist.

Theorem 2 provides a partial answer to our "general problem" in the sense that we need not look among the bounded operators for solutions to (c). It also seems to create a new philosophical problem. In addition to the various well-known and often serious differences of viewpoint, the beloved continuity of mathematicians now appears quite incompatible with the "unbounded" requirements of physicists as regards one important aspect of quantum mechanics. In the final section we shall achieve a reconciliation by establishing a correspondence between special kinds of bounded and unbounded operators. At the same time a very pleasing solution to our general problem will emerge.

4. Unitary groups and Schrödinger couples. There is a useful analogy which relates self-adjoint operators to real numbers and unitary operators to complex

numbers of modulus 1. Since the transformation

$$\tau \mapsto e^{it\tau} \quad (t \in \boldsymbol{R})$$

maps a real number τ to a one-parameter multiplicative group of complex numbers with modulus 1, analogy suggests such a transformation might convert self-adjoint operators (even unbounded ones) into one-parameter groups of unitary operators (which are of necessity bounded). All this is formalized in the following famous theorem. See Reed and Simon [**8**, p. 265] for a proof and for a definition of "strongly continuous".

THEOREM (Stone, 1932). *Every self-adjoint operator T on a Hilbert space H generates a strongly continuous one-parameter group of unitary operators e^{itT} on H. Conversely, every such one-parameter group is generated by a unique self-adjoint operator.*

The procedure for "exponentiation" of operators is of considerable interest and complexity (thus witness the 808 pages of [**4**]). Here we motivate plausibility by noting that if T is bounded, then the familiar power series expansion

$$e^{it\tau} = \sum_{k=0}^{\infty} \frac{(it\tau)^k}{k!}$$

carries over directly; and if T is unbounded, then the identity

$$e^{it\tau} = \lim_{k \to \infty} \left(1 - \frac{it\tau}{k}\right)^{-k}$$

can be generalized to obtain bounded operators. More precisely, it can be shown (with T self-adjoint) that for $t \in \boldsymbol{R}$

$$\left(I - \frac{itT}{k}\right)^{-1}$$

always exists and is bounded; and that for every $f \in H$

$$e^{itT}f = \lim_{k \to \infty} \left(I - \frac{itT}{k}\right)^{-k} f.$$

As an example of Stone's theorem we extend the unitary operators U and V on $L^2(\boldsymbol{R})$ which we defined earlier. For $t \in \boldsymbol{R}$ define operators $U(t)$ and $V(t)$ on $L^2(\boldsymbol{R})$ by

$$(U(t)f)\,(x) = f(x+t) \text{ and } (V(t)f)\,(x) = e^{itx}f(x).$$

Then $U(t)$ and $V(t)$ are strongly continuous one-parameter groups of unitary operators, and their generators are, respectively, the self-adjoint operators p and q. Thus $U(t) = e^{itp}$ and $V(t) = e^{itq}$. To motivate these claims, we show that p can be recaptured from $U(t)$ in much the same way that τ can be recaptured from $u(t)$

$= e^{it\tau}$. Indeed $u'(0) = i\tau$, and similarly

$$(U'(0)f)(x) = \lim_{h\to 0}\left(\left(\frac{U(h)-U(0)}{h}\right)f\right)(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = f'(x).$$

Hence $U'(0) = ip$, suggesting that p is the generator of $U(t)$. A similar argument exists for q and $V(t)$.

Armed with this relationship between self-adjoint operators and one-parameter groups of unitary operators, we return to our general problem of determining all pairs of self-adjoint operators satisfying the identity (c). One would expect the existence of a corresponding identity involving one-parameter unitary groups. Such expectations are confirmed by the unitary operator identity

$$\text{(C)} \qquad e^{itP}e^{isQ} = e^{its}e^{isQ}e^{itP} \quad (s,\, t\in \boldsymbol{R}),$$

which corresponds to (c) in the following way:

THEOREM 3 (Fojas, Geher, Nagy, 1960). *Given self-adjoint operators P and Q satisfying* (c) *on Ω such that $(P + iI)(Q + iI)\Omega$ or $(Q + iI)(P + iI)\Omega$ is dense in H, then P and Q satisfy* (C).

Proof. See Putnam [7, p. 76].

In attempting to obtain pairs of self-adjoint operators satisfying (c) or (C), a direct procedure is to start with p and q and any unitary transformation $U\colon L^2(\boldsymbol{R})\to H$. Then the operators P and Q on H defined by $P = UpU^{-1}$ and $Q = UqU^{-1}$ are readily seen to satisfy (c) and (C) since p and q do. Accordingly, we define a **Schrödinger couple** as a pair (P,Q) of self-adjoint operators on H such that $P = UpU^{-1}$ and $Q = UqU^{-1}$ for some unitary operator $U\colon L^2(R)\to H$.

A further procedure for generating pairs satisfying (c) and (C) is to start with two or more Schrödinger couples and to "splice" together the various P components and also the Q components in a natural fashion, thereby obtaining a new pair satisfying (c) or (C) on the direct sum of the underlying Hilbert spaces. We refer to such a pair as a **direct sum** of Schrödinger couples.

The following theorem takes the bounded operator condition (C) and states most pleasingly that the simple procedures described in the above two paragraphs account for all self-adjoint pairs satisfying (C).

THEOREM 4 (von Neumann, 1931). *Given self-adjoint operators P and Q satisfying* (C), *then (P,Q) is a Schrödinger couple or a direct sum of Schrödinger couples.*

Proof. See Putman [7, p. 65].

It is worth noting that von Neumann's beautiful and rather complicated proof seems to leave little room for improvement or simplification, despite considerable advances in the theory of groups and semigroups of operators.

The astute reader may now see that the end is in sight. Indeed, a careful look at Theorems 3 and 4 with use of the fact that p and q satisfy (c) and (C) shows that we have a very satisfactory answer to our general problem: the self-adjoint pairs P and Q satisfying (c) on a "sufficiently large" domain Ω are precisely the Schrödinger couples and their direct sums; and "sufficiently large" is given meaning in terms of the density of $(P + iI)(Q + iI)\Omega$ or $(Q + iI)(P + iI)\Omega$.

We conclude by observing that a very satisfactory rapprochement has been achieved. The theory of Hilbert space and its operators provides a most effective and elegant setting for formulating ideas of quantum mechanics, and the mathematically natural bounded operators prove to be most valuable in studying the physically indispensable unbounded ones.

References

1. D. T. Gillespie, A Quantum Mechanics Primer, Intext, New York, and London, 1970.
2. S. Goldberg, Unbounded Linear Operators, McGraw-Hill, New York, 1966.
3. J. Jauch, Foundations of Quantum Mechanics, Addison-Wesley, Reading, Mass., 1968.
4. E. Hille and R. S. Phillips, Functional Analysis and Semi-Groups, American Mathematical Society, Providence, R. I., 1957.
5. G. W. Mackey, The Mathematical Foundations of Quantum Mechanics, Benjamin, New York, 1963.
6. E. W. Packel, Functional Analysis: a Short Course, Intext, New York and London, 1974.
7. C. R. Putnam, Commutation Properties of Hilbert Space Operators and Related Topics, Springer-Verlag, New York, 1970.
8. M. Reed and B. Simon, Functional Analysis, Academic Press, New York, 1972.

DEPARTMENT OF MATHEMATICS, LAKE FOREST COLLEGE, LAKE FOREST, IL 60045.

R. W. BRINK — AN OBITUARY

J. M. H. OLMSTED

Raymond Woodard Brink was born in Newark, New Jersey, on January 4, 1890, and died in La Jolla, California, on December 27, 1973. He was an undergraduate student at Kansas State University, where he received a B. S. degree in 1908 and a B. S. E. E. degree in 1909. After a brief period of teaching, Brink entered the graduate school of Harvard University, where he earned the Ph. D. degree in 1916. His dissertation, entitled "Some Integral Tests for the Convergence and Divergence of Infinite Series," was published under a slightly different title in the *Transactions of the American Mathematical Society*. He retained an active interest in integral tests for infinite series, and published two more papers on the subject, one in the *Annals of Mathematics* and one in this MONTHLY.

Awarded a Sheldon Traveling Fellowship for postdoctoral study, Brink spent the year 1916–17 studying at the Collège de France and the Sorbonne in Paris, where he began a lifelong devotion to France and to the French language and culture. On two

later occasions, in 1924–25 and in 1932–33, he spent sabbatical leaves studying at the Sorbonne.

In 1917 Brink joined the faculty of the Department of Mathematics in the College of Science, Literature, and the Arts at the University of Minnesota, where, except for three leaves of absence, he remained until retirement in 1957, when he was awarded the title of Professor Emeritus. During the year 1919–20 he lectured at the University of Edinburgh. He was chairman of the department at Minnesota from 1928 until his retirement, except for the period from 1932 to 1939. Briefly in 1955 he also assumed additional duties as acting assistant dean for his college.

Brink was a member of several professional organizations, principally the Mathematical Association of America, the American Mathematical Society (for whom he was an associate editor of the *Transactions*), the American Association for the Advancement of Science (for whom he was a chairman of Section A on mathematics, and an invited speaker at a national meeting), Sigma Xi, the Edinburgh Mathematical Society, and the New York Academy of Sciences. His professional activities were mainly directed to the Association, of which he was a member since 1922. He wrote several book reviews and refereed many papers for the *American Mathematical Monthly*, was a member of the Board of Trustees of the Association (1934–40), and was elected vice president in 1940 and president for the period 1941–42. In addition, he was chairman of the Committee on Arrangements (for both the Association and the Society) for the summer meetings in 1931 at the University of Minnesota, and was chairman (1951) of the Association Committee on Places of Meetings. In his retiring presidential address, delivered at the twenty-seventh annual meeting of the Association in Chicago in November, 1943, he spoke on "College Mathematics During Reconstruction" with astonishing accuracy of foresight for the years immediately following the end of World War II.

Raymond Brink and Carol Ryrie were married on July 12, 1918. They had two children, David Ryrie Brink and Nora Caroline Brink. Brink was an intensely loyal husband and father, and felt special pride in the considerable writing accomplishments of his wife. His son David has said, "He was Mother's original and most devoted fan and remained always her 'best friend and (occasionally) severest critic'." Brink was an avid reader and student throughout his life, but also a lover of games and contests, both mental and physical. In addition to setting high standards intellectually, Brink was always a complete gentleman and, perhaps stemming from his upbringing as son of a Baptist minister, his life served as a model not only for his family but for all those who had the good fortune to know him well.

Many mathematicians knew Brink primarily through his many successful textbooks in trigonometry, college algebra, and analytic geometry. Others have benefited either as authors or as users of books, from his exceedingly careful and painstaking work as editor of mathematical texts. From 1944 until his death he was consulting editor for the Appleton-Century Mathematics Series. As both author and editor Brink never lost his love for good language usage and *le mot juste*.

Shortly after retirement, during the year 1958–59, Brink was a visiting professor at the University of Miami. He then established permanent residence in La Jolla, California, where, in addition to his editorial work, be pursued his hobbies of gardening, reading, photography, and music. He was a member of the La Jolla Presbyterian Church, Friends of the University of California at San Diego Library, and of the La Jolla Symphony Association.

Survivors include his wife, son, daughter, a brother, and eight grandchildren. Inurnment was at the Brinks' Wisconsin vacation home at Lake Windigo near Hayward.

DEPARTMENT OF MATHEMATICS, SOUTHERN ILLINOIS UNIVERSITY, CARBONDALE, IL 62901.

QUERIES

EDITED BY A. C. ZITRONENBAUM

This Department welcomes queries from readers about mathematics at the collegiate level, such as sources for exposition of a particular topic from a special point of view, references to vaguely remembered articles, descriptions of special kinds of courses or teaching methods, and methods for constructing illustrative examples for exercises of particular kinds (questions on research topics should, in general, be addressed to the "Queries Department" of the Notices of the American Mathematical Society). Replies will be forwarded to the questioner and may be edited into a composite answer for publication in this Department. Consequently all items submitted for consideration for possible publication should include the name and complete mailing address of the person who is to receive the reply. Queries and answers should be sent to A. C. Zitronenbaum, Department of Mathematics, Cornell University, Ithaca, NY 14850.

Reply to Query 5. A polygonal simple closed curve in 3-space bounds a polyhedral orientable surface in 3-space. This theorem is due to F. Frankl and L. Pontrjagin (Math. Ann. 102 (1930), pp. 785–789). It is also proved by H. Seifert (Math. Ann. 110 (1934), pp. 571–592). Though these two proofs are essentially the same, Seifert's proof would be more accessible at the sophomore calculus level. The construction is also discussed in R. H. Fox, "A Quick Trip Through Knot Theory" in the book *Topology of 3-Manifolds*, M. K. Fort, ed., p. 140.

If a simple closed curve in the 3-space is wild, then it may not bound an orientable surface in the 3-space (S. Kinoshita, M. E. Kidwell).

12. J. A. Cross. I have recently been investigating philosophies and methods for teaching a basic mathematics class in problem solving using electronic calculators. The basic intent is most closely related to the traditional slide rule class. I would appreciate ideas on course content, learning activities, methods of obtaining money to buy calculators for this sort of thing, and a general feeling for what is being done.

MATHEMATICAL NOTES

EDITED BY DAVID ROSELLE

Material for this Department should be sent to David Roselle, Department of Mathematics, Virginia Polytechnic Institute, Blacksburg, VA 24061.

A CURIOUS NIM-TYPE GAME

DAVID GALE

A set of mn objects is laid out in an m by n rectangular array. We denote by (i,j) the object in row i, column j. The first player P_{I} selects an object (i_1,j_1) and then removes all objects (i,j) such that $i \geqq i_1$ and $j \geqq j_1$. In other words, if i increases upward and j increases from left to right, then P_{I} removes a northeast "quadrant." Player two, P_{II}, now picks (i_2,j_2) from among the remaining objects and removes all (i,j) such that $i \geq i_2$, $j \geq j_2$. The play then reverts to P_{I} and continues in the same way until all objects have been removed. The player making the last move loses. Thus the object of the game is to make your opponent pick up $(1,1)$.

There are some trivial special cases of the game.

CASE A: P_{I} wins the $2 \times n$ $(m \times 2)$ game by selecting $(2,n)((m,2))$. Then, whatever P_{II} does, P_{I} moves so as to leave a "position" in which there is one more object in row (column) 1 than in row (coumn) 2. The reader will easily see that this is always possible and winning.

CASE B: P_{I} wins the $m \times m$ game by selecting $(2,2)$. From then on he "symmetrizes." Whenever P_{II} chooses $(1,j)$ he chooses $(j,1)$, etc. Again this is easily seen to win.

The above are the only two cases in which general winning strategies are known. The thing which makes the game interesting, however, is the following

THEOREM. *For all m and n the game is a win for P_{I}.*

The proof of this fact is typical of something which occurs quite often in game theory in that it is completely nonconstructive. Although it establishes the existence of a winning strategy for P_{I} it is of absolutely no use in finding such a strategy. Here is the argument. There are two possibilities.

CASE 1: P_{I} has a winning strategy in which his first move is to select (m,n).

CASE 2: If P_{I} selects (m,n) he loses. Then there must be a response (i_2,j_2) by P_{II} which wins for P_{II}. This means that the position of the game after P_{II}'s move is a loss for the player who must then move, in this case P_{I}. The point is, however,

that P_I could have handed this position to P_{II} if he had himself chosen (i_2,j_2). Hence (i_2,j_2) is a winning first move for P_I, and the assertion is proved.

The above argument is reminiscent of the well known one which asserts that games like tic-tac-toe cannot be a win for player II (it applies, for instance, to the unsolved Go Moku which is tic-tac-toe, 5 in a row on a 19 by 19 board.) It goes like this: suppose the game was a win for P_{II}: Then let P_I make any first move and then pretend in his mind he did not make it, and from then on behave in accordance with the alleged winning strategy for P_{II}. This will always be possible unless at some point this strategy requires him to move onto the square he chose initially. In that case, he again makes an arbitrary move. We see then that the alleged winning strategy for P_{II} is also available to P_I, but by definition of winning there cannot be winning strategies for both players. Contradiction. (Note, however, that the tic-tac-toe proof as given here is a proof by contradiction while the proof of our theorem is direct, which points up the interesting fact that a non-constructive proof is not necessarily a proof by contradiction.)

It may be of interest to observe that both Nim and this game (Gnim? Gnome?) are special cases of the following general class of games: Let S be any partially ordered set. A player moves by choosing some element of S and removing all elements greater than or equal to it, and the player moving last loses. Nim corresponds to the special case where S is the *sum* (disjoint union) of a finite number of totally ordered sets; Gnim is the case where S is the product of two totally ordered sets. The argument showing that Gnim is a win for P_I applies to any set S which has a *largest* element, thus, for example, the product of any number of totally ordered sets, but of course not to Nim.

The above is essentially all the theoretical information I have about this game (I shall give one further special result at the end.) However, with the aid of a computer some rather intriguing empirical results have been obtained. The $3 \times n$ game has been completely analyzed for $n \leqq 100$ and in all cases it has turned out that there is only one winning first move for P_I. This is also the case for 4×5 and 4×6 and of course Cases A and B discussed in the beginning.

Question. Is the winning first move unique for all m and n? (See note **"Added in proof."**)

The diagrams below give the winning first move in the $3 \times n$ game for $2 \leqq n \leqq 12$.

```
×        ×          × ×          × × ×            × × ×              × × × ×                × × × ×
× ×      ×          × ×          × × × × ×        × × ×              × × × × × × ×          × × × ×
× ×      × × ×      × × × ×      × × × × ×        × × × × × ×        × × × × × × ×          × × × × × × × ×
 2         3           4              5                 6                    7                       8
```

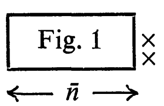
```
× × × × × ×            × × × × ×                    × × × × × ×                      × × × × × × × ×
× × × × × × × × ×      × × × × ×                    × × × × × ×                      × × × × × × × × × × × ×
× × × × × × × × ×      × × × × × × × × × ×          × × × × × × × × × × ×            × × × × × × × × × × × ×
        9                       10                            11                               12
```

There are of course two types of moves depending on whether one takes the initial

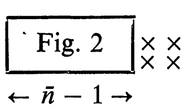
"bite" from the top row only or the top two rows. It turns out that roughly 58 percent of the moves are of this second type, as for the case $n = 3, 4, 6, 8, 10, 11$, and 42 percent of the first, e.g., $n = 2, 5, 7, 9, 12$. In general the length of the bite appears to increase with n. In fact for all $n \leqq 170$ there is only one counterexample. For $n = 87$ the bite size is 37, while for $n = 88$ the bite size is 36 (both of these are two-row bites). Phenomena like this lead one to believe that a simple formula for the winning strategy might be quite hard to come by.

We close with a conjecture: *it is never optimal to select* (m, n) *on the first move except when* $m = 2$ *or* $n = 2$. We shall prove this for the case $m = 3$ as a further illustration of the type nonconstructive argument one uses. This one requires an argument by contradiction.

First observe that in the 3×3 game $(3, 3)$ is losing (since $(2, 2)$ wins). Now assume that $(3, n)$ is losing for all $3 \times n$ games up to $\bar{n}$ and consider the case $3 \times (\bar{n} + 1)$. The argument is best given with pictures

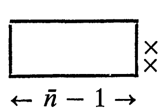
suppose 

is a winning position

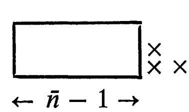
then 

must be losing,

so there must be a way of going from Fig. 2 to a winning position. Now clearly no choice (i, j), $j \leqq \bar{n}$ can give a winning position for this would leave a position which P_{II} could have presented to P_{I}, contradicting the assumption that $(3, \bar{n} + 1)$ was winning for P_{I}. The only possible choices are therefore either $(\bar{n}, 1)$ or $(\bar{n}, 2)$, but $(\bar{n}, 1)$ leaves



which is losing by the induction hypothesis and $(\bar{n}, 2)$ leaves



after which P_{II} can play (3,1) leaving

which is losing for the $2 \times (n + 1)$ game of Case A, so the proof is complete.

One can prove any number of special results of this sort by similar arguments.

For example, for $n > 4$ it is never winning to choose $(2, n-1)$. For $n > 5$ it is never winning to choose $(3, n-1)$, and presumably some general inequalities exist showing that for large rectangles the bites cannot be too small. I expect the problem of finding explicit winning strategies may be hopeless, but I should think one might find a way of settling questions like the uniqueness of the first move.

Finally, let me mention some generalizations. The first is to allow either m or n or both to be infinite. However, these games turn out to be rather trivial because (A) $1 \times \infty$ is a win for P_{I} (trivial), (B) $2 \times \infty$ is a win for P_{II} (a nice exercise for the reader) and (C) $m \times \infty$, $2 < m \leqq \infty$, is a win for P_{I}, as he can choose (2,1) leaving P_{II} with $2 \times \infty$. Of more interest are higher dimensional games, e.g. m by n by r in 3-space. Of course, any such finite game can be solved in a finite amount of time by, at worst, enumerating all possible strategies. The real challenge, it seems to me, are games like $3 \times 3 \times \infty$ or even $\infty \times \infty \times \infty$. ($2 \times m \times \infty$ is a win for P_{I}. Why?) These belong to an interesting class of games with the property that although every play of the game terminates after a finite number of moves there is no upper bound on the possible lengths of a play (as there is for example in chess). In particular, I don't know of any way to program a computer to find out, say, if $3 \times 3 \times \infty$ is a win for P_{I} or P_{II}.

Added in Proof: Since this article was submitted, a description of the game appeared in the column of Martin Gardner in the magazine "Scientific American," pp. 110–111, January 1973. In response to the article, K. Thompson of Bell Laboratories and M. Beeler at M.I.T. discovered by using computers that there exist games with more than one winning first move. The smallest known example is 8×10. Further it was learned that a numerical game isomorphic to this one was described by F. Schuh in an article entitled "The Game of Devisors" in Nieuw Tijdschrift voor Wiskunde, Vol. 39, pp. 299–304, 1952.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA, BERKELEY, CA 94720.

THE POWER MEAN AND THE LOGARITHMIC MEAN

TUNG-PO LIN

The logarithmic mean of two distinct positive numbers x and y, defined by

$$(1)\qquad L = L(x, y) = \frac{x - y}{\ln x - \ln y}, \text{ for all distinct } \quad x, y > 0,$$

is quite frequently used in some practical problems, such as in heat transfer and fluid mechanics. The power mean of two positive numbers x and y, defined by

$$(2)\qquad M_p = M_p(x, y) = \left(\frac{x^p + y^p}{2}\right)^{1/p}, \qquad x, y > 0,$$

for any real number $p \neq 0$, is a generalization of the so-called "root-mean-square average." Thus

$$M_2 = [\tfrac{1}{2}(x^2 + y^2)]^{1/2} \qquad \text{is the root-mean-square,}$$

$$M_1 = \tfrac{1}{2}(x + y) \qquad \text{is the arithmetic mean,}$$

$$M_{-1} = \frac{2xy}{x + y} \qquad \text{is the harmonic mean.}$$

Since it can be proved [1, p.63] that $M_p \to \sqrt{xy}$ as $p \to 0$, it is convenient to adopt the following convention:

$$M_0 = \sqrt{xy} = \text{the geometric mean.}$$

Another useful average is

$$M_{1/3} = \left(\frac{x^{1/3} + y^{1/3}}{2}\right)^3.$$

This is called the Lorentz combination or Lorentz mean in, for example, the theory of equation of state for gases.

A well-known result [1, p.64] about the power mean is that it is an increasing function of p, that is,

(3) *If $x \neq y$ and $p < q$, then $M_p < M_q$.*

The classical inequality: $\sqrt{xy} < \frac{1}{2}(x + y)$ is of course a special case of (3).

Another result [2], not so well-known, is that L separates M_0 and M_1, thus

(4) $$M_0 < L < M_1, \qquad \text{for any distinct } x, y > 0.$$

Recently, Carlson [3] has sharpened inequality (4) into

(5) $$M_0 < L < M_{1/2}, \qquad \text{for any distinct } x, y > 0.$$

In this note, we shall answer the following question: What are the least value q and the greatest value p such that

$$M_p < L < M_q$$

is valid for all distinct positive numbers x and y?

THEOREM 1. *If $p \geqq \frac{1}{3}$ then $L < M_p$ for any distinct $x, y > 0$.*

Proof: For any $p \neq 0$, and distinct $x, y > 0$, we have

(6) $$\frac{L}{M_p} = \frac{2^{1/p}(x - y)}{(x^p + y^p)^{1/p}\ln(x/y)} = \frac{2^{1/p}(z^{1/p} - 1)}{(z + 1)^{1/p}\ln z^{1/p}},$$

where $z = (x/y)^p$, $z > 0$, $z \neq 1$. If we let $z = (1 + w)/(1 - w)$, that is,

$$w = (z - 1)/(z + 1), \qquad 0 < |w| < 1 .$$

then $L/M_p = f(w,p)/g(w)$, where

$$f(w,p) = p[(1+w)^{1/p} - (1-w)^{1/p}]/(2w),$$
$$g(w) = [\ln(1+w) - \ln(1-w)]/(2w).$$

The Maclaurin series expansion of g in w:

$$g(w) = 1 + \tfrac{1}{3}w^2 + \tfrac{1}{5}w^4 + \tfrac{1}{7}w^6 + \cdots, \qquad 0 < |w| < 1, \tag{7}$$

shows clearly that $g(w) > 1$ for $0 < |w| < 1$. Therefore, Theorem 1 is equivalent to:

If $p \geqq 1/3$ *then* $f(w,p) < g(w)$ *for* $0 < |w| < 1$.

To prove this, we expand f also in Maclaurin series in w:

$$f(w,p) = 1 + \tfrac{1}{3}a_1 w^2 + \tfrac{1}{5}a_2 w^4 + \tfrac{1}{7}a_3 w^6 + \cdots, \qquad 0 < |w| < 1, \tag{8}$$

where

$$\begin{aligned} a_k &= \left(\frac{1}{p}-1\right)\left(\frac{1}{p}-2\right)\left(\frac{1}{p}-3\right)\cdots\left(\frac{1}{p}-2k\right)\Big/(2k)! \\ &= \left(1-\frac{1}{p}\right)\left(2-\frac{1}{p}\right)\left(3-\frac{1}{p}\right)\cdots\left(2k-\frac{1}{p}\right)\Big/(2k)! \\ &= \left(1-\frac{1}{p}\right)\left(1-\frac{1}{2p}\right)\left(1-\frac{1}{3p}\right)\cdots\left(1-\frac{1}{2kp}\right). \end{aligned}$$

Note that

$$a_1 = \left(1-\frac{1}{p}\right)\left(1-\frac{1}{2p}\right) = 1 - \frac{3}{2p^2}\left(p-\frac{1}{3}\right).$$

Therefore, $a_1 \leqq 1$ for $p \geqq 1/3$. Furthermore, $0 \leqq [1 - 1/(jp)] < 1$ for all $j \geqq 3$ and $p \geqq 1/3$. Consequently, if $p \geqq 1/3$ then $a_k < 1$ for $k = 2, 3, \cdots$, Comparison of equations (7) and (8) shows that if $p \geqq 1/3$, then

$$f(w,p) < g(w) \quad \text{for} \quad 0 < |w| < 1.$$

Thus, the theorem is proved.

Note that the first three derivatives of $f(w,\frac{1}{3})$ and $g(w)$ are all identical at $w = 0$, showing that $M_{1/3}$ serves as a very good approximation to L when x and y are not too drastically different.

THEOREM 2. *If* $p \leqq 0$ *then* $L > M_p$ *for any distinct* $x, y > 0$.

This theorem is simply a consolidation of the first inequality in (4) and the inequality (3) above. In reference [2], the first inequality in (4) was established by expanding the difference of the reciprocals of L and M_0, $L^{-1} - M_p^{-1}$, into power series. It can also be proved using our present approach, by letting

$$x/y = (1+w)^2/(1-w)^2.$$

We then have:

$$\frac{L}{M_0} = \frac{1/(1-w^2)}{g(w)} = \frac{1 + w^2 + w^4 + w^6 + \cdots}{1 + 1/3\,w^2 + 1/5\,w^4 + 1/7\,w^6 + \cdots},$$

which is obviously greater than 1 when $0 < |w| < 1$.

Now the following two theorems will establish that 1/3 and 0 are indeed the best values one can get.

THEOREM 3. *If $p < 1/3$, then there exist $x, y > 0$ such that $L > M_p$.*

Proof. If $p \leqq 0$, the conclusion is valid by Theorem 2. Thus we need to consider only the case when $0 < p < 1/3$. The expression for a_1 in the proof of Theorem 1 shows that $a_1 > 1$ when $p < 1/3$. Consequently there exists h between 0 and 1 such that $f(w,p) > g(w)$ for $0 < w < h$. Remembering the definitions of w and z in the proof of Theorem 1, one can easily prove that the conditions

$$0 < w < h \qquad \text{and} \qquad 0 < h < 1$$

are equivalent to

$$1 < z < r^p \qquad \text{and} \qquad 1 < r,$$

where $r = (1+h)^{1/p}/(1-h)^{1/p}$. Hence there exists $r > 1$ such that $L > M_p$ for $1 < (x/y) < r$ and $p < 1/3$.

THEOREM 4. *If $p > 0$, then there exist $x, y > 0$ such that $L < M_p$.*

Proof. From the last quotient in equation (6), it is easy to show that the ratio (L/M_p) approaches zero as $z \to \infty$, for a fixed positive p. Hence for large enough z, $L < M_p$.

In conclusion, we have proved in this note that the least value q and the greatest value p such that the inequality $M_p < L < M_q$ is valid for any distinct $x, y > 0$ are $p = 0$ and $q = 1/3$.

Comment by Harley Flanders. Theorem 1 above is an interesting, apparently new, inequality. A proof without MacLaurin series might be useful.

Consider

$$f(t) = \frac{3}{8}\ln t - \frac{t^3 - 1}{(t+1)^3}$$

for $t \geqq 1$. A short calculation yields

$$f'(t) = \frac{3}{8}\,\frac{(t-1)^4}{t(t+1)^4}.$$

Hence $f'(t) > 0$ for $t > 1$. Since $f(1) = 0$, we conclude that $f(t) > 0$ for $t > 1$. Now let $0 < y < x$. Apply $f(t) > 0$ to $t = x^{1/3}/y^{1/3}$.

The result is

$$\left(\frac{\sqrt[3]{x}+\sqrt[3]{y}}{2}\right)^3 > \frac{x-y}{\ln x - \ln y}.$$

Professor Beckenbach has communicated proofs along the same lines for Theorems 3 and 4.

The author wishes to thank Professor Edwin F. Beckenbach for suggesting Theorems 3 and 4 and several improvements in their proofs.

References

1. N. D. Kazarinoff, Analytic Inequalities, Holt, New York, 1961.

2. D. S. Mitrinović, Analytic Inequalities, Springer-Verlag, Berlin, 1970, pp. 272, 273 (sections 3.6.15 and 3.6.17).

3. B. C. Carlson, The logarithmic mean, this Monthly, 79 (1972) 615–618.

Department of Mathematics, California State University, Northridge, CA 91324.

RESEARCH PROBLEMS

Edited by Richard Guy

In this Department the Monthly presents easily stated research problems dealing with notions ordinarily encountered in undergraduate mathematics. Each problem should be accompanied by relevant references (if any are known to the author) and by a brief description of known partial results. Manuscripts should be sent to Richard Guy, Department of Mathematics, Statistics, and Computing Science, The University of Calgary, Calgary, Alberta, Canada, T2N 1N4.

A PROBLEM ABOUT SEQUENCES OF ZEROS AND ONES

Franz Hering

Suppose $d = (d_1, \cdots, d_m)$ is a sequence of zeros and ones. A subsequence $(d_{i_1}, \cdots, d_{i_r})$ of d is called *alternating* provided that $d_{i_v} + d_{i_{v+1}} = 1$ for $v = 1, \cdots, r-1$. Let $a_r(d)$ denote the number of such alternating subsequences of d having r elements. (E.g., $a_3((0,1,0,1,1)) = 3$, because each of the subscript sequences $(1,2,3)$, $(2,3,4)$ and $(2,3,5)$ corresponds to an alternating subsequence of $(0,1,0,1,1)$.) When D_m denotes the set of all sequences $d = (d_1, \cdots, d_m)$ of zeros and ones having m elements, then Harborth [1] proved

$$\max\{a_r(d) : d \in D_m\} = \binom{[(m+r)/2]}{r} + \binom{[(m+r-1)/2]}{r}.$$

When $D_{m,p}$ is the set of all $d \in D_m$ having exactly p zeros, then determine $\max\{a_r(d) : d \in D_{m,p}\}$. The problem has been solved by Passing [2] for $r \leqq 4$ and in some additional special cases.

References

1. H. Harborth, Alternierende 0–1-Blöcke als Teile von Dualzahlen, Arch. Math., 22 (1971) 372–378; MR 45 # 4988.

2. H. Passing, Über ein kombinatorisches Problem bei Dualzahlen, Bericht im SFB 72 des Instituts für Angewandte Mathematik der Universität Bonn (Dissertation 1973).

UNIVERSITÄT DORTMUND, ABTEILUNG STATISTIK, 46 DORTMUND-HOMBRUCH, POSTF. 500, WEST GERMANY.

AN AREA-PERIMETER PROBLEM

P. R. SCOTT

Let $\mathscr{K}$ be a convex region in the plane having area $A(\mathscr{K})$ and perimeter $P(\mathscr{K})$. A classical theorem of Minkowski [3] states that if $\mathscr{K}$ is symmetric about the origin, and contains no non-zero point of the integral lattice, then $A(\mathscr{K}) \leqq 4$. More recently, Bender [1] has shown that if $\mathscr{K}$ contains *no* points of the integral lattice, then

$$A(\mathscr{K}) \leqq \tfrac{1}{2}P(\mathscr{K}).$$

This result has been generalized and extended to higher dimensions by Hammer [2], Silver [4], Wills [5] and others.

From the point of view of the geometry of numbers, a solution to the following problem would be of interest.

Suppose that $\mathscr{K}$ is symmetric about the origin and contains no non-zero point of the integral lattice. Find the smallest real number c for which the inequality $A(\mathscr{K}) \leqq c\,P(\mathscr{K})$ is true for all regions $\mathscr{K}$.

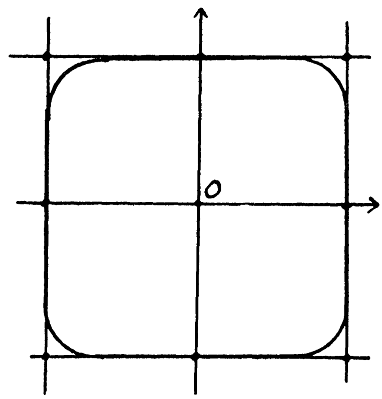
It is easy to show that $0.53 < c < 0.565$. The right inequality follows from the isoperimetric inequality:

$$\frac{A(\mathscr{K})}{P(\mathscr{K})} \leqq \sqrt{\frac{A(\mathscr{K})}{4\pi}} \leqq \sqrt{\frac{1}{\pi}} < 0.565.$$

We obtain the left inequality by maximizing the ratio $A(\mathscr{K})/P(\mathscr{K})$ when $\mathscr{K}$ is a square with rounded corners, as illustrated.

The maximum value of $2/(2+\sqrt{\pi})(\approx 0.53)$ is attained when the circular arcs have radius $2/(2+\sqrt{\pi})$.

A tentative conjecture is that $c = 2/(2+\sqrt{\pi})$.



Once the value of c is determined, we can strengthen Minkowski's Theorem in the following way.

THEOREM. *Let $\mathscr{K}$ be a convex region in the plane which is symmetric about the origin, which contains no non-zero point of the integral lattice, and which has area A and perimeter P. Then*

$$\begin{aligned} &\text{if } 0 < P \leqq 2\pi, && A \leqq P^2/4\pi; \\ &\text{if } 2\pi \leqq P \leqq 4/c, && A \leqq cP; \\ &\text{and if } 4/c \leqq P, && A \leqq 4. \end{aligned}$$

References

1. E. A. Bender, Area-perimeter relations for two-dimensional lattices, this MONTHLY, 69 (1962) 742–744.

2. J. Hammer, On a general area-perimeter relation for two-dimensional lattices, this MONTHLY, 71 (1964) 534–535.

3. H. Minkowski, Geometrie der Zahlen, Leipzig and Berlin, 1890.

4. M. Silver, Bender's theorem and associated extremal figures, this MONTHLY, 81 (1974), 382–383.

5. J. M. Wills, On lattice points and the volume/area ratio of convex bodies, this MONTHLY 78 (1971) 47–49.

DEPARTMENT OF PURE MATHEMATICS, THE UNIVERSITY OF ADELAIDE, BOX 498 D, G. P. O., ADELAIDE, SOUTH AUSTRALIA 5001.

CLASSROOM NOTES

EDITED BY DAVID ROSELLE

Material for this Department should be sent to David Roselle, Department of Mathematics, Virginia Polytechnic Institute, Blacksburg, VA 24061.

COUNTABLE ADDITIVITY FOR PROBABILITY MEASURES

DAVID T. PRICE

When presenting the axioms for probability to my class, I was asked why countable additivity was included, rather than finite additivity. After some probing, I discovered that they really wanted an example, related to the beginning course work in finite probability, where more than finite additivity occurred. The following negative binomial example and theorem grew out of my response to this question.

Consider tossing a fair coin until "heads" occurs. A suitable sample space is $S = \{1, 2, 3, \cdots, *\}$, where the integer n corresponds to the occurrence of the head on toss n, while $*$ corresponds to an infinite string of tails. Intuitively, $P(\{n, n+1, \cdots, *\}) = 2^{-(n-1)}$. Since probability should be finitely additive (the students were ready to accept this), and $\{n, n+1, \cdots, *\} = \{n\} \cup \{n+1, \cdots, *\}$, we have $P(\{n\}) = 2^{-n}$. Furthermore, as probability should be monotone (again accepted by the class),

$$P(\{*\}) \leqq P(\{n, n+1, \cdots, *\}) = 2^{-n+1}$$

for each n. Thus $P(\{*\}) = 0$.

Now suppose we want to compute $P(\{2, 4, 6, \cdots\})$. (If the model were of a tennis rally, then this would be the probability that the first player won the rally.) The most natural answer is $\sum_{i=1}^{\infty} 2^{-2i} = 1/3$. But what allows us to compute the answer as an infinite sum? Countable additivity, of course!

Let $A = \{2, 4, 6, \cdots\}$. Let us assume that there is a sensible way to compute $\operatorname{Prob}(A)$. Then $\operatorname{Prob}(A) \geqq P(\{2, 4, \cdots, 2n\})$ for every positive integer n. (Probability should be monotone.) That is, $\operatorname{Prob}(A) \geqq \sum_{k=1}^{n} 2^{-2k}$. Hence

$$\operatorname{Prob}(A) \geqq \lim_{n \to \infty} \sum_{k=1}^{n} 2^{-2k} = 1/3.$$

Similarly,

$$\begin{aligned} \operatorname{Prob}(A) &\leqq \operatorname{Prob}(\{2, 4, \cdots, 2n, 2n+1, 2n+2, \cdots, *\}) \\ &= \operatorname{Prob}(S \setminus \{1, 3, \cdots, 2n-1\}) \\ &= \operatorname{Prob}(S) - P(\{1, 3, \cdots, 2n-1\}) \\ &= 1 - \textstyle\sum_{k=1}^{n} 2^{-(2k-1)}. \end{aligned}$$

(Probability should be finitely additive, and Prob(S) should be 1.) Then,

$$\text{Prob}(A) \leqq \lim_{n\to\infty}\left(1 - \sum_{k=1}^{n} 2^{-(2k-1)}\right) = 1/3.$$

Thus, the only "sensible" value for $P(A)$ *is* 1/3. In the argument below, we show that this convergence of the approximation from within and without always occurs, and yields a countably additive probability.

THEOREM. *Let S be a countable sample space, $\mathscr{F}$ the set of finite subsets of S. Suppose a function P is defined on $\mathscr{F}$ satisfying*:

(i) $0 \leqq P(A) \leqq 1$, *for all A in $\mathscr{F}$.*

(ii) $P(A \cup B) = P(A) + P(B)$, *for all A and B in $\mathscr{F}$ with $A \cap B = \varnothing$.*

(iii) *For each $\varepsilon > 0$, there is an A in $\mathscr{F}$ with $P(A) > 1 - \varepsilon$.*

Then there is a unique extension of P to the subsets of S which is finitely additive and takes values in $[0,1]$. This extension is countably additive. If A is an infinite subset of S, then the value of the extension on A is $\sum_{a\in A} P(\{a\})$.

Proof. Let A be any subset of S. Set $\mathscr{F}(A) = \{A \cap L \mid L \in \mathscr{F}\}$. Let Q be any finitely additive extension of P to the subsets of S, taking values in $[0,1]$. Then Q is monotonic, and $Q(A) \geqq Q(L) \geqq P(L)$ for all $L \in \mathscr{F}(A)$. Hence, if we let

$$P^*(A) = \text{lub}\{P(L) \mid L \in \mathscr{F}(A)\},$$

then $Q(A) \geqq P^*(A)$.

We next show that P^* is a finitely additive extension of P to the subsets of S which has values in $[0,1]$. If $L \in \mathscr{F}$, then $0 \leqq P(L) \leqq 1$, and thus $0 \leqq P^*(A) \leqq 1$. By assumption (iii), $P^*(S) = 1$. By construction, P and P^* agree on $\mathscr{F}$. To show that P^* is finitely additive, take B in S with $A \cap B = \varnothing$. Then

$$\mathscr{F}(A \cup B) = \{L \cup M \mid L \in \mathscr{F}(A), M \in \mathscr{F}(B)\},$$

and

$$\begin{aligned} P^*(A \cup B) &= \text{lub}\{P(L \cup M) \mid L \in \mathscr{F}(A), M \in \mathscr{F}(B)\} \\ &= \text{lub}\{P(L) + P(M) \mid L \in \mathscr{F}(A), M \in \mathscr{F}(B)\} \\ &= \text{lub}\{P(L) \mid L \in \mathscr{F}(A)\} + \text{lub}\{P(M) \mid M \in \mathscr{F}(B)\} \\ &= P^*(A) + P^*(B). \end{aligned}$$

Now

$$\begin{aligned} P^*(A) \leqq Q(A) &= Q(S) - Q(S \setminus A) \\ &\leqq P^*(S) - P^*(S \setminus A) = P^*(A). \end{aligned}$$

Thus, $P^*(A) = Q(A)$, and P^* is the unique extension of P described in the theorem. (In our example, the first limit computed $P^*(A)$, and the second limit computed $P^*(S) - P^*(S\backslash A)$.)

We now show that P^* is countably additive. Let $\{A_i\}_{i=1}^{\infty}$ be a collection of pairwise disjoint subsets of S. Let $A = \bigcup_{i=1}^{\infty} A_i$. Then $A \supseteq \bigcup_{i=1}^{k} A_i$, for each integer k. Hence, $P^*(A) \geqq P^*(\bigcup_{i=1}^{k} A_i) = \sum_{i=1}^{k} P^*(A_i)$, for each integer k. Let $\varepsilon > 0$ be given. We must find an integer N so that

$$P^*(A) - \sum_{i=1}^{N} P^*(A_i) < \varepsilon.$$

Take B in $\mathscr{F}$ with $P(B) > 1 - \varepsilon$. Only finitely many of the A_i meet B. Thus, there is an integer N so that $A_i \cap B = \emptyset$ for all $i > N$. Now

$$\begin{aligned}\sum_{i=1}^{N} P^*(A_i) &\geqq \sum_{i=1}^{N} P^*(A_i \cap B)\\ &= P^*\left(\bigcup_{i=1}^{N} (A_i \cap B)\right)\\ &= P^*\left(\bigcup_{i=1}^{\infty} (A_i \cap B)\right)\\ &= P^*(A \cap B).\end{aligned}$$

Hence,

$$\begin{aligned}P^*(A) - \sum_{i=1}^{N} P^*(A_i) &\leqq P^*(A) - P^*(A \cap B) = P^*(A\backslash B)\\ &\leqq P^*(S\backslash B) < \varepsilon,\end{aligned}$$

as desired.

Let $A = \{a_k\}_{k \geqq 1}$ be any infinite subset of S. As P^* is countably additive,

$$P^*(A) = P^*\left(\bigcup_{k=1}^{\infty} \{a_k\}\right) = \sum_{k=1}^{\infty} P^*(\{a_k\}) = \sum_{k=1}^{\infty} P(\{a_k\}).$$

Advanced treatments of probability, using measure theory, include more general extension theorems (e.g. [**1**], Theorem 1.5.7). At that level, countable additivity is recognized as a desirable condition, and is therefore assumed, rather than proved as in the presentation here.

Hanish, Hirsch and Renyi [3] discuss the extension of a countably additive measure from a sigma-algebra to the power set.

Our theorem may be extended to provide a firm framework for other questions about unlimited Bernoulli trials. In Chapter 8, Feller [2] discusses the probability of obtaining a run of m heads before obtaining a run of n tails in an unlimited se-

quence of Bernoulli trials, where the probability of each head is $p \neq 0, 1$. He carefully points out that the set of unlimited trials is uncountable, and so the problem falls outside the framework of his book. Suppose, instead of unlimited trials, we agree to stop upon completing a run of m heads, or a run of n tails. Then, a suitable sample space consists of two parts: S_1, the finite sequences of H's and T's which have a terminal run of $m\,H$'s or $n\,T$'s (and no such run earlier); and S_2, the infinite sequences of H's and T's with no run of $m\,H$'s or of $n\,T$'s. S_1 is countable, while S_2 is uncountable.

Let $\mathscr{F}_0$ be the finite subsets of S_1. We can assign probabilities P to the events in $\mathscr{F}_0$ from the theory for a finite sequence of Bernoulli trials. If $A_k = \{$all strings in S_1 of length at most $kn\}$, then $P(A_k) \geqq 1 - (1 - p^n)^k$. Hence, conditions (i), (ii), and (iii) of the theorem hold in $\mathscr{F}_0$. The proof then shows that there is a unique finitely additive extension of P to the subsets of $S = S_1 \cup S_2$; the extension is countably additive; and the value of the extension on $A \subseteq S$ is $\sum_{a \in A \cap S_1} P(\{a\})$.

References

1. H. Bauer, Probability Theory and Elements of Measure Theory, Holt, Rinehart and Winston, New York, 1972.
2. W. Feller, An Introduction to Probability Theory and Its Applications, Volume 1, Third Edition, Wiley, New York, 1968.
3. H. Hanisch, W. M. Hirsch and A. Renyi, Measures in denumerable spaces, this MONTHLY, 76 (1969) 494–502.

DEPARTMENT OF MATHEMATICS, WHEATON COLLEGE, WHEATON, IL 60187.

SETS OF POSITIVE PRODUCT MEASURE IN WHICH EVERY RECTANGLE IS NULL

GARTH I. GAUDRY

In the standard treatment of product measures and the Fubini-Tonelli theorem, *measurable rectangles* play a special rôle. (If $X, \mathscr{M}, \mu)$ and $(Y, \mathscr{N}, \nu)$ are σ-finite measure spaces, a measurable rectangle is a subset of $X \times Y$ of the form $E \times F$, where $E \in \mathscr{M}$ and $F \in \mathscr{N}$.) For instance, the product σ-algebra is the smallest σ-algebra containing all the measurable rectangles. See Hewitt and Stromberg [**1**, Chapter 6] or Rudin [**2**, Chapter 7].

Accordingly, it is natural to ask whether a set in $\mathscr{M} \times \mathscr{N}$ which is of positive $\mu \times \nu$-measure must contain a measurable rectangle of positive measure. The example below shows that it need not.

EXAMPLE. Let $(X, \mathscr{M}, \mu) = (Y, \mathscr{N}, \nu) = (R, \mathscr{L}, m)$, where $\mathscr{L}$ denotes the σ-algebra of Lebesgue measurable sets, and m denotes Lebesgue measure. Let K be a Cantor-type subset of $[0, 1]$ constructed with variable ratios of dissection so that $m(K) > 0$

[1, (6.62)]. For the subset S of $\mathscr{L} \times \mathscr{L}$, take

$$S = \{(x, k - x) : x \in [0,1], k \in K\} = \bigcup_{x \in [0,1]} \{x\} \times K - x. \tag{1}$$

The set S is compact, being a continuous image of $[0,1] \times K$. Fubini's theorem shows that

$$m \times m(S) = m(K) > 0.$$

If $E \times F$ is a measurable rectangle contained in K, we see from the definition (1) that $E + F \subseteq K$; and of course both $m(E)$ and $m(F)$ are positive. We show that this is impossible.

The convolution $\xi_E * \xi_F$ of the characteristic functions ξ_E and ξ_F is defined by the formula

$$\xi_E * \xi_F = \int \xi_E(x - y)\xi_F(y)dy.$$

The function $\xi_E * \xi_F$ is everywhere defined, continuous, vanishes off $E + F$, and is nonnegative. By applying the Fubini-Tonelli theorem, one can show that its integral is $m(E)m(F)$. For a full verification of these assertions about $\xi_E * \xi_F$, refer to [1, (21.31) and (21.33)]. Since

$$\int \xi_E * \xi_F dx > 0,$$

and $\xi_E * \xi_F$ is continuous, it follows that $E + F$ must contain a nonvoid open interval. Since $K \supseteq E + F$, and K is a Cantor-type set, this is impossible.

References

1. E. Hewitt and K. Stromberg, Real and Abstract Analysis, Springer, New York, 1965.
2. W. Rudin, Real and Complex Analysis, McGraw-Hill, New York, 1966.

SCHOOL OF MATHEMATICAL SCIENCES, FLINDERS UNIVERSITY OF SOUTH AUSTRALIA, BEDFORD PARK (SA 5042) AUSTRALIA.

ADDENDUM TO "A UNIFIED PROOF OF SEVERAL BASIC THEOREMS OF REAL ANALYSIS"

PATRICK SHANAHAN

Leonard Gillman has kindly informed me that the essential idea for my note (this MONTHLY, 79 (1972) 895–898) has already appeared in a note by L. R. Ford (this MONTHLY, 64 (1957) 106–108); see also the note by W. L. Duren, Jr., in the supplementary edition of the MONTHLY dedicated to Lester R. Ford on the occasion of his seventieth birthday (this MONTHLY, 64 (1957) Part II, 19–22). Professor Ray Glenn of Tallahassee Community College has pointed out that the same idea was the subject

matter of a classroom note "A Creeping Lemma" by R. M. F. Moss and G. T. Roberts (this MONTHLY, 75 (1968) 649–652). My proof of the integrability of continuous functions does, however, appear to be new.

Professor Gillman also points out that without introducing any essentially different arguments my Proposition 1 may be replaced by the following generalization. (Here *strongly additive* means $A, B \in \mathscr{C} \Rightarrow A \cup B \in \mathscr{C}$).

PROPOSITION 1'. (A) The following are equivalent:

(i) X is compact,

(ii) X is a member of every local, strongly additive family of subsets of X.

(B) The following are equivalent:

(i) X is compact and connected,

(ii) X is a member of every local, additive family of subsets of X.

He offers the following alternative version for $\{(B), (ii)\} \Rightarrow \{X \text{ is connected}\}$ which is rather clever: if X is a disjoint union of open sets U and V, then $\{U, V\}$ is an additive family which does not contain X (!).

DEPARTMENT OF MATHEMATICS, COLLEGE OF THE HOLY CROSS, WORCESTER, MA 01610.

MATHEMATICAL EDUCATION

EDITED BY SHIRLEY HILL AND PAUL MIELKE

Material for this Department should be sent to Shirley Hill, Department of Mathematics, University of Missouri, Kansas City, MO 64110, or to Paul Mielke, Department of Mathematics, Wabash College, Crawfordsville, IN 47933.

TOWARD A MATHEMATICS MAJOR FOR THE 1980's

ALAN TUCKER

1. Introduction. Writers have recently pointed out that the interests of many mathematics majors and of their industrial employers were often neglected in the past decade (see Willcox **[10]** for a survey of this problem). This paper outlines an undergraduate program in mathematics that aims to correct this neglect. While looking towards society's future demands for mathematics, this program also reflects a characteristically pre-Sputnik concern for teaching and for problem solving (rather than for theorem proving). The undergraduate major of the SUNY at Stony Brook Applied Mathematics Department is one attempt at such a program. By student choice, its curriculum emphasizes finite mathematics and optimization. When combined with recommended mathematics and computer science courses,

ter courses, so that they can get jobs as computer programmers upon graduation. These students are also advised to defer making long-range plans until after such work experience. Indeed, most applied mathematics majors are strongly encouraged to take some job upon graduation and get away from the academic world, if only for a year. This advice is consistent with the applied mathematics program's goal of training mathematical problem-solvers.

Acknowledgement: I wish to express my thanks for the helpful comments I received from the MONTHLY editors and my colleagues in the Mathematics and Applied Mathematics Departments, especially from Paul Kumpel and Woo Jong Kim.

References

1. G. Carrier, Comments in "Education in Applied Mathematics," SIAM Review, 9 (1967) 306.
2. CUPM, A general curriculum in mathematics for colleges, Berkeley, Calif., 1965.
3. R. Feynman, New textbooks for the 'new' mathematics, Engineering and Science Magazine, March, 1965.
4. H. Flanders, A survival kit for college mathematicians, this MONTHLY, 78 (1971) 291–296.
5. H. P. Greenspan, Applied Mathematics at MIT, this MONTHLY, 80 (1973) 67–72.
6. B. B. Hughes, Survival for mathematics students, this MONTHLY, 80 (1973) 689–690.
7. B. B. Peterson, Survival for mathematicians or mathematics, this MONTHLY, 79 (1972) 70–76.
8. J. B. Rosser, Mathematics Courses in 1984, this MONTHLY, 79 (1972) 635–684.
9. A. Tucker, Polya's enumeration formula by example, MATHEMATICS MAGAZINE, 47 (1974).
10. A. B. Willcox, England was lost on the Playing Fields of Eton: A parable for mathematics, this MONTHLY, 80 (1973) 25–40.

DEPARTMENT OF APPLIED MATHEMATICS AND STATISTICS, STATE UNIVERSITY OF NEW YORK AT STONY BROOK, STONY BROOK, LONG ISLAND, NEW YORK 11790.

HISTORY IN THE MATHEMATICS CURRICULUM

KENNETH O. MAY

The history of mathematics has always played a role in the mathematics curriculum insofar as teachers have found it useful to introduce historical information in courses organized along essentially deductive lines. Separate courses in the history of mathematics have also long occupied a modest position in the undergraduate curriculum. Less common, but beginning to gain popularity, are courses in which content is arranged historically. Such courses do not ignore the logical component in mathematics, but they present it and all other aspects of the subject in historical perspective. It is the purpose of this note to describe these roles of the history of mathematics in the undergraduate and graduate curriculum at the University of Toronto.

The undergraduate curriculum at the University of Toronto continues its well-known tradition of strong, rigorous training in a new context of a curriculum based

on almost complete free choice for the student. Those interested in careers related to mathematics ordinarily take a program involving rather heavy concentration, usually at least two courses per year.

Four courses with significant historical content and orientation are offered to undergraduates. The first, entitled "Introduction to Mathematics," is designed for first-year students and has no mathematical prerequisites, although it should be kept in mind that freshmen at the University of Toronto have completed grade 13, which contains a (optional) mathematical component involving linear algebra and calculus. The course is described: "The nature and role of mathematics, illustrated primarily by the development of numerical and geometric ideas. Lectures, films, study of mathematical literature, and the writing of an essay. Tutorials will provide the opportunity for doing mathematics as well as talking about it." The course does not attempt to present a systematic chronological history of mathematics, but rather to give a survey of mathematics as a living, historically developing component of culture. Most of the students are not intending to specialize in mathematics, and the course is required by the School of Architecture. It is a common elective for students planning to teach mathematics at the secondary or elementary level, and is usually taken by them in conjunction with another freshman course in mathematics.

The second course, entitled "Development of Analysis," is intended primarily for second-year students and may be elected by those having taken at least one university mathematics course. It focuses on the development of the basic themes and concepts of calculus, including a study of 18th and 19th century rigor, and the development of the concept of the integral in recent times, with detailed examination of selected examples. Although it would serve very well as an enrichment for mathematical specialists who are already taking advanced courses in analysis at this level, it is elected primarily by future teachers and by non-specialists with a serious interest in mathematics.

The third course, intended primarily for third-year students, is entitled "20th Century Mathematics." Its purpose is to give the student "a survey of the different trends in the mathematics of this century, their interplay, social function, and effects in science and technology." The selection of topics by the lecturer is supplemented by outside speakers and student investigations. The course is motivated by the need for a broader view of mathematics than can be obtained at this level by taking the highly-specialized courses available.

At the senior level is offered the first course in the history of mathematics. Requiring six year courses in university mathematics as a prerequisite, it is taken primarily by mathematics specialists, many of whom plan to be teachers, and by a few others who have acquired the necessary mathematics background. After an initial period of general reading, the course is based primarily on student investigations using primary sources. Methods of information retrieval and historical research are emphasized, and each student writes two papers.

Three years' experience with these historically-oriented courses shows them to be popular with both the faculty who teach them and with the students (with the exception of those who are required to take them!). The teachers have found the experience challenging and interesting. The students have found the courses a welcome change of pace.

At the graduate level there is a course in the history of mathematics stressing skill in information retrieval and historical research relating to special mathematical interests. Those who wish to specialize in the history of mathematics may take further seminars and, after passing the usual preliminaries for the doctor's degree, continue by writing a thesis on a historical topic.

At the University of Toronto there is at the graduate level an alternative framework for specialization in the history of mathematics. The Institute for the History and Philosophy of Science and Technology offers a doctoral program in which the student may specialize in the history of mathematics. Perhaps the two programs might best be described by saying that one produces a mathematician who specializes in history, and the other an historian of science specializing in mathematics. At the time of writing, there are three Ph. D. candidates in each program. In addition, one doctoral degree with specialization in history of mathematics has been granted by the Institute.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF TORONTO, TORONTO 5, ONTARIO, CANADA.

PROBLEMS AND SOLUTIONS

EDITED BY EMORY P. STARKE

All problems (both elementary and advanced) proposed for inclusion in this Department should be sent to E. P. Starke, 1000 Kensington Ave., Plainfield, NJ 07060. Proposers of problems are urged to enclose any solutions or information that will assist the editors. Ordinarily, problems in well-known textbooks and results in generally accessible sources are not appropriate for this Department. No solutions (except those accompanying proposals) should be sent to Professor Starke.

ELEMENTARY PROBLEMS

Solutions of Elementary Problems should be sent to Problems Group, Mathematics Department, University of Maine, Orono, ME 04473. To facilitate their consideration, solution of Elementary Problems in this issue should be typed (with double spacing) and should be mailed before January 31, 1975.

E 2492. *Proposed by Donald Knuth, Stanford University*

For natural numbers i and j, let "i mod j" denote the nonnegative remainder when i is divided by j: i.e. $i \bmod j = i - j[i/j]$. Evaluate the following sum:

$$S_n = \sum_{k=1}^{2n^2} \left(\frac{k \bmod n}{(2k+1) \bmod (2n+1)} \right).$$

E 2493. *Proposed by C. D. H. Cooper, Macquarie University, Australia*

Prove that the sum of the (positive) divisors of the natural number n is a power of 2 if and only if n is a product of distinct Mersenne primes.

E 2494. *Proposed by J. C. Owings, Jr., University of Maryland*

Let N denote the set of natural numbers and for $A \subseteq N$ let $A + A = \{a_1 + a_2 : a_1, a_2 \in A\}$. Prove or disprove: Given any subset B of N, there exists an infinite set $A \subseteq N$ such that $A + A \subseteq B$ or $A + A \subseteq N \backslash B$.

E 2495. *Proposed by M. S. Klamkin, University of Waterloo, Ontario*

Let n be a natural number. Evaluate the following limit:

$$I_n = \lim_{x \to \infty} \left\{ \frac{(\log x)^{2n}}{2n} - \int_0^x \frac{(\log t)^{2n-1}}{1+t} \, dt \right\}.$$

E 2496. *Proposed by R. D. Whittekin, Metropolitan State College*

Show that the square matrix $M = (m_{ij})$ is nonsingular if it satisfies the following conditions:

(i) $m_{ii} \neq 0$ for all i;
(ii) If $i \neq j$ and $m_{ij} \neq 0$, then $m_{ji} = 0$;
(iii) If $m_{ij} \neq 0$ and $m_{jk} \neq 0$, then $m_{ik} \neq 0$.

E 2497. *Proposed by Jim King and Phil Hosford, New Mexico State University*

Let a_1 and a_2 be arbitrary integers and define the doubly infinite sequence $\cdots, a_{-1}, a_0, a_1, a_2, \cdots$ by $a_{n+1} = a_n + a_{n-1}$. Show that $(a_{k+2j} + a_{k-2j})$ is divisible by a_k for all integers k, j.

SOLUTIONS OF ELEMENTARY PROBLEMS

A Quadrilateral Proportion

E 1085 [1953, 551; 1973, 808]. *Proposed by Josef Langr, Prague, Czechoslovakia*

The perpendicular bisectors of the sides of a quadrilateral Q form a quadrilateral

Q_1, and the perpendicular bisectors of the sides of Q_1 form a quadrilateral Q_2. Show that Q_2 is similar to Q and find the ratio of similitude.

II. *Solution by Michael Geraghty, University of Iowa.* We obtain an expression for the ratio of similitude of Q_2 to Q, thus completing the published solution [1973, 808].

Let α, β, γ and δ denote the interior angles of $ABCD(= Q)$ and define $\alpha', \beta', \gamma', \delta'$ analogously. Let $|AC|$ be one of the diagonal lengths of $ABCD$. Then the ratio $|A''C''| : |AC|$ is the desired ratio of similitude. Consider $|B'D'|$: Since B' is the circumcenter of triangle CDA, it lies on the perpendicular bisector of AC. So also does D'. Thus the line $B'D'$ is the perpendicular bisector of AC, and $|B'D'| = |b' + d'|$, where b' is the signed distance of B' from AC (sign positive if B' is inside triangle CDA and negative if B' is outside); d' is defined analogously.

It is easy to calculate that

$$b' = \tfrac{1}{2}|AC| \cot \delta \quad \text{and} \quad d' = \tfrac{1}{2}|AC| \cot \beta.$$

Thus

$$|B'D'| = |b' + d'| = \tfrac{1}{2}|AC|\,|\cot \beta + \cot \delta|.$$

Next, applying the method to Q_1 we find that

$$|A''C''| = |a'' + c''| = \tfrac{1}{2}|B'D'|\,|\cot \alpha' + \cot \gamma'|.$$

But it is easy to see that $\alpha' = 2\pi - \gamma - \pi = \pi - \gamma$ and $\gamma' = \pi - \alpha$, and so

$$|A''C''| = \tfrac{1}{2}|B'D'|\,|\cot \alpha + \cot \gamma|.$$

Therefore

$$\frac{|A''C''|}{|AC|} = \tfrac{1}{4}|\cot \alpha + \cot \gamma| \cdot |\cot \beta + \cot \delta|$$

which is the desired result.

Also solved by George Gearhart, Albert Nijenhuis, and J. M. Quoniam (France). The ratio of similitude was discovered by Zalman Usiskin, but no derivation was submitted by him.

Editor's comment: Nijenhuis notes that if Q is a cyclic quadrilateral, then Q_1 collapses and so the ratio of similitude is 0.

Minimal Curve for Fixed Area

E 2185 [1969, 825; 1970, 531]. *Proposed by Michael Goldberg, Washington, D. C.*

Given a convex quadrilateral. Find the shortest curve which divides it into two equal areas.

III. *Comment by M. S. Klamkin, Ford Motor Company.* Although the properties of the shortest bisecting arc as given by both Ogilvy and Goldberg are correct, neither solver has really supplied a full mathematical proof. At a geometry seminar

held at Michigan State University several years ago, Branko Grunbaum raised again the more general problem of determining the shortest arc which divides a given simply connected area in a fixed ratio, and which lies wholly within the area. Grunbaum noted that Norbert Wiener [*The shortest line dividing an area in a given ratio*, Proc. Camb. Phil. Soc. 18 (1914), 56–58] proved that (if such an arc exists) it must consist of an arc of a finite or infinite circle or a chain of such arcs having the property that two successive arcs meet only on the boundary of the given area. At the end of this paper is the footnote, "It is almost self-evident that the shortest line to divide a convex area in a given ratio is a single arc of a circle, but this I have not been able to prove." This conjecture includes E 2185 as a special case.

Wiener's shortest-line conjecture went unproved for almost sixty years, but in 1973, Richard Joss, a student of Grunbaum's, announced that he had proved it [Notices A.M.S., June, 1973, Abstract 705–D1, p. A–461].

Symmedian Point of a Triangle

E 2347 [1972, 303; 1973, 321]. *Proposed by Leonard Carlitz, Duke University*

Let P denote a point in the interior of the triangle ABC. Let α, β, γ denote the angles of ABC. Let R_1, R_2, R_3 denote the distances from P to the vertices of ABC, and let r_1, r_2, r_3 denote the distances from the sides of ABC. Show that

$$R_1^2 \sin^2\alpha + R_2^2 \sin^2\beta + R_3^2 \sin^2\gamma \leqq 3(r_1^2 + r_2^2 + r_3^2)$$

with equality if and only if P is the symmedian point of ABC.

II. *Comment by Phil Tracy, Liverpool, New York.* There is a misprint in the published solution: in the equation on line 6 of p. 322, the second term should read $(r_2 \sin\alpha)^2$. The solver also does not indicate how he knows that his critical point is indeed a maximum.

III. *Solution by Leon Bankoff, Los Angeles, California.* Let G denote the centroid of triangle DEF, the pedal triangle of P. It is known (see Problem E 1744 [1965, 1130]) that

$$\begin{aligned} 3(r_1^2 + r_2^2 + r_3^2) &= 3\{(DG)^2 + (EG)^2 + (FG)^2 + 3(PG)^2\} \\ &\geqq 3\{(DG)^2 + (EG)^2 + (FG)^2\} = (DE)^2 + (EF)^2 + (FD)^2 \\ &= R_1^2 \sin^2\alpha + R_2^2 \sin^2\beta + R_3^2 \sin^2\gamma. \end{aligned}$$

Hence equality holds in the stated problem if and only if P coincides with the median point (centroid) G of its pedal triangle, that is, if and only if P is the symmedian point of triangle ABC.

The inequality holds for any point P in the plane of triangle ABC; P need not be interior to ABC.

IV. *Solution by M. S. Klamkin, Ford Motor Company.* The published solution, which is rather long, involves Lagrange multipliers, which should always be avoided whenever possible in proving elementary triangle inequalities. Furthermore, the solution is incomplete since sufficiency was not established.

We give a generalization by starting with the known inequality [**1**, p. 7],

$$xR_1^2 + yR_2^2 + zR_3^2 \geqq \frac{a^2yz + b^2zx + c^2xy}{x + y + z}, \tag{1}$$

where x, y, z are arbitrary real numbers such that $x + y + z > 0$ and where there is equality if and only if $x/F_1 = y/F_2 = z/F_3$ (F_1 denotes the area of BPC, etc.). (A physical interpretation of (1) is that the polar moment of inertia of three masses x, y, z located at A, B, C, respectively, is minimized by taking the axis through the centroid of the masses.)

For any inequality of the form $\phi(R_1, R_2, R_3, a, b, c) \geqq 0$ there is a dual inequality $\phi(r_1, r_2, r_3, R_1 \sin\alpha,\ R_2 \sin\beta,\ R_3 \sin\gamma) \geqq 0$, obtained by considering the pedal triangle of P. Here the distances from P to the vertices of the pedal triangle are r_1, r_2, r_3 and the sides of the pedal triangle are $R_1 \sin\alpha$, $R_2 \sin\beta$, $R_3 \sin\gamma$, respectively. Thus, the dual of (1) is

$$xr_1^2 + yr_2^2 + zr_3^2 \geqq \frac{yzR_1^2\sin^2\alpha + zxR_2^2\sin^2\beta + xyR_3^2\sin^2\gamma}{x + y + z}. \tag{2}$$

Then the stated inequality corresponds to the special case $x = y = z$ of (2). There is equality if and only if the point P is the centroid of the pedal triangle and consequently if and only if P is the symmedian point of ABC [**2**, Theorem 350].

Coincidentally, the stated inequality appears in the same form in [**1**, p. 10]. By applying (1) to the right hand side of (2), we obtain

$$\frac{\Sigma\, yzR_1^2\sin^2\alpha}{x + y + z} \geqq \frac{4F^2}{\dfrac{a^2}{x} + \dfrac{b^2}{y} + \dfrac{c^2}{z}}. \tag{3}$$

Inequalities (2) and (3) also provide a strengthening and a generalization of the following known inequality [**2**, Theorem 349], [**3**, Item 12.54, p. 118]:

$$r_1^2 + r_2^2 + r_3^2 \geqq \frac{4F^2}{a^2 + b^2 + c^2}. \tag{4}$$

References

1. M. S. Klamkin, Nonnegative Quadratic Forms and Triangle Inequalities, Ford Motor Company Preprint, June 1971. (Also see Notices of A. M. S., Oct. 1971, p. 966.)

2. R. A. Johnson, Advanced Euclidean Geometry, Dover, New York, 1960.

3. O. Bottema *et al.*, Geometric Inequalities, Wolters-Noordhoff, Groningen, 1969.

Nonnegative Forms

E 2348 [1972, 304; 1973, 323]. *Proposed by Leonard Carlitz, Duke University*

Let P be a point in the interior of a triangle ABC. Let R_1, R_2, R_3 denote the distances from P to the vertices of ABC and let r_1, r_2, r_3 denote the distances from P to the sides of ABC. Show that

$$(1) \qquad \Sigma\, R_1(r_1 + r_3) \geqq \Sigma\,(r_1 + r_2)(r_1 + r_3),$$

$$(2) \qquad \Sigma\,(R_1 + R_2)(R_1 + R_3) \geqq 4\Sigma(r_1 + r_2)(r_1 + r_3),$$

with equality if and only if ABC is equilateral and P is its center.

II. *Comment by Leonard Goldstone, Watervliet, N. Y.* The last inequality

$$\Sigma\, R_1^2 \geqq 4\,\Sigma\, r_1^2$$

in the printed solution [1973, 323] is incorrect. Let ABC be an isosceles right triangle with unit legs and let P be its circumcenter. Then $\Sigma R_1^2 = 3/2$, but $4\Sigma r_1^2 = 2$.

To establish (2), square the Erdös-Mordell inequality $\Sigma R_1 \geqq 2\Sigma r_1$ (Item 12.13 of O. Bottema *et al.*, *Geometric Inequalities*, Groningen, 1969) to get

$$\Sigma R_1^2 + 2\Sigma R_1 R_2 \geqq 4(\Sigma r_1^2 + 2\Sigma r_1 r_2);$$

then add $\Sigma R_1 R_2 \geqq 4\Sigma r_1 r_2$ (Item 12.21, *op. cit.*).

III. *Comment by A. Oppenheim, University of Benin, Nigeria.* The published solution to part (2) is faulty since the solver assumes incorrectly that $\Sigma R_1^2 \geqq 4\Sigma r_1^2$. Indeed, for any $\varepsilon > 0$, there exist triangles ABC and internal points P for which

$$\Sigma R_1^2 < (2 + \varepsilon)\Sigma r_1^2.$$

To prove (2), use my inequality

$$\Sigma R_2 R_3 \geqq \Sigma(r_1 + r_2)(r_1 + r_3)$$

(Item 12.22, Bottema). Since $\frac{1}{2}\Sigma(R_2 - R_3)^2 \geqq 0$, we have $\Sigma R_1^2 \geqq \Sigma R_2 R_3$ and it follows that

$$\begin{aligned}\Sigma(R_1 + R_2)(R_1 + R_3) &= \Sigma R_1^2 + 3\Sigma R_2 R_3 \geqq 4\Sigma R_2 R_3\\ &\geqq 4\Sigma(r_1 + r_2)(r_1 + r_3),\end{aligned}$$

which is (2). Equality occurs if and only if ABC is equilateral and P is its center.

IV. *Comment by M. S. Klamkin, Ford Motor Company.* We can obtain the following inequality, valid for r_1, r_2, r_3 nonnegative, by replacing (r_1, r_2, r_3) by (x^2, y^2, z^2) and then expressing G as the sum of squares of three real polynomials in x, y, z:

$$(*) \qquad G = \Sigma r_1^2 - \Sigma\left\{3 - \frac{b+c}{a}\right\} r_2 r_3 \geqq 0.$$

(This result is non-trivial; see the author's *Two nonnegative quadratic forms*, (to appear).) Since $c/b + b/c - 1 \geqq 1$, this inequality implies

$$\Sigma\left\{\frac{c}{b} + \frac{b}{c} - 1\right\} r_1^2 \geqq \Sigma\left\{3 - \frac{b+c}{a}\right\} r_2 r_3,$$

which is equivalent to equation (4) of the published solution.

If x, y, z form a triangle, then (*) can be rewritten as

$$\Sigma\left\{\frac{b+c}{a} - 1\right\} y^2 z^2 \geqq 16 F^2,$$

where F denotes the area of the triangle with sides x, y, z. This latter inequality generalizes Item 4.12 in O. Bottema *et al.*

Editor's comment: The proposer inquires if it is necessarily true that

$$\sum (R_1 + R_2)(R_1 + R_3) \geqq 4 \sum R_1 (r_2 + r_3),$$

which would strengthen (2).

The Vandermonde Matrix in a Finite Field

E 2431 [1973, 808]. *Proposed by Alan McConnell, Howard University*

Consider a finite field F with elements $a_1, a_2, \cdots, a_n$. Form the Vandermonde matrix $V(a_1, \cdots, a_n) = (v_{ij})$, where $v_{ij} = (a_j)^{i-1}$ for $i, j = 1, 2, \cdots, n$ (and where $0^0 = 1$). Find V^{-1} and evaluate $\det V$ (where the operations are carried out in F).

Solution by the Temple University Problem Solving Group. It is well known that the Vandermonde determinant is equal to the product

$$\det V = \prod_{1 \leqq i < j \leqq n} (a_j - a_i).$$

Introducing the $\binom{n}{2} = \frac{1}{2}n(n-1)$ new terms $(a_j - a_i)$ with $i > j$ we see that

$$(1) \qquad (\det V)^2 = (-1)^{n(n-1)/2} \prod_{i=1}^{n} \prod_{j \neq i} (a_j - a_i).$$

Now for each fixed i, the inner product in (1) is the product of all the non-zero elements of F. In a field, each non-zero element has a multiplicative inverse, and since the equation $x^2 = 1$ has at most two roots, only 1 and -1 have themselves for inverses. (Of course, $1 = -1$ if the field is of characteristic 2.) Hence for each i, the inner product in (1) is -1 and so

$$(\det V)^2 = (-1)^{n(n-1)/2}(-1)^n = (-1)^{n(n+1)/2}.$$

If $n \equiv 3 \pmod 4$, then $(\det V)^2 = 1$ and so $\det V = \pm 1$, depending on the order in which the terms are listed. If $n \equiv 2 \pmod 4$, then $n = 2$ since the number of elements in a finite field is a power of a prime. In this case, $(\det V)^2 = -1 = 1$ so that $\det V = 1$. Similarly, if $n \equiv 0 \pmod 4$, then $(\det V)^2 = 1$ and $\det V = 1$ since F is of characteristic 2. If $n \equiv 1 \pmod 4$, then $\det V = \pm\sqrt{-1}$ where by $\pm\sqrt{-1}$ we mean the two roots of $f(x) = x^2 + 1$ in F. Write $n = 4k + 1$ and consider the non-zero elements of F, which form a cyclic group under multiplication. If $g \in F$ is a generator of this group, then $g^{4k} = 1$ and $g^{2k} \neq 1$ so that $g^{2k} = -1$; i.e., $(g^k)^2 + 1 = 0$. Thus we can also write $\det V = \pm g^k$.

In order to find the inverse of V, suppose that $a_k = 0$. Let $B = (b_{ij})$ be defined by

$$b_{k1} = 1; \quad b_{kj} = 0 \text{ if } 1 < j < n; \quad b_{kn} = -1;$$

$$b_{i1} = 0; \quad b_{ij} = -(a_i^{-1})^{j-1} \text{ if } j > i; \text{ for } i \neq k.$$

Let $BV = C = (c_{ij})$. Then

$$c_{kk} = a_k^0 - a_k^{n-1} = 1 - 0 = 1$$

$$c_{kj} = a_j^0 - a_j^{n-1} = 1 - 1 = 0 \text{ if } j \neq k,$$

since every $x \neq 0$ in F is a root of $x^{n-1} - 1 = 0$. If $i \neq k$, then

$$c_{ii} = -\sum_{m=2}^{n} (a_i^{-1}a_i)^{m-1} = -(n-1) = 1 \qquad (\text{in } F)$$

whereas if $k \neq i \neq j$, then

$$c_{ij} = -\sum_{m=2}^{n} (a_i^{-1}a_j)^{m-1} = -\sum_{m=2}^{n} x^{m-1} = \frac{x(x^{n-1}-1)}{1-x} = 0,$$

where $x = a_i^{-1}a_j \neq 1$. Thus C is the identity matrix and so $B = V^{-1}$.

Also solved by S. C. Currier and Yasuhiko Ikeda. Partial solutions by J. Alonso, M. G. Greening (Australia), O. P. Lossers (Netherlands), and the proposer.

Editor's comment. For references dealing with inverse of the Vandermonde matrix, the reader is referred to Th. Muir, *The Theory of Determinants*, Vol. I, reprinted in Dover Publications, New York, 1960, p. 306; N. Macon and A. Spitzbart, *Inverses of Vandermonde matrices*, this MONTHLY 65 (1958), 95–100; Walter Gautschi, *On inverses of Vandermonde and confluent Vandermonde matrices* I *and* II, Numerische Mathematik 4 (1962), 117–123 and 5 (1963), 425–430.

A Converse of Kronecker's Lemma

E 2433 [1973, 943]. *Proposed by Robert T. Smythe, University of Washington*

Let $x_1, x_2, \cdots$ be a sequence of real numbers. What is sometimes called Kronecker's Lemma asserts that if $\sum_{n=1}^{\infty} n^{-1} x_n$ is convergent, then $\bar{x}_n \to 0$ as $n \to \infty$, where $\bar{x}_n = n^{-1}(x_1 + \cdots + x_n)$. (That is, $x_n \to 0$ in Cesàro mean.) The converse of this lemma is false: let $x_n = (\log(n+1))^{-1}$.

Prove the following partial converse: If $\bar{x}_n \to 0$ as $n \to \infty$, then $\sum_{n=1}^{\infty} n^{-1-\delta} x_n$ converges for all $\delta > 0$.

Solution by Allen Stenger, Student, Pennsylvania State University. The same conclusion holds under the weaker assumption that $\{\bar{x}_n\}$ is bounded. Let $S_0 = 0$ and $S_n = x_1 + \cdots + x_n$ for $n \geqq 1$. Then

$$\sum_{n=1}^{m} \frac{x_n}{n^{1+\delta}} = \sum_{n=1}^{m} \frac{S_n - S_{n-1}}{n^{1+\delta}} = \sum_{n=1}^{m-1} S_n\left\{\frac{1}{n^{1+\delta}} - \frac{1}{(n+1)^{1+\delta}}\right\} + \frac{S_m}{m^{1+\delta}}.$$

But $m^{-1-\delta} S_m = m^{-\delta} \bar{x}_m \to 0$ as $m \to \infty$ since $\{\bar{x}_m\}$ is bounded. Thus $\sum n^{-1-\delta} x_n$ converges if and only if $\sum S_n\{n^{-1-\delta} - (n+1)^{-1-\delta}\}$ is convergent. However, this latter series is absolutely convergent because an application of the Mean Value Theorem to $f(x) = x^{-1-\delta}$ on the interval $[n, n+1]$ gives the inequality

$$\left| S_n\left(\frac{1}{n^{1+\delta}} - \frac{1}{(n+1)^{1+\delta}}\right)\right| \leqq \frac{|S_n|(1+\delta)}{n^{2+\delta}} = \frac{|\bar{x}_n|(1+\delta)}{n^{1+\delta}},$$

and the series $\sum n^{-1-\delta} |\bar{x}_n|$ converges since $\{\bar{x}_n\}$ is bounded.

Also solved by K. F. Andersen, M. T. Bird, Robert Breusch, Richard Groeneveld & Dean Isaacson, Emil Grosswald, Rev. William Habakkuk, Ellen Hertz, Richard Johnsonbaugh, Robert Kopp, Eric Langford, Peter Lindstrom, O. P. Lossers (Netherlands), M. R. Modak (India), Joel Pitt, Otto Ruehr, St. Olaf College Students, T. Šalát (Czechoslovakia), J. Swetits, Ken Yocom, Philip Young, and the proposer.

Editor's comment. The Rev. William Habakkuk (Queen Adelaide College of Unreason, Erewhon) comments that this is the special case $\alpha = 1$ and $\beta = p = 0$ of Theorem A (p. 491) of L. S. Bosanquet, *Note on convergence and summability factors* (III), Proc. London Math. Soc. (2), 50 (1948), 482–496. According to Bosanquet, the special case of his Theorem A with $p = 0$ and α and β integers was originally formulated by I. Schur, *Über lineare Transformationen in der Theorie der unendlichen Reihen*, J. für Math. 151 (1921), 79–111.

It should be noted that even though the series $\sum S_n\{n^{-1-\delta} - (n+1)^{-1-\delta}\}$ is absolutely convergent, the original series $\sum n^{-1-\delta} x_n$ need not be. For example, if $x_n = (-1)^n \sqrt{n}$ then $S_n = O(\sqrt{n})$ and so $\bar{x}_n = O(n^{-1/2}) = o(1)$; yet $\sum n^{-1-\delta} |x_n| = \sum n^{-\delta-1/2}$ diverges for all $\delta \leqq \frac{1}{2}$.

Kopp and Šalát observe that the series $\sum n^{-1-\delta} x_n$ will converge for some fixed $\delta > 0$ if $\bar{x}_n = O(n^{\varepsilon})$ for some ε with $0 < \varepsilon < \delta$, and thus it converges for all $\delta > 0$ if $\bar{x}_n = O(\log n)$, say.

All solutions of Advanced Problems should be sent to J. Barlaz, Rutgers — The State University, New Brunswick, N. J. 08903. Solutions of Advanced Problems in this issue should be typed (with double spacing) on separate, signed sheets and should be mailed before January 31, 1975.

An asterisk (*) means neither the proposer nor the editors supplied a solution.

5988. *Proposed by Bertram Ross, University of New Haven*

Show that the fractional integration of order v, $0 < v < 1$, of $\log x$ equals zero at $x = e^{\psi(1+v)+\gamma}$, and the fractional derivative of order $1 - v$ of $\log x$ equals zero at $x = e^{v[\psi(v)+\gamma]}$. γ is Euler's constant and ψ is the logarithmic derivative of the gamma function.

The Riemann-Liouville definition of fractional integration of order v is

$$ {}_0D_x^{-v}f(x) = \frac{1}{\Gamma(v)} \int_0^x (x-t)^{v-1} f(t)dt, \qquad v > 0, $$

and fractional differentiation of order $1 - v$, $0 < v < 1$, is

$$ {}_0D_x^{1-v}f(x) = \frac{d}{dx}[{}_0D_x^{-v}f(x)], $$

where d/dx is ordinary differentiation.

5989*. *Proposed by R. H. C. Newton, University College of North Wales*

Prove or disprove: The kth term of the infinite sequence $\{n_k\} = \{2, 4, 11, 31, \cdots\}$ defined by

$$ h(n_k - 1) \leqq h < h(n_k) \qquad (k = 1, 2, \cdots), $$

where $h(n) = 1 + 1/2 + 1/3 + \cdots + 1/n$, is given by the integer nearest to $\exp(k - \gamma)$. Here $\gamma = 0.577 \cdots$ is Euler's constant.

5990. *Proposed by H. Kestelman, University College, London, England*

G is a group of permutations of $1, 2, \cdots, n$. For each f in G, $w(f)$ denotes the number of integers r with $f(r) = r$, and $L(f)$ denotes the transformation of R^n mapping $(x_1, x_2, \cdots, x_n)$ onto $(x_{f(1)}, x_{f(2)}, \cdots, x_{f(n)})$. Prove that the arithmetic mean of the $w(f)$ is the dimension of the set of x that are fixed points for all the $L(f)$.

5991*. *Proposed by D. S. Mitrinović, University of Belgrade, Yugoslavia*

Let $g_1, \cdots, g_n$ be linear forms in indeterminates $x_1, \cdots, x_r$ with real coefficients and let $k_1, \cdots, k_n$ be real numbers. Suppose that $\sum_{i=1}^{n} k_i |g_i| \geqq 0$ for arbitrary real numbers $x_1, \cdots, x_r$. Prove, using purely algebraic methods, that the same inequality holds when $x_1, \cdots, x_r$ are interpreted as arbitrary vectors in Euclidean space of any finite number of dimensions.

(This theorem was proved by F. W. Levi (Arch. Math. (Basel) 2, (1949), 24–26)

by the use of integrals. See also Mitrinović, *Analytic Inequalities*, Springer-Verlag, 1970, pp. 175–176.)

5992. *Proposed by H. J. Thiebaux, Boulder, Colorado*

Let

$$X = \begin{bmatrix} X_{11} & \cdots & X_{1n} \\ \vdots & & \vdots \\ X_{n1} & \cdots & X_{nn} \end{bmatrix}$$

denote an hermitian, positive definite matrix, where each block of the partitioning is $m \times m$, and define

$$Y = \begin{bmatrix} Y_{11} & \cdots & Y_{1n} \\ \vdots & & \vdots \\ Y_{n1} & \cdots & Y_{nn} \end{bmatrix} = \begin{bmatrix} X_{11} & \cdots & X_{1n} \\ \vdots & & \vdots \\ X_{n1} & \cdots & X_{nn} \end{bmatrix}^{-1} = X^{-1}$$

with corresponding partitions. Prove that $(\sum_{r,t=1}^{n} Y_{r,t})^{-1}$ is hermitian, positive definite.

5993. *Proposed by V. Dlab, Carleton University, Edward Formanek, Institute for Advanced Study, and C. M. Ringel, Universität Bonn, West Germany*

Let F be a countable field, let $K = F(X_1, X_2, \cdots)$ be the rational function field over F in countably many indeterminates, and let A be an algebraic closure of K. Show that A, as an abelian group under addition, is the sum of two proper subfields.

SOLUTIONS OF ADVANCED PROBLEMS

A Non-negative Function on $\{0, 1, 2, \cdots, p\}$

5916 [1973, 697]. *Proposed by J. G. Mauldon, Amherst College*

Let p and q be coprime integers with $0 < p < q$ and let $f(\cdot)$ be a non-negative function defined on $\{0,1,2,\cdots,p\}$ such that $f(0) = 0$, $f(p) = 1$ and, whenever $\sum_1^k m_i = q$ with $0 \leqq m_i \leqq p$ $(i = 1,2,\cdots,k)$, we have $f(m_k) \leqq \sum_{i=1}^{k-1} f(m_i)$. Is it necessarily true that $\max\{f(n): n = 0,1,2,\cdots,p\} < 3q$?

Solution by Robert Breusch, Amherst College. Let $q = up + r$, $0 < r < p$. $(p,r) = 1$ because $(q,p) = 1$. From the given inequality and the fact that $f(p) = 1$ it follows that $f(r) \leqq u \cdot f(p) = u$. For $s = 0,1,\cdots,r$, let $v_s = [sp/r]$ (greatest integer function). Thus $v_0 = 0$, $v_r = p$.

Now consider the following set of $r + p$ integers, all non-negative, and all less than p:

$$\left\{\begin{matrix} y_m = (v_m + 1)r - mp \\ (m = 0, 1, \cdots, r-1) \end{matrix}\right. \qquad \left.\begin{matrix} x_{mt} = (m+1)p - tr \\ (v_m < t \leqq v_{m+1}) \end{matrix}\right\}.$$

The p integers x_{mt} are all distinct, because $(p, r) = 1$. Thus the x_{mt} are the numbers $0, 1, \cdots, p-1$. We know that $f(y_0) = f(r) \leqq u$. Further,

$$y_m + x_{mt} + (u-1)p + (t - v_m)r = up + r = q,$$

and thus, by the given inequality,

$$f(x_{mt}) \leqq f(y_m) + (u-1)f(p) + (t-v_m)f(r),$$

that is

(a) $$f(x_{mt}) < f(y_m) + (t - v_m + 1)u.$$

In particular, for $t = v_{m+1} = M$,

(a′) $$f(x_{mM}) < f(y_m) + (v_{m+1} - v_m + 1)u.$$

Also, $y_{m+1} + x_{mM} + up = up + r = q$, and thus, again by the given inequality with (a′)

$$f(y_{m+1}) \leqq f(x_{mM}) + u < f(y_m) + (v_{m+1} - v_m + 2)u.$$

It follows by induction that for $0 \leqq m \leqq r-2$,

$$f(y_{m+1}) < f(y_0) + (v_{m+1} - v_0 + 2(m+1))u \leqq (v_{m+1} + 2m + 3)u,$$

or, replacing $m+1$ by m, that for $1 \leqq m \leqq r-1$,

(b) $$f(y_m) < (v_m + 2m + 1)u.$$

For $m = 0$, this is equally correct.

By (a) and (b), $f(x_{mt}) < (t + 2m + 2)u$, and since $t \leqq v_r = p$ and $m \leqq r-1$, this implies that

$$f(x_{mt}) < (p + 2r)u < 3pu < 3q.$$

The x_{mt} are the numbers $0, 2, \cdots, p-1$, and $f(p) = 1 < 3q$. Thus

$$\max\{f(n): n = 0, 1, \cdots, p\} < 3q.$$

Compact Topological Vector Space over an Infinite Field

5920 [1973, 697]. *Proposed by L. C. Washington, Princeton University*

Do there exist nonzero compact (Hausdorff) topological vector spaces over infinite fields?

Solution by Seth Warner, Duke University. It is a theorem in S. Warner, *Compact and finite dimensional locally compact vector spaces*, Illinois J. Math. 13

(1969), 383–393 (See Theorems 5 and 6) that there exists a nonzero compact vector space over a topological division ring K if and only if the topology of K is discrete; in this case the compact K-vector spaces are precisely cartesian products of copies of the character group of the additive group of K, made into a K-vector space in a natural way.

Also solved by I. K. Abruob, Cecilia H. Brook, and the proposer.

Pencils of Nilpotent Matrices

5921 [1973, 697]. *Proposed by Paul Cohn, Bedford College, London, England*

Over any commutative field (e.g., the complex numbers), find two square matrices of the same order, A and B, such that every matrix in the pencil $\lambda A + \mu B$ is nilpotent, but A and B cannot be simultaneously triangularized.

Solution by Olga Taussky-Todd, California Institute of Technology. A pair of such matrices had been found previously by H. Wielandt:

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}.$$

That they cannot be triangularized simultaneously follows, among other reasons, from the fact that AB is not nilpotent.

These matrices were used as an example in T. S. Motzkin and O. Taussky, *Pairs of matrices with property L*, Trans. A.M.S. 73 (1952), 108–114 (See p. 113). For simultaneously triangularizable nilpotent $n \times n$ matrices see also M. Gerstenhaber, *On nilalgebras and linear varieties of nilpotent matrices*, I, Amer. J. of Math., 80 (1958), 614, ff.

Also solved by D. Ž. Djoković, A. A. Jagers (Netherlands), Peter Ungar, and the proposer.

Reduction of Singular Matrices

5922 [1973, 814]. *Proposed by Paul Cohn, Bedford College, London, England*

A and B are two $m \times n$ matrices over an infinite field k such that rank $(A + \lambda B) \leqq$ rank A for all $\lambda \in k$. Find P and Q of orders m, n respectively such that

$$B = PA + AQ, \qquad PAQ = 0.$$

Solution by the proposer. (i) Both problem and solution are invariant under the transformation $A \mapsto UAV$, $B \mapsto UBV$, where U, V are invertible matrices of orders m, n respectively.

(ii) If we have found P, Q for a given pair A, B, we can also solve the problem for

$A' = (A\ 0)$, $B' = (B\ 0)$, where 0 is a column of zeros. We simply take $P' = P$, $Q' = \left(\begin{smallmatrix} Q & 0 \\ 0 & 0 \end{smallmatrix}\right)$; similarly for the case where A and B are bordered with a row of zeros.

(iii) If $m = n$ and $A = I$, a solution is $P = B$, $Q = 0$.

(iv) General case. By row and column operations we bring A to the form

$$A = \begin{pmatrix} 0 & 0 \\ I & 0 \end{pmatrix}. \tag{1}$$

If B is partitioned correspondingly as $B = \left(\begin{smallmatrix} B_1 & B_2 \\ B_3 & B_4 \end{smallmatrix}\right)$, then we have

$$B_2 = 0, \qquad B_1 B_3^n B_4 = 0 \qquad (n = 0, 1, \cdots). \tag{2}$$

For on applying elementary row and column operations we have

$$A + \lambda B = \begin{pmatrix} \lambda B_1 & \lambda B_2 \\ I + \lambda B_3 & \lambda B_4 \end{pmatrix} \to \begin{pmatrix} \lambda B_1 (I + \lambda B_3)^{-1} & \lambda B_2 - \lambda B_1 (I + \lambda B_3)^{-1} \lambda B_4 \\ I & 0 \end{pmatrix}.$$

Since $\mathrm{rk}(A + \lambda B) \leqq \mathrm{rk}(A)$, we conclude that $B_2 = B_1(I + \lambda B_3)^{-1} \lambda B_4$, i.e., on expanding $(I + \lambda B_3)^{-1} = \Sigma(-\lambda)^n B_3^n$ and equating coefficients of λ, we obtain (2). We note that when A has the form (1), the equations (2) are necessary and sufficient for $\mathrm{rk}(A + \lambda B) \leqq \mathrm{rk}(A)$ to hold.

By row and column operations we now bring B_4 to the form

$$B_4 = \begin{pmatrix} 0 & 0 \\ I & 0 \end{pmatrix}.$$

The column operations do not affect A, while the effect of the row operations on A can be undone by performing the corresponding column operations on A (and of course on B; this will in general change B_1 and B_3, but we keep the same notation). The columns of zeros in B_4 correspond to columns of zeros in B (since $B_2 = 0$) and in A, and so can be omitted, by (ii). Thus we may take B_4 in the form

$$B_4 = \begin{pmatrix} 0 \\ I \end{pmatrix}. \tag{3}$$

If we partition B_1 correspondingly, as $(B_1'\ B_1'')$, we have by (2) with $n = 0$, $B_1 B_4 = B_1'' = 0$; hence $B_1 = (B_1'\ 0)$. By row and column operations we can reduce B_1' to the form

$$B_1' = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix};$$

again the effect of the column operations on A can be undone by row operations, which will not affect B (because apart from transforming B_3 they act on the zero part of B_4). If we again omit the row of zeros, we obtain

$$(4) \qquad B = \begin{bmatrix} I & 0 & 0 & 0 \\ & & & 0 \\ & B_3 & & 0 \\ & & & I \end{bmatrix}, \quad A = \begin{bmatrix} 0 & 0 & 0 & 0 \\ & & & 0 \\ & I & & 0 \\ & & & 0 \end{bmatrix}.$$

Now partition B_3 into blocks C_{ij} $(i,j = 1,2,3)$. The first column and the last row of blocks in B_3 can be transformed to 0 by subtracting multiples of I, so that we have

$$(5) \qquad B_3 = \begin{bmatrix} 0 & C_{12} & C_{13} \\ 0 & C_{22} & C_{23} \\ 0 & 0 & 0 \end{bmatrix}.$$

Rewriting (2) in terms of the expressions (4) and (5) for A, B, we find

$$(6) \qquad C_{13} = 0, \qquad C_{12}C_{22}^{n}C_{23} = 0 \qquad (n = 0, 1, \cdots).$$

These equations show that the blocks from the second and third lots of rows and columns in A and B, viz.

$$A' = \begin{pmatrix} 0 & 0 \\ I & 0 \end{pmatrix}, \qquad B' = \begin{pmatrix} C_{12} & 0 \\ C_{22} & C_{23} \end{pmatrix}$$

satisfy the hypotheses imposed on A and B, i.e., $\mathrm{rk}(A' + \lambda B') \leqq \mathrm{rk}(A')$, and here A' has smaller size than A unless $A = I$, a case dealt with under (iii). We can therefore use induction on $m + n$ and find P', Q' with $B' = P'A' + A'Q'$, $P'A'Q' = 0$. It follows that P' and Q' may be taken as

$$P' = \begin{pmatrix} 0 & C_{12} \\ 0 & P_1 \end{pmatrix}, \qquad Q' = \begin{pmatrix} Q_1 & C_{23} \\ 0 & 0 \end{pmatrix},$$

$P_1 + Q_1 = C_{22}$, and $P_1Q_1 = 0$, $C_{12}Q_1 = P_1C_{23} = 0$. Hence the matrices

$$P = \begin{bmatrix} 0 & I & 0 & 0 \\ 0 & 0 & C_{12} & 0 \\ 0 & 0 & P_1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \qquad Q = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & Q_1 & C_{23} & 0 \\ 0 & 0 & 0 & I \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

satisfy the required conditions.

Also solved by David Fried.

Normal Operators on a Finite Dimensional Hilbert Space

5923 [1973, 814]. *Proposed by Emeric Deutsch, Polytechnic Institute of Brooklyn*

Let $\|\cdot\|$ be a norm on the vector space C^n of all n-tuples of complex numbers and let A be an operator on C^n such that $\|(A-\alpha I)^{-1}\| = r[(A-\alpha I)^{-1}]$ for each complex α which is not in the spectrum of A (r denotes spectral radius). Is $\|A\| = r(A)$?

Solution by A. S. Householder, University of Tennessee. In A. S. Housholder, *The Theory of Matrices in Numerical Analysis*, Blaisdell, 1964, p. 47, B is said to be of class M in case there exists a norm $\|\cdot\|$ such that $\|B\| = r(B)$. It is shown there that B is of class M if and only if the elementary divisors associated with roots of greatest modulus are all simple. If λ is any root of A, then α can be chosen so that $(\lambda-\alpha)^{-1}$ is a root of greatest modulus of $(A-\alpha I)^{-1}$. Hence all elementary divisors of A are simple and A is of class M, and is, indeed, normalizable. Suppose $(A-\alpha I)^{-1}$, hence also A, is in diagonal form. Any norm for which $\|(A-\alpha I)^{-1}\| = r[(A-\alpha I)^{-1}]$ must be an absolute norm [*loc. cit.*]. Consequently $\|A\| = r(A)$.

It is no restriction to suppose that the matrices are in diagonal form (see *loc. cit.* equation (2.2.3).)

Also solved by Fumio Kubo (Japan).

Kubo refers for his solution to S. Hildebrandt, *Über den numerischen Wertebereich eines Operators*, Math. Annalen, 163 (1966), 230–247 (Thm. 3). Kubo notes that the result fails ($\|A\| < r(A)$ for some A) if the vector space is infinite dimensional; we are also referred to J. G. Stämpfli, *Hyponormal operators and spectral density*, Trans. A. M. S. 117 (1965) when A is not compact.

REVIEWS

EDITED BY J. ARTHUR SEEBACH, JR. AND LYNN A. STEEN
with the assistance of the mathematics departments of St. Olaf and Carleton Colleges

COLLABORATING EDITOR FOR FILMS: SEYMOUR SCHUSTER, CARLETON COLLEGE

We invite readers to submit reviews of significant recent college-level mathematics books. We especially encourage reviews based on classroom use, or comparative reviews of several related books. Reviews should ordinarily not exceed two pages (per book) typed double spaced. Manuscripts of reviews as well as books submitted for review should be sent to: Book Review Editor, American Mathematical Monthly, St. Olaf College, Northfield, MN 55057.

Numerical Analysis: A Second Course. By James M. Ortega. Academic Press, New York, 1972. xiii + 201 pp. $11.00. (Telegraphic Review, August-September 1972.)

Professor Ortega's new book is well organized and carefully written, and covers relatively advanced material in an easily understood manner. However, whether it

can be used successfully as a text for your "Second Course in Numerical Analysis" is largely a matter of what you believe such a course should attempt to accomplish.

Traditionally, a first year graduate course in numerical analysis has dealt with problems deemed too advanced for its lower level counterpart. In addition, it usually provides more of an emphasis on error analysis for the methods discussed. Ortega carries the latter point to its extreme. The unifying principle behind his book involves the types of error introduced by numerical methods in general, while the methods themselves occupy a secondary position.

The reader quickly realizes this by a glance at the table of contents. The book begins with a one chapter review of linear algebra, and the remainder is divided into four parts. Part I — Mathematical Stability and Ill Conditioning, Part II — Discretization Error, Part III — Convergence of Iterative Methods, Part IV — Rounding Error.

Each part is subdivided into chapters which deal with particular applications to certain numerical problems. Specifically, the first part treats mathematical stability with regard to systems of linear equations, eigenvalue/eigenvector problems, and differential and difference equations. The next part deals with the discretization error encountered in initial and boundary value problems, while the third part covers the convergence of iterative methods for linear and nonlinear equations. The last part contains only one chapter, treating the problem of rounding error for the Gaussian elimination procedure.

Numerical Analysis: A Second Course has a number of points in its favor, in addition to those already mentioned at the beginning of this review. Each part begins with a brief introduction indicating what will be covered in the ensuing chapters, and, at the end of each chapter is a list of suggested reading and a set of well chosen exercises. Moreover, the frequent use of examples and footnotes, and the fact that the text is virtually free of errors, make for easier, more pleasant reading. Incidentally, as an aid to someone using this book as a reference, the author provides a list of "commonly used symbols" between the preface and introduction.

An interesting facet of this text, which I discovered while giving a "traditional" course in numerical linear algebra to first year graduate students, is that it makes an excellent primary reference for such a course. Most of the material that I covered could be found somewhere in the book, although it was necessary to skip around. Unfortunately, there is a complete absence of numerical methods for eigenvalue/eigenvector problems, so other references had to be given for this topic. I found the review of linear algebra (in the first chapter) to be very suitable for students at this level. It quickly dispenses with elementary notions, and concentrates on norms, eigenvalues, and canonical forms of matrices.

The prerequisites for using this book are fairly minimal. I feel that advanced calculus and introductory courses in linear algebra and numerical analysis are required. The author contends, however, that the last of these is unnecessary for "well-motivated and mature" students.

Numerical Analysis: A Second Course is one of the better non-introductory texts available dealing with numerical analysis. If the material covered fits your needs, I strongly suggest that you take a close look at it.

STEWART VENIT, California State University

Mathematics in Society. Elements of an analysis: history, education, ideology. By Else and Jens Høyrup. Gyldendal, Copenhagen, 1973. 184 pp, (P), (In Danish). (Telegraphic Review, June-July 1974.)

The stated purpose of this short book is "to sketch an analysis of the *interplay* between mathematics and the society in which the mathematics is developed and used." The book is not just for specialists, but was written "to invite debate among many sorts of people, including present and future teachers and others with a greater or lesser professional relationship to mathematics." I know of no book about mathematics at all comparable to this one, and I will say at once that the Høyrups' work has been a provoking and valuable stimulus to my thinking about the social meaning of mathematics and my own role within it.

The analysis is organized under two main headings: history and education. The historical section ranges from mathematics in antiquity to the present in less than 60 pages, and developments since the Renaissance are treated especially briefly. But it is not the authors' intention to give a history of mathematical ideas as such. The historical sketch (like the book as a whole) is meant "to be useful for the interpretation of the present-day world," and the focus is on the general ways in which mathematics has affected and been affected by the surrounding society. Many of the mechanisms are still important today, but they can be first understood most easily at a greater distance from our own situation. No doubt this part of *Mathematics in Society* offers little new to historians, but much will surely be novel and revealing to the general (mathematical) reader, as it was to me.

The second part of the book, entitled "mathematics in modern society," begins with a short chapter on "the applications of advanced mathematical theory in production and administration." One finds here few details of particular applications, either for good or ill. Of course this aspect of the subject has been much discussed elsewhere, and a more thorough treatment would require a great deal of time and space. Still, the chapter's superficiality will doubtless leave many readers unsatisfied, and I feel it is the weakest feature of the book.

The bulk of the second part concerns education in modern Western society. As in the first part, the concern is to analyse the interrelations between changes in society and the course of mathematical education, rather than just to study education *per se*.

Throughout the book, a major role is played by the concept of *ideology*. This term is carefully and somewhat narrowly defined to mean a system of beliefs and customs, not based on fact, which serves to support the existing organization and class structure of a society. It is a "collective false consciousness," with definite social functions. The authors try to show that mathematics has entered significantly

into the formation of such ideology during much of Western history. For example, the concept of proof and the abstraction (especially in geometry) which are to us the main characteristics of classical Greek mathematics had an important relation to the structure of Greek society, which after about 350 B. C. was largely based on the exploitation of slave labor. In general, the ideological aspect of mathematics is treated along with its technological aspect as a major facet of the interplay between mathematics and society.

To many readers it may at first seem strange, or even paradoxical, that a subject such as mathematics can play a role in ideology and propaganda. One of the main themes of *Mathematics in Society* is showing that this is indeed the case, and exploring the mechanisms by which the ideological side of mathematics works. To me, this is the most striking and valuable aspect of the book.

The point is especially clear in the discussion of elementary education. Perhaps other (male) readers of the *Monthly* have also recently been made conscious of the sex-role stereotyping which permeates many elementary school textbooks, with their illustrations so often showing girls as just passive spectators of the activity and initiative of the boys. Once one begins to see such things, they appear everywhere. After reading *Mathematics in Society* I am having a similar experience with school mathematics. It is not *only* that handling money, buying and selling, interest and dividends — but never, say, cooperatives — provide the great bulk of the applications and illustrations in many texts and classes. Mathematical education also has important effects on consciousness and thinking which are apart from the content of the teaching, and such effects are not always the happy ones we like to imagine. For example, teaching abstract mathematics in such a way that children do not themselves actively participate in forming the abstractions ("passive abstraction") may help to train adults who can be more easily manipulated by means of abstract terms and concepts. In addition, mathematics classes often provide the most effective place for sorting out school children into "winners" and "losers" on the basis of "ability" criteria — which none-the-less closely parallel social class background. These ideas are discussed persuasively and with insight; since reading this part of *Mathematics in Society* my own perception of mathematics teaching has changed markedly, and I think others may have the same experience.

A short final chapter is entitled "Personal conclusion: which mathematics?" Here the authors try to sketch some ideas for those "who wish for a mathematics in the interest of the majority of the people, and not a mathematics for the stabilization of the power of the upper class and the higher middle class." They believe it is only the first step to realize that "the research worker who produces mathematics to the Pentagon's order is doing executioner's work." Many of us who have avoided military applications have taken too much comfort from the supposed harmlessness of our work; the Høyrups challenge us to look more deeply.

The bulk of this "personal conclusion" is devoted to socially-conscious teaching. The discussion is general, and no specific how-to-do-it program is offered. And I

think that no simple one is possible. Surely the only prescription is that we must each find our own way — must apply to our individual circumstances all the insight we can find about mathematics, about people and about our society. I believe that *Mathematics in Society* provides very significant help in trying to do this. It certainly does not have all the answers, but it has meant a lot to me.

JOHN LAMPERTI, Dartmouth College

Computer Programming: Techniques, Analysis and Mathematics. By R. V. Andree, J. P. Andree, and D. D. Andree. Prentice-Hall, Englewood Cliffs, N. J., 1973. xvii + 549 pp. $12.95. (Telegraphic Review, June-July 1973.)

After covering most of this book in one semester, the student is well prepared for problem solving by way of the computer and FORTRAN IV. This is the authors' purpose; they succeed in a fashion which my students and I found enjoyable and stimulating. The class was composed of majors from many areas and all four college years. Although the word mathematics appears in the title it was not emphasized in my course, an approach easily taken, and there were no complaints of too much mathematics. It would be just as easy to design courses which could be taught in the mathematics department. The authors touch on many well-known, and some not so well-known, problems whose analysis is enhanced by use of the computer, but are always careful to point out that thoughtful analysis will often result in considerable machine time savings, and sometimes even solve the problem without recourse to the computer.

There is a large selection of carefully thought out problems, some with answers in the back. The sections on simulation are very well done. There are also sections on solving equations, systems of equations, area, Fibonacci numbers, the gambler's ruin problem, cryptanalysis, the Poisson process, and many number theoretic questions. One chapter is devoted to IBM 1130 Assembler Language.

I found that the students became competent programmers very quickly, had no trouble reading the book, and were often stimulated to follow paths on their own, suggested by the authors. They were asked to write programs after the first few pages, were continually reminded that there are best ways to do things ($A * A$ is better than $A ** 2$, which is much better than $A ** 2.$) and that numbers don't behave in the computer the way one expects. They were also asked to consider the philosophy that the computer should generate ideas, not numbers. Chapter 9 is devoted to good programming techniques, and should be read by every starting FORTRAN student, several times. There is a short, comprehensive, annotated bibliography. Altogether this is an excellent one-semester introductory programming book, with many bonuses.

STANLEY WERTHEIMER, Connecticut College

TELEGRAPHIC REVIEWS

Telegraphic reviews are designed to give prompt notice of new books with sufficient information to assist our readers in deciding whether to order an examination copy or to suggest library purchase. Possible uses are indicated as follows:

T = textbook P = professional reading
S = supplementary reading L = undergraduate library purchase
13 to 18 = freshman to second year graduate level usage
1 to 4 = appropriate time in semesters to cover text
Asterisks (*) or question marks (?) denote special positive or negative emphasis, respectively. Publishers are denoted by standard abbreviations; complete addresses may be found in *Books in Print*.

GENERAL, S, P, L**, *Chinese Science: Explorations of an Ancient Tradition*. Ed: Shigeru Nakayama, Nathan Sivin. MIT Pr, 1973, xxxvi + 334 pp, $12.50. Collection of articles on traditional Chinese science, in honor of Joseph Needham's *(Science and Civilisation in China)* 70th birthday. Perspectives on Needham, Chinese concept of nature, astronomy, optics, pharmacology and anesthesia. None directly concerns itself with mathematics, though the closing bibliography refers to what little there is on Chinese mathematics (p. 296). PJC

GENERAL, T(13-15), S, *Mathematik für Naturwissenschaftler*. J. Hainzl. B.G. Teubner, 1974, 311 pp, DM29 (P). Mathematics for natural scientists, particularly biologists, chemists and geologists. Assuming only high school mathematics, the book takes up analytic geometry and linear algebra, the calculus of functions of one and of several variables, Fourier series, and differential equations. About 200 problems, many with solutions. JD-B

GENERAL, S(13-14), L, *L'Arithmêtique Amusante*. Édourad Lucas. Blanchard, 1974, viii + 266 pp, 22F (P). M. Lucas était un maitre des récreation mathématiques au fin de XIXe siècle. Ce livre est un nouveau tirage d'un édition de 1895. Il y a quatre chapitres: Calcul élémentaire (exemple: le problèm des Chrêtiens et Turcs); Le Calcul rapid; Les progressions arithmétiques (exemple: une problème ici appele les trois maris jaloux--aussi connu comme les cannibales et le prêtre); Les progressions géométriques aussi un section des notes sur des problèmes comme la tour chevalier. En tout, un bon livre; il y a aussi des notes sur l'origin de tel ou tel probleme.PJM

GENERAL, T(13-14; 1, 2), L, *Mathematics: A World of Ideas*. Bevan K. Youse. Allyn, 1974, xi + 468 pp, $11.95. A good book with a wide variety of topics and minimal interdependence of chapters. Differs from most books of this type in the inclusion of calculus. Other topics: topology, number theory, modern algebra, probability, computer programming, logic. This book calls to my mind Courant and Robbins' classic, and seems a suitable updating of that book. PJM

GENERAL, T(13), S, L, *How to Solve Problems: Elements of a Theory of Problems and Problem Solving*. Wayne A. Wickelgren. Freeman, 1974, xi + 262 pp, $10; $4.95 (P). Occasionally wordy description of seven general problem-solving methods with detailed illustrations; directed to college students of mathematics, science and engineering. Principal emphasis is on method (influenced by work in artificial intelligence and computer simulation of thinking) as opposed to Polya's work on the heuristics of problem solving. LCL

BASIC, T(13: 1). *Developmental Arithmetic: An Individualized Approach.* James C. Curl. McGraw, 1973, $5.95 (P). Arithmetic review for personalized instruction. Includes golden mean, prime numbers, and Fibonacci numbers. LLK

BASIC, T(13: 1, 2). *Basic Mathematics for College Students, Fifth Edition.* Edwin I. Stein. Allyn, 1974, xii + 659 pp, $10.95. Arithmetic, informal geometry, numerical trigonometry, basic algebra, and consumer applications. FLW

BASIC, T(13: 1). *Business Mathematics.* Charles D. Miller, Stanley A. Salzman. Scott F, 1974, 334 pp, $7 (P). Applications of basic arithmetic in business: discounts, markups, interest, installment buying, depreciation. A semiprogrammed work-text with many exercises. PJM

BASIC, T(13: 1). *Intermediate Algebra for Today.* Robert E. Mosher. Har-Row, 1974, x + 436 pp, $9.95. Written for college students, this text covers the traditional topics of high school algebra with emphasis on set theory, notation and informal axiomatics. JNC

BASIC, S(13). *Mathematical Review for the Physical Sciences.* Jerry B. Marion, Ronald C. Davidson. Saunders, 1974, vi + 112 pp, $3.95 (P). A few remedial topics, e.g., units, averages, distance, graphs. LLK

PRECALCULUS, T*(13: 1). *Algebra and Trigonometry: A Functions Approach.* Mervin L. Keedy, Marvin L. Bittinger. A-W, 1974, viii + 693 pp, $9.95 (P); *College Algebra: A Functions Approach*, vi + 513 pp, $8.95 (P); *Trigonometry: A Functions Approach*, viii + 354 pp, $7.95 (P). Good format for individualized instruction. The text contains pre-tests, developmental exercises in the margins (answers in back), exercise sets, and chapter tests. An Instructor's Manual contains two alternate forms for each chapter test and final examination. The *Trigonometry* volume contains eleven and the *College Algebra* volume contains seven of the fifteen chapters in the combined text. LLK

PRECALCULUS, T(13: 2). *Introductory Mathematical Analysis, Fourth Edition.* Edgar D. Eaves. Allyn, 1974, ix + 678 pp, $11.95. The major change from third edition (TR, January 1970) is the inclusion of an appendix on matrices. Most other changes are expository--rewriting for sake of clarity. Several problems have been added. SG

PRECALCULUS, T(13: 1). *Topics in Precalculus Mathematics.* Donald R. Burleson. P-H, 1974, xii + 484 pp, $10.50. Usual assortment of precalculus topics. LLK

PRECALCULUS, T(13: 1). *Topics in College Algebra.* Donald R. Burleson. P-H, 1974, xii + 338 pp, $8.95. Same as *Topics in Precalculus Mathematics* with Chapters 8 and 9 (on circular and trigonometric functions) omitted. LLK

PRECALCULUS, T(13: 1). *Modern Trigonometry (Complete Course).* John P. Ashley, E.R. Harvey. Glencoe Pr, 1974, ix + 422 pp, $7.95 (P); *Progress Tests for Modern Trigonometry*, 61 pp, $7.95 (P). Trigonometry introduced with vector methods. Presented in frames for individualized instruction. Has complete set of Spirit Duplicating Masters with two forms for each chapter test and final exam. LLK

PRECALCULUS, T*(13: 1). *An Elementary Approach to Functions.* Henry R. Korn, Albert W. Liberi. McGraw, 1974, x + 498 pp, $10.95. This is one of the best pre-calculus books I've seen. Starting with the real line and properties of the real numbers, the authors move on to linear and quadratic functions, then inequalities, then polynomials and rational functions. Only after all of these examples are at hand do the authors treat algebraic operations and composition of functions. The book concludes with trigonometry and exponential functions and two chapters on conics. Problems are good, but not extensive. Index; answers to odd numbered problems. PJM

PRECALCULUS, T(13: 1, 2). *Integrated Algebra and Trigonometry: A Modified Programmed Approach.* Dennis Bila. Rinehart Pr, 1974, ix + 582 pp, $12.95 (P). A modified programmed approach means "stop lines" and a question and answer format, but no multiple choice branching. The topics are standard: sets, functions and graphing, rational functions, exponential functions, trig functions, complex numbers, conic sections, sequences, series and induction. Lots of exercises, well written. Only possible problem is doing algebraic operations on functions before having many examples at hand, i.e., before the chapter on polynomials and rational functions. No index, but the table of contents is sufficient. PJM

FINITE MATHEMATICS, T(14: 1, 2), S, L. *Mathematics for the Biological Sciences.* Stanley I. Grossman, James E. Turner. Macmillan, 1974, xi + 512 pp, $10.95. You can sense the influence of the editor Carl Allendoerfer in this pleasing mixture of motivation, example and illustration, theory, exercise and problem. Topics as presented (all with a strong dose of biological flavoring): discrete probability, vectors and matrices, linear programming, Markov chains, game theory, difference equations, differential equations, continuous probability, mathematical models in biology. LCL

FINITE MATHEMATICS, T(13: 1, 2). *Finite Mathematics from Sets to Game Theory.* Adelbert F. Hackert. Heath, 1974, 390 pp, $10.95. Written for liberal arts students or prospective teachers of elementary mathematics. Treats sets, logic, postulational systems, probability and statistics, linear programming, and two-person games. Assumes only elementary algebra. Its treatment of number systems is not nearly complete enough for prospective teachers, and its mathematical unsoundness, though it would probably be noticed by few students, makes it seem to this reviewer unsuitable for either of its intended uses. JD-B

EDUCATION, T(13-14: 2). *Fundamental Mathematics for Elementary Teachers: A Behavioral Objectives Approach.* A. Richard Polis, Earl M.L. Beard. Har-Row, 1973, xix + 454 pp, $9.95. Meets CUPM recommendations for mathematics of the real number system. Includes final chapters on logic and development of Euclidean geometry based on transformations. Six "Capsules", with bibliographies, spaced throughout text provide methods material. Behavioral objectives preceding each chapter are keyed to text material by marginal notation. PSJ

EDUCATION, T(13-14: 1, 2). *Mathematics for Elementary Education.* Donald F. Devine, Jerome E. Kaufmann. Wiley, 1974, xviii + 609 pp, $10.95. Logic, sets, the number systems, geometry (standard and translational), modular arithmetics, elementary probability, and descriptive statistics. FLW

EDUCATION, S, L, *Neater by the Meter; An American Guide to the Metric System.* Anton Glaser, 1237 Whitney Road, Southampton, PA 18966. 1974, iv + 112 pp, $3.50 (P); $6.50. A light, informal yet thorough introduction to metric measurement. Concentrates on thinking (e.g., estimating, calculating) in metric terms rather than on conversion between metric and British-American units. Suggestions for appropriate school classroom projects. LAS

EDUCATION, S(14), P, *Modèles Finis.* André Myx. CEDIC, 1973, 168 pp, (P). Text in concrete group theory for French school teachers, who must teach simple geometrical groups in elementary school (!). Largely examples and models, plus informal presentation of Cayley's theorem, isomorphisms and homomorphisms and finite congruence rings of integers. PJC

EDUCATION, S, L, *Teacher-made Aids for Elementary School Mathematics: Readings from the Arithmetic Teacher.* Ed: Seaton E. Smith, Jr., Carl A. Backman. NCTM, 1974, vi + 186 pp, $3 (P). 51 brief reprints with ideas for simple home-made arithmetic aids. LAS

HISTORY, S, P, *Message d'un mathématicien: Henri Lebesgue.* Lucienne Felix. Blanchard, 1974, vi + 259 pp, 65F (P). The book is divided into three parts: Part 1 is a biography and some recollections of Lebesgue by some of his students. Part II investigates Lebesgue's mathematical contributions as a researcher. Part III discusses his views on the teaching of mathematics. Written by one of Lebesgue's students, the book does have a definite bias, but Lebesgue was a great mathematician, and this book presents a good summary of his life and work. The French is moderately hard; a translation would be nice. Includes fascimiles of manuscripts, and a list of articles quoted from, but no complete bibliography of Lebesgue. Would be good supplemental reading in a history course. PJM

HISTORY, P, *Das Kontinuum, und Andere Monographien.* H. Weyl, *et al.* Chelsea, 1973, v + 381 pp, $8.50. A reprint of the following four historically important works, grouped solely for economic reasons: *Das Kontinuum,* H. Weyl, Berlin, 1917. *Mathematische Analyse des Raumproblems,* H. Weyl, Berlin, 1923. *Darstellung und Begründung einiger Neuerer Ergebnisse der Funktionentheorie,* E. Landau, 2nd ed., Berlin, 1929. *Über die Hypothesen, welche der Geometrie zu Grunde liegen,* B. Riemann; H. Weyl, editor, Berlin, 1923. JAS

FOUNDATIONS, S(16-18), P*, *Notes on Constructive Mathematics.* Per Martin-Löf. Almqvist & Wiksell, 1970, 109 pp, (P). Lectures given at the universities of Aarhus and Stockholm, 1966-68. An excellent introduction to the foundations and general principles of constructive mathematics written for mathematicians with no previous background in logic. Complements Bishop's more detailed, pragmatic approach. LCL

FOUNDATIONS, T(16-17: 1, 2), S, P, L, *Sets and Transfinite Numbers.* Martin M. Zuckerman. Macmillan, 1974, xvi + 423 pp, $11.95. Can be used for a variety of courses. First three chapters culminate in the construction of the real and complex numbers within the framework of Zermelo-Fraenkel set theory. Final three chapters give a rather extensive and detailed introduction to ordinals, cardinals, axiom of choice equivalences. Although accessible to beginners, the presentation is directed to mature students. LCL

FOUNDATIONS, T(15-16), L, *Récursivité*. Jean-Pierre Azra, Bernard Jaulin. Gauthier-Villars, 1973, xv + 218 pp, 98F (P). Two books in one: Part 1 is lecture notes from a course on recursive functions, Part 2 is a discussion of mathematical theories and the problem of undecidability. The French is of intermediate difficulty. A good background in logic is necessary for easy reading. PJM

FOUNDATIONS, T(18), P, *Lecture Notes in Mathematics-354: Aspects of Constructibility*. Keith J. Devlin. Springer-Verlag, 1973, xii + 240 pp, $8.50 (P). First draft of a projected book containing most of the existing information on constructibility. Intended as a graduate text. LCL

FOUNDATIONS, T*(14-15: 1), S(13-15), L*, *Set Theory: An Intuitive Approach*. Shwu-Yeng T. Lin, You-Feng Lin. HM, 1974, ix + 164 pp, $8.95. Non-axiomatic, informal presentation--intended to eliminate repetition and duplication of lectures in advanced courses and to provide once and for all a working knowledge of cardinals, ordinals, Zorn's Lemma, axiom of choice, Hausdorff principle, transfinite induction. Also appropriate for secondary teachers. The pace and style is leisurely; the text should be easily understood by the intended audience. This outstanding book makes a sophomore-level set theory course a curricular possibility that should be seriously considered. LCL

FOUNDATIONS, P, L, *Towards a Definitive Solution of Zeno's Paradoxes*. F.A. Shamsi. Hamdard Academy, Karachi, 1973, 84 pp, $2 (P). Primarily a philosophical critique of the "mathematicians' solution" via an attack on the logical foundations of Cantorian set theory. Includes an extensive, up-to-date bibliography. LAS

COMBINATORICS, T(15-16: 1), S, L, *A First Course in Combinatorial Mathematics*. Ian Anderson. Oxford U Pr, 1974, viii + 123 pp, $13.50. Very suitable for an undergraduate course or for independent study. Topics: counting, selection, binomial coefficients, pairing problems (e.g., marriage theorem), recurrence, inclusion-exclusion, generating functions, rook polynomials, block designs, error-correcting codes, Steiner systems, sphere packing, Leech's lattice. The writing is clear and simple--very readable. Problems are both interesting and challenging. SG

COMBINATORICS, P, *Probabilistic Methods in Combinatorics*. Paul Erdos, Joel Spencer. Prob. and Math. Stat., No. 17. Acad Pr, 1974, 106 pp, $11.75. An exposition of the probabilistic approach to combinatorial problems. While this method has been widely used in recent years, its applications have been littered throughout the literature. This interesting monograph presents many of these applications with emphasis on Ramsey-type theorems. While the price is a bit high, the reader can win up to $300 for solutions to various unsolved problems listed throughout the book. In any case, a valuable reference. SG

COMBINATORICS, T*(14-17: 1, 2), S, P, L, *Graph Theory with Applications to Engineering and Computer Science*. Narsingh Deo. P-H, 1974, xvii + 478 pp, $17.95. A rich, inviting introductory text (with problems and references in each chapter) emphasizing applications (e.g., coding theory, network analysis, operations research) and algorithms (including a few computer programs). Its many and varied references would make it attractive as the backbone in an introductory seminar.LAS

NUMBER THEORY, T(15-16: 1), L. *Elementary Number Theory: A Computer Approach.* Allan M. Kirch. Intext, 1974, xi + 339 pp, $11.75. Stresses problems of number theory susceptible to computer investigation; 28 such problems are considered, e.g., Euclidean algorithm; primes less than 10,000; division of large numbers; Euler's function; perfect numbers; quadratic residues. Introduction to Fortran IV is included. SG

LINEAR ALGEBRA, T(15-16: 1, 2). *Linear Algebra: An Introductory Approach, Third Edition.* Charles W. Curtis. Allyn, 1974, ix + 337 pp, $12.95. Third edition of a text first published in 1963. The chief change from the second edition is the addition of considerable material on dual spaces, multilinear algebra, the rational and Jordan canonical forms, and applications. JD-B

LINEAR ALGEBRA, T*(13-14: 1). *Computational Matrix Algebra.* David I. Steinberg. McGraw, 1974, viii + 280 pp, $11.95. Not as "computational" as the title might lead one to believe. Basic theory is included and problems include some proofs. A very useable text with good references for further numerical methods. LLK

LINEAR ALGEBRA, T(14-15: 1). *A Basis for Linear Algebra.* Warren Brisley. Wiley, 1973, viii + 189 pp, $12.95. Intended as a text for a first course in abstract algebra but actually an introduction to linear algebra with a few pages devoted to fields, rings and groups. Informal in style, elementary in level. Has no great advantages over many other texts on this subject. JD-B

LINEAR ALGEBRA, T(16-17: 1), P, L. *Systemes de Polynomes.* A. Robert. Papers in Pure and Appl. Math., No. 35. Queen's U, 1973, 108 pp, $3.75 (P). This book's topics would normally be classified numerical analysis, but the author's approach is primarily that of linear algebra. A system of polynomials is a basis for the vector space of polynomials with real coefficients. By choosing different bases, one obtains interesting analogues of Taylor's series; or, by requiring bases to be orthogonal with respect to some inner product, one gets other representation theorems. The author shows in particular that all classical examples can be described in this way, and presents results by methods that are delightfully free of the usual messy calculations one encounters. A good text for a senior seminar. Some complex variables and differential equations are assumed along with affine algebra. Index and references. PJM

ALGEBRA, T(17: 2). *Algebra.* Thomas W. Hungerford. HR&W, 1974, xix + 502 pp, $17. Intended as text for first graduate course in algebra. Essentially self-contained, but most students would need to have had a more gentle introduction to the subject. Introduces the language of categories early, but the chapter dealing more deeply with the subject can be omitted. Carefully and clearly written, with many problems. Competes with such texts as those by Lang and MacLane and Birkhoff, but might be slightly easier reading for students than either of these. JD-B

ALGEBRA, P. *An Introduction to Lie Algebra for Lie Group.* Morikuni Goto. Aarhus U, 1973, iii + 129 pp, (P). A good introduction to the subject, but no sign of Lie groups. Includes definitions, representations, cohomology, study of semi-simple Lie algebras. Some exercises. PJM

ALGEBRA, T(14-16: 3). *Algebra, Volume 1.* P.M. Cohn. Wiley, 1974, xi + 321 pp, $19.50. Combines in one book largely independent introductions to abstract algebra and finite-dimensional linear spaces. Carefully written, with many exercises, both routine and otherwise. Defines categories, but they are not central to the development. Suitable for good undergraduates. JD-B

ALGEBRA, T(14-15: 1). *Introductory Modern Algebra.* Elwyn H. Davis. Merrill, 1974, xii + 241 pp, $9.95. Written for the average mathematics major. Treats, in this order, permutation groups, abstract groups, rings, polynomials and fields. Conventional in topics and methods. Probably too easy for potential graduate students. JD-B

ALGEBRA, T**(15-16: 1, 2), S, L. *A Course in Modern Algebra.* Peter Hilton, Yel-Chiang Wu. Wiley, 1974, x + 249 pp, $16.95. An excellent book. After discussing groups and abelian groups, so the student will have some feeling for what is being generalized, the authors devote a chapter to categories, with the emphasis on terminology and examples rather than a large amount of "general abstract nonsense." The terminology is then available for the final four chapters: Modules, Integral Domains, Semisimple Rings, and Ext and Tor. Exercises are numerous; the only lack is a good bibliography. PJM

ALGEBRA, T(15-17: 1), S, P, L. *Modern Algebra: An Introduction.* Tim Anderson. Merrill, 1974, vii + 215 pp, $9.95. A rather classical approach--Galois theory replaced by Lagrange's theory of equations, intuition buttressed by geometrical considerations (fields and affine geometries, groups and symmetries, integral domains and irreducible curves and surfaces); ideals are not explicitly mentioned. Format on a level roughly comparable to Fraleigh: number theory, polynomials, fields, groups, algebraic geometry. More than enough algebra for one term; fairly tightly written. LCL

ALGEBRA, S(17-18), P**, L**. *Elements of Mathematics: Algebra I.* Nikolas Bourbaki. A-W, 1974, xxiii + 708 pp, $37.50. The long-awaited translation of Chapters 1-3 of *Elements de Mathematique, Algebre:* algebraic structures, linear algebra, tensor algebras, exterior algebras, symmetric algebras. Remaining chapters 4-9 will be published anon as *Algebra II.* Some readers may be surprised to find that this epitome of rigorous, abstract, formal mathematics begins with the disarming observation that "Algebra is essentially concerned with *calculations...*" Ironically, this concern has moved algebra in the two decades since these chapters were written towards problems of automata and algorithms, thus dating Bourbaki as a heroic monument to the just-past golden age of "pure" mathematics. LAS

CALCULUS, T(13-14: 2, 3). *Calculus with Analytic Geometry, Fifth Edition.* R.E. Johnson, F.L. Kiokemeister. Allyn, 1974, x + 839 pp, $14.95; *Study Guide for Calculus with Analytic Geometry, Fifth Edition.* Joseph Cunsolo, Joseph P. Mokanski. Allyn, 1974, 272 pp, $5.95 (P); *Instructor's Supplement,* 228 pp, free (P). The revisions are an improvement on an already widely used text. Unwieldy notation has been deleted, derivatives and integrals are introduced earlier, a new chapter on line integrals, Green's theorem, and change of variable is very good. An attractive edition with teacher and student supplements. LLK

CALCULUS, T(13-14: 3), *Calculus and Analytic Geometry, Third Edition*. Abraham Schwartz. HR&W, 1974, xv + 1140 pp, $15. Many changes from the second edition, including the introduction of the logarithmic function before its inverse. Almost all earlier exercises have been rewritten, and many new ones added. An attractively illustrated, usable book which allows freedom in selection of topics and theoretical complexity. DFA

CALCULUS, T(13-14: 2, 3), *Calculus: One and Several Variables with Analytic Geometry, Second Edition*. Saturnino L. Salas, Einar Hille. Xerox, 1974, xiii + 904 pp, $14.50. Additions to the first edition (TR, June-July 1971) include text and exercises on harmonic motion, vector geometry, quadric surfaces, applications to economics. Looks at vector calculus and functions of several variables, but not linear algebra. Available in two volumes whose intersection studies infinite series, l'Hospital's rule, improper integrals. DFA

CALCULUS, T(14: 2), *Multivariable Mathematics: Linear Algebra, Differential Equations, Calculus*. Richard E. Williamson, Hale F. Trotter. P-H, 1974, ix + 630 pp, $15.95. A nicely coordinated set of topics for the year following a first course in calculus. Outlines for several combinations of topics are suggested. Includes vector field theory through the divergence theorem, and existence and uniqueness theorems in differential equations. LLK

CALCULUS, T(14: 2), *A Second Course in Calculus*. Harley Flanders, Robert R. Korfhage, Justin J. Price. Acad Pr, 1974, xii + 687 pp, $10.95. Standard non-rigorous presentation of topics to follow one-variable calculus. Enough vector calculus to cover line integrals, Green's theorem, and change of variable. Includes some series and introduction to complex analysis. LLK

CALCULUS, T(15-16: 2), S, P, L, *Advanced Calculus of Real-Valued Functions of a Real Variable and Vector-Valued Functions of a Vector Variable*. Hans Sagan. HM, 1974, xv + 671 pp, $15.95. Topics, written for prospective mathematicians, are dealt with rigorously, beginning with establishing the real number system as a complete ordered field. The coverage and treatment is slightly more extensive and more detailed than Buck's classic. LCL

CALCULUS, T(13-14: 1, 2), *Calculus of Several Variables*. Serge Lang. A-W, 1973, viii + 376 pp, $9.95. To conclude the calculus sequence. Lang's old *Second Course* with just that introductory linear algebra needed for use in a short chapter on functions of several variables. Multiple integrals, surface integrals. The text style and the exercises are straightforward. DFA

CALCULUS, T(13: 2), *Elementary Applied Calculus: A Short Course*. Raymond Coughlin. Allyn, 1974, viii + 280 pp, $11.95. A typical "short calculus" distinguished by a wealth of applications (handily referenced inside the front cover) to biology, ecology, sociology, physics, psychology and business. Unfortunately, most of the applications are too brief to give insight into the process or complexities of building mathematical models and none includes references to the primary sources in which its full context might be revealed. LAS

CALCULUS, T(13-14: 3), *Calculus and Analytic Geometry, Second Edition*. Douglas F. Riddle. Wadsworth, 1974, xii + 848 pp, $15.95. Adds sections and exercises devoted to applications, plus examples,

illustrations, review problems and historical notes to the 1970 edition (TR, April 1970). Vectors appear earlier and are now used in the solid analytic geometry. The multitude of problems and three-level approach to limits and continuity make it a definite contender for any calculus sequence. DFA

DIFFERENTIAL EQUATIONS, T*(15-16), L*. *Differential Equations, Dynamical Systems, and Linear Algebra.* Morris W. Hirsch, Stephen Smale. Pure and Appl. Math., No. 60. Acad Pr, 1974, xi + 358 pp, $14.95. A clearly written, sophisticated, fascinating introduction to dynamical systems, suitable for strong undergraduates. Concepts are well-motivated; interplay between mathematics and its applications (esp. mechanics) is stressed. Among the applications discussed: Kepler's Laws, electrical circuits (including Van der Pol's equation), ecology (predator-prey, competing species), classical mechanics((n-body problem, Hamiltonian mechanics). Other topics: linear systems of DEs with constant coefficients, exponentials of operators, canonical forms of operators, existence and uniqueness, phase portrait analysis, Poincaré-Bendixson, perturbation theory and structural stability. Highly recommended for an advanced undergraduate seminar. SG

DIFFERENTIAL EQUATIONS, T(18), P. *An Introduction to Nonlinear Boundary Value Problems.* Stephen R. Bernfeld, V. Lakshmikantham. Math. in Sci. and Eng., V. 109. Acad Pr, 1974, xi + 386 pp, $18.50. Details methods employed in studying these problems: shooting type, topological, functional analytic, and ones involving differential inequalities. Concludes with consideration of functional differential equations and of special topics. Assumes background in ordinary differential equations, real and functional analysis. Instructive exercises throughout. DFA

DIFFERENTIAL EQUATIONS, P. *Lecture Notes in Mathematics-324: Oscillation Theory.* Kurt Kreith. Springer-Verlag, 1973, vi + 109 pp, $6 (P). Focuses on recent developments. From lectures at Chelsea College, London, 1972. Classical theory, generalization to certain partial differential equations, relations to other topics in analysis. Abstract oscillation theory. Consideration of systems of equations of even order. DFA

DIFFERENTIAL EQUATIONS, T(17: 1, 2). *The Hamilton-Jacobi Theory in the Calculus of Variations: Its Role in Mathematics and Physics.* Hanno Rund. Krieger, 1973, xi + 440 pp, $15. Non-homogeneous and homogeneous single integral problems, multiple integral problems, the problem of Lagrange. Contains much theoretical physics, which can be omitted for a shorter, "pure" course. Reprint of 1966 edition, with supplementary appendix (extending the material on multiple integral problems) and bibliography. DFA

FUNCTIONAL ANALYSIS, P. *Integral Equations and Stability of Feedback Systems.* Constantin Corduneanu. Math. in Sci. and Eng., V. 104. Acad Pr, 1973, ix + 238 pp, $19.50. Chapters on admissibility and Hammerstein equations, frequency techniques and stability, Wiener-Hopf equations, further topics (the energy method, positive-definite kernels, tauberian theorems, the dynamics of nuclear reactors). Applications, exercises, bibliographical notes in each chapter. 200 citations. DFA

FUNCTIONAL ANALYSIS, P. *Variational Method and Method of Monotone Operators in the Theory of Nonlinear Equations*. M.M. Vainberg. Transl: A. Libin, D. Louvish. Halsted Pr, 1973, xi + 356 pp, $35. Results concerning the variational method obtained since the author's 1956 book on the subject, plus a study of the method of monotone operators. Application of the latter method to nonlinear Hammerstein integral equations, boundary-value problems for quasilinear partial differential equations, nonlinear differential equations in Banach spaces. 250 references. Note price. DFA

OPTIMIZATION, T(15-17: 1, 2), S. *Methods and Applications of Linear Programming*. Leon Cooper, David Steinberg. Saunders, 1974, ix + 434 pp, $14. The simplex method, duality theory and related algorithms, current computational techniques, integer programming, and applications. Presupposes linear algebra. FLW

OPTIMIZATION, S(16-17), P. *Lecture Notes in Economics and Mathematical Systems-4: Branch and Bound: Eine Einführung*. Franz Weinberg. Springer-Verlag, 1973, vii + 174 pp, $7 (P). Papers by different authors collected for use in a course on branch and bound methods of optimization. The discussion is largely in terms of more or less practical problems. JD-B

OPTIMIZATION, T(16-17: 1, 2), S(15-18), P, L. *Industrial Systems: Planning, Analysis, Control*. David D. Bedworth. Ronald Pr, 1973, x + 504 pp, $12.50. A comprehensive study of quantitative techniques of industrial systems (including network planning, forecasting (regression analysis), optimization, scheduling, inventory control) using fully documented case studies to stress particular applications and effective problem-solving techniques. Assumes introductory background in calculus and statistics; otherwise self-contained. LCL

ANALYSIS, S(17-18), P, L. *Compact Lie Groups and Their Representations*. D.P. Želobenko. Transl. of Math. Mono., V. 40. AMS, 1973, viii + 448 pp, $35.70. An excellent "elementary" exposition focused on applications to theoretical physics. Includes a full discussion of finite dimensional representations of compact, semisimple and reductive Lie groups together with some aspects of infinite dimensional representation theory. Many specific illustrations of the general theory and innumerable notes pointing to connections with applications, history or other parts of mathematics make this a rich resource for serious students of Lie theory. LAS

ANALYSIS, T(16-18: 2), S, P, L*. *Lie Groups, Lie Algebras, and Some of Their Applications*. Robert Gilmore. Wiley, 1974, xx + 587 pp, $24.95. A relatively gentle, historically and physically motivated introductory text covering basic theory and examples at a leisurely pace. Concludes with classification theory and examples of how Lie groups can be altered. Does not include any representation theory. A valuable transition volume for both physicists and mathematicians. LAS

ANALYSIS, T(18: 3), P. *Lecture Notes in Mathematics-371: Analyse Différentielle*. Valentin Poenaru. Springer-Verlag, 1974, 228 pp, $7.70 (P). A two-semester graduate course on the structure of the spaces $C^\infty(X,Y)$ for general Y, and when $Y = R$. PJM

ANALYSIS, P. *American Mathematical Society Translations, Series 2, V. 103: Nine Papers in Analysis*. AMS, 1974, iii + 203 pp, $17.70.

TOPOLOGY, P, *Complex Actions of Lie Groups*. Connor Lazarov, Arthur Wasserman. Memoirs No. 137. AMS, 1973, ii + 82 pp, $2.90 (P). A study of bordism for stable almost complex manifolds acted on by a compact Lie group. An almost complex structure on a manifold is a map $J:M \to M$ with $J^2 = -id$. The action $G \times M \to M$ of the Lie group G is required to preserve the structure. PJM

GEOMETRY, T(14-17: 1, 2), S*, L*, *Regular Polytopes, Third Edition*. H.S.M. Coxeter. Dover, 1973, xiv + 321 pp, $4 (P). Corrected republication of second edition, with new preface by author. Sample correction: it is possible for an inorganic substance to have icosahedral symmetry, viz. the element boron in its molecular form B_{12}. As Coxeter says, the original claim of impossibility must now be taken "with a grain of borax." PJC

GEOMETRY, P, *Spaces of Constant Curvature, Third Edition*. Joseph A. Wolf. Publish or Perish, 1974, xv + 408 pp, $10 (P). Third edition with corrections of Wolf's excellent book. SG

GEOMETRY, P, *The Action of a Real Semisimple Lie Group on a Complex Flag Manifold II: Unitary Representations on Partially Holomorphic Cohomology Spaces*. Joseph A. Wolf. Memoirs No. 138. AMS, 1974, iii + 152 pp, $3.60 (P). A study of the geometric realization of unitary representations of semisimple groups using the group orbits of the decomposition introduced in part one of this work (Bull. A.M.S., 75 (1969) 1121-1237). JAS

PROBABILITY, T(15-16: 1, 2), *Applications of Probability and Random Variables, Second Edition*. George P. Wadsworth, Joseph G. Bryan. McGraw, 1974, xv + 448 pp, $13.50. Intended as a text for a first course in probability theory at the post-calculus level. Includes: basic probability; difference equations; standard, discrete, continuous and joint distributions; expectation; generating functions; Markov processes; some statistical tests. Emphasis is on applications with extensive problem sets and examples. An attractive text. TAV

PROBABILITY, P, *Random Integral Equations with Applications to Life Sciences and Engineering*. Chris P. Tsokos, W.J. Padgett. Math. in Sci. and Eng., V. 108. Acad Pr, 1974, x + 278 pp, $24. Existence, uniqueness, stochastic stability, and approximation of random solutions of nonlinear stochastic integral equations of the Volterra and Fredholm types, with some attention to equations of the Ito-Doob type. Emphasizes stochastic modeling, and obtains models for various phenomena in the biological, engineering, and physical sciences. DFA

PROBABILITY, P, *Fourier Transforms of Distributions and Their Inverses: A Collection of Tables*. Fritz Oberhettinger. Acad Pr, 1973, ix + 167 pp, $18. Primarily a set of tables of the Fourier transforms (characteristic functions) of absolutely continuous distributions, and the inverse transforms, arranged by type of function. A brief introduction lists the properties of characteristic functions which are important in probability and explains how to use the tables, while an appendix gives the familiar forms of the characteristic functions of the most common distributions in statistical literature. Contains several obvious typographical errors. RSK

PROBABILITY, T(18: 1), P, *Concentration Functions*. W. Hengartner, R. Theodorescu. Acad Pr, 1973, xii + 139 pp, $12. Introduced by Lévy in his famous 1937 monograph, concentrations functions play an

increasingly important role in modern probability theory. This book begins with Lévy's work and develops the properties of these functions, showing their power in the treatment of essential convergence. TAV

PROBABILITY, P, *Lecture Notes in Mathematics-321; Séminaire de Probabilités VII* Ed: C. Dellacherie, P.A. Meyer, M. Weil. Springer-Verlag, 1973, vi + 322 pp, $9.70 (P). The presentations from the 1971-72 probability seminar at the Mathematics Institute of the University of Strasbourg. JAS

STATISTICS, T?(16-17: 1), P, *Econometric Estimation*. J.C.R. Rowley. Wiley, 1973, ix + 234 pp, $14.50. By and large a specialized statistics book, with very little indication of the connections with econometrics. Fortunately, references to current literature occur frequently. Topics: the linear model, stochastic difference equations, interdependent systems and identification, structural estimation and maximum likelihood. Prerequisites: linear algebra and two terms of mathematical statistics, including testing of linear hypotheses. PJC

STATISTICS, S, P, L, *Statistical Methods and Scientific Inference, Third Edition*. Sir Ronald A. Fisher. Hafner Pr, 1973, viii + 182 pp, $9.95. Published posthumously, this edition contains numerous small changes from the 1959 *Second Edition*, based on notes left by the author. Gives the author's view of the logical principles of statistical reasoning. Stresses the role of statistics as an aid in understanding the natural world, and opposes the formalism of Neyman and Pearson. RSK

STATISTICS, P, *Statistical Methods for Rates and Proportions*. Joseph L. Fleiss. Wiley, 1973, xiii + 223 pp, $12.95. In the Wiley Series in Probability and Mathematical Statistics. Concerned with procedures and problems in comparing qualitative or categorical data. Contains many good medical examples and references. Prerequisites are a year of applied statistics and high school algebra. RSK

STATISTICS, T(13: 1), *Statistics and Society: Data Collection and Interpretation*. Walter T. Federer. Dekker, 1973, ix + 399 pp, $14.50 (P). Designed for a course in learning *about* statistics rather than learning statistical techniques. Emphasis is on methods of processing meaningful and accurate data and hence the major portion is on design concepts, with virtually nothing on statistical inference. Price is high for a photo-offset paperback. RSK

STATISTICS, P, *Mathematics and Statistics: Essays in Honour of Harald Bergström*. Ed: Peter Jagers, Lennart Råde. U of Goteborg, 1973, 121 pp, 35 Skr (P). Ten papers in mathematical statistics by authors from around the world. JAS

STATISTICS, P, *Multivariate Error Analysis*. A.A. Clifford. Halsted Pr, 1973, ix + 112 pp, $12.75. A handbook which outlines procedures used to obtain error estimates when two or more parameters are simultaneously calculated from an equal or greater number of experimental observables. Requires familarity with elementary statistical concepts, multivariable calculus, and, for the case of more than two parameters, elementary linear algebra. Includes computer programs in both Algol and Fortran for the methods described. RSK

STATISTICS, P. *Statistical Sequential Analysis: Optimal Stopping Rules.* A.N. Sirjaev. Transl. of Math. Mono., V. 38. AMS, 1973, iv + 174 pp, $18.50. Major portion is devoted to the theory of optimal stopping rules for Markov random processes in both the discrete and continuous time cases. Last chapter applies these results to two problems in statistics: testing two simple hypotheses, and detecting a "disruption", or change in underlying distribution. RSK

STATISTICS, P. *Poisson's Exponential Binomial Limit.* E.C. Molina. Krieger, 1973, 92 pp, $4.50 (P). Two extensive tables: values of $\lambda^x e^{-\lambda}/x!$ for λ from .0001 to $\lambda = 100$, x ranging over values for which the terms exceed 10^{-7}, and the sums of such terms for x > various values. Reprint of a frequently cited source. TAV

STATISTICS, T(17-18: 2), S, P. *Linear Statistical Inference and Its Applications, Second Edition.* C. Radhakrishna Rao. Wiley, 1973, xx + 625 pp, $22.50. In the Wiley Series in Probability and Mathematical Statistics. Updated version of the author's 1965 book, containing much new material. Two extensive introductory chapters on matrix theory and probability precede the major topics of statistical inference. Very thorough presentation of theoretical results, together with realistic examples providing practical applications. RSK

STATISTICS, T(14-17: 2), L. *Statistics for the Social Sciences, Second Edition.* William L. Hays. HR&W, 1973, xxi + 954 pp, $13.95. Revision of the author's well-known 1963 text *Statistics for Psychologists.* Examples are still psychological in nature, and hypothetical, but can be easily translated into other areas. Some ideas from calculus are treated intuitively, and in general some sophistication is required. Covers the standard topics in a very thorough, detailed manner, so may be used as a reference. Concludes with a new chapter on Bayesian methods. RSK

COMPUTER SCIENCE, P, L. *Textile Graphics/Computer Aided.* Janice R. Lourie. Fairchild, 1973, xi + 297 pp, $15. The algebra and algorithms of weaving patterns, as told in a personal, autobiographical style by a weaver who is a senior staff member at IBM's Systems Science Institute. Includes description of computer systems used to design weaving patterns and to control loom operations. LAS

COMPUTER SCIENCE, P. *Lecture Notes in Computer Science- 3 & 4: 5th Conference on Optimization Techniques.* Ed: R. Conti, A. Ruberti. Springer-Verlag, 1973. *Part I:* xiii + 565 pp, $14.70 (P); *Part II:* xiii + 389 pp, $10.80 (P). Proceedings of a May, 1973 conference in Rome. Part I emphasizes methodological issues; Part II deals with applications. LAS

COMPUTER SCIENCE, T(15-17: 1, 2). *Theory of Computation.* Walter S. Brainerd, Lawrence H. Landweber. Wiley, 1974, xxi + 336 pp, $19.95. Focuses on higher-order machine-independent languages (similar to FORTRAN) rather than Turing machines or machine-dependent languages (such as assembly languages), as a level for algorithms to take place on. Partial recursive functions are defined using repetition and exponentiation (corresponding to FOR/DO loop programming statements) rather than traditional primitive recursion and minimization. Features equivalence of various algorithmic languages, unsolvability and undecidability, formal languages, reducibility, complexity theory, and some combinatory logic. PJC

COMPUTER SCIENCE, T(15-17: 1), S, P, L. *Computational Methods for Matrix Eigenproblems.* A.R. Gourlay, G.A. Watson. Wiley, 1973, xi + 132 pp, $9.95. Short (one or two lecture), single topic descriptions of the more common techniques; little is said about error analysis.LCL

COMPUTER SCIENCE, T(15-16: 1), S, L. *Introduction to the Theory of Computation.* Erwin Engeler. Acad Pr, 1973, viii + 231 pp, $12.95. An easy-to-read introduction. Not intended to be thorough. Finite automata, corresponding languages, examples include building an acceptor and truth functions. Recursive functions, computability and complexity are approached using universal calculators. Context free grammars, generation and translation. Good examples. Too few exercises. RWN

COMPUTER SCIENCE, P. *Effective vs. Efficient Computing.* Ed: Fred Gruenberger. P-H, 1973, vi + 153 pp, $6.95. Papers from a symposium held in March, 1972 sponsored by Informatics and UCLA. Mostly oriented toward management and evaluation of data processing. RWN

COMPUTER SCIENCE, S(15-17), L. *Computability.* Martin Davis. New York U, 1974, v + 248 pp, $6.25 (P). Lecture notes from a course on "Computers and Computability": programs and computable functions, recursive functions, Turing machines, Thue and normal systems. LAS

SYSTEMS THEORY, P. *System Identification: Methods and Applications.* Harriet H. Kagiwada. Appl. Math. and Comp., No. 4. A-W, 1974, xix + 293 pp, $8.50 (P); $16. Develops numerical methods for nonlinear "inverse" problems which arise in systems identification. Several applications are described: multiple scattering, neutron transport, wave propagation, nonlinear filtering, physiology. SG

SYSTEMS THEORY, T(17), P. *Simulation: Statistical Foundations and Methodology.* G. Arthur Mihram. Acad Pr, 1972, xv + 526 pp, $24.50. Concerned with the dynamic, stochastic, simulation model--a general category covering symbolic, but not formalized, models which contain random phenomena and vary with time. Emphasis is on the stochastic element in recognition of an "Uncertainty Principle of Modeling" which states that "refinement in modeling eventuates a requirement for stochasticity." RSK

SYSTEMS THEORY, T(16-17: 1, 2), S, P, L. *System Theory: A Unified State-Space Approach to Continuous and Discrete Systems.* Louis Padulo, Michael A. Arbib. Saunders, 1974, xvii + 779 pp, $17.50. "A third-generation textbook, integrating the first and second transform and state-variable methods into a framework which unifies continuous-time and discrete-time systems and embraces both classical linear systems and automata." Presupposes only calculus and matrix algebra. FLW

APPLICATIONS (BUSINESS), T(13-14: 1, 2). *Modern Mathematics for Business Decision-Making.* Donald R. Williams. Wadsworth, 1974, xiv + 513 pp, $11.95. Touches on calculus, probability, linear programming, matrix algebra, interest and annuities. Many good examples. Problems emphasize applications and decision making but are limited to straight drill. Glossary of terms and tables in back. LH

APPLICATIONS (CONTROL THEORY), P. *Lecture Notes in Economics and Mathematical Systems-93 and 94; 4th IFAC/IFIP International Conference on Digital Computer Applications to Process Control.* Ed: M. Mansour, W. Schaufelberger. Springer-Verlag, 1974. *Part I:* xvii + 544 pp, $13.90 (P); *Part II:* xvii + 546 pp, $13.90 (P). Papers presented at a March, 1974 conference at Zurich on the use of digital computers to operate and supervise complex industrial processes. LAS

APPLICATIONS (CONTROL THEORY), P. *Control and Dynamic Systems: Advances in Theory and Applications, V. 11.* Ed: C.T. Leondes. Acad Pr, 1974, xv + 516 pp, $24.50. Six survey papers in filters, smoothing, control theory and differential games. LAS

APPLICATIONS (DECISION THEORY), S*(16-17), P, L*. *Mathematics of Decision Theory.* Peter C. Fishburn. Mouton, 1972, 104 pp, $5.10 (P). A rapid-fire survey of sophisticated mathematical topics (e.g., Zorn's lemma, ordered topological spaces, choice functions) with brief discussion and careful references to their applications in decision theory. An attractive resource for a seminar in contemporary applied mathematics. LAS

APPLICATIONS (DEMOGRAPHY), P. *Lecture Notes in Operations Research and Mathematical Systems-44: Stochastische Modelle demographischer Prozesse.* G. Feichtinger. Springer-Verlag, 1971, ix + 404 pp, $7.70 (P). A major broad spectrum work designed to provide German speaking researchers with a significant reference work. Special features: depth of the mathematical work and emphasis on stochastic models. JAS

APPLICATIONS (ECONOMICS), P. *Frontiers in Econometrics.* Ed: Paul Zarembka. Acad Pr, 1974, ix + 252 pp, $17.50. Eight chapters by different authors on model selection, linear models and multiple-equation models intended to define certain frontiers of econometrics. LAS

APPLICATIONS (ECONOMICS), S*(15-17), P, L. *Macroeconomic Theory: A Mathematical Introduction.* Paul Burrows, Theodore Hitiris. Wiley, 1974, xiii + 210 pp, $11.95. An attempt to bridge "the widening gap between the verbal/diagrammatic exposition of macroeconomic theory in introductory and intermediate texts and the mathematical treatment commonly found in journal articles." The repeated juxtaposition of verbal, diagrammatic and mathematical exposition of conventional theory make this concise volume an excellent entrance to mathematical economics. LAS

APPLICATIONS (ECONOMICS), T(17-18: 2), S, P*, L. *Theory of General Economic Equilibrium.* Trout Rader. Acad Pr, 1972, xx + 362 pp, $18. A wide ranging survey including optimality in social ordering (e.g., Arrow's impossibility theorem), bargaining, competitive equilibria, dynamics. An impressive display of sophisticated applied mathematics. LAS

APPLICATIONS (ECONOMICS), P. *Topics in Applied Econometrics.* Kenneth F. Wallis. Lect. in Econ., No. 5. Gray-Mills, 1973, 136 pp, £1.25 (P). Case studies which illustrate problems of formulating and empirically testing economic hypotheses in consumption, production, and investment functions and in macroeconomic models. LAS

APPLICATIONS (ECONOMICS), P, L, *Lecture Notes in Economics and Mathematical Systems-92: Topological Methods in Walrasian Economics.* E. Dierker. Springer-Verlag, 1974, v + 130 pp, $6.20 (P). Analysis of competitive equilibria in a decentralized economy leads to the study of singularities of a tangent field of a Euclidean manifold. These notes offer a mathematically clear introduction to such economic models, begun by Debreu and refined by the differential topology of Morse, Milnor and Smale. LAS

APPLICATIONS (ENGINEERING), S, *Robotics.* John F. Young. Halsted Pr, 1973, 303 pp, $18.95. Interesting for its philosophical discussions on the future and direction of robotics. Essentially a review of various ways of obtaining humanoid characteristics from a machine, the book touches only briefly on the engineering development required. JJ

APPLICATIONS (ENGINEERING), S, *Cybernetic Engineering.* John F. Young. Halsted Pr, 1973, 153 pp, $11.95. More detailed technical development than the companion volume *Robotics*, the book investigates various concepts concerning a robot "brain." Devices for the physical implementation of these concepts are expanded as mathematical models. The presentation emphasizes the engineering, as opposed to the mathematical, aspects of the subject. JJ

APPLICATIONS (ENGINEERING), T(17-18: 2), P, *Graph Theory in Modern Engineering.* Ernest J. Henley, R.A. Williams. Math. in Sci. and Eng., V. 98. Acad Pr, 1973, xvi + 303 pp, $16. Applications of graphs and digraphs to engineering. Flow graph analysis and applications to engineering mathematical problems, and many direct applications in computer-aided design, control and optimization. SS

APPLICATIONS (ENGINEERING), P, *Probleme în Teoria Filtratiei.* Horia I. Ene, Sorin Gogonea. Editura Academiei Romania, 1973, 438 pp. Problems of porous media in theoretical hydrodynamics. In Romanian. JAS

APPLICATIONS (ENGINEERING), S(17-18), P, L, *Lecture Notes in Economics and Mathematical Systems-84: Stochastic Differential Systems I, Filtering and Control: A Function Space Approach.* A.V. Balakrishnan. Springer-Verlag, 1973, v + 252 pp, $8.20 (P). Functional analysis applied to stochastic filtering and control. Feedback control of systems subject to random disturbances, stochastic differential games, and a 3-martingale approach to linear filter theory. Rigorous mathematical presentation using Wiener measure and Ito integrals. LH

Reviewers Whose Initials Appear Above

David F. Appleyard, Carleton; Paul J. Campbell, St. Olaf; Judith N. Cederberg, St. Olaf; John Dyer-Bennet, Carleton; Steven Galovich, Carleton; Loren Haskins, Carleton; James Johnson, St. Olaf; Paul S. Jorgensen, Carleton; Lorraine L. Keller, St. Olaf; Richard S. Kleber, St. Olaf; Loren C. Larson, St. Olaf; Pierre J. Malraison, Carleton; R.W. Nau, Carleton; Seymour Schuster, Carleton; J. Arthur Seebach, Jr., St. Olaf; Lynn A. Steen, St. Olaf; T.A. Vessey, St. Olaf; Frank L. Wolf, Carleton.

NEWS AND NOTICES

EDITED BY RAOUL HAILPERN, SUNY at Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to Mathematical Association of America, 1225 Connecticut Avenue, NW, Washington, D. C. 20036. Items must be submitted at least two months before publication can take place.

PERSONAL ITEMS

Professor Edgar Franz, Illinois College, received the Dunbaugh Distinguished Professor Award for excellency in teaching for 1973–74, at a recent convocation.

Instructor Barry Jones, Keuka College, has been promoted to Assistant Professor.

Assistant Professor C. S. Kim, Indiana University Southeast, has been promoted to Associate Professor.

Dr. Len Pikaart, Professor of Mathematics Education at the University of Georgia, has been named to the Robert L. Morton Professorship in Mathematics Education at Ohio University.

Assistant Professor J. E. Wolfe, Northwestern University, has been appointed Assistant Professor at Oklahoma State University.

Professor Emeritus Grover C. Bartoo, Western Michigan University, died on July 19, 1973, at the age of 88. He was a member of the Association for thirty-two years.

Dr. W. D. Baten, Wichita Falls, Texas, died on April 4, 1974, at the age of 82. He was a member of the Association for fifty-four years.

Mr. William H. Cain, Western Michigan University, died on July 17, 1973, at the age of 87. He was a member of the Association for thirty years.

Mr. Howard Harding, Rochester, New York, died on November 11, 1973, at the age of 88. He was a member of the Association for fifty-three years.

Associate Professor Emeritus C. B. Helms, Widener College, died on November 6, 1973, at the age of 71. He was a member of the Association for thirty-four years.

Professor Richard J. Kohlmeyer, Hartwick College, died on April 1, 1974, at the age of 54. He was a member of the Association for twenty-five years.

Dr. Hyman L. Laden, Brooklandville, Maryland, died on December 21, 1973, at the age of 58. He was a member of the Association for nineteen years.

Professor Charles W. Lytle, Drew University, died on January 9, 1974, at the age of 46. He was a member of the Association for sixteen years.

Professor Emeritus Clifford N. Mills, Illinois State University, died on March 5, 1974, at the age of 87. He was a Charter Member of the Association.

THE GREATER METROPOLITAN NEW YORK MATH FAIR

Fair Date: April 13, 1975 — Pace University, Manhattan.

Eligibility: Students who have completed, or are currently taking, mathematics at the eleventh year level or higher in the public, private and parochial high schools in New York City, Westchester, Putnam, Dutchess, or Rockland Counties are eligible to participate in the FAIR. Exceptional students who do not meet these requirements but wish to submit papers for consideration by the FAIR Committee are welcome to do so.

Purpose: The FAIR encourages a student to pursue, in depth, some phase of mathematics to which he is drawn.

Procedure: 1. Research a topic in mathematics. 2. Write a paper on this topic. 3. Give a talk (of no more than 15 minutes duration) on your paper to a group of judges.

Nature of Paper: The work need not be completely original, but the paper should reveal scholarship appropriate to the course level of the student. Teachers may suggest topics but the presentation should be voluntary and based entirely upon the student's own readings and investigations.

Further details and application forms may be obtained from: Dr. Theresa J. Barz, Secretary, MATH FAIR Committee, Department of Mathematics and Computer Science, St. John's University, Jamaica, New York 11439.

SYMPOSIUM AT INDIANA UNIVERSITY — BLOOMINGTON

The 1975 International Symposium on Multiple-Valued Logic will be held at Indiana University, Bloomington, May 21–23, 1975. Authors are invited to submit papers on the theory and applications of multiple-valued logics, or nonclassical logics connected with multiple-valued logics, including algebraic aspects. Authors should submit 3 copies of a first draft, 5 pages minimum and 30 pages maximum ,by December 1, 1974, to the Conference Chairman, G. Epstein, Computer Science, Indiana University, Bloomington, Indiana 47401, USA. Accepted papers will appear with the invited papers in the SYMPOSIUM PROCEEDINGS.

MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

MARCH MEETING OF THE FLORIDA SECTION

The Seventh Annual Spring Meeting of the Florida Section of the MAA was held on March 8 and 9, 1974, at the University of Florida in Gainesville, Florida.

Eight invited addresses were presented as follows: "Calculus with Computers," Professor Garrett Birkhoff, Harvard University; "On Some Problems of Complexity, Analogies, and Metrics," Professor Stanislaw Ulam, University of Colorado; "Non-standard Analysis and Compactification," Professor J. L. Kelley, The University of California; "The Creeping Lemma," Professor Osiefield Anderson, Florida A & M University; "Mathematics Courses for the Non-Mathematician," Professor Pamela Ferguson, University of Miami; "L'Hôpital's Rule and Functions of Type $[f(x)]^{g(x)}$," Professor R. C. Meacham, Eckerd College; "Invariants Don't Change," Professor G. W. Medlin, Stetson University; "The Mathematics of Music," Professor A. D. Snider, University of South Florida.

In conjunction with the meeting there was a State Articulation Conference which began with a report on progress in articulation and an open discussion on current questions. The following talks were presented to the Conference: "An Innovative Approach to Pre-Calculus Mathematics," Nick Passell and Mark Hale, University of Florida; "General Education Mathematics — A Cultural Approach," Bill Kirshner, Florida Atlantic University; "Mini-

mester — Modmester — Openmester — Repeatmester — Continumester — Second Chancemester...," George Cash, Manatee Junior College; "Combinatorial Computing at the Junior College," Frank Hadlock, Florida Atlantic University; "Today's Probability and Statistics — Methodology and Purpose," C. P. Tsokos, University of South Florida, and George Schultz, St. Petersburg Junior College.

The Southeastern Section of the American Mathematical Society also had a spring meeting March 7 and 8, preceding the Association meeting.

The following papers were presented:

1. *Linear Inequality Systems and Lloyd L. Dines*, by L. E. Bragg, Florida Institute of Technology.
2. *Inconsistent Logics*, by John Grant, University of Florida.
3. *Theorem of Pappus in n-Dimensions*, by Paul McDougle, University of Miami.
4. *On Holomorphs and Nakano's Characterization of Automoration in Groups*, by Frederick Hoffman, Florida Atlantic University.
5. *Walks In the Twilight Zone*, by H. W. Thwing, Stetson University.
6. *Teaching the Inverse of a Function*, by Alan Wayne, Cooper Union in New York, Retired.
7. *The Inequality* $e^{\pi} > \pi^{e}$, by J. G. MacCarthy, Hollywood, Florida.
8. *Endomorphisms of the Circle*, by Louis Block, University of Florida.
9. *A Simple Constructive Proof of the Jordan Matrix*, by Joseph Wilkerson and S. H. L. Kung, Jacksonville University.
10. *Euler's Pentagonal Recurrence Formula for* $\sigma(n)$ *In the Chiliagonal Case*, by J. G. MacCarthy, Hollywood, Florida.
11. *The Teaching of Mathematics in the Congo (Zaire)*, by D. M. Hill, Florida A & M University.
12. *A General Education Course for Business Majors*, by Ignacio Bello, Hillsborough Community College.

The luncheon-business meeting was held Saturday, March 9, 1974. Chairman James Brooks presided at the meeting. Committee reports were presented and Professor Bill Rice of St. Petersburg Junior College was elected chairman-elect; Professor Elizabeth A. Magarian of Stetson University and A. M. Dutton of Florida Technological University were elected Vice-Chairmen.

F. L. CLEAVER, *Secretary*

MARCH MEETING OF THE MISSOURI SECTION

The annual Spring meeting of the Missouri Section was held at the University of Missouri-Rolla on March 29 and 30, 1974; 90 members and 10 students registered.

The officers for the next year are: Chairman, Jerry Wilkerson, Missouri Western College; Vice Chairman, Jim Downing, Southwest Missouri State University; Secretary-Treasurer, Keith Stumpff, Central Missouri State University; Past Chairman, Edward Andalafte, University of Missouri-St. Louis.

The following presentations were made at the Friday session:

1. *On difference equations*, by Charles Hatfield, University of Missouri-Rolla.
2. *One answer to the challenge of the open door policy*, by Frances S. Mangan, Meramec Community College.
3. *Mathematical modeling*, by Carolyn T. MacDonald, University of Missouri-Kansas City.

4. *Convergence, summability, and applications,* (invited address) by S. M. Shah, University of Kentucky.

5. *Employment prospects in mathematics,* (invited banquet address) by John Jewett, Oklahoma State University.

The Saturday session consisted of the Business Meeting and the following presentations:

1. *Study of small sample estimators,* by S. K. Katti, University of Missouri-Columbia.
2. *A simple proof of Cauchy's group theorem and applications,* by R. Friedlander, University of Missouri-St. Louis.
3. *Innovative ways of teaching undergraduate mathematics,* (invited address) by Alex Rosenberg, Cornell University, Editor of the MONTHLY.

T. L. HICKS, *Secretary-Treasurer*

APRIL MEETING OF THE MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA SECTION

The annual Spring meeting of the Maryland-District of Columbia-Virginia Section of the MAA was held April 27, 1974, at George Washington University, Washington, D. C. One hundred seventeen persons attended of whom ninety-four were members of the Association. Professor Geraldine Coon, chairman of the Section, presided.

During the morning, there was a session of contributed papers, a short business meeting, and a second session of contributed papers. In the business meeting, the following officers were elected: Chairman-Elect, Professor Ronald Davis, Northern Virginia Community College; Vice-Chairman for Programs, Professor Kenneth Berg, University of Maryland; Treasurer, Professor Hewitt Kenyon, George Washington University.

Following lunch, the invited speaker, Dr. C. R. Johnson of the Applied Mathematics Division, National Bureau of Standards, spoke on "Solving Equations Exactly." The invited talk was followed by a third session of contributed papers.

The contributed papers presented were:

1. *Rings and Catalan numbers,* by L. W. Shapiro, Howard University.
2. *The implications of short-term memory research for the learning of mathematics,* by H. B. Tunis, University of Maryland.
3. *Applications of non-Euclidean geometry to physics,* by Hanna Nekvasil, Washington, D. C.
4. *Covering a square by equal circles,* by Michael Goldberg, Washington, D. C.
5. *Invariance of the essential spectra of ordinary differential operators,* by T. W. Prevatt, The Johns Hopkins University.
6. *A simple set — theoretic characterization for proximity spaces,* by R. A. Herrmann, U. S. Naval Academy.
7. *Working with large scale transportation and communications networks in the computer,* by Judith F. Gilsinn, National Bureau of Standards.
8. *Tests of randomness of Maryland Lottery numbers,* by B. A. Knoppers, Mathematica, Inc.
9. *Oscillation matrices and partial differential systems,* by L. A. Kurtz, Hollins College.
10. *The holor representation of raw materials measures,* by Vivian E. Spencer, University of Connecticut.

11. *The four-color theorem for small maps*, by Walter Stromquist, Department of the Treasury.
12. *The multiplication of determinants*, by D. C. Lewis, Jr., The Johns Hopkins University.
13. *The algebraic structure of network problems*, by D. R. Shier, National Bureau of Standards.
14. *Dense groups of primes*, by A. L. Brown, Alexandria, Virginia.
15. *How many non-Euclidean geometries are there?*, by C. J. Maloney, Bethesda, Maryland.

J. M. SMITH, *Secretary*

APRIL MEETING OF THE NEBRASKA SECTION

The fiftieth annual meeting of the Nebraska Section of the MAA was held on April 19–20, 1974, at the University of South Dakota at Vermillion. There were 70 persons present of whom 33 were members of the Association.

Officers for 1974–75 were elected as follows: Chairman, Professor Gerald Johnson, University of Nebraska-Lincoln; Chairman Elect, Professor L. M. Larsen, Kearney State College; Secretary-Treasurer, Professor H. M. Cox, University of Nebraska-Lincoln. Professor Dorothy Bernstein represented the MAA and brought greetings to the group. Professor Larry Stephens was selected to succeed Professor Charles Warden as Chairman of the Nebraska-South Dakota High School Contest Committee.

The following papers were presented:

1. *The Mathematician in Air Force Civil Engineering*, by Colonel W. A. Orth, Offut Air Force Base.
2. *Solution of an RLC Series Circuit Using Stieltjes Mean Sigma Integral Represented Linear Operators*, by K. P. Smith, University of Nebraska at Omaha.
3. *A Conjecture on Isotone Functions of a Finite Partially Ordered Set*, by M. C. Thornton, University of Nebraska — Lincoln.
4. *The Relationship of Subclasses to Finite Group Representation Theory*, by John Karlof, University of Nebraska — Omaha.
5. *The Buffon Needle Problem*, by David Ballew, South Dakota School of Mines and Technology.
6. (Invited Address) *How to Make and Break Codes*, by Dorothy Bernstein, Goucher College.
7. *The Basic Subset of an Irreducibly Connected Space*, by Edwin Halfar, University of Nebraska — Lincoln.
8. *Existence and Uniqueness of Solutions of Boundary Value Problems*, by Dwight Sukup, University of Nebraska — Lincoln.
9. *Truncating Terms of Power Series*, by Dale Mesner, University of Nebraska — Lincoln.
10. *An Epsilon of History Makes the Calculus Go Down*, by Alexander Mehaffey, Jr., University of South Dakota.
11. *Recursion Formulas for the Problem of Montmort*, by Emil Knapp, Augustana College.
12. *Integer Representation Using Recursion Sequences*, by Leslie Miller, University of South Dakota.
13. *Dynamic Programming and Scheduling of Power Generators*, by John Powell, University of South Dakota.
14. *Applying Data Structures to Information Processing Systems*, by Richard LaRue and Robert Wood, University of South Dakota.
15. *Report on Nebraska-South Dakota Mathematics Contest*, by Charles Warden, University of Nebraska — Omaha.

16. *Relative Efficiency of Different Estimates of the Mean of a Rectangular Population*, by Wayne Gutzman, University of South Dakota.

17. *The Twenty-Fifth Annual High School Mathematics Examination*, by H. M. Cox, Executive Director, University of Nebraska (by title).

18. (Panel Discussion) *Why Johnny Can't Add*, Dorothy Bernstein, Panel Chairwoman.

H. M. Cox, *Secretary*

APRIL MEETING OF THE NORTH CENTRAL SECTION

The Spring meeting of the North Central Section of the MAA was held at Macalester College, St, Paul, on April 26–27, 1974. On Friday evening Professor Dale Varberg, Hamline University, spoke on the topic of "Convexity" (as related to functions).

The invited speaker for the Saturday morning session was Professor Harley Flanders, retiring editor of the MONTHLY, whose topic was "Some Mathematical Aspects of Editing the Monthly." Other papers presented at that session were:

1. *Cos nx and some related sequences*, by Gerald Bergum, Sr., South Dakota State University.
2. *On the probability that the k-th alternate delegate will be seated*, by Peter Treuenfels, Honeywell.
3. *The queen of the sciences and the king of games*, by Loren C. Larson, St. Olaf College.
4. *Exact interim reserves vs. interpolated values*, by F. C. Smith, Stennes and Associates.

Invited speakers for the Saturday afternoon session were Professor Sabra S. Anderson, University of Minnesota, Duluth, and Professor Seymour Schuster, Carleton College, who each gave 30-minute presentations: Professor Anderson on the topic "Finite Projective Planes," and Professor Schuster on the topic "Generalized Ramsey Theory." Other presentations at the afternoon session were:

1. $\sum_{n=1}^{\infty} n^n/n!e^n$, *convergent or divergent*, by Arthur Lindberg, Mankato State College.
2. *Three films produced at Carleton College NSF Summer Institute* — R. B. Kirchner, Director: Non-Uniform Convergence and Vibrating String, George Abdo and Jerry Caldwell, $W=Z$ Exp (2), Michael Collins.

The following officers were elected to serve for the coming year: Chairman, Professor Sylvan Burgstahler, University of Minnesota, Duluth; Chairman-elect, Professor K. F. Carlson, St. Cloud State College; Secretary-Treasurer, Professor Louis Guillou, St. Mary's College; Members-at-large, Professor Elaine D. Nelson, Bemidji State College, and Professor Jayaseetha Premanand, Normandale Community College.

H. M. ANDERSON, *Secretary-Treasurer*

APRIL MEETING OF THE SOUTHWESTERN SECTION

The annual meeting of the Southwestern Section of the MAA was held at New Mexico State University, Las Cruces, New Mexico, on April 5–6, 1974. Forty-nine persons registered their attendance.

The invited speaker was Professor R. D. Anderson of Louisiana State University. He gave a talk entitled "Where The Jobs Are" at the banquet on Friday night and also spoke on Saturday morning on "Intuition in Infinite Dimensional Topology".

On Saturday morning there was a panel discussion on the topic "Should The Southern Section be Split?" Panelists were E. L. Walter, Northern Arizona University, and James Nyman, University of Texas at El Paso. Their remarks were followed by an open discussion on the subject and the final consensus seemed to be that the section should not be split.

The following contributed papers were presented:

1. *A New Look at Principal Ideal Domains*, by Fred Richman, New Mexico State University.
2. *The Generalized Riemann Integral*, by Robert McLeod, Kenyon College and New Mexico State University.
3. *Calculus Made Easier With Topology*, by Richard Woodcock, New Mexico Institute of Mining and Technology.
4. *Generalised Inverses for Infinite Dimensional Matrices*, by G. S. Rogers, New Mexico State University.
5. *Moments of Log Rayleigh Distributions*, by W. L. Shepherd, White Sands Missile Range.
6. *A Theorem of Kronecker — Factoring Polynomials*, by Ray Mines, New Mexico State University.
7. *Groups and Fields in Z_n*, by James Nyman, University of Texas at El Paso.
8. *Another Look at Hardy's Inequality for Series*, by Richard Bagby, New Mexico State University.
9. *A Radical Table of Sines*, by R. A. Knoebel, New Mexico State University.
10. *Curvature Tensors and Cross Ratios*, by A. Swimmer, Arizona State University.

The chairmen of the contributed paper sessions were: Charles Swartz, New Mexico State University, and Carl Hall, University of Texas at El Paso.

A. SWIMMER, *Secretary-Treasurer*

1974 CONTRIBUTING MEMBERS AND SPECIAL GIFTS

The Association expresses its deep appreciation to 181 of its members who have elected to be Contributing Members, Sponsors or Patrons for 1974 by making contributions beyond the normal dues. There are 162 Contributing Members in 1974. In addition, the following members are 1974 Sponsors and Patrons:

Patrons

Nathaniel Chafee
Richard D. Edwards
Leonard Gillman
W. S. Lewis
D. D. Miller
H. J. Ryser
One anonymous Patron

Patron Life Member

G. B. Thomas, Jr.

Sponsors

R. H. Bing
W. H. Bluhm
Herbert Carus
A. M. Gleason
W. E. Hartnett
Bill Hassinger, Jr.
W. S. Loud
Alex Rosenberg
Maria W. Steinberg
E. F. Wilde
R. E. Zink

The Association also acknowledges with deepest gratitude a gift of $2000 received from Margaret Gehman Dodge in memory of her mother Marian Gehman.

CALENDAR OF FUTURE MEETINGS

Fifty-eighth Annual Meeting, Washington, D. C., January 25–27, 1975.

Fifty-fifth Summer Meeting, Western Michigan University, Kalamazoo, August 18–20, 1975.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Editorial Director.

ALLEGHENY MOUNTAIN, Duquesne University, Pittsburgh, Pennsylvania, April 25–26, 1975.

FLORIDA, Manatee Junior College, Bradenton, March 7–8, 1975.

ILLINOIS, Rockford College, Rockford, May 9–10, 1975.

INDIANA, Indiana University — Purdue University at Indianapolis, November 30, 1974.

IOWA, Iowa State University, Ames, April 18–19, 1975.

KANSAS

KENTUCKY

LOUISIANA-MISSISSIPPI, Centenary College, Shreveport, Louisiana, February 1975.

MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA

METROPOLITAN NEW YORK

MICHIGAN

MISSOURI, Missouri Western College, St. Joseph, Spring 1975.

NEBRASKA, Nebraska Wesleyan University, Lincoln, April 18–19, 1975.

NEW JERSEY

NORTH CENTRAL, Hamline University, St. Paul, Minnesota, April 28, 1975.

NORTHEASTERN, Lowell Technological Institute, Lowell, Massachusetts, November 30, 1974.

NORTHERN CALIFORNIA, Menlo College, Menlo Park, February 8, 1975.

OHIO, University of Cincinnati, November 1–2, 1974.

OKLAHOMA-ARKANSAS, Central State University, Edmond, Oklahoma, April 4–5, 1975.

PACIFIC NORTHWEST

PHILADELPHIA, Swarthmore College, Swarthmore, Pennsylvania, November 23, 1974.

ROCKY MOUNTAIN, Mesa College, Grand Junction, Colorado, April 11–12, 1975.

SEAWAY, St. John Fisher College, Rochester, New York, November 1–2, 1974.

SOUTHEASTERN, University of South Alabama, Mobile, March 21–22, 1975.

SOUTHERN CALIFORNIA

SOUTHWESTERN

TEXAS, Angelo State University, San Angelo, April 1975.

WISCONSIN, University of Wisconsin — Superior, April or May 1975.

FUTURE MEETINGS OF OTHER ORGANIZATIONS

AMERICAN ASSOCIATION FOR THE ADVANCEMENT OF SCIENCE

AMERICAN MATHEMATICAL SOCIETY, Washington, D. C., January 23–26, 1975.

AMERICAN SOCIETY FOR ENGINEERING EDUCATION, Colorado State University, Fort Collins, June 16–19, 1975.

ASSOCIATION FOR COMPUTING MACHINERY, San Diego, California, November 11–13, 1974.

ASSOCIATION FOR SYMBOLIC LOGIC, Shoreham Hotel, Washington, D. C., January 23–24, 1975.

ASSOCIATION FOR WOMEN IN MATHEMATICS, Washington, D. C., January 24, 1975.

FIBONACCI ASSOCIATION

INSTITUTE OF MATHEMATICAL STATISTICS

MU ALPHA THETA

NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS, Washington, D. C., January 25–26, 1975 (joint meeting with MAA).

OPERATIONS RESEARCH SOCIETY OF AMERICA, Chicago, April 30–May 2, 1975.

PI MU EPSILON, Western Michigan University, Kalamazoo, August 19–20, 1975.

SCHOOL SCIENCE AND MATHEMATICS ASSOCIATION, Sheraton-Gibson Hotel, Cincinnati, Ohio, November 7–9, 1974.

SOCIETY FOR INDUSTRIAL AND APPLIED MATHEMATICS

THE CARUS MATHEMATICAL MONOGRAPHS

The Monographs are a series of expository books intended to make topics in pure and applied mathematics accessible to teachers and students of mathematics and also to non-specialists and scientific workers in other fields.

These numbers are currently available:

1. *Calculus of Variations,* by G. A. Bliss.
2. *Analytic Functions of a Complex Variable,* by D. R. Curtiss.
3. *Mathematical Statistics,* by H. L. Rietz.
4. *Projective Geometry,* by J. W. Young.
6. *Fourier Series and Orthogonal Polynomials,* by Dunham Jackson.
7. *Vectors and Matrices,* by C. C. MacDuffee.
8. *Rings and Ideals,* by N. H. McCoy.
9. *The Theory of Algebraic Numbers,* by Harry Pollard.
10. *The Arithmetic Theory of Quadratic Forms,* by B. W. Jones.
11. *Irrational Numbers,* by Ivan Niven.
12. *Statistical Independence in Probability, Analysis and Number Theory,* by Mark Kac.
13. *A Primer of Real Functions* (Second edition), by Ralph P. Boas, Jr.
14. *Combinatorial Mathematics,* by H. J. Ryser.
15. *Noncommutative Rings,* by I. N. Herstein.
16. *Dedekind Sums,* by Hans Rademacher and Emil Grosswald.
17. *The Schwarz Function and its Applications,* by Philip J. Davis.

One copy of each Carus Monograph may be purchased by individual members of the Association for $4.00 each; additional copies and copies for nonmembers are priced at $8.00 each.

Orders with remittance should be sent to:

MATHEMATICAL ASSOCIATION OF AMERICA
1225 Connecticut Avenue, N.W.
Washington, D.C. 20036

THE AMERICAN MATHEMATICAL MONTHLY

(FOUNDED IN 1894 BY BENJAMIN F. FINKEL)

THE OFFICIAL JOURNAL OF

THE MATHEMATICAL ASSOCIATION OF AMERICA

VOLUME 81 NUMBER 9

CODEN: AMMYAE

CONTENTS

(Continued on inside cover)

NOVEMBER **1974**

NOTICE TO AUTHORS

Research papers per se are unsuitable; see statement of policy (Vol. 81, p. 1).

Please follow the format in current issues of the MONTHLY. Manuscripts must be legibly typewritten or reproduced from typewritten copy, double spaced with wide margins. Three copies should be submitted to the appropriate editor and one kept by the author as protection against loss. The author's full address *must* appear at the end of the manuscript.

Backlog: Main Articles 15 months, Math. Notes 19 months, Research Problems 11 months, Classroom Notes 23 months, Math. Education 20 months.

EDITORIAL CORRESPONDENCE AND MAIN ARTICLES: to ALEX ROSENBERG, Department of Mathematics, Cornell University, Ithaca, N.Y. 14853; NOTES, etc.: to the corresponding Associate Editor; REPRINT PERMISSION: to LEONARD GILLMAN, Mathematical Association of America, University of Texas, Austin, Texas 78712 (see also the copyright notice below); ADVERTISING CORRESPONDENCE: to RAOUL HAILPERN, Mathematical Association of America, SUNY at Buffalo, Buffalo, N. Y. 14214; CHANGE OF ADDRESS and SUBSCRIPTIONS: to A. B. WILLCOX, Mathematical Association of America, 1225 Connecticut Ave., N.W., Washington, D.C. 20036.

Annual dues for members of the Association (including a subscription to the American Mathematical Monthly) are $14.00 for each of the first two years of membership and $18.00 thereafter. Student Membership is available with annual dues of $9.00. For nonmembers the subscription price is $18.00.

PUBLISHED BY THE ASSOCIATION at Washington, D. C., and Menasha, Wisconsin, during the months of January, February, March, April, May, June-July, August-September, October, November, December.

Second-class postage paid at Washington, D. C., and additional mailing offices.

PRINTED IN THE UNITED STATES OF AMERICA

PLATEAU'S PROBLEMS AND THEIR MODERN RAMIFICATIONS

JOHANNES C. C. NITSCHE*

Today's special session within the National Meeting of the American Chemical Society is held in honor of Joseph-Antoine-Ferdinand Plateau whose magnum opus, *Statique expérimentale et théorique des liquides soumis aux seules forces moléculaires*, was published a century ago, in 1873. This book, in the main unifying the contents of a series of eleven long scientific papers which appeared during the period 1843–1869, may be described as the condensation of Plateau's experimental observations and measurements regarding the phenomena caused by capillary forces. It contains a complete historical account and a summary of theoretical insights as well. The book's impact on its contemporaries and on subsequent generations cannot be exaggerated. His reports were studied in many quarters. His experiments were repeated by many scientists, among them Faraday, Brewster and Boys. In present days Plateau's work has become rather lost in obscurity. Surface chemists who, with a few notable exceptions, now often shun the use of mathematical approaches, are generally content with a brief reference to Plateau in their text books, and it is really in mathematics where his name lives on.

1. Liquids in contact with their own vapors or air possess a surface tension which, unless opposed by external forces—for example, gravitation, centrifugal forces, the influence of an electric field, etc.—, causes the interface to assume the configuration of minimum area. The laws which govern the behavior of surfaces separating one medium from another clearly belong in the domain of physicists and chemists. Once these laws are conceived and formulated, however, it is not hard to understand why the investigation of the shape and the stability of such interfaces, which are, after all, geometrical objects subject to variational principles, differential equations and various boundary conditions, has over the years stimulated mathematical activities of considerable consequences. Some of the most beautiful purely mathematical developments and, more than incidentally, the mobilization of results from a variety of mathematical fields can be traced back to the study of capillary phenomena and also to the influence of Plateau. The theory of minimal surfaces, the study of surfaces of prescribed mean curvature, as well as the whole complex of questions which mathematicians describe as Plateau's problem, are prominent, but by far not the only examples. Mathematics, in turn, has served its users well although even now substantial problems remain unsolved, posing a challenge which guarantees further fruitful interaction. Without doubt, here is a case where a mathematician need not be ashamed of a lack of "relevance", a term menacingly applied by

* Address delivered at the American Chemical Society National Meeting in Dallas, April 10, 1973. The speaker is Professor of Mathematics and Head, School of Mathematics, University of Minnesota.

Joseph-Antoine-Ferdinand Plateau

the (often self-appointed and often misunderstanding) custodians of today's society. It is well known, and helpful beyond the realm of frivolity, that portions of the surfaces under consideration, at least as long as these portions remain in stable equilibrium, can often be experimentally realized with soap films or soap bubbles bounded by suitable fixed, or movable, frames. From time immemorial such experiments have been recorded in many sources and have even been depicted in paintings (*e.g.*, by Murillo, v. Oost, Chardin, Hamilton, Spitzweg, Manet). Leonardo da Vinci already knew about capillary phenomena; subsequent attempts aimed at their explanation and at their utilization have occupied great minds ever since. Capillary action is at work in the processes of wetting, displacing, dyeing, coalescing, flotation, emulsification—all of utmost significance in nature and technology. It seems unnecessary to elaborate further on the importance of our subject in a city like Dallas which lies in the heartland of the oil industry, whose research efforts must include the mastery of surface forces. A decrease of the surface tension between water and oil by several orders of magnitude, which is now possible through the injection of newly discovered chemicals, will in fact extend the useful life of the Texas oil fields by several decades.

The organizers of this session, my colleague Dr. Scriven from the University of Minnesota and Dr. Melrose from the Mobile Research and Development Corporation, both distinguished experts in the field and enthusiastic connoisseurs of Plateau, have asked me, a mathematician by profession, to speak about Plateau, to elucidate the influence of his work on mathematical thought and to pursue, in anecdotal form or in mathematical guise, whatever would seem most suitable, a few of the problems raised to their modern habitat in mathematics. Appreciating a mathematician's shortcomings, particularly when communication with scientists in other disciplines is at stake, and charged with—here I quote—"being as intriguing and as off-beat as possible," I am aware of the difficulties of my task. Quite recently I have completed a voluminous monograph on the theory of minimal surfaces which will soon appear as number 199 in the "yellow series" of the Springer Publishing Company. I have thus had many occasions to look into Plateau's treatise. Only in the last weeks, however, as in preparation for this lecture I went through the book again more systematically, did I come to fully appreciate the wealth of the material treated and the complexity of the experiments which Plateau and his disciples conducted with remarkable skill and perception. This experience in conjunction with the recognition of the abundance of problems initiated and of the intricacies inherent to mathematical techniques, which in analysis unfortunately often tend to distract from the essence of a problem and thus mar its conceptual simplicity, made me uncertain to the point that I would have preferred not even to attempt to do justice to our hero and to relate instead some of the interesting particulars of his rather colorful life. Realizing the inappropriateness of such a course, but mindful of the limited time available, I decided to proceed as follows. After briefly reminiscing about Plateau's life and about his general activities, I shall select a few specific

topics from his writings, five or six at most, whose substance will then be explained and whose fate will be discussed. Each case will be concluded with the statement of open problems. It must be kept in mind that the problems chosen are nothing but a small sample. An observation which to me came as a surprise may already be mentioned here. Originally it had been my expectation that my main objective should consist first of my pointing to the forces of mathematical generalization and then of the demonstration that, with suitable safeguards, generalization may lead better than anything to the heart of a mathematical idea and, at the same time, to the definitive solution of a problem. This, of course, is one of the strengths and virtues of the mathematical process, and we shall see it here at work in the molding of the concept of a surface. It turned out, however, that another, equally important, part of Plateau's discussions leads to questions, among them hard problems of geometry and analysis (often referred to as problems of hard analysis), whose formulation today is as classical as it was a hundred years ago and for whose solution new insights are mandatory while a generalization of concepts appears altogether useless. References to the literature will be suppressed*, and names will be mentioned only sparingly. Although our subject, more than other mathematical fields, invites, and greatly gains by, experimental illustrations, I shall be forced to keep such illustrations to a minimum. The few models which I have brought along, pretty as they are, can be looked at outside of my lecture.

2. Plateau was born in Brussels in 1801 and died in 1883, an 82-year-old man. From his father he had inherited artistic skills. Already in elementary school he displayed a lively interest in physical experiments and in mechanical contrivances as well as a love of nature. The boy's chasing of butterflies led to a fine collection of these insects later in life. The misfortune of his losing in quick succession first his mother and then his father and becoming an orphan at age fourteen, made Joseph severely ill. A stay in the country, in a little village near Waterloo, appeared to be

---

*) The reader interested in references may consult the following sources:

On capillarity and stability: H. Minkowski: Kapillarität. Enc. d. math. Wiss. 5.1.9; Teubner, 1903–21. — N. K. Adam: The physics and chemistry of surfaces. 3rd ed., Oxford Univ. Press, 1941. — K. L. Wolf: Physik und Chemie der Grenzflächen. Springer, Vol. 1, 1957; Vol. 2, 1959. — Further also J. C. Maxwell's article "Capillary action" in the 11th edition (Vol. 5, pp. 256–275) and A. W. Porter's article "Surface tension" in recent editions of the Encyclopaedia Britannica.

On minimal surfaces: R. Courant: Dirichlet's principle, conformal mapping, and minimal surfaces. Interscience, 1950. — J. C. C. Nitsche: Vorlesungen über Minimalflächen. Springer, 1974.

On geometric measure theory: F. J. Almgren: Plateau's problem; an inviation to varifold geometry. Benjamin, 1966. — H. Federer: Geometric measure theory. Springer, 1969. — W. K. Allard: On the first variation of a varifold. Ann. of Math. (2) **95** (1972), 417–491.

On the calculus of variations and Morse theory: O. Bolza: Vorlesungen über Variationsrechnung. Teubner, 1909. — M. Morse and C. Tomkins: The existence of minimal surfaces of general critical types. Ann. of Math. (2) **40** (1939) 443–472; (2) **42** (1941), 331. — M. Shiffman: The Plateau problem for non-relative minima. Ann. of Math. (2) **40** (1939), 834–854. — C. B. Morrey: Multiple integrals in the calculus of variations. Springer, 1966.

beneficial for him in the eyes of his uncle, a lawyer, who became his guardian. By an unforeseen coincidence the journey of Plateau and his two younger sisters fell on the eve of the battle of Waterloo. Together with other villagers they hurried to hide in the woods around Soignies. Young Plateau remained oblivious to the fearful events of those days and nights. While bombardments rumbled all around he enjoyed himself eating country-fried potatoes and catching butterflies. Recovery followed. Upon his return to Brussels his schooling continued with best results. Again there are reports about his experimental skills and interests. In the Atheneum, Plateau impressed Quetelet, a renowned scientist and later secretary of the Royal Academy of Brussels over four decades (1834–1874), and became one of Quetelet's protégés for life. In 1822 Plateau entered the University at Liège. His uncle, full of admiration for the profession of the devoted barrister who defends widows and orphans, would have liked to see him become a lawyer but eventually agreed to let him study natural sciences and mathematics and to cultivate the art of observation. Material needs, among them Plateau's responsibility for his sister, Joséphine, led him to accept a teaching position first in Liège, where he received his doctor's diploma in 1829, and later in Brussels.

Plateau's first scientific interests concerned physiological optics, particularly the study of the sensations produced in the human eye by fixed or moving light sources, a subject in which he made substantial discoveries and which, as a matter of record, led him to the invention of various optical toys. A special experiment, without doubt related to these interests, caused him once (in 1829) to view the sun with his naked eyes for longer than 25 seconds. This proved to be a fateful incident. Painful treatments could not prevent his irrevocable blindness in 1843, which his biographer considers to be a direct consequence of the 1829 experiment. (An ophthalmologist has advised me, however, that the exposure to the sun alone could not have been the cause of a total loss of sight so many years later.) In the meantime Plateau had gotten married. In this connection the following little incident is reported. Visiting Paris on his honeymoon, he seized the opportunity to look up a few of the French professors with whom he had cultivated scientific contacts. As he stayed away from his hotel far longer than anticipated, his wife got extremely worried. Finally, upon his return he admitted to having forgotten that he was married. His loss of sight, of course, opened depressing perspectives. Fortunately, in 1844 Plateau was named ordinary (full) professor, and a royal edict soon to follow relieved him of his teaching duties and of all material worries. Having thus become what today would be called a research professor he could devote all his efforts to his work which now had begun to concentrate on a systematic study of capillary phenomena, an endeavor which culminated in his celebrated book. Plateau's accomplishments and his perseverance in performing delicate experiments under so adverse circumstances deserve our admiration. To be sure, he enjoyed the assistance of members of his family and of dedicated pupils, substantial scholars and later professors themselves, among them Lamarle and Van der Mensbrugghe. (The latter became his

son-in-law in 1871.) Whether the relationship between Plateau and his selfless younger helpers was always untroubled, whether his collaborators ever pursued their joint investigations while gritting their teeth or feeling impeded in their independence and originality, these are questions worth asking.

Much has been passed on concerning Plateau's later years, his family life, and his working and dictating habits. Illuminating as these things might be, they would lead us too far. Thus, before proceeding to the next part of my lecture let me merely state in a summary fashion that Plateau's total scientific activities did cover rather diverse areas, not all equal in importance, and let me mention two instances. According to a popular superstition of the time, the prophet Mohammed's tomb was said to be held in mid-air by the action of strong magnets. Plateau took on this myth and conclusively proved the impossibility of a stable equilibrium for the tomb in any arrangement. For his argument it is essential that the distance between elements appears with exponent -2 in the laws of electromagnetism. Were one to substitute another exponent in these laws, the situation would be different. Experiments were conducted to illustrate this.

The following observation belongs to the realm of mathematical recreation. Given any odd number q, not divisible by 5, say $q = 7$, and any integer d between 1 and 9, say $d = 9$. Then a digital number formed by a suitable repetition of d, that is, a number of the form $ddd \cdots d$, is divisible by q. In our case we find that $999999 = 7 \cdot 142857$. In other words: For a correctly chosen natural number m, here $m = 6$, the number $10^m - 1$ is divisible by q.

3. The normal pressure on an interface, due to the existence of surface tension is, per unit area, equal to

$$p = \sigma\left(\frac{1}{R_1} + \frac{1}{R_2}\right), \quad \sigma = \text{surface (or interfacial) tension.}$$

Thus the conditions of equilibrium for the boundary surface of a liquid which is free from the influence of gravity are expressed by the equation

$$H \equiv \frac{1}{2}\left(\frac{1}{R_1} + \frac{1}{R_2}\right) = \text{const.}$$

Here R_1 and R_2 are the principal curvature radii of the surface and the quantity H is called its mean curvature. If the surface has a representation of the form $z = f(x, y)$, then we have

$$H = \frac{(1 + f_y^2)f_{xx} - 2f_x f_y f_{xy} + (1 + f_x^2)f_{yy}}{2[1 + f_x^2 + f_y^2]^{3/2}}.$$

In Figure 1 various types of surface curvature are illustrated.

Plateau's investigations of the effects of capillary forces may well be viewed as grand variations on the theme $H = \text{const}$, which, in the framework of analysis,

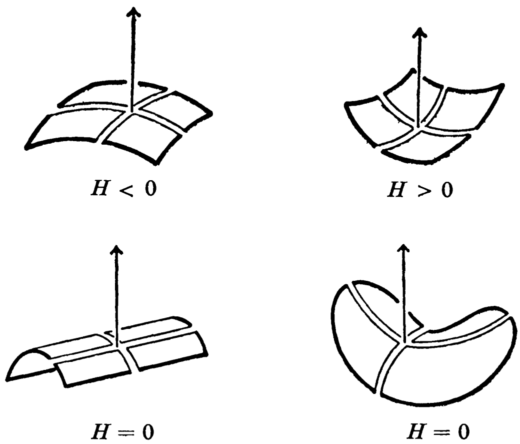


FIG. 1

turns out to be a non-linear second order elliptic partial differential equation, often called the Laplace-Young equation by surface chemists. Taking up a suggestion made by Segner in 1751, Plateau got rid of the effects of gravity in the following way. He formed a mixture of alcohol and water of precisely the same density as olive oil and then introduced a quantity of oil into the mixture. Under the action of surface tension alone the portion of oil assumes the form of a sphere, never any other form. This fact seems to substantiate the mathematical theorem that a closed surface of constant mean curvature must be a sphere. Plateau was well aware of the difficulties connected with this proposition. Only much later, in 1900, Liebmann proved the theorem that every *convex* surface of constant mean curvature is a sphere. A more general earlier attempt of Jellet, who already in 1853 considered starshaped surfaces, seems to have been overlooked by the mathematicians. Neither theorem, of course, answers the question about the shape of a general closed surface of constant mean curvature. Aside from the sphere (of genus $g = 0$) could there be such ring-type surfaces (of genus $g = 1$), or pretzel-type surfaces (of genus $g = 2$) etc.? See Figure 2.

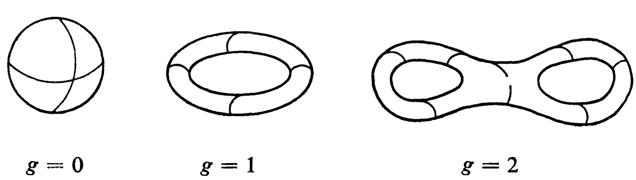


FIG. 2

In physical terms: Are there bodies, different from spheres, in equilibrium under the sole influence of surface forces? Experimental evidence seems to negate this

question. At this point a distinction has to be brought out between physical and mathematical surfaces. A physical surface appears as boundary of a domain, i.e., as the interface separating a quantity of matter from its outside, and thus obviously cannot possess self-intersections. Physical surfaces are also called simple. A mathematical surface, on the other hand, may well intersect itself. The difference is illustrated in Figure 3.

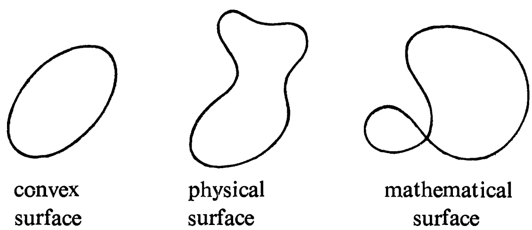


FIG. 3

In 1951, H. Hopf proved that a closed (mathematical) surface of genus zero, i.e., a sphere-like surface, having constant mean curvature must in fact be a sphere. A few years later the Russian mathematician A. D. Alexandrov, using ingenious arguments which were often copied in later years, demonstrated that any physical surface of constant mean curvature and of *arbitrary* genus by necessity is a sphere. It takes Alexandrov's proof to answer the physical question. The general question, however, continues to remain open: It is not known today whether there are closed surfaces of constant mean curvature other than the sphere. Here we have one of the outstanding problems of global differential geometry. Most mathematicians do not even try any more to answer it.

Instead of characterizing our surfaces by the constancy of their mean curvature we could have, as we did at the outset of this lecture, described them as surfaces of minimum area. There is indeed a close connection with the isoperimetric problem. In its classical form this problem calls for the determination of a region of prescribed volume whose boundary has minimal surface area. It can be shown that a closed surface has constant mean curvature if, and only if, its area is stationary with respect to volume-preserving variations. For non-simple closed surfaces it is, of course, first necessary to generalize the notion of volume. What has to be done can best be explained on the example of a closed curve in the plane and the area enclosed by it. For an oriented closed curve $\mathscr{C}$ we define the order of a point p not situated on $\mathscr{C}$ as the algebraic number of times $\mathscr{C}$ winds around p. For each point in a connected component R of the complement of $\mathscr{C}$ the order is the same integer, as indicated in Figure 4.

Denoting the various components by R_i, their areas by $|R_i|$ and the corresponding orders by d_i, the area enclosed by $\mathscr{C}$ is now defined as the weighted sum $\sum d_i |R_i|$. The definition of the volume of a closed surface is now evident.

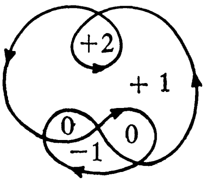


FIG. 4.

4. Leaving the case of equilibrium in the presence of surface forces only, we shall now briefly consider a situation in which external influences, say gravitational forces, are present. Then the mean curvature of the interface under consideration is no longer a constant but rather a (linear) function of the space variables. Typical examples are liquid drops, supported by or suspended from a horizontal plane, liquids in tubes and containers, bubbles, menisci, etc. See Figure 5. The liquid contacts the supporting

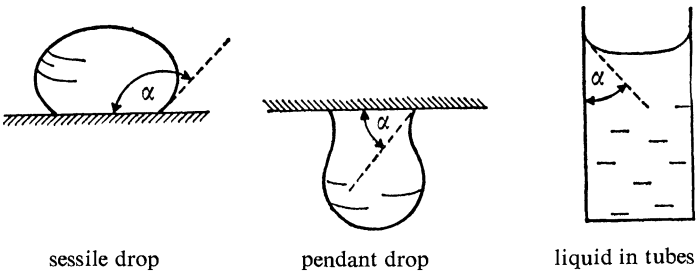


FIG. 5

boundary at a certain contact angle α. This angle varies from almost zero for liquids that "wet" the solid boundary to about 140° for mercury on glass. For fats on wool as an example we have $\alpha = 0$, incidentally, a problem for stain removers. Under ideal circumstances the contact angle, however elusive its actual experimental determination may be, can be considered as a material constant.

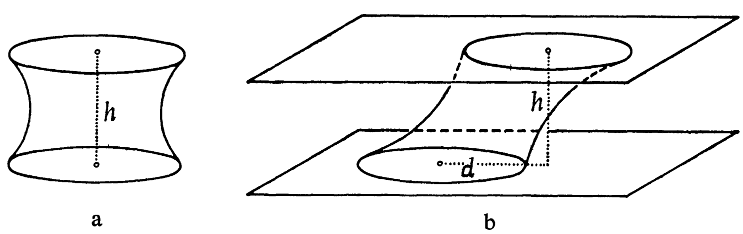
Let us look at the sessile drop. What is its shape? In a coordinate system with vertical z-axis (parallel to the direction of gravitational forces) the boundary of the drop satisfies the differential equation

$$H = az + b$$

in which a and b are certain constants. The solution of this harmless looking elliptic partial differential equation poses formidable difficulties, particularly if one, as he must, tries to satisfy the correct boundary conditions. Were we allowed to assume the drop to be rotationally symmetric the partial differential equation above could be transformed into an ordinary differential equation. Even in order to solve the latter, one has to take refuge in numerical computations. So far, however, it has never been demonstrated that the assumption of rotational symmetry is warranted. Such a

demonstration might be feasible with the help of a symmetrization process. Consider a sessile drop which is not axisymmetric. Replacing each of its horizontal cross sections by a circular slice of equal area in the same height and suitably shifting these slices, a new shape with rotational symmetry could be obtained. Its volume as well as the potential energy of its parts are the same while its surface area has been diminished. For a drop whose total potential (surface and gravitational) energy is already an absolute minimum this would not be possible. To complete the argument, however, one would have to convince oneself that the symmetrization could be performed in a way which would not change the contact angle between the drop and its supporting plane. Trying to avoid energy considerations and a discussion of the physical aspects altogether (large drops may be unstable and disintegrate, etc.), I shall formulate the problem in a purely geometrical form as follows: Let S be a (mathematical) surface of the type of the disk (a precise definition will be given later, in section 7) whose mean curvature at each of its points (x, y, z) is a linear function of the vertical coordinate z and whose boundary meets the plane $z = 0$ at a fixed angle $\alpha, 0 \leqq \alpha \leqq \pi$. Prove that S has rotational symmetry. The special case where the surface is assumed to have a non-parametric representation $z = f(x, y)$ and where $0 \leqq \alpha < \pi/2$ has been settled by J. Serrin.)

5. One of the most celebrated of Plateau's activities is of course connected with what is commonly called Plateau's problem. Imagine the following experiment. Take a frame fashioned of one or several thin wires and dip it into a (suitably prepared) soap solution. Upon withdrawal of the frame a soap film spanning the wires may develop. This film is extremely thin so that the influence of gravitation can be neglected and, although it is actually bounded by two surfaces, it presents the image of an ideal surface. Since the latter is generally open, i.e., encloses no volume, both of its sides are subject to the same pressure and its mean curvature must be zero everywhere. Surfaces of vanishing mean curvature are called minimal surfaces. Our experiment shows one way to realize them physically. As anybody who has ever tried his hand at such experiments knows, it is not every time that a soap film develops in the frame. There can be two different reasons for this failure: either a lack of skill on the part of the experimenter, or, more deeply seated, the mathematical fact that no minimal surface bounded by the curves of the frame exists at all. A classical example will illustrate



a b

FIG. 6

our point. Two coaxial unit circles in parallel planes will bound a ring-type minimal surface, in fact the well-known catenoid, as long as they are not too far apart; more precisely, as long as $h \leqq 1.325 \approx 4/3$. See Figure 6a. If we now move these circles away from each other, either vertically or sideways, there will arrive a moment when the minimal surface tears. It has become unstable. From this moment on our circles are not capable any more of bounding a minimal surface of the type of the circular annulus. That the catenoid disintegrates when the circles are moved apart vertically, a case of rotational symmetry in which the pertinent partial differential equation reduces to a much more amenable ordinary differential equation, has been known for more than a century. The limit value, $h_{\text{lim}} \approx 4/3$, had been determined experimentally by Plateau, and its exact value was computed by Lindelöf. The intuitively obvious fact that the minimal surface tears, however, the circles — or, for this matter, any two boundary contours — are moved apart has only recently been stated and studied by the speaker. For the situation illustrated in Figure 6b a detailed numerical table has been computed relating the limit values of h to the lateral distance d. A graph of this relationship is given in Figure 7. Note that the circles cannot bound a

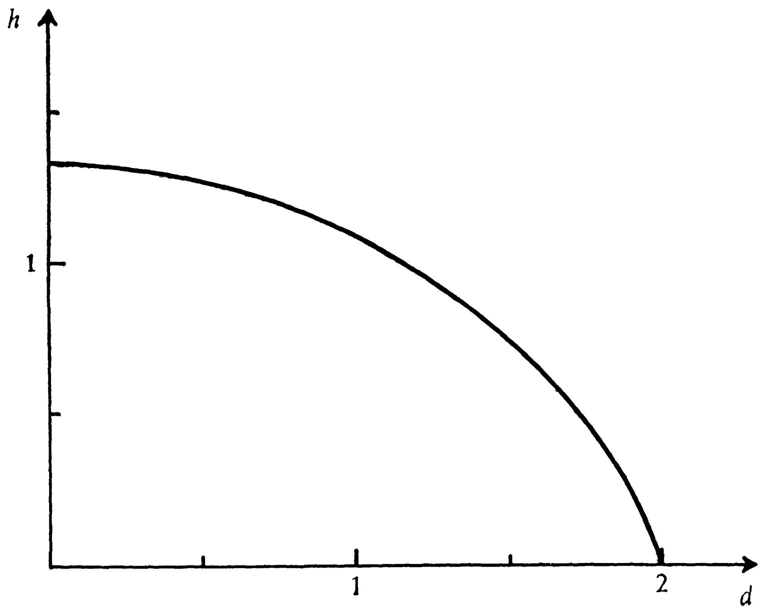


FIG. 7

ring-type minimal surface, no matter how close they are, if their projections do not overlap.

Having been alerted to the disquieting fact that boundary value problems for minimal surfaces may have no solutions at all, let us now mention a favorable case.

A single contour — mathematically speaking, a simple closed curve (or Jordan curve), no matter how bizarre, *always bounds a disk-type minimal surface.* Plateau arrived at this conclusion which he expressed in precise words, on the basis of his elaborate experiments. What he had formulated, of course, was in effect a mathematical statement: that a certain geometric boundary value problem always possesses a solution.

The precise mathematical formulation of "Plateau's problem" is as follows. Given a Jordan curve $\Gamma = \{\boldsymbol{x} = \boldsymbol{y}(\theta);\ 0 \leqq \theta \leqq 2\pi\}$ in Euclidean 3-space. To determine a vector $\boldsymbol{x}(u, v)$ defined in the closure $\bar{P}$ of the unit disk $P = \{u, v;\ u^2 + v^2 < 1\}$ such that

(i) $\boldsymbol{x}(u, v) \in C^2(P) \cap C^0(\bar{P})$;

(ii) $\Delta \boldsymbol{x} = 0$, $\boldsymbol{x}_u^2 = \boldsymbol{x}_v^2$, $\boldsymbol{x}_u \boldsymbol{x}_v = 0$ in P;

(iii) $\boldsymbol{x}(u, v)$ maps the boundary ∂P onto Γ topologically.

Any surface $\{\boldsymbol{x} = \boldsymbol{x}(u,v);\ (u, v) \in \bar{P}\}$ whose position vector satisfies (i), (ii), (iii) is called a solution of Plateau's problem.

If one considers the efforts (and failures) of the leading geometers in providing a rigorous existence proof, efforts which bore fruits only in 1930 through the pioneering work of Douglas and of Radó, one will appreciate the importance of Plateau's observation. It must be mentioned that for a special situation the problem had already been formulated by Lagrange in 1762 (the "birth date" of the theory of minimal surfaces) and that it had been brought again to the attention of the mathematicians by Gergonne in 1816. The only case completely settled for a long time was that of the minimal surface through the sides of a skew quadrilateral whose shape is depicted in Figure 8 and whose equations were derived by H. A. Schwarz in the explicit form

$$x = \operatorname{Re} \int^w (1 - w^2)R(w)dw \qquad w = u + iv$$

$$y = \operatorname{Re} \int^w i(1 + w^2)R(w)dw, \qquad R(w) = \frac{1}{\sqrt{1 - 14w^4 + w^8}}$$

$$z = \operatorname{Re} \int^w 2wR(w)dw.$$

Note that the function $R(w)$ has singularities at the points $w = \pm(\sqrt{3} \pm 1)/\sqrt{2}$, $\pm i(\sqrt{3} \pm 1)/\sqrt{2}$. (The detailed study of the three hyperelliptic integrals and their periodicity properties occupies half of the first volume of Schwarz's *Collected Works.*)

The difficulties in the case of more general, even polygonal, boundaries proved insurmountable for another sixty years. It was necessary first to separate the question of mere existence from the goal of actually determining the solution surface, say, with the help of explicit equations, before progress was possible. Precisely this step, the isolation of the pure existence problem from everything else, has often been cited as a mark of modern mathematics. It can be found in many diverse developments. (Hilbert gained first fame through a comparable step, as he solved Gordan's problem in invariant theory in 1888.)

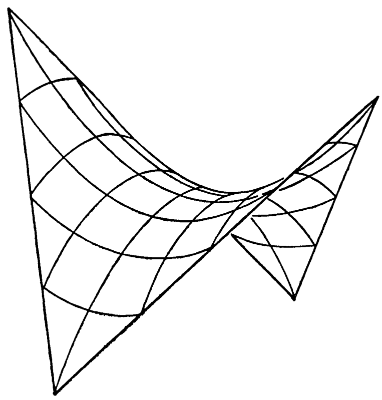


FIG. 8

6. The minimal surfaces which appear as solutions of Plateau's problem possess many special characteristics whose detailed study seemed mandatory. After all, the charge to a mathematician is the same as the charge to the witness in a court of law: To tell the truth, the whole truth and nothing but the truth. Consequently many questions were raised in connection with the attempts to understand all ramifications of Plateau's problem and to fully describe the properties of the solution surfaces, their regularity, their uniqueness or non-uniqueness, their geometrical features and so on. For instance, while the minimal surfaces are analytic in their interior, their precise boundary behavior had been uncertain until five years ago. From the investigations of H. Lewy, the speaker and others we now know that they are, roughly speaking, as regular on their boundary as the Jordan curves which they span. More precisely: If the position vector $y(\theta)$ of the Jordan curve Γ belongs to a certain differentiability class — $C^{m,\alpha}$, or C^{∞}, or C^{ω} — then the vector $x(u,v)$ defining a solution of Plateau's problem is a member of the same differentiability class in the entire closed disk $\bar{P}$.

Another problem, in my opinion currently the most important question connected with the classical Plateau problem, is still open: Can a given contour ever bound infinitely many minimal surfaces, and if not, can one estimate the number of possible solutions in terms of geometric quantities of their boundary?

7. All these questions would lead us too far. Trying rather to concentrate next on one particular aspect in the evolution of Plateau's problem, I shall now make a jump. The existence proofs utilize, in one version or another, the so-called direct methods of the calculus of variations. In order to obtain a surface of least area bounded by a given contour one takes a sequence of comparison surfaces whose areas converge to the minimum value possible. It is then hoped that not only these areas but also the comparison surfaces themselves converge (in some sense) to a limit, hopefully a solution surface. Unfortunately, even in the case of well-behaved comparison surfaces, the most unforeseen things can happen in this limiting process. This fact, among others,

has motivated the mathematicians to reconsider the customary definition of a surface, as well as the objects and quantities related to it (its boundary, its area, etc.). The classical concept of a parameter surface — a definition with which we all (consciously or unconsciously) have become acquainted in college — is often found ill-suited. Such a surface is defined as the mapping into space, effected by a continuous vector, of a domain in the parameter plane. Thus we speak of a surface of the type of the disk if the parameter domain is a disk, a surface of the type of the annulus if the parameter domain is an annulus, etc. Trying to solve Plateau's problem we have as a first step to choose a parameter domain for the comparison surfaces. In doing this we may prejudge the topological type of the solution surface of least area which may well turn out to be different from that of the comparison surfaces. A number of alternate definitions have been suggested designed to avoid this difficulty. All of these definitions are rather abstract. For the purpose of illuminating their crucial ingredient, I shall for a moment digress and turn my attention to a much simpler, yet analogous, concept, that of a function, which has undergone a similar abstraction process.

I am sure we all think we know what a function $y = f(x)$ is: a law which associates with every value of the independent variable x a value of the dependent variable y. We may have heard about objections by logicians and we may be aware of rather unusual "functions," as for instance the δ-function $\delta(x)$ which is zero for all $x \neq 0$ and infinite for $x = 0$. (This "function" has even been differentiated by physicists!) Let me now interject a little anecdote. Assume there are some questions about Mr. X who used to live in Y-city, or Mr. X died, and we want to find out what kind of a man he was. Information is solicited from his former employer, from his neighbors, his former wife, his schoolmates, his girlfriends, his students, the FBI and so forth. Having available all this information we might then be inclined to believe that we know or can determine who and what Mr. X is, or was. For all practical purposes this may in fact be the case. Naturally, I am convinced that there is more to man than this. There may not be more to a function. Instead of these sources of information before (FBI agents, girlfriends, etc.), take now a class of very well-behaved functions $\phi(x)$ — also called test functions (these are infinitely often differentiable functions which vanish outside of a finite interval; we say: functions of class C_0^∞) — and integrate to obtain an expression

$$\int_{-\infty}^{+\infty} f(x)\phi(x)dx \equiv L(\phi).$$

For every test function $\phi(x)$ the integration produces a specific numerical value. In mathematical terms this makes L a functional which is linear since obviously $L(\phi + \psi) = L(\phi) + L(\psi)$. A second important property of L is its continuity (suitably defined) over the space of test functions. It is easy to see that any two reasonable, say, continuous or integrable, functions $f(x)$ and $g(x)$ must be identical if the corresponding functionals are the same, i.e., have the same values for all test functions. Since every unctfion is thus uniquely characterized by the actions upon it of all test functions,

we may actually identify our functions with such functionals. Indeed, forgetting functions altogether, we can now work with these (linear and continuous) functionals regardless of their origin. We then speak of distributions, and our traditional functions simply appear as special distributions, namely, continuous linear functionals which happen to possess a representation utilizing an integral as before. The δ-function is a distribution; it associates to every test function $\phi(x)$ its value $\phi(0)$ at $x = 0$, since formally

$$\int_{-\infty}^{+\infty} \delta(x)\phi(x)dx = \phi(0).$$

In spite of, or rather because of, their generality distributions have many nice properties and are therefore a convenient tool. They can for instance be arbitrarily often differentiated. In view of a formal integration by parts according to which

$$\int_{-\infty}^{+\infty} f'(x)\phi(x)dx = -\int_{-\infty}^{+\infty} f(x)\phi'(x)dx$$

(note that no boundary terms appear, and that $\phi'(x)$ is again a test function), we define the derivative L' of a distribution by the stipulation $L'(\phi) = -L(\phi')$. It is often advantageous to seek the solution to a problem in the framework of distribution theory. Later on it may be possible to prove that the solution actually is a function.

For the purpose of illustrating the last remark let us mention here the simplest example possible. Generalizing the trivial fact that a differentiable function whose derivative vanishes in an interval must be a constant there, we try to characterize all distributions of vanishing derivative L'. In view of the above, this means that $L(\phi') = 0$ for all test functions $\phi(x)$. Denote by $\phi_0(x)$ a fixed test function whose integral is equal to one and consider an arbitrary test function $\phi(x)$. Clearly, the function $\psi(x) = \phi(x) - \lambda\phi_0(x)$, where $\lambda = \int_{-\infty}^{+\infty}\phi(x)dx$, is of class C_0^∞ and $\int_{-\infty}^{+\infty}\psi(x)dx = 0$. Thus, setting $\chi(x) = \int_{-\infty}^{x}\psi(\xi)d\xi$, we see that also $\chi(x)$ is a test function and $\chi'(x) = \psi(x)$. Consequently

$$L(\phi) = L(\lambda\phi_0 + \psi) = \lambda L(\phi_0) + L(\chi') = \lambda L(\phi_0) = \int_{-\infty}^{\infty} c\phi(x)dx.$$

Here $c = L(\phi_0)$. In other words: A distribution L, whose derivative vanishes, is a constant, i.e., L is generated by a constant function.

The preceding remarks about functions will have served to illustrate our point, and we now return to surfaces. Given a regular surface S (rather than a function f as before) we introduce an appropriate class of differential forms Φ (in place of the test functions ϕ before). Each form Φ can be integrated over S leading to a numerical value. Thus we are again encountering a linear functional, and by the same abstraction process we arrive at a class of objects, called currents, which contains our surfaces as special elements but which incorporates more general entities as well. Related to currents are the so-called varifolds. In technical language their definition reads as

follows: "A varifold is a Radon measure on the bundle over an m-dimensional Riemannian manifold M whose fiber at each point p of M is the Grassmann manifold of k-dimensional linear subspaces of the tangent space to M at p." A new powerful branch of mathematics, called geometric measure theory, created by Federer, Fleming, Almgren, Allard, a.o., has scored first successes by attacking Plateau's problem in this general framework.

8. Let us assume that a solution of Plateau's problem has been found in the form of a current or a varifold. One must then set one's sight on obtaining a full description of this structure, primarily on the demonstration that it is really as close to being a classical surface as possible. In the process of this demonstration whose technical intricacies lie beyond the scope of a description here, it turns out that two kinds of points on the solution current must be distinguished, regular points and singular points. In the neighborhood of a regular point the current is manifold-like; that is to say, that part of the current which is contained in a sufficiently small ball about one its regular points is in fact a surface portion in the classical sense. As this regular picture is disturbed near the singular points, any information concerning the size of the "singular set," i.e., the totality of all singular points on the solution structure, becomes of interest. For the simplest case of Plateau's problem (and for two-dimensional surfaces in three-space) the singular set is empty and the solution structure is therefore everywhere regular. Globally it may be quite wild, possessing a number of, even infinitely many, handles, as depicted in Figure 9. Locally, however, it is a

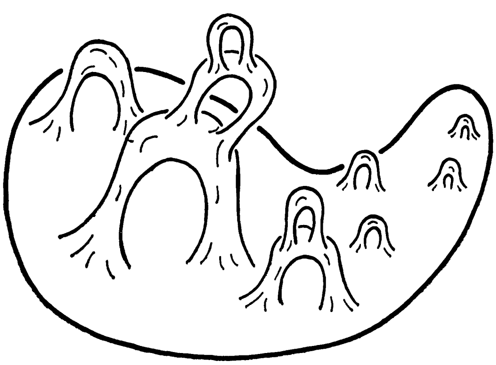


FIG. 9

smooth regular manifold. In more general cases, particularly if we are considering Plateau's problem in its higher dimensional versions, the situation is more complex. It is known that the dimension of the singular set is lower than that of the solution structure. Precisely, how voluminous it can be under the most unfavorable circumstances is an open question today. For structures of codimension one, i.e., for n-dimensional currents in $(n + 1)$-dimensional space a most striking fact was brought out a few years ago through the efforts of a number of mathematicians: If we have

such a current, its singular set will be empty as long as $n \leqq 7$. For $n > 7$ the (so-called Hausdorff) dimension of the singular set cannot be larger than $n - 7$. The appearance of the limit dimension seven is most surprising. Why seven, can be explained by the properties of a certain variational problem, but is still hard to understand intuitively.

More elaborate experiments suggest that the singular set may often possess a rather special shape. For instance, if one dips a frame made up of the edges of a cube in the soap solution one obtains upon withdrawal of the frame a system of thirteen membranes as shown in Figure 10. The singular set is one-dimensional, consisting of

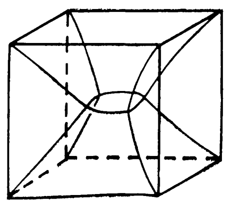


FIG. 10

the branch lines, also called liquid edges, along which several surfaces meet. Plateau's experiments (and the theoretical considerations of Lamarle and others) led him to the following conclusion which has been theoretically substantiated in concrete cases: In a stable configuration it is not possible that more than three membranes come together along a branch line; and if there are three, they meet mutually at equal angles of 120°. As far as the branch lines themselves are concerned, at most four can issue from a point where they then mutually intersect each other at equal angles of 109.47°. Mathematical existence proofs for surface systems similar to those depicted in Figure 10 as well as the investigations of the nature and the regularity of the branch lines are still in their infancy. First progress has been made in the 1972 Princeton dissertation of J. Taylor.

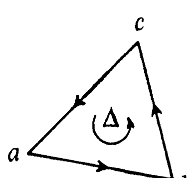
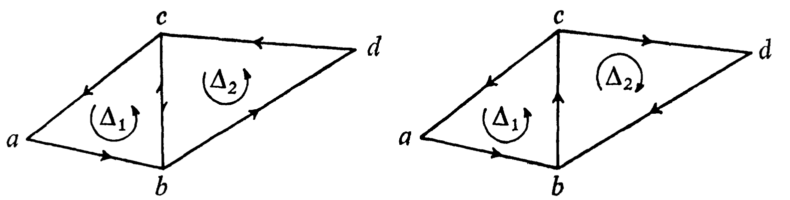


FIG. 11

9. A few words might be in order to explain the methods dealing with such surface systems. In studying manifolds it is a customary procedure of algebraic topology to triangulate, i.e., to consider a manifold as made up of triangles. Each triangle is given an orientation. (In Figure 11 we have employed arrows for this purpose.) An in-

dividual oriented triangle Δ with vertices a, b, c as in Figure 11 is assigned as boundary $\partial\Delta$ the sum $ab + bc + ca$ of oriented segments. Here we use the convention $ab = -ba$, so that $ab + ba = 0$. The boundary of a collection of triangles is obtained by addition, as illustrated in Figure 12, where two choices of orientations are depicted. Depending on the orientations chosen, the boundary of the complex of triangles has different forms. Actually the segment bc, being an interior segment of the



$\partial(\Delta_1 + \Delta_2) = ab + bd + dc + ca$ $\quad$ $\partial(\Delta_1 + \Delta_2) = ab - bd - dc + ca + 2bc$

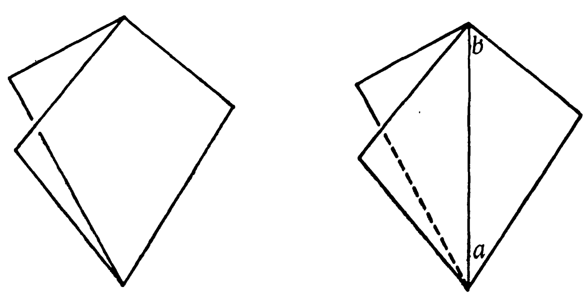
$\equiv ab + bd + dc + ca \,(\mathrm{mod}\ 2)$

FIG. 12

quadrilateral triangulated in Figure 12, should not appear in the boundary expression at all. This can be arranged, even for the "wrong" case, if we agree to reduce every integer coefficient of a segment "modulo two," i.e., if we replace every integer by the remainder it leaves when divided by two. Thus

$$\cdots, -1 \equiv 1 (\mathrm{mod}\ 2),\ 0 \equiv 0 (\mathrm{mod}\ 2),\ 1 \equiv 1 (\mathrm{mod}\ 2),\ 2 \equiv 0 (\mathrm{mod}\ 2), \cdots$$

and $-bd \equiv bd(\mathrm{mod}\ 2)$, $-dc \equiv dc(\mathrm{mod}\ 2)$, $2bc \equiv 0(\mathrm{mod}\ 2)$. Then, regardless of the orientations chosen for the individual triangles, the boundaries are the same modulo two.



a b

FIG. 13

Let us now consider a frame of three wires as shown in Figure 13a. From soap film experiments we know that this frame spans a system of three surfaces which meet along a branch line connecting the common end points a and b of the arcs as schematically sketched in Figure 13b. In Figure 14 in which the segment ab appears with three specimens (actually to be identified) an orientation is suggested for the triangulation of the system.

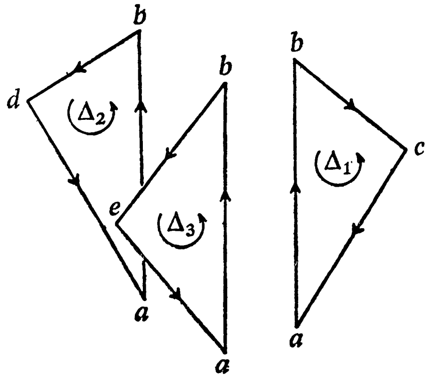


FIG. 14

From

$$\begin{aligned}\partial\Delta_1 &= ab + bc + ca\\ \partial\Delta_2 &= ab + bd + da\\ \partial\Delta_3 &= ab + be + ea\end{aligned}$$

we find that

$$\partial(\Delta_1 + \Delta_2 + \Delta_3) = 3ab + bc + \cdots + ea.$$

Thus the segment ab appears in the expression for the boundary of the surface system. In our concrete problem, however, ab should be an interior segment of the surface system and the boundary should consist exclusively of the wires of the frame, i.e., of all the segments with the exception of the segment ab. This can be achieved if we this time work modulo three. Since $3 \equiv 0 \pmod 3$, the boundary of our surface system becomes

$$\partial(\Delta_1 + \Delta_2 + \Delta_3) \equiv bc + ca + bd + da + be + ea \pmod 3,$$

which is precisely the desired expression.

These remarks may suffice to indicate how one, working modulo a suitable integer, can adapt the solution procedure to various concrete problems. Whether for a given frame the determination of currents of absolutely least area, but modulo different integers, can lead to different solutions (different also in area), is a question which has not yet been fully answered in all cases.

10. In conclusion of the preceding remarks concerning the modern approaches to the solution of Plateau's problem it must be pointed out that these approaches until now have succeeded only in the determination of those solution structures whose area represents the absolute minimum among the areas of all competing structures

The existence proofs for other solutions, those whose area is a relative minimum only and, in particular, altogether unstable solutions, are more elusive. Here the powerful topological theory, named after its founder Marston Morse, has been applied with success only in the more classical setting of the 1930's. Interesting theorems assuring the existence of unstable minimal surfaces which span a given frame have been proved in 1939 by Morse and Tompkins and by Shiffman. (Peculiarly enough, the subject has not yet been taken up again since then.)

In order to explain these theorems it has to be understood that a surface bounded by the frame or, more precisely, the position vector of such a surface, can be regarded as an element of a certain general function space and that the integral for the area then becomes a functional defined on all the elements of this space. Minimal surfaces are characterized by the property that their area is stationary, i.e., for them the first variation of the area functional vanishes. In this sense minimal surfaces are the critical points for the area functional. It is true that our functional is merely semi-continuous but not continuous, and this fact is the cause of considerable difficulties.

For the purpose of giving a simple illustration, however, let us now instead of position vectors in a function space consider points (x, y) in the Cartesian plane and let us focus our attention on a continuous, better still, a sufficiently often differentiable, function $f(x, y)$ rather than on a (semi-continuous) functional. Stationary, or critical, points for $f(x, y)$ are those points (x, y) in which the first partial derivatives $f_x = \partial f/\partial x$ and $f_y = \partial f/\partial y$ are zero. Critical points describe minima and maxima for $f(x, y)$, but also saddle points and ridges. In Figure 15 a topographical map is reproduced showing

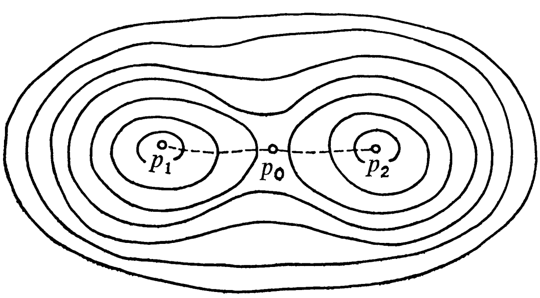


FIG. 15

level lines which may be thought of as the loci of constancy for a function $z = f(x, y)$. We observe that there are valleys, or pits, at p_1 and p_2. In these points our function possesses absolute minima so that there $f_{xx} > 0$, $f_{xx}f_{yy} - f_{xy}^2 > 0$. Searching for further stationary points of $f(x, y)$ let us walk from p_1 to p_2 along a certain path, and let us mark the point of highest elevation which we have to pass on our journey. In the attempt to minimize our efforts we will, of course, choose a path on which the highest elevation is as low as feasible. Such a path is shown as a dotted line in Figure 15. It is intuitively clear, and can also be proved mathematically, that the point p_0 of

highest elevation on *this* path is a stationary point for our function, in the case at hand, a saddle point.

The experience gained from the foregoing example motivates the theorems about minimal surfaces alluded to earlier. A typical statement is the following: Assume that a given contour Γ bounds two minimal surfaces, and that the areas of both represent strict relative minima. Then Γ must still bound a third, generally unstable, minimal surface. The proof for this statement is highly technical.

The degree of sophistication reached by the abstract Morse theory has so far not been fully transferred to the concrete, but difficult, application which Plateau's problem represents. A classification of minimal surfaces according to their critical types as well as an estimate of their number are still out of reach today.

11. The end of my lecture is in sight. My attempts to illustrate by examples the influences of Plateau's work on mathematical developments could at best provide a few glimpses. There are so many mathematical subjects of current interest which should have been, but could not be, discussed:

Minimal surfaces bounded by a flexible, but inextensible, string (the string will assume the shape of a curve of constant space curvature) —

Surfaces of least area which are forced to lie on one side of a fixed obstacle —

Free boundary value problems where the boundary of a solution surface is required to lie on a given manifold —

The intimate relations to the calculus of variations and to the theory of partial differential equations —

The still incomplete existence and non-existence problems for surfaces of prescribed (but variable and not vanishing) mean curvature —

The behavior of interfaces under varying gravity conditions —

Investigations concerning the oscillations of liquid masses —

to mention but a few. I should also have given an account of the powerful numerical methods which were created in connection with the attempts to solve the Young-Laplace equation, but which, far transcending their original goal, have led to new developments in the whole area of ordinary differential equations. By a fine coincidence the basic theoretical and numerical study of Bashforth and Adams regarding the shape of liquid drops was published in 1883, the year Plateau died. With the advent of the computer a classification of all axisymmetric solutions of the Young-Laplace equation has become feasible. Surveying the print-outs Huhs and Scriven as well as Concus and Finn discovered that the (non-parametric) equation of the pendant drop

$$H \equiv \frac{(1 + z_y^2)z_{xx} - 2z_x z_y z_{xy} + (1 + z_x^2)z_{yy}}{2(\sqrt{1 + z_x^2 + z_y^2})^3} = az + b$$

possesses one particular solution $z = z(x, y)$ with an isolated singularity, describing an (unstable) equilibrium configuration, a "pendant spike." Subsequently Finn and Concus have substantiated this interesting fact by a mathematical proof.

12. It seems only fitting, however, that I now conclude this lecture with a few remarks concerning the question of stability to which Plateau and others have devoted a considerable amount of experimental ingenuity and theoretical attention. In practical importance this question may well surpass that of the other problems mentioned. In his experiments with liquid cylinders of variable length, Plateau observed the phenomenon of instability. Instability occurs as soon as the length l of the cylinder exceeds its circumference $2\pi r$. Then even the smallest perturbations, caused by vibrations, convection currents, an electric field, or other unavoidable disturbances, lead to ever increasing bulgings and contractions. The cylinder changes into intermediate unduloidal shapes and finally disintegrates into a sequence of disconnected spheres appearing in a rather regular arrangement. The speed of this decomposition process depends on the viscosity of the liquid and may be very slow. The process itself can be observed in many disparate situations — in liquid jets, in spider threads, in the melting of an electrically overheated wire, etc. Plateau gave several derivations of the stability limit $l = 2\pi r$; none can be accepted as rigorous. A mathematical approach to the question of (mechanical) stability proceeds as follows. One subjects the cylindrical shape to small, but volume preserving, distortions and studies the change of the surface area under these distortions. If any one particular perturbation is capable of decreasing the surface area, then the cylinder cannot be in stable equilibrium. Strictly speaking, we have here a variational problem with subsidiary conditions. When we consider that the first variation of this problem always vanishes for a figure in equilibrium, we are led to a study of its "second variation." There are close ties between the theory of the second variation and the eigenvalue problem for a second order elliptic partial differential equation. This was first observed by H. A. Schwarz in 1885, who showed in his pioneering investigation of a concrete problem that the size of the smallest eigenvalue is crucial for the determination of stability limits. An extraordinary development in the calculus of variations, which cannot be discussed here, has taken place since.

While the question of stability for a liquid cylinder in equilibrium has in the meantime become an elementary exercise, it can still serve to illustrate one point, namely, the specification of physically realistic boundary conditions if the contact line is permitted to move. Particularly in view of problems of a more general nature, this specification deserves a careful examination. Consider a circular cylinder

$$\Sigma = \{x = u,\ y = \cos v,\ z = \sin v;\ 0 \leqq u \leqq l,\ 0 \leqq v \leqq 2\pi\}$$

of radius one and length l which is suspended between the vertical planes $x = 0$ and $x = l$. A small perturbation leads to the distortion of Σ into a surface

$$\Sigma' = \{x = u,\ y = [1 + \zeta(u,v)]\cos v,\ z = [1 + \zeta(u,v)]\sin v;\ 0 \leqq u \leqq l,\ 0 \leqq v \leqq 2\pi\}.$$

Here $\zeta(u,v)$ denotes a function which together with its first derivatives is assumed to be sufficiently small, so that $|\zeta| < \varepsilon$, $|\zeta_u| < \varepsilon$, $|\zeta_v| < \varepsilon$. The perturbation will be

volume preserving, if it satisfies the condition

$$2\int_0^l \int_0^{2\pi} \zeta(u,v)dudv + \int_0^l \int_0^{2\pi} \zeta^2(u,v)dudv = 0.$$

The area $A(\Sigma)$ of the lateral surface of Σ is equal to $2\pi l$, while the corresponding area of Σ' comes to

$$\begin{aligned} A(\Sigma') &= A(\Sigma) + \tfrac{1}{2}\int_0^l \int_0^{2\pi} [\zeta_u^2 + \zeta_v^2 + \zeta^2]dudv + O(\varepsilon^3) \\ &= A(\Sigma) + \tfrac{1}{2}\int_0^{2\pi} [\zeta(l,v)\zeta_u(l,v) - \zeta(0,v)\zeta_u(0,v)]dv \\ &\quad + \tfrac{1}{2}\int_0^l \int_0^{2\pi} [\Delta\zeta + \zeta]\zeta\, dudv + O(\varepsilon^3). \end{aligned}$$

In this formula the volume constraint is incorporated, and the expression $O(\varepsilon^3)$ embraces all terms whose order of smallness is ε^3 at least. The boundary conditions normally considered are of one of the forms

(i) $\zeta(0,v) = \zeta(l,v) = 0$,

(ii) $\zeta_u(0,v) = \zeta_u(l,v) = 0$,

(iii) $\zeta(0,v) = \zeta(l,v)$, $\zeta_u(0,v) = \zeta_u(l,v)$.

All three cause the disappearance of the single integral in the second expression for $A(\Sigma')$. The first set of boundary conditions corresponds to the case of a cylinder with fixed end circles, an experimentally realizable situation. The third set is appropriate for a configuration with periodicity features. It is the second choice of boundary conditions, expressing a transversality property, which appears to be in need of some interpretation. Visualize an experiment in which our cylinder Σ is freely suspended between vertical plates situated in the planes $x = 0$ and $x = l$. We assume, of course, that the interfacial tensions of the materials involved allow for a contact angle of 90° at the plates in the first place. A perturbation, described by a function $\zeta(u,v)$ as above, will distort the originally circular curves in which the lateral surface of Σ meets the planes $x = 0$ and $x = l$. Owing to the hysteresis which is experimentally observed in connection with the contact line between media, the boundary conditions $\zeta_u(0,v) = \zeta_u(l,v) = 0$, according to which the lateral surface of Σ' intersects the bounding plates at the angle of 90° also, and which thus are based on the assumption of an instantaneous reaction to disturbances of the contact line, merely represent a (more or less justified) idealization. It would be more realistic to take into account the course of the hysteresis as well as the time dependence of the disturbance. The corresponding physical law, if it would be established, might then impose a dependence between ζ and ζ_u, in the simplest case possibly a linear relationship of the form

(iv) $\zeta_u(0,v) = a\,\zeta(0,v), \zeta_u(l,v) = -a\zeta(l,v)$.

The search for suitable boundary conditions and the determination of stability limits for equilibrium configurations subject to these conditions — in the case of a cylinder

as well as in more general cases (drops, menisci, etc.) — would seem to be of physical and mathematical interest.

For a number of equilibrium configurations, among them planes, spheres, cylinders as well as unduloids and nodoids, whose stability properties have also been extensively discussed by Plateau, the equations of the undisturbed surfaces are known — a fact which allows the explicit use of surface coordinates. Often, particularly in more complex arrangements in which the influence of gravity cannot be neglected (a special example is briefly mentioned by Plateau in §424 of his book), the investigator is less fortunate. Here the energy function to be minimized consists of two parts stemming from the surface tension and from the gravitational forces, and the shape of equilibrium figures cannot generally be expressed with the help of explicit equations. Scriven and his collaborators, Huh and Pujado, have employed the following approach. In conjunction with the stepwise numerical generation of the (axisymmetric) shape of a liquid drop or meniscus, they computationally test at every step the validity of the Jacobi condition which, in controlling the occurrence of a conjugate point, constitutes a well-known criterion for stability. For a one-parameter family of pendant drops of increasing volume they find that an axisymmetric drop with a fixed, or freely movable, contact line cannot be stable, i.e., cannot any more be suspended from a horizontal plane and rather disintegrates, if its meridional profile contains a point of inflexion. The crucial role of an inflexion point, which will always be present in a sufficiently voluminous drop, has been suspected for a long time. It would be worthwhile to develop a purely mathematical proof of this result which only relies on the fact that the meridional profile of the drop satisfies a certain non-linear differential equation as well as suitable boundary conditions, but which does not require the (unattainable) knowledge of this profile in an explicit form.

[**Added in proof, August 1, 1974:** Such an analysis has now been carried out by E. Pitts (J. Fluid Mech., 63 (1974) 487–508). The axisymmetric drop is stable (i.e., the second variation of the energy function remains positive) as long as its volume increases with its height. The occurrence of an inflexion point seems to have no bearing on the situation.]

What has been said here, of course, is not intended to provide a full picture of the stability problem. Commenting on the general investigations concerned with the determination of stability limits for liquid masses in equilibrium, Plateau states: "These investigations appear to me not devoid of interest, even from the purely mathematical point of view. They will probably present very great difficulties, and I shall leave the trouble of carrying them out to the geometers." Now, a hundred years later, many problems have been settled, while others still wait for their solution; but all are important.

School of Mathematics, University of Minnesota, Minneapolis, MN 55455.

STRONG DERIVATIVES AND INVERSE MAPPINGS

ALBERT NIJENHUIS

To Carl Allendoerfer: For *not* taking the Implicit Function Theorem for granted.

The notion of derivative as a linear approximation is strengthened ("strong derivative") with the result that strongly partially differentiable functions are strongly totally differentiable, and functions are locally invertible if their strong derivative at one point is invertible (Jacobian $\neq 0$). A new version of the Implicit Mapping Theorem (classically the Implicit Function Theorem) is formulated which implies the standard Picard existence theorem for differential equations $y' = f(x, y)$ with only the Lipschitz condition with respect to y.

This work follows and extends an idea of E. B. Leach.

1. Introduction. The notion of derivative as providing a linear approximation to a function is by now well established in the mathematics curriculum: a function f is **(Frechet-) differentiable** at x_0 with derivative l, a linear function, if for every $\varepsilon > 0$ there is $\delta > 0$ such that $|x - x_0| < \delta$ implies

$$|f(x) - f(x_0) - l(x - x_0)| \leqq \varepsilon |x - x_0|. \tag{1.1}$$

This definition is valid for functions whose domain and range are contained in the real line, or in a finite, or even infinite, dimensional vector space. Of course, $|\cdots|$ must be defined, so the vector spaces must be normed. Also, l must be a continuous linear map for a sensible theory; this is a limitation only in infinite dimensional situations.

Derivatives may be denoted l, as above, or $(Df)_{x_0}, f'(x_0)$, etc. In any case, they are continuous linear maps of one normed vector space to another; the first of these contains the domain, the second the image of f.

A theory of differentiation, with theorems for derivatives of sums, products and compositions of differentiable functions, follows naturally. The textbooks which give elegant proofs of these facts in great generality include H. Cartan [**1**], J. Dieudonné [**2**], and S. Lang [**3**]; many others give a good treatment in at least the finite-dimensional case.

For the somewhat deeper theorems it is necessary, or at least customary, to consider the derivative at more than one point, and to make continuity assumptions about the dependence of $(Df)_x$ on x. For example, take a function whose values $f(x, y)$ depend on two vectors (each with its own domain) and suppose the partial derivatives $D_1 f$ and $D_2 f$ of f with respect to the separate variables x and y exist at (x_0, y_0). Then f is not necessarily totally differentiable at (x_0, y_0). That is, setting $l_1 = (D_1 f)_{(x_0, y_0)}$, and $l_2 = (D_2 f)_{(x_0, y_0)}$, the existence of $\delta > 0$ for any $\varepsilon > 0$, such that

$|x - x_0| + |y - y_0| < \delta$ implies

$$|f(x,y)-f(x_0,y_0)-l_1(x-x_0)-l_2(y-y_0)| \leqq \varepsilon(|x-x_0| + |y-y_0|),$$

does not follow. However, the hypothesis that $(D_1f)_{(x,y)}$ exists at all (x,y) near (x_0,y_0) and that the map $(x,y) \mapsto (D_1f)_{(x,y)}$ is continuous at (x_0,y_0), will do the trick. For reasons of convenience, textbooks frequently assume this continuity for both D_1f and D_2f.

Another example with similar continuity hypotheses for Df is the inverse mapping theorem for mappings f between complete normed vector spaces; in order that f be locally and differentiably invertible at x_0 one requires not only that the linear map $(Df)_{x_0}$ is an isomorphism (i.e., is continuous with continuous inverse) but also that the map $x \mapsto (Df)_x$ is defined for all x near x_0, and is continuous at x_0. At least, that is how it is usually presented in the textbooks.

The literature contains a number of variations on the notion of derivative as stated above. Most are weaker. In this instance we plead for a stronger notion of differentiability, called strong differentiability. The author found it in papers **[4,5]** by E. B. Leach, which seem to have gone unnoticed by textbook writers. The exposition here is a partially weakened and partially extended but hopefully more accessible presentation of some of Leach's ideas.

2. Strong differentiability. In what follows, E, F, G, etc., with or without subscripts, denote normed vector spaces. U, V, W, with or without subscripts, denote open sets. X, Z, with or without subscripts, denote topological spaces which may or may not be normed vector spaces. Completeness of a normed vector space is stated any time it is assumed.

DEFINITION. Let $f: U \to F$, with $U \subset E$, be a map and $l: E \to F$ be a continuous linear map. Then f is **strongly differentiable** at x_0 with derivative l if for every $\varepsilon > 0$ there is $\delta > 0$ such that $x_1, x_2 \in U$ and $|x_1 - x_0| < \delta$, $|x_2 - x_0| < \delta$ imply

$$|f(x_2) - f(x_1) - l(x_2 - x_1)| \leqq \varepsilon |x_2 - x_1|.$$

Remarks. The case $x_1 = x_0$ is permitted above; hence strong differentiability implies differentiability. It also follows that the strong derivative, if it exists, equals the derivative.

ELEMENTARY PROPERTIES:

(1) *If f is strongly differentiable at x_0, then f satisfies a Lipschitz condition in a neighborhood of x_0.*

(2) *If $f: U \to F$ is differentiable in a neighborhood of $x_0 \in U$ and if $x \mapsto (Df)_x$ is continuous at x_0, then f is strongly differentiable at x_0.*

(3) *If $f\colon U \to F$ is strongly differentiable at $x_0 \in U$, and the derivative $(Df)_x$ exists for all $x \in A \subset U$ and if x_0 is an accumulation point of A, then $x \mapsto (Df)_x$ $(x \in A)$ is continuous at x_0.*

Remarks. 1. The values of the function $x \mapsto (Df)_x$ lie in the vector space $L(E, F)$ of continuous linear maps $\lambda\colon E \to F$. The norm $|\lambda|$ is defined as the smallest number c such that $|\lambda(x)| \leqq c|x|$ for all $x \in E$.

2. A version of the Mean Value Theorem states

$$|f(x_2) - f(x_1)| \leqq |x_2 - x_1| \sup_{x \in [x_1, x_2]} |(Df)_x|,$$

where $[x_1, x_2]$ is the line segment connecting x_1 and x_2. Applied to $x \mapsto f(x) - l(x)$, where l is any continuous linear map, this yields

$$|f(x_2) - f(x_1) - l(x_2 - x_1)| \leqq |x_2 - x_1| \sup_{x \in [x_1, x_2]} |(Df)_x - l|.$$

We give proofs of properties 2 and 3.

Proof of 2: Take $l = (Df)_{x_0}$ in Remark 2; and let $\varepsilon > 0$ be given. Then there is a δ-ball $B_\delta(x_0)$ about x_0 such that $|(Df)_x - (Df)_{x_0}| < \varepsilon$ for $x \in B_\delta(x_0)$. Let $x_1, x_2 \in B_\delta(x_0)$, then $[x_1, x_2] \subset B_\delta(x_0)$ and the inequality in Remark 2 yields

$$|f(x_2) - f(x_1) - (Df)_{x_0}(x_2 - x_1)| \leqq \varepsilon |x_2 - x_1|,$$

which shows that $(Df)_{x_0}$ is a strong derivative at x_0.

Proof of 3. Pick $\varepsilon > 0$ and let $\delta > 0$ be such that $|x_1 - x_0| < \delta$, $|x_2 - x_0| < \delta$ implies

$$|f(x_2) - f(x_1) - (Df)_{x_0}(x_1 - x_1)| \leqq \tfrac{1}{2}\varepsilon |x_2 - x_1|.$$

Now, let $x_1 \in A \cap B_\delta(x_0)$; then there is $\delta' > 0$ such that $|x_2 - x_1| < \delta'$ implies

$$|f(x_2) - f(x_1) - (Df)x_1(x_2 - x_1)| \leqq \tfrac{1}{2}\varepsilon |x_2 - x_1|.$$

Let $\delta^* = \min(\delta', \delta - |x_0 - x_1|)$; then both inequalities hold for all x_2 such that $|x_2 - x_1| < \delta^*$. Hence, for these x_2 we have

$$|(Df)_{x_1}(x_2 - x_1) - (Df)_{x_0}(x_2 - x_1)| \leqq \varepsilon |x_2 - x_1|.$$

By a standard homogeneity argument this proves $|(Df)_{x_1} - (Df)_{x_0}| \leqq \varepsilon$; hence $x \mapsto (Df)_x$ is continuous at x_0, with $x \in A$.

A simple example shows that strongly differentiable functions are not necessarily differentiable in a neighborhood of x_0. Take for example $f(x)$ (f and x real) defined on $[-1, 1]$ by $f(0) = 0$, $f(1/n) = n^{-2}$ for integer $n \neq 0$, and f linear and continuous in the segments $[1/n, 1/(n+1)]$. (Take $x_0 = 0$.) On the other hand, the function $g(x)$ with $g(0) = 0$, $g(x) = \frac{1}{2}x + x^2 \sin(1/x)$ for $x \neq 0$ is differentiable at 0, but not strongly;

its derivative is discontinuous at 0. Note that $g'(0) \neq 0$, yet g is not monotone in any neighborhood of 0. This does not happen with strongly differentiable functions.

The usual rules hold for strongly differentiable functions: closure under addition, scalar multiplication and composition. Also, closure under products provided, as usual, a bilinear product map $\pi: F_1 \times F_2 \to G$ is given to multiply $f_1: U \to F_1$ and $f_2: U \to F_2$ ($U \subset E$).

3. Strong partial differentiability. If f depends on a parameter, f: $U \times X \to F$, $U \subset E$, 'partial' differentiability with respect to the first variable is meaningful.

DEFINITION. f is called **strongly partially differentiable** at (u_0, x_0) if

(i) there is a continuous linear map $l: E \to F$ (also denoted $(D_1 f)_{(u_0\, x_0)}$) such that to every $\varepsilon > 0$ there are $\delta > 0$ and a neighborhood $N(x_0)$ of x_0 in X such that $|u_1 - u_0| < \delta$, $|u_2 - u_0| < \delta$, $x \in N(x_0)$ imply

$$|f(u_2, x) - f(u_1, x) - l(u_2 - u_1)| \leqq \varepsilon |u_2 - u_1|,$$

(ii) the map $x \mapsto f(u_0, x)$ of X to F is continuous at x_0.

These hypotheses imply that f is continuous at (u_0, x_0). The definition applies, in particular, for $f: U \to F$ when U is an open set in $E_1 \times E_2$. Then there may be strong partial derivatives $D_1 f$ and $D_2 f$. The next theorem shows that strong partial differentiability *does* imply total differentiability. Furthermore, the proof is almost trivial!

THEOREM 1. *Let $f: U \to F$, where $U \subset E_1 \times E_2$, be strongly partially differentiable with respect to both the factors E_1, E_2 at (x_0, y_0). Then f is strongly differentiable at (x_0, y_0) and*

$$(Df)_{(x_0, y_0)}(u, v) = (D_1 f)_{(x_0, y_0)} u + (D_2 f)_{(x_0, y_0)} v.$$

Proof. Let $l_1: E_1 \to F$ and $l_2: E_2 \to F$ be the two strong partial derivatives, and let $\varepsilon > 0$. Then there are δ_1, $\delta_2 > 0$ such that

$$|x_1 - x_0| < \delta_1, \; |x_2 - x_0| < \delta_1, \; |y - y_0| < \delta_2$$

imply

$$|f(x_2, y) - f(x_1, y) - l_1(x_2 - x_1)| \leqq \varepsilon |x_2 - x_1|,$$

and

$$|x - x_0| < \delta_1, \; |y_1 - y_0| < \delta_2, \; |y_2 - y_0| < \delta_2$$

imply

$$|f(x, y_2) - f(x, y_1) - l_2(y_2 - y_1)| \leqq \varepsilon |y_2 - y_1|.$$

Now consider all x_1, x_2, y_1, y_2 for which

$$|x_1 - x_0| < \delta_1, \; |x_2 - x_0| < \delta_1, \; |y_1 - y_0| < \delta_2, \; |y_2 - y_0| < \delta_2;$$

then

$$\begin{aligned}&|f(x_2,y_2)-f(x_1,y_1)-l_1(x_2-x_1)-l_2(y_2-y_1)|\\ &\quad\leqq |f(x_2,y_2)-f(x_1,y_2)-l_1(x_2-x_1)|+|f(x_1,y_2)-f(x_1,y_1)-l_2(y_2-y_1)|\\ &\quad\leqq \varepsilon(|x_2-x_1|+|y_2-y_1|),\end{aligned}$$

which was to be shown.

Recall that in the classical theorems for the existence of a total derivative (not strong) the continuity at (x_0, y_0) of one of the partials was required. The above inequality shows that it suffices that one partial derivative is strong (e.g., set $x_1 = x_0$, $y_1 = y_0$); it need exist only at (x_0, y_0). Hence, we have

THEOREM 1′. *Let $f: U \to F$, where $U \subset E_1 \times E_2$, be partially differentiable at $(x_0, y_0) \in U$ with respect to both factors, and let one of these be strong. Then f is differentiable at (x_0, y_0).*

4. Inverse mappings. One of the strongest points about Leach's notion of strong derivative is the inverse mapping theorem with only a one-point condition on the (strong) derivative. Let again $f: U \to F$ be strongly differentiable at x_0, where $U \subset E$ and where E is assumed to be complete. One can show that, if $(Df)_{x_0}: E \to F$ is an isomorphism, then f is locally invertible at x_0. In the version below we allow an extra parameter for good measure; that will produce a new version of the implicit mapping theorem which, though seemingly weak, is surprisingly powerful.

THEOREM 2. *Let E, F be complete normed vector spaces, Z a topological space. Suppose $f: U \to F$ is a map, $U \subset E \times Z$, which is strongly partially differentiable at $(x_0, z_0) \in U$ with respect to the first factor. Suppose, further, that*

$$(D_1 f)_{(x_0,z_0)}: E \to F$$

is an isomorphism. Then there exist a neighborhood N_1 of (x_0, z_0) in $E \times Z$, and a neighborhood N_2 of $(y_0, z_0) = (f(x_0, z_0), z_0)$ in $F \times Z$ which are in $(1,1)$ correspondence under the map $(x, z) \mapsto (f(x, z), z)$; the inverse map $(y, z) \mapsto (\phi(y, z), z)$ is strongly partially differentiable at (y_0, z_0) with respect to the first factor.

In more intuitive terms, the theorem says that a family $\{f_z\}_{z \in Z}$ of maps from E to F is locally invertible if the family is strongly partially differentiable, and if the linear approximation to the member f_{z_0} of the given family is an isomorphism. The family $\{\phi_z\}$ of inverses is also strongly partially differentiable.

Note that N_1 and N_2 are generally not open neighborhoods as the maps between them are not continuous in z, except at (x_0, z_0). Of course, $x \mapsto f(z, x)$ is continuous in x for fixed z when $(x, z) \in N_1$; and $y \mapsto \phi(y, z)$ behaves similarly in N_2.

The *implicit mapping theorem* asks for the full solution of the equation $f(x, z) = 0$ for x near x_0 and z near z_0, when (x_0, z_0) is given to be one solution. Theorem 2 with $y_0 = 0$ provides a complete local solution, with x parametrized by z, namely $x = \phi(0,z)$.

This is a typical way in which "implicit function theorems" are obtained from "inverse function theorems." Therefore, there is little need to mention each time the "implicit" theorem that corresponds to an "inverse" theorem.

The remainder of this section is devoted to a detailed proof of Theorem 2. It is a rather straightforward adaptation of proofs of a more traditional version. The first part establishes the existence of ϕ and its continuity at (y_0, z_0), and depends heavily upon the completeness of E. The second part uses this ϕ, but makes no further appeal to the completeness of the normed vector spaces. This approach makes possible separate use of these parts in the proof of Theorem 3″.

PART I. Define f^* by

$$f^*(\xi, z) = \lambda^{-1}(f(x_0 + \xi, z) - y_0); \ \lambda = (D_1 f)_{(x_0, z_0)},$$

then f^* maps the open neighborhood $U^* = \{(x - x_0, z) \mid (x, z) \in U\}$ of $(0, z_0) \in E \times Z$ into E; and $f^*(0, z_0) = 0$. Furthermore, f^* is strongly partially differentiable at $(0, z_0)$ and the derivative is id_E. Hence, there exist $r > 0$ and an open neighborhood W of z_0 in Z such that, for all $z \in W$ and for all ξ_1, ξ_2 in the closed r-ball $\bar{B}_r$ about 0 in E we have $(\xi_i, z) \in U^*$ $(i = 1, 2)$ and

$$|f^*(\xi_2, z) - f^*(\xi_1, z) - (\xi_2 - \xi_1)| \leqq \tfrac{1}{3}|\xi_2 - \xi_1| \tag{4.1}$$

and also

$$|f^*(0, z)| \leqq r/3. \tag{4.2}$$

Let $\xi \in \bar{B}_r$, $\eta \in B_{r/3}$ (open $r/3$ ball), and define

$$h_{\eta,z}(\xi) = \xi - f^*(\xi, z) + \eta; \tag{4.3}$$

then $h_{\eta,z}$ maps $\bar{B}_r$ into itself, because, setting $\xi_1 = 0$, $\xi_2 = \xi$ in (4.1) we find

$$|h_{\eta,z}(\xi)| \leqq |\eta| + |f^*(0, z)| + \tfrac{1}{3}|\xi| < r;$$

we have used (4.2) in the second estimation.

Also, $h_{\eta,z}$ is a shrinking map, as by (4.1,3)

$$|h_{\eta,z}(\xi_2) - h_{\eta,z}(\xi_1)| \leqq \tfrac{1}{3}|\xi_2 - \xi_1|.$$

By the contraction property for complete metric spaces, such as $\bar{B}_r$, $h_{\eta,z}$ has a unique fixed point ξ; this defines a function ϕ^*; $\phi^*(\eta, z) = \xi$ for $(\eta, z) \in N_2^* = B_{r/3} \times W$. So we have $\xi = h_{\eta,z}(\xi)$; or $\eta = f^*(\xi, z)$; or

$$\eta = f^*(\phi^*(\eta, z), z). \tag{4.4}$$

The uniqueness of $\phi^*(\eta, z)$ as solution of (4.4) implies $\phi^*(0, z_0) = 0$. The continuity of f^* at $(0, z_0)$ implies that the inverse image N_1^* of N_2^* under $(\xi, z) \mapsto (f^*(\xi, z), z)$ is a neighborhood of $(0, z_0)$, though not necessarily an open one. The uniqueness property

implies that $(\xi, z) \mapsto (f^*(\xi, z), z)$ gives a one-to-one correspondence between N_1^* and N_2^*; the inverse map is, of course, $(\eta, z) \mapsto (\phi^*(\eta, z), z)$.

Let $(\eta_i, z) \in N_2$ $(i = 1,2)$; then (4.1) holds for $\xi_i = \phi^*(\eta_i, z)$, and we have

$$\left|\eta_2 - \eta_1 - \phi^*(\eta_2, z) - \phi^*(\eta_1, z)\right| \leqq \tfrac{1}{3}\left|\phi^*(\eta_2, z) - \phi^*(\eta_1, z)\right|$$

from which follows (by $|a - b| \geqq \left| |a| - |b| \right|$)

$$\left|\phi^*(\eta_2, z) - \phi^*(\eta_1, z)\right| \leqq \tfrac{3}{2}\left|\eta_2 - \eta_1\right|. \tag{4.5}$$

We also have

$$f^*(\phi^*(0, z), z) = 0.$$

By (4.1) with $\xi_1 = 0$, $\xi_2 = \phi^*(0, z)$ this implies

$$\left|-f^*(0, z) - \phi^*(0, z)\right| \leqq \tfrac{1}{3}\left|\phi^*(0, z)\right|;$$

hence

$$\left|\phi^*(0, z)\right| \leqq \tfrac{3}{2}\left|f^*(0, z)\right|. \tag{4.6}$$

Properties (4.5,6) imply that ϕ^* is continuous at $(0, z_0)$.

Now define

$$\phi(x, z) = y_0 + \lambda\phi^*(x - x_0, z); \tag{4.7}$$

then (4.4) becomes

$$y = f(\phi(y, z), z), \tag{4.8}$$

and the maps $(x, z) \mapsto (f(x, z), z)$ and $(y, z) \mapsto (\phi(y, z), z)$ are one-to-one correspondences between the sets

$$\begin{aligned} N_1 &= \{(x, z) \mid (x - x_0, z) \in N_1^*\} \\ N_2 &= \{(y, z) \mid (\lambda^{-1}(y - y_0), z) \in N_2^*\}, \end{aligned} \tag{4.9}$$

which are, respectively, neighborhoods of (x_0, z_0) and (y_0, z_0). The map ϕ is continuous at (y_0, z_0).

Part II. We now prove that ϕ is strongly partially differentiable at (y_0, z_0), and that $(D_1\phi)_{(y_0, z_0)} = \lambda^{-1}$. For $(y_i, z) \in N_2$ $(i = 1,2)$ we have (4.8); hence, setting $x_i = \phi(y_i, z)$

$$\begin{aligned} &\left|\phi(y_2, z) - \phi(y_1, z) - \lambda^{-1}(y_2 - y_1)\right| \\ &\qquad = \left|\lambda^{-1}\{f(x_2, z) - f(x_1, z) - \lambda(x_2, x_1)\}\right| \\ &\qquad \leqq \left|\lambda^{-1}\right| \left|f(x_2, z) - f(x_1, z) - \lambda(x_2 - x_1)\right|. \end{aligned}$$

Let $\varepsilon > 0$, $\varepsilon < \frac{1}{2}\left|\lambda^{-1}\right|^{-1}$, then there are $\delta > 0$ and a neighborhood $N(z_0)$ of z_0 as in the definition of strong partial differentiability of f. There are δ' and $N'(z_0) \subset N(z_0)$ such that, if $\left|y - y_0\right| < \delta'$, $z \in N'(z_0)$ then $\left|\phi(y,z) - x_0\right| < \delta$. Let $\left|y_i - y_0\right| < \delta'$ $(i = 1,2)$, $z \in N'(z_0)$; then we have from the above:

$$\left|\phi(y_2,z) - \phi(y_1,z) - \lambda^{-1}(y_2 - y_1)\right| \leqq \left|\lambda^{-1}\right|\varepsilon(\left|\phi(y_2,z) - \phi(y_1,z)\right|).$$

This implies first (again use $\left|a - b\right| \geqq \left|\,\left|a\right| - \left|b\right|\,\right|$) the crude estimate

$$\left|\phi(y_2,z) - \phi(y_1,z)\right| \leqq (1-\varepsilon\left|\lambda^{-1}\right|)^{-1}\left|\lambda^{-1}\right|\,\left|y_2-y_1\right| \leqq 2\left|\lambda^{-1}\right|\,\left|y_2-y_1\right|$$

which then yields the finer estimate

$$\left|\phi(y_2,z) - \phi(y_1,z) - \lambda^{-1}(y_2 - y_1)\right| \leqq 2\varepsilon\left|\lambda^{-1}\right|^2\left|y_2 - y_1\right|.$$

This, together with the continuity of $z \mapsto \phi(y_0,z)$ at z_0 proves the strong partial differentiability of ϕ at (y_0,z_0).

The proof is noticeably simpler, of course, when no parameter z is involved. The reader is urgently advised to write up the simpler proof of that case, as the simplified argument should be very clarifying.

5. Additional hypotheses. The classical implicit mapping theorem considers the equation $f(x,z) = 0$, with hypotheses to assure that x is a differentiable function of z. This raises the more general question of strengthening Theorem 2 by adding to the hypotheses. We consider only two cases, whose treatment is not completely routine:

(1) f is partially differentiable with respect to z;

(2) f is strongly partially differentiable with respect to z;

in this latter case we allow an additional parameter for good measure, and make a small change in the notation.

THEOREM 3′. *Let E, F be complete normed vector spaces, Z a normed vector space. Suppose $f: U \to F$ is a map, $U \subset E \times Z$, which is partially differentiable at (x_0,z_0) with respect to both factors; suppose the first of these is strong. Suppose further, that $(D_1f)_{(x_0,z_0)}: E \to F$ is an isomorphism. Then there exist a neighborhood N_1 of (x_0,z_0) in $E \times Z$ and a neighborhood N_2 of $(y_0,z_0) = (f(x_0,z_0),z_0)$ in $F \times Z$ which are in $(1,1)$ correspondence under the map $(x,z) \mapsto (f(x,z),z)$; the inverse map $(y,z) \mapsto (\phi(y,z),z)$ is partially differentiable with respect to both factors at (y_0,z_0); the first of the partial derivatives is strong.*

Proof. By Theorem 1′ the hypotheses imply that $(x,z) \mapsto (f(x,z),z)$ is totally differentiable at (x_0,z_0), its derivative is an isomorphism and by Theorem 2 it has an inverse $(x,y) \mapsto (\phi(x,z),z)$ which is strongly partially differentiable, hence continuous at (x_0,z_0). It suffices to show that this inverse is differentiable. The proof of this is the classical prototype of Part II.

THEOREM 3″. *Let E_1 and F be complete normed vector spaces, E_2 a normed*

vector space and Z a topological space. Suppose $f: U \to F$ is a map, $U \subset E_1 \times E_2 \times Z$, which is strongly partially differentiable at (x_0, u_0, z_0) with respect to $E_1 \times E_2$. Suppose further, that $(D_1 f)_{(x_0,u_0,z_0)}: E_1 \to F$ is an isomorphism. Then there exist a neighborhood N_1 of (x_0, u_0, z_0) in $E_1 \times E_2 \times Z$ and a neighborhood N_2 of (y_0, u_0, z_0) (where $y_0 = f(x_0, u_0, z_0)$) in $F \times E_2 \times Z$ which are in $(1,1)$ correspondence under the map $(x, u, z) \mapsto (f(x, u, z), u, z)$; the inverse map

$$(y, u, z) \mapsto (\phi(y, u, z), u, z)$$

is strongly partially differentiable at (y_0, u_0, z_0) with respect to $F \times E_2$.

The proof follows that of Theorem 2. First, following Part I we consider $(x, u, z) \mapsto (f(x, u, z), u, z)$ as a family of maps $E_1 \to F$ depending on the parameter (u, z). The contraction property, for which the completeness of E_1 is essential, assures the existence of $(y, u, z) \mapsto (\phi(y, u, z), u, z)$ and its continuity at (y_0, u_0, z_0). Then we apply Part II to $(x, u, z) \mapsto (f(x, y, z), u, z)$ as a family of maps $E_1 \times E_2 \to F \times E_2$ depending on the parameter z. This family is strongly partially differentiable; and it has an inverse family $(y, u) \mapsto (\phi(y, u, z), u)$ depending on z; Part II shows that the inverse family is strongly partially differentiable.

6. The Picard Theorem for differential equations. We are concerned with the existence and uniqueness of solutions of the initial value problem

$$u' = \mathfrak{F}(t, u), \ u(0) = u_0 \tag{6.1}$$

assuming that $\mathfrak{F}$ is continuous and satisfies a Lipschitz condition in u. Although these hypotheses do not assume any differentiability of $\mathfrak{F}$, it is nevertheless possible to prove the Picard theorem as a consequence of the Implicit Mapping Theorem as given in Theorem 2. In fact, if one traces through the proof of Theorem 2 with the Picard theorem in mind, one finds pretty much the traditional proof; the main difference lies in a somewhat smoother wording. In any case, the realization that both theorems are essentially the same is a positive gain.

It suffices to assume (without loss of generality) that $\mathfrak{F}$ is defined on a set $[a, b] \times B_R$, where the real interval $[a, b]$ contains 0 (it may be an endpoint) and where B_R is the R-ball about the origin in a complete normed vector space E_1. The Lipschitz constant for this domain is denoted L. The values of $\mathfrak{F}$ lie in E_1 (or in the naturally isomorphic space $L(\mathbb{R}, E_1)$; u' may be considered as having values in either one).

The Picard theorem asserts the existence of a $\delta > 0$ and a neighborhood V of 0 in E_1 such that a unique solution u satisfying (6.1) exists with domain

$$(-\delta, \delta) \cap [a, b]$$

and initial value $u_0 \in V$. In addition, the dependence of u on u_0 is Lipschitz.

There may also be additional parameters on which $\mathfrak{F}$ depends; this matter is briefly discussed at the end.

Let α be real, then the conclusion of the Picard theorem is equivalent to the assertion that there exists $\delta > 0$ such that, for $|\alpha| < \delta$, $\alpha \in [a, b]$ the initial value problem

$$u' = \alpha\, \mathfrak{F}(\alpha t, u), \qquad u(0) = u_0 \tag{6.2}$$

has a unique solution on the unit interval $I = [0, 1]$. The equivalence is obtained by the change of variable $t \to \alpha t$.

The proof is accomplished by constructing an appropriate function $f(x, z)$, and inverting it by Theorem 2. Using the same notation as in Theorem 2, E is the space $C^1(I, E_1)$ of C^1 maps $u: I \to E_1$ with C^1 norm

$$|u| = \sup_{t \in I} |u(t)| + \sup_{t \in I} |u'(t)|;$$

U is the set of u with $|u| < R$.

Z is $[a, b]$; so for $\alpha \in Z$ the segment $[0, \alpha]$ belongs to $[a, b]$. The image space F is $E_1 \times C^0(I, E_1)$; the second factor with sup norm. Thus, both E and F are complete normed vector spaces, and

$$\beta: u \mapsto (u(0), u') \tag{6.3}$$

is an isomorphism of E to F. (The proofs of the statements in this sentence are instructive exercises.)

The map f is now defined as a pair (f_1, f_2), where f_1 has values in the first factor E_1, and f_2 has values in $C^0(I, E_1)$; so the values of f_2 are continuous functions $I \to E_1$:

$$f_1(u, \alpha) = u(0), \qquad f_2(u, \alpha)\,(t) = u'(t) - \alpha\, \mathfrak{F}(\alpha t, u(t)). \tag{6.4}$$

For x_0 and z_0 in Theorem 2, we take the zero elements of $C^1(I, E_1)$ and of $\mathbb{R}$. Thus, the known solution (x_0, z_0) of $f(x, z) = 0$ is the trivial solution $u \equiv 0$ of the initial value problem (6.2) with $\alpha = 0$, $u_0 = 0$.

To apply Theorem 2 its hypotheses have to be verified; i.e., we have to show that $(D_1 f)_{(0,0)}$ exists, is strong, and is an isomorphism. The resulting local inverse map ϕ will satisfy $f(\phi(y, z), z) = y$; where $y = (y_1, y_2)$; we take $y_1 = u_0 \in E_1$,

$$y_2 = 0 \in C^0(I, E_1)$$

(constant map), $z = \alpha$; then $u = \phi(u_0, 0, \alpha)$ satisfies $f_1(u, \alpha) = u_0$, $f_2(u, \alpha) = 0$; i.e., by (6.4)

$$u(0) = u_0, \; u'(t) - \alpha\, \mathfrak{F}(\alpha t, u(t)) = 0,$$

which means it solves Picard's problem (6.2). The strong partial differentiability of ϕ at the zero element of $F \times Z$ results in a Lipschitz dependence of u and u' on u_0 for $|\alpha|$ small but not zero.

To prove the hypotheses of Theorem 2 we first show that β of (6.3) is the derivative $(D_1 f)_{(0,0)}$; we set $\beta_1(u) = u(0)$, $\beta_2(u)\,(t) = u'(t)$. We choose $\varepsilon > 0$ and try to find δ_1 and $\delta_2 > 0$ such that

$$|u_1| < \delta_1, \quad |u_2| < \delta_1, \quad |\alpha| < \delta_2$$

imply

$$|f_1(u_1,\alpha) - f_1(u_2,\alpha) - \beta_1(u_1 - u_2)| \leqq \varepsilon |u_1 - u_2|$$

and

$$|f_2(u_1,\alpha) - f_2(u_2,\alpha) - \beta_2(u_1 - u_2)| \leqq \varepsilon |u_1 - u_2|.$$

The first of these is trivial, as $f_1(u,\alpha) = \beta_1(u) = u(0)$. The left side of the second inequality is equal to

$$\begin{aligned}
&\sup_{t \in I} |u_1'(t) - \alpha\mathfrak{F}(\alpha t, u_1(t)) - u_2'(t) + \alpha\mathfrak{F}(\alpha t, u_2(t)) - (u_1'(t) - u_2'(t))| \\
&= \sup_{t \in I} |\alpha\mathfrak{F}(\alpha t, u_1(t)) - \alpha\mathfrak{F}(\alpha t, u_2(t))| \leqq |\alpha| \cdot L \sup_{t \in I} |u_1(t) - u_2(t)| \\
&\leqq |\alpha| \cdot L \cdot |u_1 - u_2|,
\end{aligned}$$

and this is $\leqq \varepsilon |u_1 - u_2|$ when $\delta_1 = R$, $\delta_2 = \varepsilon / L$.

Last, we show that $f(0,\alpha)$ is continuous in α at $\alpha = 0$:

$$\begin{aligned}
|f_1(0,\alpha) - f_1(0,0)| &= 0, \\
|f_2(0,\alpha) - f_1(0,0)| &= \sup_{t \in I} |\alpha\mathfrak{F}(\alpha t, 0)| \\
&\leqq |\alpha| \sup_{t \in [a,b]} |\mathfrak{F}(t,0)|.
\end{aligned}$$

As $\sup|\mathfrak{F}|$ is finite, the proof is complete.

The matter of dependence on parameters (classically one takes $\mathfrak{F}(t,u,\lambda)$ instead of $\mathfrak{F}(t,u)$) can very eleganty be handled by considering $\mathfrak{F}$ itself as a parameter; i.e., one considers *all* maps $\mathfrak{F}$ near a map $\mathfrak{F}_0$ in a parameter space Z' with an appropriate topology. For example, Z' could consist of all continuous maps $\mathfrak{F}$ on $[a,b] \times B_R$ with values in E_1 and with Lipschitz constant $\leqq L$; distance defined by

$$\sup |\mathfrak{F}_1(t,u) - \mathfrak{F}_2(t,u)|.$$

Then f and ϕ will depend continuously on $\mathfrak{F}$.

Another interesting possibility lies in a vector space E_2 of functions $\mathfrak{F}$ (same domain and range as above) with norm

$$|\mathfrak{F}| = \sup |\mathfrak{F}(t,u)| + \mathrm{Lip}\,(\mathfrak{F});$$

then f and ϕ depend strongly partially differentiably on $\mathfrak{F}$ at $\alpha = 0$. The reader may want to work out some consequences.

7. **Additional comments.** The work of Leach [**4**, **5**] introduces the strong derivative, and proves the implicit mapping theorem (our Theorem 2; without the parameter), in addition to other results. The strong partial derivative seems to be new, and is vital in obtaining Theorems 1 and 2. In fact, it would be most interesting to build up a theory just for strongly partially differentiable functions $f: U \to F$, $U \subset E \times X$ for various E, X, F, etc., where X is visualized as some immense parameter space of ways in which f can be varied. Theorem 1 would then be generalized to the assertion that if $f: U \to F$, $U \subset E_1 \times E_2 \times X$ is strongly partially differentiable at (u_0, v_0, x_0) with respect to E_1 and also with respect to E_2, then f is strongly partially differentiable with respect to $E_1 \times E_2$. Together with Theorems 2 and 3″ this would be the foundation for a very elegant theory of functions of several variables. Functions of a single variable would fit in well; also: the formulas for strong derivatives of the functions of elementary calculus are deduced just like the usual ones.

The method of proof of the Picard theorem from an implicit mapping theorem is not new; applications of this type came up after 1960 in several places stimulated by books such as the first editions of [**1**, **2**, **3**]. A proof of the Picard theorem for *differentiable* f along these lines is found in a paper by J. W. Robbin [**6**]. What is new in the present paper is the return of the Lipschitz condition.

Research partially supported by an N.S.F. grant at the University of Pennsylvania. A lecture covering part of this material was given by the author at Swarthmore College in 1969.

References

1. H. Cartan, Differential Calculus, Hermann, Paris; Houghton Mifflin, Boston, 1971.

2. J. Dieudonne, Treatise on Analysis I (Foundations of Modern Analysis), Academic Press, New York and London, 1969.

3. S. Lang, Differentiable Manifolds, Addison-Wesley, Reading, 1972.

4. E. B. Leach, A note on inverse function theorems, Proc. Amer. Math. Soc., 12 (1961) 694–697.

5. ———, On a related function theorem, *ibid.*, 14 (1963) 687–689.

6. J. W. Robbin, On the existence theorem for differential equations, Proc. Amer. Math. Soc., 19 (1968) 1005–1006.

Department of Mathematics, University of Pennsylvania, Philadelphia, PA 19174.

HOW I BECAME A MATHEMATICIAN

K. MAHLER

As I am now slightly over 70, my memory is not particularly good, and some of the dates may be slightly wrong; but the facts will be essentially correct.

My twin sister and I were the 7th and 8th children of my parents; four of my brothers and sisters had died before we two were born on July 26, 1903. My parents were middle class Jews, and, with all my four grandparents, came from the lower Rheinland, Prussia, Germany. My birthplace, Krefeld, not far from the Dutch border and on the western side of the river Rhein, was then a rather pretty town of about 100,000 inhabitants. The main industry in those days had to do with silk and its uses.

My father and several of my uncles worked in printing and bookbinding. We had a small printing firm dealing mainly with commercial work, and my father had started at the bottom since none in our family was well to do. The printing firm lasted till the coming of the Nazis in 1933, and my only surviving brother was due to take over the firm when this happened.

There was no academic tradition in our family or in those of my relatives, and it is doubtful whether I should have ended at a university if I had been in good health. We all were great readers, and we read anything we could get, without much discrimination. In those days (but not after the First World War), our home was still run on strictly orthodox Jewish lines, and we were also all good German patriots. Krefeld belonged to Prussia, was in a mainly Catholic district, and there was even then a certain amount of anti-Semitism, although more as a reminder of the past.

My twin sister was very healthy, good at languages, and later an excellent business woman, but mathematics was not in her line. She had one child, but died early in the Nazi period; her husband is still alive. My brother was less intellectually inclined. He served in the First World War in France until he was very badly wounded and therefore released from the army with the Iron Cross and a promotion. He and his wife, and many more of my relatives, were put to death in German concentration camps during the Second World War. My elder sister, now over 80, is still alive. She, her husband (recently deceased), and her only daughter have been living for many years in the Netherlands. Both she and her husband were mainly interested in music. In contrast, although I can somewhat appreciate the great composers from Bach to Beethoven, I find most of the earlier and the later music mainly noise and nuisance! I have always preferred things I can see to things I can hear.

My health never was good. At about 5, I became a victim of tuberculosis in my right knee. There was no good treatment in those days, and after an operation when I was about 8, the knee became stiff, the leg bent at the knee, and there was for many years an open wound. I was thus very much hindered in walking. This remained so

until 1937 when the doctors finally could remove the infection, straighten the leg, and once and for all get rid of the disease.

Between my 7th and my 11th year I was at school only for short periods, but had some private tuition at home in the three R's. Then, until I was not quite 14, I went to the *Volksschule* (elementary school), a Jewish one, where in spite of the small staff, the teaching in the true Prussian spirit was surprisingly good. It involved only elementary subjects and did not include mathematics, apart from arithmetic. My worst subject was *Schönschreiben* (calligraphy), but in later years I was to acquire a very satisfactory way of writing.

When I left the *Volksschule* some three months before my 14th birthday, my parents thought I should try to become a *Fein-Mechaniker* (precision mechanic) because this would allow me to work sitting. To prepare for this, I went for the next two years to elementary technical schools in Krefeld. It was there that I made my first contacts with elementary algebra and geometry, and I think that this was the moment when I decided to become a mathematician. In the summer vacation of 1917, on my 14th birthday, I bought a logarithm table and spent the vacation most pleasantly by doing all kinds of calculations with it. Soon afterwards, I obtained books on trigonometry and analytic geometry, and a little later, books on calculus. By about 1919 I began the study of books of university standard[1], learning about groups, invariants, sets, and a little later, about analytic functions, elliptic functions, modular functions, number theory, non-Euclidean geometry, etc. I had no teachers to help me, and I therefore tried to teach a couple of my fellow scholars at the technical school some of the pretty things I was attempting to understand!

About this time, in 1919, I entered a machine factory in Krefeld as an apprentice. The reason for this was that, if I could conclude the apprenticeship, I might be allowed to enter a *Technische Hochschule* (technical college), without first passing the very difficult *Abiturienten-Examen* (university entrance examination). At such a *Technische Hochschule* I should then be able to study mathematics.

The machine factory made machines for producing screws and bolts, some of them quite large, but all rather simple mechanically. I spent one year at their drawing office and not quite two in the factory itself, learning to handle simple instruments and machines. The technical drawing proved later very useful in my own work and also enabled me to help colleagues with such drawings, e.g., L. J. Mordell[2].

I was in the factory not quite three years. Then a happy accident happened. The director of the local *Realschule* (i.e., headmaster of a middle school), a very good mathematican and former student of Christoffel, learned of my studies and sent some of my attempts to Felix Klein in Göttingen[3]. Klein, who was already old and in bad health, gave these to C. L. Siegel who was then doing research in Göttingen, for a report. Siegel found a number of errors, but recommended that I should be helped to pass the *Abiturienten-Examen* so that I could proceed to a university.

As a consequence, several teachers at the *Realschule* helped me in the next years

with languages (French and English) and other subjects, and at last, in the summer of 1923, I was admitted as an outsider to the *Abiturienten-Examen.*

At this time, under the Weimar Republic, Prussia had a very liberal government, and they allowed me, against the usual rules, to take the examination at the *Ober-Realschule* in my home town, rather than elsewhere. This was of great importance, for it was the time of the great inflation in Germany, of the French occupation of the Ruhr, and of the pro-French separatist movement in the Rheinland; there was even some shooting in Krefeld between the separatists and the police.

The subjects of my examination were German, French, English, Mathematics, Physics, Chemistry, History, Geography, and possibly some more which I have forgotten. For five successive days I was given written tests in the different subjects, and then I had to endure an oral examination. I just passed in most of these subjects, and I obtained good marks only in mathematics, physics, and history. But I was successful, and hence a couple of months later I was able to enter the University of Frankfurt am Main as a full student, shortly after I had reached the age of 20.

I stayed in Frankfurt for three semesters until early 1925, and from then until the coming of Hitler in 1933 studied and did research at the University of Göttingen[4]. Then came the exodus, and, except for two years at the University of Groningen in the Netherlands, I stayed for more than a quarter of a century at the University of Manchester, England. But by this time it had of course become clear that I had become a mathematician!

Notes

1 My main advanced books before I went to the University were as follows: Pascal, *Repertorium der Mathematik*, Vol. 1, Section 1; Cesáro-Kowalewski, *Algebraische Analysis;* Clebsch-Lindemann, *Geometrie;* Klein, and Klein and Fricke, *Das Ikosaeder; Modul-Funktionen; Automorphe Funktionen;* Hilbert, *Grundlagen der Geometrie;* Bachmann, *Zahlentheorie* (several volumes); Landau, *Primzahlen;* Knopp, *Analytische Funktionen.*

2 See Mordell's papers on geometry of numbers, mainly J. London Math. Soc., between 1941 and 1945.

3 My notes dealt with irrationality of certain series; the solution of a quartic equation by means of Klein's Oktaeder equation; and with three-dimensional hyperbolic geometry. They were sent to Klein at different dates.

4 My teachers in Frankfurt were Dehn, Epstein, Hellinger, Szász, and in particular Siegel, who taught me a lot privately. In Göttingen, I was particularly influenced by Courant and Emmy Noether, and more indirectly by Landau. Although then at Göttingen, I submitted my Ph. D. thesis *Über die Nullstellen der unvollständigen Gammafunktionen* in Frankfurt where Szász acted as the examiner. This was in 1927.

DEPARTMENT OF MATHEMATICS, THE AUSTRALIAN NATIONAL UNIVERSITY, P. O. BOX 4, CANBERRA, ACT. 2600, AUSTRALIA.

REPORT OF THE COMMITTEE ON NEW PRIORITIES FOR UNDERGRADUATE EDUCATION IN THE MATHEMATICAL SCIENCES

This is a condensation of the report presented to the Board of Governors at the August 1973 meeting in Missoula, Montana. Full copies of the report are available free of charge to members through the Washington office of the MAA.

The Board of Governors, at the August 1971 meeting of the MAA, was acutely aware that there are many problems now faced by the educational community. One of the pervasive problems is the financial crisis now faced by almost all colleges and universities. Another is the recent change from a rapidly expanding national student body to a stable or decreasing one and the effect of this change on employment prospects. With these concerns and others in mind, the Board of Governors passed a resolution requesting President Klee to appoint an *ad hoc* Committee on New Priorities for Undergraduate Education in the Mathematical Sciences. This Committee was asked to "identify the most important problems that have arisen in mathematical education and to recommend initial steps leading to their solution." The task is not to find masks to disguise the problems, but to assess the present situation and to suggest ways we might go from here.

The Committee finds more problems than solutions, and most of the problems lack the precise formulation that mathematicians desire. The Committee also finds that CUPM has already considered and made important inroads into a number of these problems, but many of them admit no permanent solution. While this Committee does not believe that it has identified all of the problems or all of the solutions, we do believe that we have identified some very serious problems now facing the mathematical community.

Some Problems and Recommendations.

We face a host of educational, social, professional and economic problems.

(A) *The image of science and mathematics to the student.* Many students regard physical science and mathematics to be either irrelevant or dangerous to society. (See references [9], [11], [14], [15], [16].)

(i) *We urge continuing development of courses on the nature of mathematics and its application to society.* Experiences of teachers who have given such courses would be helpful.

(ii) *We need to improve the image of mathematics and mathematicians.* The Corporation of Public Broadcasting or other agencies may help us reach the public. Articles suitable for the popular media should be encouraged.

(B) *The mathematician as an inhabitant of the ivory tower.* Many people who recognize that mathematics is important believe that much current mathematical

training is not particularly useful. (See **[5]** and **[10]**.) We see three likely faults in our program. First, students often lack exposure to applied problems and modeling. Second, there is insufficient exposure to other disciplines. Finally, our students have little experience in written exposition. *We must reorient our students toward meaningful applications, convincing them that the very formulation of a problem in mathematical terms is an achievement and often the major step toward its solution.* In the past, students in mathematics were required to minor in physics or another science. Today, few programs require an outside minor. Without taking a stand on the issue of formal outside minors, we urge that

(i) *significant applications be included in our courses when appropriate,*

(ii) *a course in modeling be required of all mathematics majors,*

(iii) *undergraduate and graduate programs be reviewed in light of this problem.*

Thus, we should expect to require more problem-oriented courses as well as courses in model-building, computing, numerical analysis, statistics, probability, operations research, and other sciences. Individuals with experiences to report should share them through the MONTHLY or the Clearing Agents (to be described below).

(C) *Training of high school teachers.* While it is important for teachers of mathematics to study foundational aspects of mathematics, they should also learn applications of mathematics at a level appropriate to what they will be teaching. *We join CUPM in urging that teacher-training programs include significant applications of mathematics.*

(D) *Relations with our non-mathematical colleagues* (cf. [15, p. 36]). We recommend that

(i) *a series of articles entitled "Mathematics and Subject X" be written for the Monthly outlining the mathematical background useful in Subject X,*

(ii) *articles be published describing experiences in interdisciplinary projects.*

We also urge that more mathematicians study outside subjects — for example, by participating in seminars in Subject X for two or three years. The idea is to identify problems which might yield to mathematical analysis as well as to provide a bridge between the Department of Mathematics and the Department of Subject X. We should seek opportunities to teach joint courses.

(E) *Educational problems.* NSF has recommended more emphasis on science courses for the non-science major [**13**, p. 27]. However, we should guard that these are honest courses with real content. The same NSF report states:

> "Narrowing down to the science major himself, we reiterate the need for scientists who will be at home in society. Let us add the hope that these scientists will find a welcome home in society. This requires education beyond a single science discipline and indeed beyond science."

When projects involve mathematics, they should involve mathematicians. *We urge the NSF and other governmental agencies to increase their consultation with the mathematical community about educational problems which concern mathematics.*

There are many problems in education.

(i) *Two-year colleges.* We need to be attentive to the needs of the student at the two-year colleges, the problems of the transfer student, the staffing problems, and whether recent Ph. D.'s can be utilized as staff members. THE TWO-YEAR COLLEGE MATHEMATICS JOURNAL and the MONTHLY provide suitable forums for discussion of these problems.

(ii) *Continuing Education.* There is increasing interest in making education a "continuing process," with alternate periods of study and work. There is also a need to retrain and offer refresher courses to midcareer engineers, etc. Individuals with ideas and experience in such courses should share their wisdom.

The Carnegie Commission, in their pamphlet "Less Time, More Options" [**8**] recommends wider use of the Associate of Arts, Master of Philosophy, and Doctor of Arts degrees. We feel that even though the present Ph. D. may not meet all of our needs, this is an inopportune time to advocate a new degree at approximately the same level. Instead, Ph. D. training should be broadened to include seminars on teaching and its relation to subject matter.

"Open Universities" are being created. We should be active in this area to ensure that decisions about content and credit in mathematical subjects will be made, at least in part, by mathematicians. Perhaps the MAA should oversee the development of proficiency examinations

(iii) *Teaching techniques.* While most of us have strong opinions on teaching, we have little, if any, scientific information. It seems time to investigate carefully such questions as the effectiveness of large lecture sections, programmed materials, teaching machines, TV, and tapes. However, we must be wary of measurements of short-term results, for it is the long-range effect that is important.

In the past, CUPM has been concerned with problems of the type reported here, and has made many recommendations. We think it would be useful to study the impact of these recommendations. To fill this need as well as other information gathering services, *we propose that the MAA appoint Clearing Agents for specific educational areas.* Each Clearing Agent should stay informed of the current state of knowledge, collect and circulate information, possibly assemble a descriptive pamphlet, and generally endeavor to keep the community and the MAA informed of ideas of merit. The names of the Clearing Agents should appear on the inside cover of the MONTHLY.

Descriptions of important panel discussions at national MAA meetings should be available through the Washington office.

(F) *The problems of society.* We are not likely to find easy solutions to any of the problems of society. However, some problems are amenable to mathematical

modeling, and Clearing Agents should be appointed to help stimulate efforts. More mathematicians should become involved in the RANN program of NSF. In addition, mathematicians should join colleagues in other disciplines to formulate proposals concerning national problems. As an experiment *we recommend the formation, under experienced guidance and federal funding, of a few interdisciplinary projects to work on important societal problems.*

(G) *Employment problems.* A great deal has been written about the job market ([**1**]–[**4**], [**6**], [**12**], [**16**], [**17**]). We would like to see some changes in existing educational programs with more of them directed toward modern applications and potential non-academic employment [7]. We all agree that we should continue to train the very best graduate students; otherwise, we might wipe out an entire generation of American mathematics ([**11**], [**12**]). *We should press for a substantial number of postdoctoral fellowships for young Ph.D's to enable them to broaden their education and usefulness.*

(H) *Our relations with government.* Decisions affecting mathematics have often been made by governmental agencies without adequate consultation with mathematicians ([**11**], [**12**], [**18**]). *We should press for a strengthened influence in governmental decisions.* While we must speak as responsible citizens, we should summon whatever political muscle we have in order to press our points. It is important that *serious efforts be made to consolidate the voices of the various mathematical organizations.* In particular, the activities of CBMS are of great importance and should be continued.

(I) *The future of CUPM.* CUPM's future is of deep concern. Virtually all problems reviewed here have been considered by CUPM, and many of our recommendations duplicate theirs. *It is very important to keep CUPM operating.*

Conclusion.

The recommendations we have made are largely of an advisory (and sometimes very general) nature. Some of them, we think, *can* be implemented by individuals and by individual departments directly, but others will require larger and more unified efforts. To assist in these efforts, we call for an increased use of the MONTHLY as a forum on innovation, the introduction of a *Queries* section in the MONTHLY, and the appointment of *Clearing Agents* in various areas of concern. The Washington office of the MAA should oversee and coordinate much of the above activity; because of these and other added duties, we recommend the appointment of an *Associate Director*. As long as CUPM survives, we shall look to it for guidance, although we should not depend solely on it for the generation of ideas or for their implementation.

Ultimately, we feel that what is needed is improved communication within the

mathematical community, and with other groups in society. As individuals and as a community, we must not only keep alive mathematically, but we must be flexible and sensitive to change and to the needs of society. Projects, committees, and panels can often be extremely valuable in suggesting desirable changes of emphasis and direction for our teaching. However, if these changes are to take place, in the long run they must be accomplished by us in our studies and classrooms. And, if they do not take place, then it is likely that mathematics will drift into a narrow ivory tower.

RICHARD D. ANDERSON
RALPH P. BOAS
WILLIAM G. CHINN
BURTON H. COLVIN
JOHN W. GREEN
BRUCE E. MESERVE
WALTER E. MIENTKA
HENRY O. POLLAK
ROBERT G. BARTLE, *Chairman*

References

1. R. D. Anderson, Are there too many Ph.D.'s?, this MONTHLY, No. 6, 77 (1970) 626–641.

2. ———, Employment of new Ph.D.'s, Notices of the Amer. Math. Soc., No. 7, 18 (1971) 1021–1026.

3. ———, Academic employment prospects for September 1972, Notices of the Amer. Math. Soc., No. 2, 19 (1972).

4. ———, et al., Statement on employment of Ph.D.'s in mathematics, Notices of the Amer. Math. Soc., No. 3, 18 (1971) 486–490.

5. E. H. Bareiss, The college preparation for a mathematician in industry, this MONTHLY No. 9, 79 (1972) 972–984.

6. W. L. Duren, Are there too many Ph.D.'s in mathematics?, this MONTHLY, No. 7, 77 (1970) 641–646.

7. H. D. Greenspan, Applied Mathematics at Massachusetts Institute of Technology, this MONTHLY, No. 1, 80 (1973) 67-72.

8. *Less Time, More Options*, A special report and recommendations by the Carnegie Commission on Higher Education, McGraw-Hill, Hightstown, N. J., January, 1971.

9. D. W. Lick, New directions and commitments for mathematics, Notices of the Amer. Math. Soc., No. 6, 19 (1972) 272–276.

10. C. V. Newsom, The image of the mathematician, this MONTHLY, No. 8, 79 (1972) 878–882.

11. L. J. Paige, Public understanding of science and its implication for mathematics, this MONTHLY, No. 2, 78 (1971) 130–142.

12. ———, The problem of support for the mathematical sciences, CBMS Newsletter, No. 4, 7, (1972) 1-7.

13. *Science Education — The Task ahead for the National Science Foundation*, Report of the Advisory Committee for Science Education, published by the National Science Foundation, March 1970, Washington, D.C.

14. W. G. Spohn, Jr., Can mathematics be saved?, Notices of the Amer. Math. Soc., No. 6, 16 (1969) 890–894.

15. A. B. Willcox, England was lost on the playing fields of Eton: a parable for mathematics, this MONTHLY, No. 1, 80 (1973) 25–40.

16. M. B. Williams, Letter to the editor, Notices of the Amer. Math. Soc., No. 3, 18 (1971) 502–503.

17. G. S. Young, The problems of employment in mathematical sciences, Notices of the Amer. Math. Soc., No. 5, 18 (1971) 718–722.

18. ———, The crises of the mathematical sciences, and why no one group can solve them, this MONTHLY, No. 9, 78 (1971) 980–987.

QUERIES

Edited by A. C. Zitronenbaum

This Department welcomes queries from readers about mathematics at the collegiate level, such as sources for exposition of a particular topic from a special point of view, references to vaguely remembered articles, descriptions of special kinds of courses or teaching methods, and methods for constructing illustrative examples for exercises of particular kinds (questions on research topics should, in general, be addressed to the "Queries Department" of the Notices of the American Mathematical Society). Replies will be forwarded to the questioner and may also be edited into a composite answer for publication in this Department. Consequently all items submitted for consideration for possible publication should include the name and complete mailing address of the person who is to receive the reply. Queries and answers should be sent to A. C. Zitronenbaum, Department of Mathematics, Cornell University, Ithaca, NY 14853.

13. E. I. Sheide. In order to supply role models for my women students, I would very much like to obtain a list of outstanding women mathematicians, both pure and applied, who are currently active. Brief descriptions of their accomplishments would be especially appreciated.

14. P. Malraison, P. Campbell. We would like to know of any unusual (non-Modern Learning Aids, non-International Film Board) sources of mathematical films.

15. P. Mielke. What success can be reported in using media other than textbooks, for example: slides, tape cassettes, movies, etc., in the teaching of the Calculus?

MATHEMATICAL NOTES

EDITED BY DAVID ROSELLE

Material for this Department should be sent to David Roselle, Department of Mathematics, Virginia Polytechnic Institute, Blacksburg, VA 24061.

APOLLONIUS AND INNER PRODUCTS

DESMOND FEARNLEY-SANDER AND J. S. V. SYMONS

The theorem of Apollonius in Euclidean geometry asserts that the sum of the squares of the lengths of the two diagonals of a parallelogram is equal to the sum of the squares of the lengths of the four sides. This is a generalization of the more famous theorem of Pythagoras, which asserts the same thing for rectangles. An intriguing feature of the Apollonius theorem is that, unlike Pythagoras, it makes sense in affine geometry; in affine geometry one has the concepts of parallelism and a rudimentary translation-invariant distance which are needed to state Apollonius, while one does not have the concept of a right angle which is needed to state Pythagoras. Of course it is one thing to say that Apollonius makes sense in an affine geometry and another to say that it is valid; its validity depends on how distances along non-parallel lines are related to one another. We claim that if Apollonius is valid then the metric, and hence the geometry, is Euclidean.

We must be a bit more explicit. It is enough to say that an *affine geometry* is determined in all essential respects by a real vector space $\mathscr{V}$ (the space of translations), and that the distance referred to corresponds to a rudimentary length $\| \ \| : \mathscr{V} \to \mathbb{R}$ which satisfies

(1) *for x in $\mathscr{V}$*

$$\|x\| \geqq 0, \text{ and } \|x\| = 0 \Rightarrow x = 0, \text{ and}$$

(2) *for α in $\mathbb{R}$ and x in $\mathscr{V}$*

$$\|\alpha x\| = |\alpha| \, \|x\|.$$

We say that $\| \ \|$ (and the corresponding distance) is *Euclidean* if there is an inner product $\langle \ , \ \rangle : \mathscr{V} \times \mathscr{V} \to \mathbb{R}$ such that

(3) *for x in $\mathscr{V}$*

$$\langle x, x \rangle = \| x \|^2;$$

by imposing this inner product on $\mathscr{V}$ we get a *Euclidean geometry*. To validate our claim we must show that for $\| \ \|$ satisfying (1) and (2) to be Euclidean it is sufficient to have the Apollonius property:

(4) *for* x *and* y *in* $\mathscr{V}$

$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2 .$$

This was proved by von Neumann and Jordan [1] in the case where $\| \;\|$ is actually a norm. In fact, one does not need the triangle inequality.

Suppose then that we have a function $\| \;\| : \mathscr{V} \to \mathbb{R}$ which satisfies (2) and (4). We shall show that $\langle \; , \; \rangle : \mathscr{V} \times \mathscr{V} \to \mathbb{R}$ defined by

(5) $$\langle x, y\rangle = \tfrac{1}{2}(\|x+y\|^2 - \|x\|^2 - \|y\|^2) \textit{ for } x \textit{ and } y \textit{ in } \mathscr{V}$$

is a symmetric bilinear form satisfying (3) (and hence an inner product, if (1) is added to our hypotheses).

We briefly give the ingenious proof of Jordan and von Neumann that $\langle \; , \; \rangle$ is bi-additive. Using (4) several times we see that for x, y and z in $\mathscr{V}$

$$\begin{aligned} 4\langle x, y\rangle + 4\langle z, y\rangle &= 2\|x+y\|^2 + 2\|z+y\|^2 - 2\|x\|^2 - 2\|z\|^2 - 4\|y\|^2 \\ &= \|x+2y+z\|^2 - \|x+y\|^2 - 4\|y\|^2 \\ &= 4\langle x+z, y\rangle . \end{aligned}$$

Symmetry is obvious and hence $\langle \; , \; \rangle$ is additive in its second argument too.

Let m and n be natural numbers. From additivity it is clear that for x and y in $\mathscr{V}$ we have $\langle nx, y\rangle = n\langle x, y\rangle$; hence $m\langle m^{-1}nx, y\rangle = \langle nx, y\rangle = n\langle x, y\rangle$. Using (4), which says precisely that $\langle -x, y\rangle = -\langle x, y\rangle$, we conclude that for all rational ρ (including 0)

$$\langle \rho x, y\rangle = \rho\langle x, y\rangle .$$

To get this for all real ρ we must first prove the Cauchy-Schwartz inequality. For given x and y in $\mathscr{V}$, define a polynomial p by

$$p(\rho) = \|x\|^2\rho^2 + 2\langle x, y\rangle\rho + \|y\|^2 .$$

For all rational ρ

$$p(\rho) = \|\rho x + y\|^2$$

so that $p(\rho)$ is non-negative; since p is continuous, $p(\rho)$ must be non-negative for all real ρ. It follows that the discriminant of p is negative, and so

$$|\langle x, y\rangle|^2 \leqq \|x\|^2 \, \|y\|^2 .$$

For any real α and x and y in $\mathscr{V}$

(6) $$|\langle \alpha x, y\rangle - \alpha\langle x, y\rangle| \leqq |\langle(\alpha-\rho)x, y\rangle| + |(\rho-\alpha)\langle x, y\rangle|$$

for all rational ρ; using the Cauchy-Schwartz inequality we see that the right-hand side of (6) can be made arbitrarily small by choosing ρ to be close enough to α,

and we conclude that

$$\alpha\langle x, y\rangle = \langle \alpha x, y\rangle .$$

This is the one point of our argument at which we need to use (2); and even here one can get by with the weaker hypothesis

(2′) *for every x in $\mathscr{V}$ the function $\alpha \mapsto \|\alpha x\|$ on $\mathbb{R}$ is continuous at* 0.

Careful readers will notice that to do this one needs the fact that $\|0\| = 0$; this is easily deduced from (4) without using (2).

One can of course derive (3) directly from (2), but here again (4) will do the job. Let $x \in \mathscr{V}$ and write $y = x/4$; then

$$\begin{aligned}\langle x, x\rangle &= -4\langle -2y, 2y\rangle \\ &= 2\|2y\|^2 + 2\|-2y\|^2 - 2\|0\|^2 \\ &= \|4y\|^2 \\ &= \|x\|^2 .\end{aligned}$$

An immediate consequence is non-negativity of $\langle\ ,\ \rangle$.

We have proved our claim, and in fact something stronger:

THEOREM. *Let $\mathscr{V}$ be a real vector space and let $\|\ \|: \mathscr{V} \to \mathbb{R}$ be a function satisfying* (2′) *and* (4). *Then the function $\langle\ ,\ \rangle: \mathscr{V} \times \mathscr{V} \to \mathbb{R}$ defined by* (5) *is a non-negative symmetric bilinear form which satisfies* (3).

There is a complex version of this theorem: if $\mathscr{V}$ is a complex vector space and $\|\ \|: \mathscr{V} \to \mathbb{R}$ is a function satisfying (4) and (2), then the function $\langle\ ,\ \rangle: \mathscr{V} \times \mathscr{V} \to \mathbb{C}$ with real part given by

$$\operatorname{Re}\langle x, y\rangle = \tfrac{1}{2}(\|x + y\|^2 - \|x\|^2 - \|y\|^2)$$

and imaginary part given by

$$\operatorname{Im}\langle x, y\rangle = \operatorname{Re}\langle x, iy\rangle$$

is a non-negative sesquilinear Hermitian form which satisfies (3). The Theorem shows that the real part of $\langle\ ,\ \rangle$ is a non-negative symmetric bilinear form over $\mathbb{R}$; one can prove the complex version by using this fact and repeatedly applying (4).

The Apollonius property (4) played so strong a part in proving the theorem that one wonders whether the very weak continuity condition (2′) can be omitted from the hypotheses. This is not so, since even in the case $\mathscr{V} = \mathbb{R}$, a function $\|\ \|$ on $\mathscr{V}$ may satisfy (4) without $\langle\ ,\ \rangle$ given by (5) being homogeneous. To see this, select a Hamel basis $\mathscr{B}$ for $\mathbb{R}$ considered as a vector space over the rationals; we may assume that 1 and $\sqrt{2}$ belong to $\mathscr{B}$. Define $\|\ \|: \mathbb{R} \to \mathbb{R}$ by

$$\|x\| = \left(\sum_{i=1}^{n} \xi_i^2\right)^{1/2},$$

where $x = \sum_{i=1}^{n} \xi_i b_i$ is the expression for x as a rational linear combination of the elements b_i of $\mathscr{B}$. Clearly $\| \ \|$ satisfies (4) (and (1) as well); but

$$\| \sqrt{2} \|^2 = 1 = \| 1 \|^2$$

and so $\langle \ , \ \rangle$ is not homogeneous.

It follows easily from our theorem that for a scalar function Q on $\mathscr{V}$ to be a quadratic form it is sufficient that

(1′) *for x in $\mathscr{V}$*

$$Q(x) \geqq 0,$$

(2″) *for every x in $\mathscr{V}$ the function $\alpha \mapsto Q(\alpha x)$ on $\mathbb{R}$ is continuous at* 0, *and*

(4′) *for x and y in $\mathscr{V}$* $Q(x+y) + Q(x-y) = 2Q(x) + 2Q(y)$.

In an interesting paper, Gleason asks whether the purely algebraic conditions (4′) and

(2‴) *for x in $\mathscr{V}$ and α in $\mathbb{R}$*, $Q(\alpha x) = \alpha^2 Q(x)$

are sufficient and finds that they are not: these two conditions guarantee homogeneity of the associated bi-additive form with respect to algebraic scalars but not with respect to transcendental scalars.

References

1. P. Jordan and J. von Neumann, On Inner Products in Linear Metric Spaces, Ann. of Math. (2), 36 (1935) 719–723.

2. A. M. Gleason, The Definition of a Quadratic Form, this MONTHLY 73 (1966) 1049–1056.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF WESTERN AUSTRALIA, NEDLANDS, W. AUSTRALIA 6009.

DEPARBMENT OF MATHEMATICS, MONASH UNIVERSITY, CLAYTON, VICTORIA 3168, AUSTRALIA.

A CHARACTERIZATION OF COMPLETELY MULTIPLICATIVE ARITHMETIC FUNCTIONS

T. B. CARROLL

An arithmetic function f which is not identically zero is called *multiplicative* if $f(mn) = f(m)f(n)$ whenever $(m, n) = 1$, and it is called *completely multiplicative* (or linear) if $f(mn) = f(m)f(n)$ for all m and n.

Apostol [1] has given numerous characterizations of completely multiplicative arithmetic functions in his 1971 survey article in this MONTHLY. Two such are:

(1) $f(p^a) = f^a(p)$ for all primes p and all integers $a \geqq 1$;

(2) $f^{-1}(p^a) = 0$ for all primes p and all integers $a \geqq 2$.

In (2), f^{-1} is defined by

$$[1/n] = \sum_{d|n} f(d)f^{-1}(n/d).$$

It is the purpose of this note to exhibit another characterization not in the literature heretofore. It is based on a generalization of the von Mangoldt Λ-function of prime number theory given by Rearick [2]. Rearick's logarithm operator L is defined as follows: If $f \in P$ (the set of realvalued arithmetic functions such that $f(1) > 0$), then

$$Lf(n) = \begin{cases} \sum_{d|n} f(d)f^{-1}(n/d)\log d & \text{if } n > 1, \\ \log f(1) & \text{if } n = 1. \end{cases}$$

THEOREM. *If $f \in P$, then f is completely multiplicative if and only if, for all primes p and all integers $a \geqq 1$,*

$$Lf(n) = \begin{cases} (\log p)f^a(p) & \text{if } n = p^a, \\ 0 & \text{elsewhere.} \end{cases}$$

To facilitate the proof we state the following two propositions.

LEMMA. *For p a prime, if $f(1) = 1$ and $f(p^i) = f^i(p)$ for $2 \leqq i \leqq n$, then $f^{-1}(p^i) = 0$ for $2 \leqq i \leqq n$.*

THEOREM (Rearick, Theorem 4). *If $f \in P$, then f is multiplicative if and only if $Lf(n) = 0$ whenever n is not a power of a prime.*

Proof of Characterization. If f is completely multiplicative, then f is multiplicative and hence by Rearick's theorem $Lf(n) = 0$ whenever n is not a power of a prime. If $n = p^a$ where p is a prime and a is a positive integer, then by characterizations (1) and (2)

$$\begin{aligned} Lf(n) &= \sum_{i=0}^{a} f(p^i)f^{-1}(p^{a-i})\log p^i \\ &= f^a(p)\log p^a + f^{a-1}(p)f^{-1}(p)\log p^{a-1} \\ &= af^a(p)\log p - (a-1)f^{a-1}(p)f(p)\log p \\ &= f^a(p)\log p. \end{aligned}$$

Conversely, f is multiplicative by Rearick's theorem. Hence by characterization (1) it suffices to show $f(p^a) = f^a(p)$ for every prime p and every positive integer a. Let p be an arbitrary prime. Since f is multiplicative $f(1) = f^{-1}(1) = 1$ and $f^{-1}(p) = -f(p)$. Thus

$$\begin{aligned} Lf(p^2) = (\log p)f^2(p) &= \sum_{i=0}^{2} f(p^i)f^{-1}(p^{2-i})\log p^i \\ &= f(p^2)\log p^2 + f(p)f^{-1}(p)\log p. \end{aligned}$$

This implies $2(\log p)f^2(p) = 2(\log p)f(p^2)$. Hence

$$f^2(p) = f(p^2).$$

Assume a is any positive integer greater than two and $f^i(p) = f(p^i)$ for $2 \leqq i \leqq a-1$. By the above lemma $f^{-1}(p^i) = 0$ for $2 \leqq i \leqq a-1$. So

$$\begin{aligned} Lf(p^a) &= (\log p)f^a(p) \\ &= \sum_{i=0}^{a} f(p^i)f^{-1}(p^{a-i})\log p^i \\ &= f(p^a)\log p^a + f^{a-1}(p)f^{-1}(p)\log p^{a-1} + f^{-1}(p^a)\log 1. \end{aligned}$$

This implies $a(\log p)f^a(p) = a(\log p)f(p^a)$. Hence $f(p^a) = f^a(p)$ for each prime p and each positive integer a.

References

1. T. M. Apostol, Some properties of completely multiplicative arithmetical functions, this MONTHLY, 78 (1971) 266–271.

2. D. Rearick, Operators on algebras of arithmetic functions, Duke Math. J., 35 (1968) 761–766.

DEPARTMENT OF MATHEMATICS, WESTERN MICHIGAN UNIVERSITY, KALAMAZOO, MI 49001.

A NOTE ON THE INTERMEDIATE VALUE PROPERTY

D. A. NEUSER AND S. G. WAYMENT

A function f defined on $[a, b]$ is said to have the intermediate value property provided the closed interval from $f(x)$ to $f(y)$ is contained in the image of the closed interval from x to y for each x and y in $[a, b]$.

One of the more important and well known properties of continuous functions is that they possess the intermediate value property. Darboux's theorem states that if f' exists on $[a, b]$, then f' possesses the intermediate value property. This is a striking result since f' may fail to be continuous. If f and g are continuous on $[a, b]$, then $f + g$ has the intermediate value property since $f + g$ is continuous. If f' and g' exist on $[a, b]$, then $f' + g'$ has the intermediate value property since $f' + g'$ is the derivative of $f + g$. In [2] the following example is given (for another purpose) in which two functions f and g have the intermediate value property but $f + g$ does not. Let

$$F(t) = \begin{cases} t^2 \sin(1/t) & \text{if } t \neq 0 \\ 0 & \text{if } t = 0, \end{cases} \quad \text{and } G(t) = \begin{cases} t^2 \cos(1/t) & \text{if } t \neq 0 \\ 0 & \text{if } t = 0. \end{cases}$$

Then

$$F'(t) = \begin{cases} 2t \sin(1/t) - \cos(1/t) & \text{if } t \neq 0 \\ 0 & \text{if } t = 0 \end{cases}$$

and

$$G'(t) = \begin{cases} 2t\cos(1/t) + \sin(1/t) & \text{if } t \neq 0 \\ 0 & \text{if } t = 0. \end{cases}$$

If $f(t) = [F'(t)]^2$ and $g(t) = [G'(t)]^2$, then f and g have the intermediate value property because F' and G' do. But $(f + g)(t) = 4t^2 + 1$ if $t \neq 0$ and $(f + g)(0) = 0$. Hence $f + g$ does not possess the intermediate value property on any interval containing 0.

This raises the following elementary question. If f is continuous on $[a, b]$ and g has the intermediate value property, must $f + g$ have the intermediate value property? The authors conjecture that most persons would be inclined to answer yes. It may be somewhat of a surprise that the following example, whose construction is very complicated compared to the elementary nature of the question, answers the question in the negative. The techniques used in the construction of this example are somewhat standard in topology and were suggested to the authors by J. W. Cannon. Although somewhat involved, the proof that follows essentially uses only advanced calculus results.

We first generate a sequence $\{C_i\}$ of Cantor sets in the following way. Construct C_1 on $[0,1]$ and let L_1 be the largest number for which there exists at least one open interval (a_1, b_1) in $\sim C_1$ with length L_1. Construct C_2 on the closed middle third of (a_1, b_1). Let L_2 be the largest number for which there exists at least one open interval (a_2, b_2) in $\sim (C_1 \cup C_2)$ with length $L_2 \leqq L_1$. Construct C_3 on the closed middle third of (a_2, b_2). In general, construct C_n on the closed middle third of the open interval in $\sim (\bigcup_{i=1}^n C_i)$ with length L_n, where L_n is the largest number for which there exists at least one open interval in $\sim (\bigcup_{i=1}^n C_i)$ with length $L_n \leqq L_{n-1}$.

For each i, let g_i be a continuous, monotone increasing function which maps C_i onto $[0,1]$. The construction of such functions can be found in [**1**]. Define,

$$g(x) = \begin{cases} g_i(x) & \text{if } x \in C_i \\ 0 & \text{if } x \in [0,1] - \bigcup_{i=1}^\infty C_i. \end{cases}$$

We note that g has the intermediate value property, for given any arbitrary open interval (a, b) contained in $[0, 1]$, there is an i such that $C_i \subset (a, b)$ and $g(C_i) = [0, 1]$.

For a real valued function $p(x)$ defined on $[0, 1]$, let $\| p \| = \sup | p(x) |$. We next define a sequence $\{f_i\}$ of continuous functions on $[0, 1]$ such that $\sum_{i=1}^\infty \| f_i \| < \infty$ and, hence $f = \sum_{i=1}^\infty f_i$ is continuous on $[0, 1]$. The $\{f_i\}$ will be constructed in such a way that $(f + g)(x) \neq \frac{1}{2}$ for any value of x in $[0, 1]$. Since g is continuous on C_1, there is a finite collection U_1 of open intervals whose closures are pairwise disjoint which covers C_1 and such that if x and y are in $0_{1i} \in U_1$ for some i, then $| g(x) - g(y) | < \varepsilon_1 < \frac{1}{3}$. Let V_1 be the collection of those open intervals in U_1 which contain values of x satisfying $g(x) = \frac{1}{2}$. Define $f_1(x) = \varepsilon_1$ for the values of x covered by V_1, and let $f_1(x) = 0$ for the values of x which are covered by U_1 but not covered by V_1.

Extend f_1 to be continuous on $[0,1]$ with functional values between 0 and ε_1. If $x \in C_1$ and is covered by V_1, then $g(x) > \frac{1}{2} - \varepsilon_1$ and hence $(f_1 + g)(x) > \frac{1}{2}$. Thus $f_1 + g$ is continuous on C_1 and $(f_1 + g)(x) \neq \frac{1}{2}$ for any $x \in C_1$. Let δ_1 be the distance from $\frac{1}{2}$ to the image of C_1 under $f_1 + g$, that is, $\delta_1 = \rho(\frac{1}{2}, [f_1 + g](C_1))$. We note that $\delta_1 \leqq \varepsilon_1$ and choose ε_2 such that $0 < \varepsilon_2 < \delta_1/3$. Let $h_2 = f_1 + g$ on C_2. Since h_2 is continuous on C_2, there is a finite collection U_2 of open intervals whose closures are pairwise disjoint which covers C_2 and such that if x and y are in $0_{2i} \in U_2$ for some i, then $|h_2(x) - h_2(y)| < \varepsilon_2$. Let V_2 be the collection of those open intervals in U_2 which contain values of x satisfying $h_2(x) = \frac{1}{2}$. Define $f_2(x) = \varepsilon_2$ for the values of x covered by V_2, and let $f_2(x) = 0$ for the values of x which are covered by U_2 but not covered by V_2. Extend f_2 to be continuous on $[0,1]$ with functional values between 0 and ε_2. If $x \in C_2$ and is covered by V_2, then $h_2(x) > \frac{1}{2} - \varepsilon_2$ and hence $(f_2 + h_2)(x) > \frac{1}{2}$. Thus $f_2 + h_2$ is continuous on C_2 and $(f_2 + h_2)(x) \neq \frac{1}{2}$ for any $x \in C_2$. Let $\delta_2 = \rho(\frac{1}{2}, [f_2 + h_2](C_2))$. We note that $\delta_2 \leqq \varepsilon_2$ and choose ε_3 such that $0 < \varepsilon_3 < \delta_2/3$. Let $h_3 = f_2 + h_2$ on C_3. We proceed inductively to define $\{\varepsilon_i\}, \{\delta_i\}, \{f_i\}$, and $\{h_i\}$ such that $\delta_i = \rho(\frac{1}{2}, [f_i + h_i](C_i))$, noting that $\delta_i \leqq \varepsilon_i$, and $0 < \varepsilon_{i+1} < \delta_i/3$. If $f = \sum_{i=1}^{\infty} f_i$, then since

$$\rho(\tfrac{1}{2}, [f_j + h_j](C_j)) = \rho(\tfrac{1}{2}, [g + \textstyle\sum_{i=1}^{j} f_i](C_j)) = \delta_i$$

and $\sum_{i=j+1}^{\infty} \| f_i \| < \sum_{i=1}^{\infty} \delta_j (1/3^i) = \delta_j/2$, it follows by the triangle inequality that $(f+g)(x) \neq \frac{1}{2}$ for any x in C_j. If $x \in \sim \bigcup_{i=1}^{\infty} C_i$, then $f(x) \leqq \sum_{i=1}^{\infty} \| f_i \| < \sum_{i=1}^{\infty} 1/3^i = \frac{1}{2}$.

References

1. H. L. Royden, Real Analysis, Macmillan, New York, 1968, pp. 47–48.
2. J. E. Valentine and S. G. Wayment, On mean value theorems for a metric derivative; Revue Roumaine Math. Pures Appl., Tome XV, No. 1, Tirage A Part (1970) 153–156.

DEPARTMENT OF MATHEMATICS, UTAH STATE UNIVERSITY, LOGAN, UT 84321.

ON THE RATIOS OF INTEGER-VALUED POLYNOMIALS OVER ANY ALGEBRAIC NUMBER FIELD

DEMETRIOS BRIZOLIS

Let K be an algebraic number field and A its ring of integers. Define the *ring of integer-valued polynomials over* K to be

$$R_K = \{f \in K[X] \,|\, f(A) \subseteq A\}.$$

If $K = Q$, the rational numbers, then R_Q is free over Z and a basis is given by the polynomials

$$(1) \qquad f_n(X) = \frac{X(X-1)(X-2)\cdots(X-n+1)}{n!}, \qquad n = 0, 1, 2, \cdots.$$

A minimal generating set for R_K over A where K is any algebraic number field was given by G. Polya [2].

According to D. A. Lind [1] it is known that if $f, g \in R_Q$ and $f(n) \mid g(n)$ for all $n \in Z$, then $f \mid g$ in R_Q. The proof relies on the basis given in (1). In this note the more general result is proved for $f, g \in R_K$ where K is any algebraic number field. Furthermore, the proof does not make use of any basis for R_K.

THEOREM. *Let K be an algebraic number field, A its ring of integers and R_K the ring of integer-valued polynomials over K. If $f, g \in R_K$ satisfy $f(a) \mid g(a)$ for all $a \in A$, then $f \mid g$ in R_K.*

Proof. Without loss of generality we can assume K is normal over Q since we can always embed K in its normal closure. Let G be the Galois group of K, and for $\alpha \in K$, define the *norm* of α, $N(\alpha) = \prod_{\sigma \in G} \alpha^\sigma$. For $h(X) \in K[X]$, $h(X) = \sum_{j=0}^n \alpha_j X^j$, define $h^\sigma(X) = \sum_{j=0}^n \alpha_j^\sigma X^j$. Then $\prod_{\sigma \in G} h^\sigma(X) = (Nh)(X) \in Q[X]$. When $x \in Q$ we see that $(Nh)(x) = N(h(x))$. Suppose now that $\deg f > \deg g$. Then since Nf, $Ng \in Q[X]$, we have

$$\frac{(Ng)(x)}{(Nf)(x)} \to 0 \text{ as } x \to \infty,\ x \in Z.$$

Thus, by the remarks above and by the multiplicative property for the norm,

$$N\left(\frac{g(x)}{f(x)}\right) \to 0 \text{ as } x \to \infty,\ x \in Z. \tag{2}$$

But by hypothesis, $g(x)/f(x) \in A$ for all $x \in Z$ and so $|N(g(x)/(f(x)))| \geqq 1$, whenever $f(x) \neq 0$, $g(x) \neq 0$, which contradicts (2). Thus $\deg f \leqq \deg g$. We proceed with the proof by induction on the degree of g. If $\deg g = 0$, then $g \equiv c \in A$ and $f \equiv d \in A$ and so $c/d \in A \in R_K$ and the theorem is true.

Suppose now the theorem is true for polynomials in R_K of degree less than n. Let

$$g(X) = a_n X^n + a_{n-1} X^{n-1} + \cdots + a_0,$$
$$f(X) = b_m X^m + b_{m-1} X^{m-1} + \cdots + b_0,$$

where $a_n \neq 0 \neq b_m$, and $n \geqq m$. Let

$$h(X) = g(X) - \frac{a_n}{b_m} X^{n-m} f(X).$$

There is an $r \in A$ such that $r(a_n/b_n) = s \in A$. Therefore

$$rh(X) = rg(X) - sX^{n-m} f(X). \tag{3}$$

Then $\deg h < \deg g$ and so $\deg(rh) < \deg g$. From (3) and by the hypothesis, $f(a) \mid rh(a)$ for all $a \in A$, and so by the induction hypothesis, $f \mid rh$ in R_K. Thus by (3) $f \mid rg$ in $K[X]$ and so $g/f \in K[X]$. Since $(g/f)(A) \subset A$, by hypothesis, we have $f \mid g$ in R_K.

References

1. D. A. Lind, Which polynomials over an algebraic number field map the algebraic integers into themselves? this Monthly, 78 (1971) 179–181.

2. George Polya, Über ganzwertige Polynome in algebraischen Zahlenkörpern, J. Reine und Angew. Math., (Crelle's Journal), 148 (1918) 97.

Mathematics Department, California State Polytechnic University, Pomona, CA 91768.

BETTER-THAN-UNIFORM APPROXIMATION OF CONTINUOUS FUNCTIONS BY C^{∞} FUNCTIONS

Artin Boghossian

Let C^{∞} be the set of all continuous real valued functions in m variables which have continuous partial derivatives of all orders in all the variables.

According to a classical theorem of Weierstrass, a continuous function on a compact subset of R^m (the m-dimensional Euclidean space) can be approximated uniformly by polynomials in m variables. If we give up the restriction of boundedness on the domain, then we know that uniform approximation by polynomials in m variables may not be possible since, e.g., the exponential function cannot be approximated uniformly on R^m by polynomials in m variables. However, we shall prove in this article that any continuous function on any closed subset of R^m can be approximated uniformly by C^{∞} functions and the degree of approximation gets better as we go out towards infinity. Specifically, we shall prove the following theorem:

Theorem. *Let f be a real valued continuous function on R^m. If u is any real valued function on R^m which has a positive infimum on each compact subset of R^m, then there exists g in C^{∞} satisfying $|g(x)-f(x)| < u(x)$ for all x in R^m.*

In the special case when $m = 1$, T. Carleman [1] has shown that g can actually be chosen to be an entire function.

In the proof of our theorem, we shall need a well-known fact which we state as a lemma and indicate the essentials of its proof for completeness.

Lemma. *Let A and B be two closed subsets of R^m which are a positive distance apart (i.e., $d(A,B) = 3c > 0$). There exists h in C^{∞} such that $0 \leqq h(x) \leqq 1$ for all x in R^m, $h(x) = 1$ for all x in A and $h(x) = 0$ for all x in B.*

Proof of lemma: For each x in R^m, define

$$v(x) = \begin{cases} \exp -1/(1-|x|^2) & \text{if } |x| < 1 \\ 0 & \text{if } |x| \geqq 1 \end{cases}$$

clearly v is in C^{∞} having $\{x \in R^m : |x| \leqq 1\}$ as its support; thus the function whose values are $v(x/c)$ for each x in R^m is in C^{∞} having $\{x \in R^m : |x| \leqq c\}$ as its support;

hence the m-dimensional Lebesgue integral

$$\int_{R^m} v\left(\frac{x}{c}\right)dx = \int_{|x|\leqq c} v\left(\frac{x}{c}\right)dx = L > 0$$

exists. Defining

$$f(x) = \frac{v(x/c)}{L}$$

for all x in R^m, f is in C^∞ and it is easily verified that for each x in R^m, $\int_{R^m} f(x-y)dy = 1$. Let $G = \{x \in R^m\colon d(x, A) \leqq c\}$ and define $h(x) = \int_G f(x-y)dy$ for each x in R^m. The function h satisfies the claim of the lemma; in fact, since f is in C^∞ and f has compact support, we may differentiate under the integral sign any number of times which shows that h is in C^∞; since f is non-negative, we have

$$0 \leqq \int_G f(x-y)dy = h(x) \leqq \int_{R^m} f(x-y)dy = 1$$

as required; for all x in A, $\{y \in R^m\colon |x-y| \leqq c\}$ is contained in G, hence $h(x) = 1$; and, for all x in B, $\{y \in R^m\colon |x-y| \leqq c\}$ has an empty intersection with G, hence $h(x) = 0$.

Proof of theorem. By the classical Weierstrass theorem, for all n, $n = 1, 2, 3, \cdots$, there exists a polynomial in m variables p_n in C^∞ satisfying

$$|p_n(x) - f(x)| < \tfrac{1}{3} \underset{|y|\leqq n+1}{\text{infimum}}\, u(y) \leqq \frac{u(x)}{3} \tag{1}$$

for all x where $|x| \leqq n + \frac{1}{2}$. Using the above lemma, there exists a sequence $\{H_n\}_{n=1}^{n=\infty}$ in C^∞ satisfying for all n, where $n = 1, 2, 3, \cdots$, the following two conditions:

(i) $0 \leqq H_n \leqq 1$.

(ii) $H_n(x) = 1$ if $n-1 \leqq |x| \leqq n$, and $H_n(x) = 0$ if $|x| \leqq n - \frac{3}{2}$ or $|x| \geqq n + \frac{1}{2}$.

Letting $H_0 \equiv 0$, we claim: $g = \sum_{n=1}^{\infty} (1 - H_{n-1})H_n p_n$ is the required function. It is easily seen that g is locally a finite sum of C^∞ functions; in fact, $g(x) = p_1(x)$ if $|x| \leqq \frac{1}{2}$, $g(x) = p_n(x)$ if $n - \frac{1}{2} \leqq |x| \leqq n$ for some $n \geqq 1$, and $g(x) = H_n(x)p_n(x) + (1-H_n)(x)p_{n+1}(x)$ if $n \leqq |x| \leqq n + \frac{1}{2}$ for some $n \geqq 1$. Hence g is in C^∞. Now, for all x in R^m, we shall estimate $|g(x) - f(x)|$:

(a) If $|x| \leqq \frac{1}{2}$ or if $n - \frac{1}{2} \leqq |x| \leqq n$ for some $n \geqq 1$, then (ii) and (1) imply $|g(x) - f(x)| < u(x)$ as required.

(b) If $n < |x| < n + \frac{1}{2}$ for some $n \geqq 1$, then using (i), (ii) and (1) we get:

$$\begin{aligned} |g(x) - f(x)| &= |H_n(x)p_n(x) + (1 - H_n)(x)p_{n+1}(x) - f(x)| \\ &\leqq |p_n(x) - f(x)| + 2|p_{n+1}(x) - f(x)| \\ &< u(x) \text{ as required.} \end{aligned}$$

Thus, (a) and (b) imply $|g(x) - f(x)| < u(x)$ for all x in R^m.

An immediate consequence of this theorem along with Tietze's extension theorem is that any continuous real valued function f defined on any closed subset F of R^m can be approximated uniformly on F by C^∞ functions. Moreover, any such f can actually be approximated by C^∞ functions in the "mean of order p," where $1 \leqq p < \infty$ for any σ-finite countably additive measure on F.

Reference

1. T. Carleman, Sur un théorème de Weierstrass, Ark. Mat., Astr., Fysik, 20 (1927) No. 4, 1–5.

MATHEMATICS DEPARTMENT, SYRACUSE UNIVERSITY, SYRACUSE, N.Y. 13210.

A NOTE ON PRIMES, WITH ARBITRARY INITIAL OR TERMINAL DECIMAL CIPHERS, IN DIRICHLET ARITHMETIC PROGRESSIONS

L. J. BORUCKI AND J. B. DIAZ

1. In 1951, W. Sierpinski proved that, among the natural numbers, one can find infinitely many primes starting with an arbitrary string of decimal digits $c_1c_2 \cdots c_r$; or ending with such a string, when $c_r = 1, 3, 7$, or 9, where r is a given positive integer. The object of this note is the extension of these two curious results to the more general case of a Dirichlet progression.

THEOREM 1. *If $c_1c_2 \cdots c_r$ is any sequence of decimal digits and $P = \{Dn + l\}$ is a Dirichlet progression with $(D, l) = 1$, then P contains an infinity of primes starting with $c_1c_2 \cdots c_r$.*

Proof. From the theory of the Dirichlet progression, it is known that $\lim_{x\to\infty} \theta'(x)/x = 1/\phi(D)$, where $\theta'(x) = \sum \log(p)$, summed over all primes p in $\{Dn + l\}$ not exceeding x, and ϕ is the function of Euler. This theorem can be restated in the following form:

$$\theta'(x) = x/\phi(D) + o(x). \tag{1}$$

Let n be a positive integer. If, in decimal notation, we set $a = c_1c_2 \cdots c_r$, then all the numbers with $r + n$ digits that begin with a are between $a10^n$ and $(a + 1)10^n$. Applying the estimate (1),

$$\theta'((a + 1)10^n) - \theta'(a10^n) = 10^n/\phi(D) + o((a + 1)10^n),$$

and

$$\frac{\theta'((a + 1)10^n) - \theta'(a10^n)}{(a + 1)10^n} = \frac{1}{\phi(D)(a + 1)} + o(1).$$

Therefore,

$$\lim_{n\to\infty} \frac{\theta'((a+1)10^n) - \theta'(a10^n)}{(a+1)10^n} = \frac{1}{\phi(D)(a+1)} \neq 0. \tag{2}$$

If the number of primes in $\{Dn + l\}$ beginning with a were finite, the limit on the left side of (2) would be zero. Hence the theorem is true.

THEOREM 2. *If $c_1c_2 \cdots c_r$ is any sequence of decimal digits ending in $c_r = 1$, 3, 7, or 9, and P is a Dirichlet progression as in the first theorem, then P contains an infinity of primes ending in $c_1c_2 \cdots c_r$, if it contains just one number ending in $c_1c_2 \cdots c_r$.*

Proof. By hypothesis, we have that $b \equiv l \bmod D$ for some b ending in $c_1c_2 \cdots c_r$. Suppose that b has n digits. Then, since we have $10^nDk \equiv 0 \bmod D$ for all $k > 0$, it follows that $10^nDk + b \equiv l \bmod D$. This means that every term of the progression $P' = \{10^nDk + b\}$ is also in P. In addition, every member of P' ends in $c_1c_2 \cdots c_r$.

But, in P', we have that $(10^nD, b) = 1$, for $(10^n, b) = 1$, because b ends in 1, 3, 7 or 9, and $(D, b) = 1$ follows from the facts that $(D, l) = 1$ and $b \equiv l \bmod D$. Hence, P' is itself a Dirichlet progression, and, so, according to Dirichlet's theorem, must contain an infinity of primes. Since each of these primes is in P, and each ends in $c_1c_2 \cdots c_r$, the theorem is true.

2. In 1959, Sierpinski generalized his earlier result of 1951, by assigning, simultaneously, *both* the initial and terminal ciphers. The following theorem is an extension of this result:

THEOREM 3. *If $d_1d_2 \cdots d_s$ and $c_1c_2 \cdots c_r$ are two strings of decimal digits of arbitrary positive integer lengths s and r, respectively, where $c_r = 1$, 3, 7, or 9, then the Dirichlet progression $\{Dk + l\}$ contains an infinity of primes, both beginning in $d_1d_2 \cdots d_s$ and ending in $c_1c_2 \cdots c_r$, if it contains one number terminating with $c_1c_2 \cdots c_r$.*

Proof. Let b denote the postulated member of $\{Dk + l\}$ which ends in $c_1c_2 \cdots c_r$. Suppose, as before, that b has n digits. Then, from the last theorem, $\{Dk + l\}$ contains the Dirichlet progression $P' = \{10^nDk + b\}$ whose members all terminate in $c_1c_2 \cdots c_r$. By the first theorem, P' contains an unlimited number of primes which begin with $d_1d_2 \cdots d_s$. Thus, the proposition is true.

References

1. W. Sierpinski, Sur l'existence des nombres premiers avec une suite arbitraire de chiffres initiaux, Matematiche: Rivista Quadrimestrale. Catania, Italy. 6 (1951) 135–137.

2. W. Sierpinski, Sur les nombres ayant des chiffres initiaux et finals donnés, Acta Arith., 5 (1959) 265–266.

DEPARTMENT OF MATHEMATICS, RENSSELAER POLYTECHNIC INSTITUTE, TROY, N. Y. 12181.

RESEARCH PROBLEMS

EDITED BY RICHARD GUY

In this Department the Monthly presents easily stated research problems dealing with notions ordinarily encountered in undergraduate mathematics. Each problem should be accompanied by relevant references (if any are known to the author) and by a brief description of known partial results. Manuscripts should be sent to Richard Guy, Department of Mathematics, Statistics, and Computing Science, The University of Calgary, Calgary, Alberta, Canada, T2N 1N4.

CHARACTERIZING A CURVE WITH THE DOUBLE MIDSET PROPERTY

L. D. LOVELAND AND S. G. WAYMENT

The **midset** $M(a,b)$ of two points a and b in a metric space (X,d) is the set of all points x in X for which the distances $d(a,x)$ and $d(b,x)$ are equal. In the Euclidean plane E^2 for example, each midset $M(a,b)$ is simply the perpendicular bisector of the segment joining a and b. A metric space X is said to have the **double midset property** (DMP) if, for each two points a and b of X, the midset $M(a,b)$ consists of two points. The circle and the ellipse in E^2 are examples of spaces having the DMP.

CONJECTURE. *If a continuum X has the* DMP, *then X is homeomorphic to a simple closed curve.*

A **continuum** is a nondegenerate (contains more than one point), compact, connected, metric space.

It is well known that a continuum is a simple closed curve if each pair of its points separates it [**5**; Theorem 28.14, p. 207]; thus to establish the conjecture it would be sufficient to show that each pair of points in X lies in the midset of some two points of X. To see that this approach is doomed to failure, consider X to be the circle in E^2.

We present here some tools with which to attack the problem, and we also present some partial answers. The separation of X by a midset, as described in Lemma 1, is called the **standard separation.**

LEMMA 1. *Let X be a connected metric space, and let a and b be points of X. If $A = \{x \mid d(a,x) < d(b,x)\}$ and $B = \{x \mid d(a,x) > d(b,x)\}$, then A and B are disjoint open sets and $X - M(a,b) = A \cup B$.*

The proof of Lemma 1 is easily obtained using the continuity of the metric function $d\colon X \times X \to R$. The next lemma is stated for a continuum with the DMP, but it is evident from the proof that any finite midset property would suffice. It follows from Lemma 2 that each two points of X are the endpoints of an arc in X [**5**; Theorem 31.2, p. 219].

LEMMA 2. *If X is a continuum with the DMP, then X is locally connected.*

Proof. Suppose that X is not locally connected. Then there must exist a nondegenerate continuum K_0 and a sequence $\{K_i\}$ of continua in X such that $\{K_i\}$ converges to K_0 [4; Theorem 44, p. 111]. Let a and b be two points of K_0. It follows from Lemma 1 that $M(a,b) \cap K_0 \neq \varnothing$. Also $X - M(a,b) = A \cup B$, the standard separation, and each K_i (except K_0 and possibly one other K_i) is a connected subset of $X - M(a,b)$. Thus we may assume that every element of some subsequence $\{K_{n_i}\}$ of $\{K_i\}$ lies in A. But this is a contradiction since b is a limit point of $\bigcup_1^\infty K_{n_i}$ and b belongs to B.

A **triod** is a continuum T containing a point v, called the vertex of T, such that T is the union of three arcs A_1, A_2, and A_3 where $A_i \cap A_j = \{v\}$ if $i \neq j$.

LEMMA 3. *If X is a locally connected continuum that contains no triod, then X is either an arc or a simple closed curve.*

Proof. This is Theorem 75 of [4; p. 218]. However, a short proof can be obtained from the familiar characterization of a simple closed curve as a continuum separated by each doubleton subset [5; Theorem 28.14, p. 207]. For suppose X is not a simple closed curve. Then some doubleton subset $\{a,b\}$ of X fails to separate X. Since a locally connected continuum is arcwise connected [5, Theorem 31.2, p. 219], there is an arc C in X with endpoints a and b. If there is a point x in $X - C$, then the connected open set $X - \{a,b\}$ contains an arc joining x to a point of $C - \{a,b\}$ (see the statement following the proof of Theorem 31.2 in [5; p. 221]). Thus X would contain a triod, contrary to the hypothesis. This shows that $X = C$, and the lemma follows.

A point p of a continuum X is called a **cut point** of X if $X - \{p\}$ is not connected.

Notice that no continuum hypothesis is needed on X in Theorem 1 or in Corollary 1.

THEOREM 1. *If X is a metric space with the DMP and X contains a continuum J with no cut points, then X is a simple closed curve.*

Proof. Suppose there exists a point x in $X - J$. Let $g\colon J \to R$ be defined by $g(t) = d(x,t)$. If there do not exist two points a and b of J equidistant from x then g is an injective mapping of the compact set J into the real line R. In this case g must be a homeomorphism [5; Theorem 17.14, p. 123]. But each continuum in R has cut points, so there must exist points a and b in J such that $d(a,x) = d(b,x)$.

By Lemma 1 we see that $M(a,b)$ must separate a from b in J. However, x belongs to $M(a,b)$ and x is not in J. Since $M(a,b)$ consists of two points and at most one of these points belongs to J, we see that a and b are not separated in J by $M(a,b)$. This contradiction shows that $X = J$.

The continuum X is locally connected by Lemma 2, and X has no cut points since $X = J$. Thus it follows from Lemma 3 that either X contains a triod or X

is a simple closed curve. Suppose that X contains a triod. Since no point of X is a cut point of X and X minus a point is arcwise connected, we see that X must contain a simple closed curve J'. Now the first two paragraphs of this proof, where J' plays the role of J, show that $X = J'$, and the proof is complete.

COROLLARY 1. *If a metric space X has the DMP and contains a simple closed curve, then X is a simple closed curve.*

COROLLARY 2. *If a continuum X has the DMP and has no cut points, then X is a simple closed curve.*

A metric space X is said to have the **strong double midset property** (SDMP) if, for each two points a and b of X, the sets A and B of Lemma 1 are each connected.

THEOREM 2. *If a continuum X has the SDMP, then X is a simple closed curve.*

Proof. We may suppose that X contains no simple closed curve since otherwise the result follows from Corollary 1. By Lemmas 2 and 3, either X contains a triod or X is an arc. However, X cannot be an arc because the midset of its endpoints would separate X into three disjoint open sets, contrary to the SDMP.

Suppose X contains a triod T with vertex v. By the definition, T is the union of three arcs C_1, C_2, and C_3 where $\{v\} = C_i \cap C_j$ when $i \neq j$. There must exist two points $a \in C_1$ and $b \in C_2$ with $d(a,v) = d(b,v)$. From the hypothesis and Lemma 1 we see that $X - M(a,b)$ is the union of two disjoint, open, connected sets A and B where $a \in A$ and $b \in B$. We may assume for convenience that A contains a point q of C_3. Since A is a connected open subset of the arcwise connected space X, A is also arcwise connected [**5**; p. 221]. This means that an arc D exists in A joining q to a point of C_1, and it follows that $C_1 \cup C_3 \cup D$ contains a simple closed curve. This is a contradiction, and the result follows.

REMARKS. The above conjecture is false if X is not assumed to be connected (a four-point space with the discrete metric is a counterexample). We have not been able to answer the following related questions:

(1) If X is a nondegenerate, connected, metric space with the DMP, then is X compact?

(2) Can an arc have the DMP?

(3) Can a continuum have the DMP if it contains a triod?

(4) Can a continuum have the triple midset property?

Notice that negative answers to both of questions (2) and (3) would establish the conjecture by use of Lemma 3.

In most of the work involving midsets the metric is assumed to be convex. One exception to this is Berard's work in [**1**] where he proves that a connected metric space with the unique midset property is homeomorphic to an interval on the real line. It follows from [**2**] and [**3**] that a continuum with a convex metric that has the DMP is isometric to a circle in the plane with the "shorter arc" metric.

References

1. Anthony D. Berard, Jr., Characterizations of metric spaces by use of their midsets: Intervals, Fund. Math., 73 (1971) 1–7.
2. ———, Characterizations of metric spaces by use of their midsets: One-spheres, Fund. Math., (to appear).
3. L. D. Loveland and J. E. Valentine, Convex metric spaces with 0-dimensional midsets, Proc. Amer. Math. Soc., 37 (1973) 568–571.
4. R. L. Moore, Foundations of Point Set Theory, revised edition, Amer. Math. Soc. Colloq. Publ., 13, 1962.
5. Stephen Willard, General Topology, Addison-Wesley, Reading, Mass., 1970.

DEPARTMENT OF MATHEMATICS, UTAH STATE UNIVERSITY, LOGAN, UT 84321.
DEPARTMENT OF MATHEMATICS, UNIVERSITY OF TEXAS, SAN ANTONIO, TX 78212.

CLASSROOM NOTES

EDITED BY DAVID ROSELLE

Material for this Department should be sent to David Roselle, Department of Mathematics, Virginia Polytechnic Institute, Blacksburg, VA 24061.

DIFFERENTIALS IN ONE DIMENSION

MICHAEL A. GOLBERG

The recent increase in the use of the Fréchet approach to calculus is a valuable trend in mathematics teaching [2, 3, 5]. However, little use is made of the general theory, since one quickly appeals to the standard case of functions between R^n and R^m where finding differentials is reduced to the problem of computing Jacobians (derivatives when $m = n = 1$). As a consequence, students are frequently confused as to the necessity for such abstraction. Of course, if one is willing to use function space examples of the type that appear in elementary calculus of variations problems [5] the general method is shown to pay immediate dividends. The purpose of this note is to present some examples, based on the multiplicative vector space structure of the positive reals, showing that this approach can have surprising consequences even in one dimension.

The author claims no particular originality for the following examples. They appear without computation in the important article of Averbukh and Smolyanov [1]. However, a search of calculus texts and reference material such as the MAA collected papers on calculus [5] and conversations with colleagues indicate that as far as pedagogical applications are concerned, they are unknown.

Recall now for convenience some of the basic notions of the Fréchet calculus.

SOME APPLICATIONS OF GALOIS THEORY TO NORMAL POLYNOMIALS

TOM PARKER

In his classical text *Algebra* [**8**, p. 124], van der Waerden introduces the concept of a **normal polynomial** $f(X)$ over a field K — an irreducible polynomial over K such that $K(\theta)$ is a splitting field of $f(X)$ over K for each root θ of $f(X)$ in an extension field of K. But like other authors (see, for example, [**3**, p. 259]) van der Waerden does little with the concept, and indeed, at the point at which the definition of a normal polynomial is introduced in [**8**], the student has little to work with — no Galois theory and only the rudiments of field theory. On the other hand, once the Fundamental Theorem of Galois Theory has been proved and cyclotomic fields have been studied in Section 8.4 of [**8**], some nice applications of these results to the theory of normal polynomials can be given. The purpose of this note is to present some of these applications. We make no claims for originality of the results, although we have not been able to find some of them in the literature. Also, to be a bit more concrete, we restrict to the case where the base field is the field Q of rational numbers. Our main purpose is to outline proofs of the following two theorems.

THEOREM 1. *If n is a positive integer, then there is a normal polynomial $f(X)$ in $Q[X]$ of degree n such that all the roots of $f(X)$ are real.*

THEOREM 2. *If n is an even positive integer, then there is a normal polynomial in $Q[X]$ of degree n with no real roots.*

It is clear, of course, that if one root of a normal polynomial over Q is real, then each of its roots is real. Theorem 2 is easy to prove; in view of the remark just made, we need only show that if n is even, then there is a nonreal normal extension field F of Q such that $[F:Q]=n$, and this will follow easily from our proof of Theorem 1. To prove Theorem 1, we propose to show that some cyclotomic field $Q_d = Q(\zeta_d)$, where $\zeta_d = e^{2\pi i/d}$, contains a real subfield K of degree n over Q. If this be proved, then Theorem 1 follows, for K is a primitive extension of Q — say $K = Q(\theta)$ — and K/Q is normal since the Galois group G_d of Q_d/Q is abelian [**8**, p. 174], and hence each subgroup of G_d is normal. Consequently, the minimal polynomial for θ over Q is a normal polynomial of degree n with real roots.

The fact that some Q_d contains a real subfield of degree n over Q can be based on the following special case of Dirichlet's Theorem: *for each positive integer a, there aré infinitely many primes in the arithmetic progression $a+1, 2a+1, 3a+1, \cdots$.* Let us stress that we do not base our proof of Theorem 1 on Dirichlet's Theorem, which is a result much more profound than any of the topics previously mentioned. Rather, a proof of the special case of the theorem that we need can be obtained from the following two results, which are reasonable to assign as exercises at this level.

RESULT 1. (See [**1**, Theorem 1], [**7**].) *If f is a nonconstant polynomial with*

integer coefficients, then the set of primes that divide $f(n)$, for some integer n, is infinite.

RESULT 2. (See [**5**, Theorem 317, p. 512], [**2**].) *For each positive integer n, let $F_n(X)$ denote the nth cyclotomic polynomial. If p is prime and $p \nmid n$, then $F_n(X)$ factors modulo p into a product of $\phi(n)/f$ distinct irreducible polynomials, each of degree f, where f is the order of p modulo n. If r is a positive integer, then $F_{p^r n}(X) \equiv F_n(X)^{\phi(p^r)} (\text{mod } p)$.*

COROLLARY. (See [**4**, Theorem 94, p. 164].) *The polynomial $F_n(X)$ has a root modulo p (that is, p divides $F_n(k)$ for some integer k) if and only if $p \equiv 1 \pmod{n}$. Consequently, there are infinitely many primes of the form $nt + 1$.*

We proceed to give a proof of Theorem 1. If n is odd, we choose a prime q of the form $kn + 1$. The Galois group of $Q_q = Q(\zeta_q)$ is cyclic of order $q - 1$, and hence it contains a unique subgroup of index n. The field K corresponding to this subgroup has degree n over Q, is normal over Q, and since n is odd, K is a subfield of the reals. If θ is a primitive element for K/Q — that is, $K = Q(\theta)$ — then the minimal polynomial for θ over Q is a normal polynomial of degree n with real roots. If n is even, say $n = 2^k m$, where k is positive and m is odd, then we take a normal real field L of degree m over Q, and the field $F = Q(\sqrt{p_1}, \sqrt{p_2}, \cdots, \sqrt{p_k})$, where $p_1 < p_2 < p_3 < \cdots$ is the sequence of positive primes. The field F is real, normal over Q, and $[F : Q] = 2^k$ (for an easy proof of this equality, see [**6**]). The composite $LF = L(\sqrt{p_1}, \cdots, \sqrt{p_k})$ of L and F is a normal extension of Q, it is a subfield of the reals, and since $(m, 2^k) = 1$, $[LF : Q] = 2^k m = n$. Therefore we obtain, as in the case already considered, a normal polynomial over Q of degree $2^k m$ with real roots.

Since the polynomials $f(X)$ and $f(X + a)$, for $a \in Q$, are simultaneously normal, Theorem 1 yields, upon a suitable translation of roots, the following corollary.

COROLLARY. *If r and s are nonnegative integers such that $r + s > 0$, then there is a normal polynomial $g(X)$ over Q such that $g(X)$ has exactly r positive real roots and s negative real roots.*

A final remark is in order. If q is prime, then van der Waerden's technique in Section 8.4 of [**8**] yields an algorithmic process for determining a normal polynomial h_n of degree n over Q for each positive divisor n of $q - 1$; h_n is the minimal polynomial over Q of the $(q - 1)/n$-term period (see Section 8.4 of [**8**] for the definition) of the cyclotomic field Q_q.

References

1. I. Gerst and J. Brillhart, On the prime divisors of polynomials, this MONTHLY, 78 (1971) 250–266.

2. W. J. Guerrier, The factorization of the cyclotomic polynomial mod p, this MONTHLY, 75 (1968) 46.

3. H. Hasse, Higher Algebra, Ungar, New York, 1954.
4. T. Nagell, Introduction to Number Theory, Wiley, New York, 1951.
5. L. Redei, Algebra, Vol. I, Pergamon, London, 1967.
6. R. L. Roth, On extensions of Q by square roots, this MONTHLY, 78 (1971) 392–393.
7. I. Schur, Über die Existenz unendlich vieler Primzahlen in einigen speziellen arithmetischen Progressionen, S.-B. Berlin Math. Ges., 11 (1912) 40–50.
8. B. L. van der Waerden, Algebra, Vol. I, Ungar, New York, 1970.

1317–C ORTIZ DRIVE, ALBUQUERQUE, NM 87108.

ANOTHER PROOF OF AN ESTIMATE FOR e

R. F. JOHNSONBAUGH

At the national meeting of the Mathematical Association of America in January, 1973, Professor Paul Halmos suggested that publication of mathematical folklore would be a worthwhile contribution to mathematical literature. I thus submit an old proof of the monotonicity and boundedness of the sequence $\{(1 + 1/n)^n\}$ which I believe is the simplest that I have ever seen.

The proof uses the simple inequality

$$\frac{b^{n+1} - a^{n+1}}{b - a} < (n + 1)b^n$$

which we rewrite as

$$b^n[(n + 1)a - nb] < a^{n+1}$$

valid for $0 \leqq a < b$. Taking $a = 1 + (1/(n + 1))$ and $b = 1 + (1/n)$, the term in brackets reduces to 1 and we have

$$\left(1 + \frac{1}{n}\right)^n < \left(1 + \frac{1}{n + 1}\right)^{n+1}. \tag{*}$$

Taking $a = 1$ and $b = 1 + (1/2n)$, the term in brackets reduces to $\frac{1}{2}$ and we have

$$\left(1 + \frac{1}{2n}\right)^n \frac{1}{2} < 1.$$

Multiplying by 2 and squaring we have

$$\left(1 + \frac{1}{2n}\right)^{2n} < 4. \tag{**}$$

Inequalities (*) and (**) imply that the sequence $\{(1 + (1/n))^n\}$ is increasing and bounded above by 4.

In a similar way, one can use the inequality

$$\frac{b^{n+1} - a^{n+1}}{b - a} > (n + 1)a^n$$

with $a = 1 + (1/(n+1))$ and $b = 1 + (1/n)$ to derive

$$\left(1+\frac{1}{n}\right)^{n+1} > \left(1+\frac{1}{n+1}\right)^{n+2}\left[\frac{n^3+4n^2+4n+1}{n(n+2)^2}\right].$$

Since the term in brackets is at least 1, we have shown that $(1+(1/n))^{n+1}$ is decreasing.

DEPARTMENT OF MATHEMATICS, CHICAGO STATE UNIVERSITY, CHICAGO, IL 60628.

TO e VIA CONVEXITY

H. SAMELSON

1. The exponential function often receives a somewhat left-handed treatment in textbooks on real variables. The general function a^x is introduced quite early; but to meet e, e^x, and $\ln x$, one has to wait till integration or differential equations appear. In this note we outline a variant that gets there a bit sooner, provides an example for limits or derivatives, and motivates the usual formula for e. Our approach does not seem to be in the literature, except for the recent note [7] by J. Tull which uses pretty much the same idea (the present note was written independently of [7] and submitted before it appeared).

2. As usual, we first develop a^x (for $a > 0$) from first principles (i.e., existence of sup in $\mathbb{R}$), to get the following properties: a^x is defined for all x; $a^x > 0$; $a^{x+y} = a^x \cdot a^y$; $(a^x)^y = a^{x \cdot y}$; $a^x \cdot b^x = (ab)^x$; a^x is strictly increasing for $a > 1$; it is a continuous function (this amounts to $a^{1/n} \to 1$), with image $(0, \infty)$ and continuous inverse $\log_a y$ if $a \neq 1$. (See, e.g., [**1**], pp. 87–89.)

3. Now the main observation: a^x is *convex*; for any two x-values the graph of a^x is below the chord. We show this first for the midpoint. We want $a^{\frac{1}{2}(x+x')} \leqq \frac{1}{2}(a^x + a^{x'})$. With $x' = x + 2h$ (say $h > 0$) this becomes $a^{x+2h} - a^{x+h} \geqq a^{x+h} - a^x$. But this is clear on factoring a^h from the L.H.S. By repeated halving we get all points $x + m(x'-x)/2^n$; and by continuity we get the result for all points on $[x, x']$.

4. Knowing $a^x \to 1$ as $x \to 0$, it makes sense to ask how $a^x - 1$ compares to x; e.g., does

$$\lim_{x\to 0}\frac{a^x-1}{x}$$

exist? (Is a^x differentiable at $x = 0$?) By convexity of a^x the difference quotient $(a^x - 1)/x$, defined for all $x \neq 0$, is an increasing function (look at the slope of the secant lines of the graph of a^x, from $(0,1)$ to (x, a^x)). Therefore $\lim_{x\to 0^+}$ and $\lim_{x\to 0^-}$ exist. For $x < 0$ we rewrite $(a^x - 1)/x$ as

$$\frac{1}{a^{|x|}} \cdot \frac{a^{|x|}-1}{|x|}.$$

From $a^{|x|} \to 1$ we see that the two limits are equal, and that $\lim_{x\to 0}(a^x - 1)/x$ *exists*. We call the limit value $\lambda(a)$, thus defining a function λ on $(0, \infty)$. (From $a^{x+y} = a^x \cdot a^y$ one finds that a^x is differentiable at each x, and that $(a^x)'$ equals $\lambda(a) \cdot a^x$.) From $(a^t)^x = a^{t \cdot x}$ one finds $\lambda(a^t) = t \cdot \lambda(a)$ (using $(a^{tx} - 1)/x = t \cdot (a^{tx} - 1)/tx$). Similarly one shows $\lambda(a \cdot b) = \lambda(a) + \lambda(b)$, from $a^x \cdot b^x = (ab)^x$ (using $(a^x \cdot b^x - 1)/x = (a^x \cdot b^x - b^x)/x + (b^x - 1)/x$). Thus λ begins to behave like log.

λ is 0 for $a = 1$ and only for $a = 1$: If $a > 1$, then, taking $x = -1$, we have

$$\lambda(a) \geqq \frac{a^{-1}-1}{-1} = 1 - \frac{1}{a} > 0;$$

similarly for $a < 1$ we get $\lambda(a) < 0$. In fact λ is strictly monotone (if $a > b$, write $a = b \cdot c$ with $c > 1$).

5. And now: Clearly the "best" number is the one, to be called e, for which $\lambda(e) = 1$. From the above laws one sees easily that there is one and only such value. We have $\lambda(e^t) = t \cdot \lambda(e) = t$, so that λ is just $\log_e = \ln$, the inverse of e^x; i.e., we have $x = e^{\lambda(x)}$. (And we have $(e^x)' = e^x$.)

The relation

$$\lim_{x\to 0} \frac{e^x - 1}{x} = 1,$$

specialized to $\lim_{n\to\infty} n \cdot (e^{1/n} - 1) = 1$, suggests the relation $e = \lim_{n\to\infty}(1 + (1/n))^n$; namely, $n \cdot (e^{1/n} - 1) \sim 1$ suggests $e^{1/n} \sim 1 + 1/n$ and $e \sim (1 + (1/n))^n$. This is of course vague. But from convexity we know $n \cdot (e^{1/n} - 1) \geqq 1$ and so $e \geqq (1 + (1/n))^n$. A standard proof shows $(1 + (1/n))^n$ monotone increasing (see [**2**], pp. 24–25, or [**4**], p. 146). Calling the limit of this sequence c, for a moment, we have $e \geqq c$. From $c \geqq (1 + (1/n))^n$ or $n \cdot (c^{1/n} - 1) \geqq 1$ we get $\lambda(c) \geqq 1$, and so $c \geqq e$. Therefore $c = e$. In other words, $e = \lim_{n\to\infty}(1 + (1/n))^n$. We are now far enough along.

6. The bibliography contains a few other references concerned with the problem of "introducing e".

References

1. A. Devinatz, Advanced Calculus, Holt Rinehart and Winston, New York, 1968.
2. H. G. Eggleston, Elementary Real Analysis, Cambridge, 1962.
3. K. Menger, Methods of Presenting e and π, this MONTHLY, 52 (1945) 28–33.
4. J. F. Randolph, Basic Real and Abstract Analysis, Academic Press, New York, 1968.
5. SMSG, Elementary Functions, Yale Press, 1959.
6. Selected Papers on Calculus, edited by T. M. Apostol *et al.*, Math. Assoc. of Amer., 1969.
7. J. Tull, A Discovery Approach to e, this MONTHLY, 80 (1973) 193–194.

DEPARTMENT OF MATHEMATICS, STANFORD UNIVERSITY, STANFORD, CALIFORNIA 94305.

ANOTHER PROOF THAT CONVEX FUNCTIONS ARE LOCALLY LIPSCHITZ

A. W. ROBERTS AND D. E. VARBERG

The Wayne State Mathematics Department Coffee Room recently brewed the following result [this MONTHLY, vol. 79 (1972), 1121–1124]. *Every convex function f defined on an open convex set in R^n is locally Lipschitz.* A different recipe yields the same result with less work and applies in much more general spaces. It goes like this: (1) control the size of f by showing (local) boundedness, (2) mix boundedness with convexity to obtain a Lipschitz condition, (3) embellish with desired generalizations. Here are the details.

LEMMA A. *A convex function f, defined on an open convex set U in R^n, is locally bounded; that is, it is bounded in a neighborhood of each point x_0 in U.*

Proof. Choose a cube K in U centered at x_0 and with vertices $v_1, v_2, \cdots, v_m$ $(m=2^n)$. Since a cube is the convex hull of its vertices, we may for any x in K find scalars λ_i satisfying

$$x = \sum_1^m \lambda_i v_i, \qquad \lambda_i \geqq 0, \qquad \sum_1^m \lambda_i = 1.$$

By convexity (Jensen's inequality for convex functions),

$$f(x) \leqq \sum_1^m \lambda_i f(v_i) \leqq \max_{1 \leqq i \leqq m} f(v_i) \equiv M,$$

so f is bounded above on K.

On the other hand, for x in K we may choose y in K so that $x_0 = \frac{1}{2}x + \frac{1}{2}y$. Thus,

$$f(x_0) \leqq \tfrac{1}{2}f(x) + \tfrac{1}{2}f(y),$$

or

$$f(x) \geqq 2f(x_0) - f(y) \geqq 2f(x_0) - M,$$

and f is also bounded below on K. ∎

THEOREM A. *Let f be convex on an open convex set U in R^n. Then f is locally Lipschitz on U; that is, it is Lipschitz on a neighborhood of each point x_0 of U. Consequently, f is Lipschitz on any compact subset of U.*

Proof. According to the lemma, f is locally bounded; so given x_0, we may find a spherical neighborhood $N_{2\varepsilon}(x_0)$ of radius 2ε on which f is bounded, say by M. For distinct x_1 and x_2 in $N_\varepsilon(x_0)$, set $x_3 = x_2 + (\varepsilon/\alpha)(x_2 - x_1)$ where $\alpha = \|x_2 - x_1\|$ and note that x_3 is in $N_{2\varepsilon}(x_0)$. If we solve for x_2, we obtain

$$x_2 = \frac{\varepsilon}{\alpha + \varepsilon} x_1 + \frac{\alpha}{\alpha + \varepsilon} x_3$$

and so by convexity,

$$f(x_2) \leqq \frac{\varepsilon}{\alpha+\varepsilon} f(x_1) + \frac{\alpha}{\alpha+\varepsilon} f(x_3).$$

Then

$$f(x_2) - f(x_1) \leqq \frac{\alpha}{\alpha+\varepsilon} [f(x_3) - f(x_1)] \leqq \frac{\alpha}{\varepsilon} |f(x_3) - f(x_1)|,$$

which combined with $|f| \leqq M$ and $\alpha = \| x_2 - x_1 \|$ yields

$$f(x_2) - f(x_1) \leqq (2M/\varepsilon) \| x_2 - x_1 \|.$$

Since the roles of x_1 and x_2 can be interchanged, we have

$$|f(x_2) - f(x_1)| \leqq (2M/\varepsilon) \| x_2 - x_1 \|,$$

that is, f is Lipschitz on $N_\varepsilon(x_0)$. We conclude that f is locally Lipschitz on U.

Now let D be a compact subset of U. The collection $\{N_\varepsilon(x_0)\}$ of neighborhoods obtained above covers D, as does some finite subcollection $N_1, N_2, \cdots, N_m$. Let $K = \max\{K_1, K_2, \cdots, K_m\}$ where K_i is the Lipschitz constant corresponding to N_i, $i = 1, 2, \cdots, m$. Finally let $x \in N_i$ and $y \in N_j$ be any two distinct points of D and choose a segment $[w, z]$ containing segment $[x, y]$ in its interior so that $w \in N_i$ and $z \in N_j$. From the convexity of f on segment $[w, z]$,

$$-K \leqq \frac{f(x) - f(w)}{\| x - w \|} \leqq \frac{f(y) - f(x)}{\| y - x \|} \leqq \frac{f(z) - f(y)}{\| z - y \|} \leqq K$$

which yields the conclusion $|f(y) - f(x)| \leqq K \| y - x \|$. ∎

Now for the embellishments. The definitions of convex, bounded, and Lipschitz all extend without modification to an arbitrary normed linear space. So does the proof of Theorem A; only the lemma offers any difficulties, but they are real. A convex function on an infinite dimensional normed linear space may be locally unbounded. For example, the linear functional $f : p \to p'(0)$ on the space of polynomials normed by

$$\| p \| = \max_{-1 \leqq x \leqq 1} |p(x)|$$

has this property. A slight additional condition fixes everything up.

LEMMA B. *Let f be convex on an open convex set U in a normed linear space. If f is bounded above in a neighborhood of just one point, then f is locally bounded on U.*

Proof. For convenience of notation, we suppose that the given point is the origin and that f is bounded above by M on a spherical neighborhood $N = N_\varepsilon(0)$. Let y be any other point of U and choose $\rho > 1$ so that $z = \rho y$ is in U. If $\lambda = 1/\rho$, then

$$V = \{v : v = (1 - \lambda)x + \lambda z, x \text{ in } N\}$$

is a neighborhood of $y = \lambda z$ with radius $(1 - \lambda)\varepsilon$. Moreover,

$$f(v) \leqq (1 - \lambda)f(x) + \lambda f(z) \leqq M + f(z).$$

Thus, f is bounded above in some neighborhood of each point y in U. A repetition of the second paragraph in the proof of Lemma A shows that it is also bounded below on each such neighborhood. ∎

We have all the ingredients for a tangy generalization.

THEOREM B. *Let f be convex on an open convex set U in a normed linear space. If f is bounded above in a neighborhood of one point of U, then f is locally Lipschitz on U, hence Lipschitz on any compact subset of U.*

Compactness is a strong requirement, often missing, especially for sets in infinite dimensional spaces. We can make a substitute for it; and the proof of the resulting theorem is still essentially that of Theorem A.

THEOREM C. *Let f be convex with $|f| \leqq M$ on an open convex set U in a normed linear space. If U contains an ε-neighborhood of a subset V, then f is Lipschitz (with Lipschitz constant $2M/\varepsilon$) on V.*

DEPARTMENT OF MATHEMATICS, MACALESTER COLLEGE, ST. PAUL, MN 55101.
DEPARTMENT OF MATHEMATICS, HAMLINE UNIVERSITY, ST. PAUL, MN 55104.

ON POLARS OF CONVEX POLYGONS

ROBERT H. LOHMAN AND TERRY J. MORRISON

In discussions concerning convexity and linear inequalities, it is often necessary to find the polar of a convex set in Euclidean space. The purpose of this note is to give a very elementary method for completely determining the polars of certain convex polygons in R^2. We feel this is worthwhile for two reasons. First, it is an interesting geometric result that can be easily understood by students with a minimal background in geometry. Second, while it is usually stated that the polar of a convex polyhedron is a convex polyhedron (cf. [**1**, p. 174]), no mention is made of how the vertices of the polar can be explicitly found, and this is the content of our result.

Given a set U in the real linear space R^2, the polar of U is defined by

$$U^\circ = \{(u, v) \in R^2 : |ux + vy| \leqq 1 \text{ for all } (x, y) \in U\}.$$

If $z = (a, b)$ and $(a, b) \neq (0, 0)$, it is simple to show that $\{z\}^\circ$ is the infinite strip

bounded by, and including, the parallel lines $au + bv = \pm 1$. If A is a subset of R^2, we let $\langle A \rangle$ denote the convex hull of A; that is, $\langle A \rangle$ is the smallest convex subset of R^2 which contains A. A is symmetric if $-A = A$. It is always the case that A° is symmetric. The double polar theorem states that if U is a closed, convex, symmetric subset of R^2, then $U = U^{\circ\circ}$. A proof of the double polar theorem in a general setting can be found in [2, p. 238]. Let

$$H = \{(x, y): x < 0 \text{ and } y > 0 \text{ or } x \geqq 0 \text{ and } y \geqq 0\}.$$

For the remainder of the discussion, we assume that U is a convex, symmetric polygon containing $(0,0)$ as an interior point. We consider all the vertices $z_i = (x_i, y_i)$, $i = 1, \cdots n$, of U that lie in H and assume that they are labelled as they are encountered while traversing the boundary of U in H in a clockwise direction. The vertices of U are therefore, $\pm z_i$, $i = 1, \cdots, n$. Let $z_0 = (x_0, y_0)$ be the first vertex on the boundary of U lying below z_1, i.e., $z_0 = -z_n$.

THEOREM. *U° is the convex polygon with vertices $\pm w_i$, $i = 1, \cdots, n$, where*

$$w_i = \left(\frac{y_i - y_{i-1}}{x_{i-1}y_i - x_i y_{i-1}}, \frac{x_{i-1} - x_i}{x_{i-1}y_i - x_i y_{i-1}} \right).$$

Proof. Note that z_{i-1} and z_i do not both lie on the same line through the origin, nor can either point be the origin. Therefore, $x_{i-1}y_i - x_i y_{i-1} \neq 0$ so that w_i is well defined. Also, $w_i \neq (0,0)$ since z_{i-1} and z_i are distinct. Let $W = \langle \{\pm w_i : i = 1, \cdots, n\} \rangle$. Then, by facts 1 and 7 in ([2], p. 237) and since $\{z\}^\circ = \{-z\}^\circ$ for any point z,

$$W^\circ = \langle \{\pm w_i : i = 1, \cdots, n\} \rangle^\circ = \{\pm w_i : i = 1, \cdots, n\}^\circ = \bigcap_{i=1}^{n} \{w_i\}^\circ.$$

But $\{w_i\}^\circ$ is the infinite strip bounded by, and including, the parallel lines

$$(y_i - y_{i-1})x + (x_{i-1} - x_i)y = \pm(x_{i-1}y_i - x_i y_{i-1}).$$

The latter lines are precisely the lines through the pairs of vertices z_{i-1}, z_i and $-z_{i-1}$, $-z_i$ of U for $i = 1, \cdots, n$. Since U is the intersection of the strips bounded by these lines, it follows that $U = W^\circ$. On the other hand, W is a closed, convex, symmetric set in R^2. By the double polar theorem, $U^\circ = W^{\circ\circ} = W$.

As the convex hull of a finite set of points, U° is a convex polygon. Moreover, each vertex of U° must lie in this finite set. Therefore, since U° is symmetric, the proof will be complete if it is shown that each w_i is a vertex of U°. Assume, to the contrary, that some w_i is not a vertex of U°. Without loss of generality, $i = n$. Then $w_n \in \langle \{\pm w_i : i = 1, \cdots, n-1\} \rangle$, implying $U^\circ = \langle \{\pm w_i : i = 1, \cdots, n-1\} \rangle$. Therefore

$$U = U^{\circ\circ} = \bigcap_{i=1}^{n-1} \{w_i\}^\circ.$$

If z_n lies in the interior of each of the strips $\{w_i\}^\circ$, $i = 1, \cdots, n-1$, then it lies in the interior of their intersection, which (as inferred from the assumption) is the interior of U, an impossibility. Thus for some i, $1 \leqq i \leqq n-1$, z_n lies on the line through z_{i-1} and z_i or on the line through $-z_{i-1}$ and $-z_i$. This contradicts the fact that z_{i-1}, z_i, z_n or $-z_{i-1}, -z_i, z_n$ are distinct vertices of U. Therefore, each point w_i is a vertex of U°.

References

1. R. T. Rockafellar, Convex Analysis, Princeton, 1970.
2. A. Wilansky, Functional Analysis, Blaisdell, New York, 1964.

Department of Mathematics, Kent State University, Kent, OH 44242.

A NEARLY DISCRETE METRIC

D. D. Rothmann

An elementary metric d, having many interesting properties, can be defined on the set R of real numbers by setting $d(x,y) = 0$ if $x = y$ and setting $d(x,y) = \max\{|x|,|y|\}$ if $x \neq y$.

The open sets in this metric space are the elements of $\{B \mid B \subset R \text{ and } 0 \notin B\} \cup \{(-\varepsilon,\varepsilon) \cup A \mid \varepsilon > 0 \text{ and } A \subseteq R\}$. The title of this article is due to the fact that if a is a nonzero real number, then $\{a\}$ is an open set. It is easy to see that this metric space is disconnected and complete.

It is interesting to note that if $r > 0$, then the closure of $\{x \mid d(a,x) < r\}$ need not equal $\{x \mid d(a,x) \leqq r\}$. To see this, note that both $\{x \mid d(1,x) < 1\}$ and the closure of $\{x \mid d(1,x) < 1\}$ are given by $\{1\}$, whereas

$$\{x \mid d(1,x) \leqq 1\} = \{x \mid -1 \leqq x \leqq 1\}.$$

Furthermore, if $\lim x_n$ and $\lim y_n$ both exist, it is not necessary for $\lim(x_n + y_n)$ to equal $\lim x_n + \lim y_n$. For let $x_n = 1$ and $y_n = -1/n$, in which case, $\lim x_n = 1$ and $\lim y_n = 0$. Since $d(1-1/n, 1) = 1$ for every natural number n, $\lim(x_n + y_n) = \lim(1 - 1/n)$ fails to converge to 1.

As a consequence of the above paragraph, it is clear that the addition function $(x,y) \to x + y$ from $R \times R$ onto R fails to be continuous. This follows since the sequence $\{(1, -1/n)\}$ converges in the domain to $(1,0)$, but the image sequence fails to converge.

Note, however, that the inversion function $x \to -x$ from R onto R is continuous. To see this, assume the sequence $\{x_n\}$ converges to a. If $a = 0$, then

$$d(x_n, a) = |x_n| = |-x_n| = d(-x_n, -a)$$

and hence, $\{-x_n\}$ converges to $-a$. On the other hand, if $a \neq 0$, it follows that

$x_n = a$ for all n greater than some natural number k. Consequently, $-x_n = -a$ for all n greater than k and hence $\{-x_n\}$ converges to $-a$.

It is evident that both addition and inversion cannot be continuous in this metric topology, since otherwise R would be a topological group and, hence, homogeneous in this topology.

MATHEMATICAL DEPARTMENT, MOORHEAD STATE COLLEGE, MOORHEAD, MN 56560.

MATHEMATICAL EDUCATION

EDITED BY SHIRLEY HILL AND PAUL MIELKE

Material for this Department should be sent to Shirley Hill, Department of Mathematics, University of Missouri, Kansas City, MO 64110, or to Paul Mielke, Department of Mathematics, Wabash College, Crawfordsville, IN 47933.

PRECALCULUS MINI-COURSES, SELF PACING WITH CONSTRAINTS

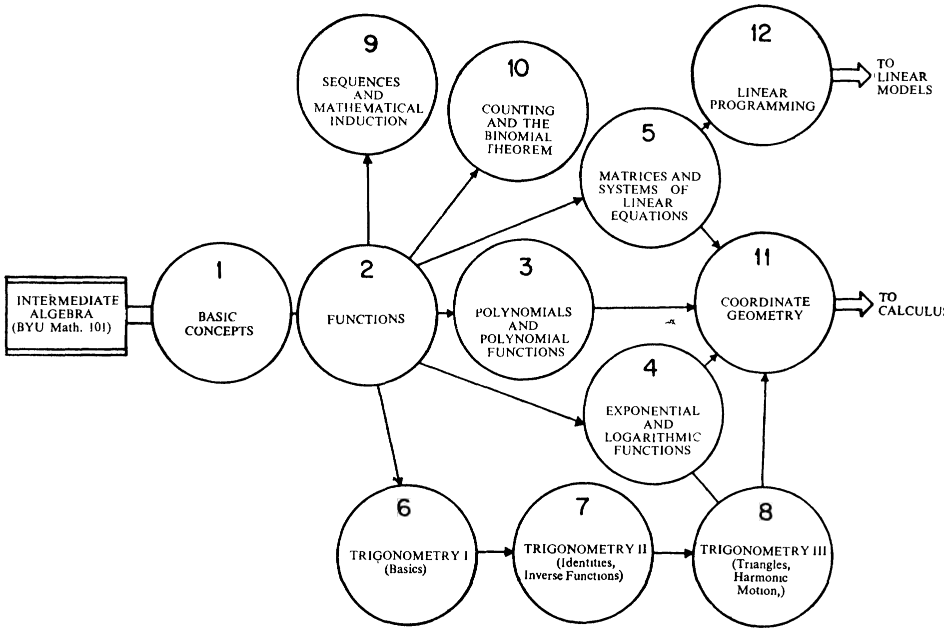
H. G. MOORE

Introduction. The past several years have seen the extension of self-paced and individualized instruction in mathematics to colleges and universities. For a variety of intricate reasons, the traditional university mathematics lecture is being replaced, at least for beginning undergraduates, by various adaptations of the self-pacing concept. [See Refs.]

One of the obvious weaknesses of total self-pacing is manifested by the very high attrition rate in these programs. In many cases almost 75% of the students who begin the course will fail to complete it in an entirely self-paced program. In an effort to overcome this weakness, and to combine the best features of self-pacing with those of performance-criteria grading, the BYU Mathematics Department began this fall to experiment with a modular approach to teaching precalculus mathematics. This note outlines some of the details of that program.

Mathematics 110. In attempting to meet the demands of various mathematical consumers (Engineers, Physical Scientists, Biological Scientists, Behavioral and Ma̋nagement Scientists, Technicians, etc.) a multiple track program of precalculus mathematics has grown up over the years. At Brigham Young, this involved at least five separate courses in the usual college-algebra-trigonometry-elementary functions courses. Each had its own flavor and particular combination of ingredients. Because this material is largely a collection of only loosely related topics anyway, it seemed natural to investigate modularization of these courses. Mathe-

matics 110 is an experimental course consisting of twelve mini-courses (modules) covering the usual content of these precalculus tracks. The diagram below indicates these modules and their interrelationship with each other. A brief table of contents for each module is appended to this note.



Some of our reasons for using a modular approach in place of the more traditional courses were stated as follows:

(i) This approach allows each student to select those topics which are relevant to his major. At present, for example, a student must take 3 semester hours of mathematics 105 even though he needs only a portion of the topics covered in that course.

(ii) This approach allows the student to study only those topics in which he is unprepared. If one has studied part of a traditional course elsewhere he must, in the old system, take the entire course to acquire those topics he needs.

(iii) This approach allows each student to proceed at his own pace. Each module is presented several times a semester. If a student cannot learn the material in the allotted time, he may repeat the module in the same semester.

(iv) This approach allows a student to study on his own and "test out" of any given module whenever he feels ready, rather than having his progress determined by the rest of the class.

(v) This approach allows students to enter or leave the course at several different times during the semester.

(vi) This approach allows students to sample a larger variety of mathematics than

the traditional system. For example, in the old system, if a student needs to know some trigonometry, he must take either Mathematics 106 (3 hours) or Mathematics 111 (5 hours). If his schedule does not allow enough time for one of these courses, he remains unprepared in trigonometry, which may well be detrimental to his professional progress.

Mechanics of the course. Each module is studied individually as a short mathematics mini-course. With cooperation from our Registrar each student merely approximates the number of modules he will complete as he registers at the beginning of the semester. (A sop to the VA and other number counters.) The mini-courses run for seven class days and are scheduled at various times throughout the semester. A schedule is provided to the student together with an instruction booklet describing the course. Each student is free to select his own program by studying, either on his own or in class, those modules he needs. He may attend any mini-course he wishes, subject only to the constraint that once he has begun that module he must attempt the examination at the end of the mini-course. Class attendance is encouraged particularly when the material is new to the student.

Whenever the student feels that he is sufficiently well prepared, before, during, or after the mini-course, he may take the final examination for that module. These exams are given in class at the end of the module, or at any time during the school day in the Testing Center. The exams are machine generated, and a computer system maintains the records. The examination is graded immediately and, if passed, the student receives 0.5 semester hours of credit in Mathematics 110 with the grade earned on the examination. The grade and credit which will appear on the student's permanent record is the composite of these individual modular grades earned during the semester. If the examination is failed, no grade or credit is recorded. At the end of the semester, the rare student who has passed nothing will receive 0.5 credit hours of failure in Mathematics 110. It is possible, therefore, for one student to earn 0.5 hours of B in Mathematics 110 during the semester while another student earns 6.0 hours of A in the same course.

A student may retake an examination for a given module as many times as he wishes during the semester, either to eventually pass it, or to raise his grade. A different exam is administered each time. It is the last *passing* grade that determines the student's grade on the module. This retaking privilege is constrained only by the semester calendar. The end of the semester ends the game for those modules passed that semester. However, additional modules may be studied during a subsequent semester.

A fairly high level of performance is required to pass a given modular examination. This is usually at the 75% level; over 90% is required for an A grade. This is significantly higher performance than is expected to pass a traditional course.

Students may study a given module on their own. They are free to attend the mini-course as often as they feel is necessary to prepare themselves to pass the examina-

tion. Study guides have been prepared for each module. Each was written by a different member of the faculty. Each guide consists of a brief table of contents, a list of behavioral objectives, a pre-test (to determine readiness for the material), expository material with examples, exercises, and a sample final examination.

RESULTS. The experiment is now* in its second semester and is still facing some difficulties. These include the following:

(1) The unevenness of the textual material (showing their camel-like nature).

(2) Some mechanical problems with the computerized record systems.

(3) The need for multi-media assistance to supplement the mini-courses.

(4) Some problems of teacher morale owing to the shortness of the mini-course and the student turn-over.

Nevertheless, whereas in a traditional mathematics course at this level upwards of 20% of the students receive no credit at all due to failure or withdrawal, while an additional 10% receive D grades, with this program 88% of the students received passing grades and some credit. They demonstrated some mathematical skill.

In a survey conducted with a small sample of last semester's participants, over 70% reported a willingness to recommend this procedure to their friends.

Last semester's participants were anything but volunteers. All of the students who initially registered in some of the traditional mathematics sections were summarily selected to participate in the program. Of the 382 students so drafted, the mean number of credit hours earned was 2.51. The average grade on this course (composite) was 3.14 (4.00 = A). In contrast to this, of the 543 students enrolled in the traditional college algebra course, Mathematics 105, 22.5% (122 students) received no credit while the remaining 421 students received 3 hours credit for a mean credit of 2.25 hours per student. The average grade was 1.93 in the traditional course.

This current semester, students were allowed to choose between the Mathematics 110 modular program and the traditional courses. Data on these students will be reported later. At the half-way point this semester, it seems that the program is accomplishing many of its goals. While there is inherent danger in the fragmentation of a college course, this is only an extension of the danger already inherent in the college calendar itself. It seems that this modular approach does a better job of meeting the realities of students' precollege preparation in mathematics. This procedure is in general agreement with the current trend to grant college credit based on knowledge or skill wherever it was obtained, rather than for sitting through an academic exercise.

On the other hand, it may be true that 1/2 credit hour modules are *too* fragmented. The problems of teacher morale and disciplined self-pacing may be better achieved if some modules were combined into larger units. The committee is investigating these ideas along with its evaluation of current procedure.

---

* March 1973.

Contents of the twelve modules.

MODULE 1: BASIC CONCEPTS

1. Sets and Operations on Sets
2. The Real Numbers and Their Properties
3. The Fundamental Operations, Linear Equations, Factoring
4. Algebraic Fractions, Exponents and Radicals
5. Simultaneous Linear Equations, Quadratic Equations
6. Inequalities and Absolute Value
7. The Complex Numbers

MODULE 2: FUNCTIONS

1. The Rectangular Coordinate System and Relations
2. Functions
3. Some Functions and Their Graphs
4. Another Look at Functional Notation
5. New Functions From Old: The Sum, Product, Quotient, and Composition of Two Functions
6. Inverse Functions

MODULE 3: POLYNOMIALS

1. Introduction to Polynomials
2. The Basic Algebra of Polynomials
3. Zeros, Degree, and The Fundamental Theorem of Algebra
4. Division of Polynomials
5. Synthetic Division of Polynomials
6. Theory of Polynomials
7. Rational Roots of Polynomials

MODULE 4: EXPONENTIAL AND LOGARITHMIC FUNCTIONS

1. Exponents
2. Exponential Functions
3. Logarithmic Functions
4. Computation with Logarithms

MODULE 5: MATRICES AND SYSTEMS OF LINEAR EQUATIONS

1. Equivalent Systems of Linear Equations
2. Substitution and Graphing
3. Solutions in Echelon Form
4. Matrix Method
5. Matrix Algebra
6. Determinants and Cramer's Rule
7. The Inverse of a Matrix

MODULE 6: TRIGONOMETRY 1

1. Angular Measure and Unit Circle Diagrams
2. Circular Functions
3. Right Triangle Relations

4. Values of the Circular Functions for Special Numbers
5. Graphs of the Circular Functions
6. Basic Identities

MODULE 7: TRIGONOMETRY 2

1. Identities and Reduction Formula
2. Identiti s — Logic — Proof
3. More Graphs of Circular Functions — Periodicity
4. Inverses of Circular Functions
5. Open Sentences
6. Solution of Simple Trigonometric Equations
7. Solution of More Complex Trigonometric Equations

MODULE 8: TRIGONOMETRY 3

1. The solutions of Right Triangles
   (a) Use of the Trigonometric tables
   (b) Linear interpolation
2. The Law of Cosines
3. The Law of Sines — The ambiguous case
4. Cartesian and Polar Representations of Complex Numbers
5. DeMoivre's Theorem and Roots of Complex Numbers
6. Harmonic Motion
7. Summary

MODULE 9: SEQUENCES AND MATHEMATICAL INDUCTION

1. Sequences and their definition
2. The Positive Integers and Mathematical Induction
3. Proofs by Mathematical Induction
4. Arithmetic Progressions
5. Geometric Progressions
6. The Binomial Theorem

MODULE 10: COUNTING AND THE BINOMIAL THEOREM

1. Cardinality of Sets
2. Permutation
3. Combinations
4. The Binomial Theorem
5. Partitions and the Multinomial Theorem
6. Summary

MODULE 11: ANALYTIC GEOMETRY

1. Introduction
2. Plane Coordinates
3. The Line
4. The Circle
5. The Parabola
6. The Ellipse
7. The Hyperbola

MODULE 12: LINEAR PROGRAMMING

1. Systems of linear inequalities
2. The max-min problems — geometric solutions and applications
3. Max. problems — the simplex method
4. Min. problems — the duality principle
5. Allocation problems — applications of linear programming

References

1. L. J. Ablon, A modular approach to preparatory mathematics, this MONTHLY, 79 (1972) 1126–1131.

2. Ronald D. Jamison, A program of personalized and individualized instruction in mathematics education, MLS Newsletter, 3 (May 1971).

3. Ellen Oliver, A course in two-week units, The Math. Teacher, 65 (1972) 145.

4. John Riner, Individualizing mathematics instruction, this MONTHLY, 79 (1972) 77–86.

5. Bert K. Waits, Individualized instruction in large enrollment mathematics courses, this MONTHLY, 80 (1973) 307–310.

MATHEMATICS DEPARTMENT, 292 TMCB, BRIGHAM YOUNG UNIVERSITY, PROVO, UT 84602.

PROBLEMS AND SOLUTIONS

All problems (both elementary and advanced) proposed for inclusion in this Department should be sent to E. P. Starke, 1000 Kensington Ave., Plainfield, NJ 07060. Proposers of problems are urged to enclose any solutions or information that will assist the editors. Ordinarily, problems in well-known textbooks and results in generally accessible sources are not appropriate for this Department. No solutions (except those accompanying proposals) should be sent to Professor Starke.

ELEMENTARY PROBLEMS

Solutions of Elementary Problems should be sent to Problems Group, Mathematics Department, University of Maine, Orono, ME 04473. To facilitate their consideration, solutions of Elementary Problems in this issue should be typed (with double spacing) and should be mailed before February 28, 1975.

Note by the Editors. In view of the considerable interest in a number of the

older unsolved problems, it has been decided to reprint some of them every now and then. Solutions are solicited.

E 570 [1943, 260]. *Proposed by L. M. Kelly*

If the six conics, determined by each five of a set of six points, are congruent must they coincide?

E 585 [1943, 454]. *Proposed by A. H. Stone*

Let a circle with center O meet the sides BC, CA, AB of a triangle in the pairs of points X and X', Y and Y', Z and Z'. Let M be the Miquel point of XYZ (i.e., the point of concurrence of circles AYZ, BZX, CXY), and M' be that of $X'Y'Z'$. Prove that $OM = OM'$.

If, further, the lines AX, BY, CZ concur, say in P, and consequently the lines AX', BY', CZ' concur, say in P', prove that PP' and MM' are parallel.

E 2498. *Proposed by R. E. Smith, Bishop Union High School, Bishop, California*

Given triangle ABC, find the locus of all points R (not necessarily in the plane of ABC) with the property that the three triangles RAB, RBC, and RCA have the same area.

E 2499. *Proposed by S. R. Conrad, B. N. Cardozo High School, Bayside, New York*

Construct the triangle ABC, given side a, angle A, and the bisector t_a of angle A.

E 2500. *Proposed by P. Richard Herr, University Park, Pennsylvania*

Find all natural numbers n with the property that both n and $\sigma(\sigma(n))$ are perfect numbers, or prove that none exist.

E 2501. *Proposed by S. R. Conrad, B. N. Cardozo High School, Bayside, New York*

Let ABC be a right triangle with $\sphericalangle C > \sphericalangle B \geqq \sphericalangle A$, and let O and I be its circumcenter and incenter respectively. Show that the triangle BIO is a right triangle if and only if $BC : CA : AB = 3:4:5$.

E 2502. *Proposed by Jean-Marie De Koninck, Université Laval, Quebec*

For each natural number n, let

$$f(n) = \sin \pi \left(\frac{(n-1)! + 1}{n}\right).$$

Prove that

$$\sum_{n=1}^{N} f(n) = \pi \log \left(\frac{N}{\log N}\right) + O(1).$$

E 2503. *Proposed by R. F. Jackson, University of Toledo*

A fixed disk C_0 of unit radius is centered at $(-1, 1)$. Beginning with the disk C_1, centered at $(1, 1)$ and tangent to the x-axis and to C_0, an infinite chain of disks $\{C_k\}$ is constructed, each tangent to the x-axis, to C_0, and to C_{k-1}. Find the sum of their areas.

SOLUTIONS OF ELEMENTARY PROBLEMS

Primes q not Satisfying $q^{p-1} \equiv 1 \pmod{p^2}$

E 2435 [1973, 943]. *Proposed by Wells Johnson, Bowdoin College*

Let $p \geqq 5$ be a prime number. Prove that there exist at least two distinct primes q_1, q_2 satisfying $1 < q_i < p - 1$ and $(q_i)^{p-1} \not\equiv 1 (\text{mod } p^2)$.

Solution by Irving Gerst, SUNY at Stony Brook. For $p = 5$, the primes 2 and 3 satisfy the conditions. Hence assume $p \geqq 7$. Call an integer n with $(n, p) = 1$ *proper* if $n^{p-1} \equiv 1$, and *improper* otherwise (all congruences are to be taken modulo p^2). Since the product of two proper integers is again proper, the following result holds: (A) If $n > 1$ is improper, then n has at least one improper prime divisor. Also, if n is proper and k is an integer with $(k, p) = 1$, then

$$(kp - n)^{p-1} \equiv n^{p-1} - (p - 1)kpn^{p-2} \equiv 1 + kpn^{p-2} \not\equiv 1,$$

since $p \nmid kn^{p-2}$. Thus: (B) If n is proper and $(k, p) = 1$, then $kp - n$ is improper.

There are now two cases to consider:

Case I: $p - 2$ is improper. Then, by (A), there exists an improper prime q with $q \mid (p - 2)$. Also, since 1 is proper, (B) with $n = 1$, $k = 1$ implies that $p - 1$ is improper, so that, again by (A), there is an improper prime r with $r \mid (p - 1)$. The primes r and q are distinct since $(p - 2, p - 1) = 1$, and each is less than $p - 1$.

Case II: $p - 2$ is proper. Then (B), applied with $n = p - 2$ and $k = 1$, shows that 2 is improper. Now $(p - 2)^2$ is proper which implies that $-4p + 4$ is proper. It now follows from (B) with $n = -4p + 4$ and $k = -3$ that $p - 4$ is improper, so that, by (A), there is an improper prime s with $s \mid (p - 4)$. Since s is odd and less than $p - 1$, it follows that 2 and s are the required primes.

Also solved by the Bennett College Team, D. M. Bloom, Leonard Carlitz, Stephen Currier, O. P. Lossers (Netherlands), L. E. Mattics, Allen Stenger, Temple University Problem Solving Group, and the proposer.

Matrices which Commute with all Doubly Stochastic Matrices

E 2436 [1973, 943]. *Proposed by E. T. H. Wang, University of Waterloo*

An $n \times n$ matrix with nonnegative entries is *doubly stochastic* if each row sum and each column sum is 1. Characterize those doubly stochastic $n \times n$ matrices that commute with all doubly stochastic $n \times n$ matrices.

I. *Solution by D. M. Bloom, Brooklyn College.* An $n \times n$ matrix A (doubly

stochastic or not) commutes with all $n \times n$ doubly stochastic matrices if and only if (1) all diagonal entries are equal (to α, say) and (2) all off-diagonal entries are equal (to β, say). If A is restricted to be doubly stochastic, then $\alpha, \beta \geqq 0$ and $\alpha + (n-1)\beta = 1$ so that these conditions are equivalent to (3) $A = (1-\gamma)I + \gamma J$, where $0 \leqq \gamma \leqq 1$, where I is the identity matrix and where J is the matrix with all entries equal to $1/n$.

The fact that every matrix satisfying (1) and (2) commutes with all $n \times n$ doubly stochastic matrices follows by direct matrix multiplication. Conversely, if A commutes with all $n \times n$ doubly stochastic matrices, then it commutes with all permutation matrices P. (The matrix $P = (p_{ij})$ is a *permutation matrix* if there exists a permutation π of $\{1, 2, \cdots, n\}$ such that $p_{ij} = 1$ if $j = i\pi$ and 0 otherwise.) For such a matrix, $(AP)_{ij} = a(i, j\pi^{-1})$ and $(PA)_{ij} = a(i\pi, j)$ where $a(i,j) \equiv a_{ij}$. If $AP = PA$ for all π, then $a(i,j) = a(i\pi, j\pi)$ for all i, j, π. For all i, k there is a permutation π such that $i\pi = k$ and so $a_{ii} = a_{kk}$ for all i, k and (1) follows. Similarly, if $i \neq j$ and $r \neq s$, then there is a permutation π such that $i\pi = r$ and $j\pi = s$ and so $a_{ij} = a_{rs}$ and (2) follows.

II. *Comment by K. F. Andersen, University of Alberta.* There is an analog of this result for doubly stochastic operators on the space L of Lebesgue integrable functions on the unit interval $[0, 1]$. A (linear) operator $T : L \to L$ is *doubly stochastic* if

(1) $Tf \geqq \theta$ whenever $f \geqq \theta$,

(2) $T\mathbf{1} = \mathbf{1}$,

(3) $\int_0^1 Tf\,d\mu = \int_0^1 f\,d\mu$ for all $f \in L$.

(In the above θ represents the zero function $\theta(x) \equiv 0$, $\mathbf{1}$ represents the constant function $\mathbf{1}(x) \equiv 1$, and μ is Lebesgue measure.) By considering first simple functions, we can show that a doubly stochastic operator T which commutes with all doubly stochastic operators on L must be of the form

$$T = (1-\gamma)I + \gamma J$$

for some $0 \leqq \gamma \leqq 1$ where I is the identity operator ($If = f$) and J is the averaging operator ($Jf = \{\int_0^1 f\,d\mu\}\mathbf{1}$.)

(This result is not surprising since an $n \times n$ doubly stochastic matrix can be considered as a doubly stochastic operator on the real n-dimensional space R^n, and R^n can be considered as $L^1(X, A, \nu)$ where $X = \{1, 2, \cdots, n\}$, $A = P(X)$, and ν = counting measure. —Ed.)

Also solved by K. F. Andersen, Ethan Bolker, J. V. Brawley, A. W. Briggs, Jr., Floyd Christian, Jr., Peter de Buda, D. Ž. Djoković, T. E. Elsner, Jane Evans, Clark Givens, G. A. Heuer & Karl Heuer (Germany), Yasuhiko Ikeda, R. A. Jacobson, S. D. Kerr, J. C. Kieffer, Detlef Laugwitz (Germany), H. S. Lieberman, O. P. Lossers (Netherlands), Carolyn MacDonald, J. G. Mauldon, Robert Patenaude, Ken Rebman, Emma Riddle, D. D. Rothmann, P. C. Shields (England), Richard Sinkhorn, W. C. Stone, Temple University Problem Solving Group, M. R. Vitale, William Watkins, R. D. Whittekin, W. W. Williams, Ken Yocom, and the proposer.

The Termwise Minimum of Two Divergent Series Can Converge

E 2437 [1973, 943]. *Proposed by David McLean, Warren, Michigan*

Let $\{a_n\}$ and $\{b_n\}$ be two sequences of real numbers that decrease monotonically to 0. If the series $\Sigma\, a_n$ and Σb_n both converge, then so does $\Sigma \max(a_n, b_n)$. Suppose that the series $\Sigma\, a_n$ and $\Sigma\, b_n$ both diverge. Does it follow that the series $\Sigma \min(a_n, b_n)$ must diverge?

Comment by Richard Poppen, Stanford University. A practically identical problem occurred as Advanced Problem 4278 by Peter Ungar [1948, 34; 1949, 423]. The stated problem also appears in the appendix (p. 201) of the 1967 corrected printing of G. Polya, *Mathematical Discovery*, Vol. 2 (Wiley, New York).

It is interesting to note that the problem is slightly easier in its 1948 statement, since there the reader was asked to construct an example, whereas in the 1973 statement, the reader was asked to settle the question of whether an example existed. Yet the problem has been "demoted" from an Advanced to an Elementary Problem. Perhaps this indicates the advance of mathematics in the last 25 years.

Solved by K. F. Andersen, John Annulis, Nicolas Artémiadis, Larry Bennett & Ken Yocom, Robert Breusch, George Crofts, S. C. Currier, Jr., D. Ž. Djoković, Roger Eggleton (Israel), D. W. Erbach (England), P. K. Garlick, A. M. Gendler ,G. A. Heuer (Germany), R. A. Jacobson, D. Z. Kilhefner, Robert Kopp, L. Kuipers, P. W. Lindstrom, O. P. Lossers (Netherlands), J. G. Mauldon, C. P. McCarty, Glen Meyers, Joel Pitt, J. W. Reed & J. P. Mayberry, Bruce Reznick, St. Olaf College Students, T. Šalát (Czechoslovakia), Kenneth Schilling, SCSC Problem Solving Group, Wolfe Snow, Joel Spencer, A. H. Stein, Allen Stenger, Temple University Problems Group, R. H. Warren, Charles Wexler, Gordon Williams, J. K. Yates, and the proposer.

Editor's comment: It was also noted by Peter Ungar that this problem had occurred previously as 4278. Professor Ungar modestly declined to observe that it was his problem originally.

The result is surprising since the dual result with "divergent" replaced by "convergent" and "min" replaced by "max" is true. Suppose that $a_n \downarrow 0$ and $b_n \downarrow 0$ and that $m_n = \min(a_n, b_n)$ and $M_n = \max(a_n, b_n)$. Then always $a_n + b_n = m_n + M_n$. If $\sum a_n$ and $\sum b_n$ both converge, then so do $\sum(a_n + b_n)$ and $\sum m_n$ and since $\sum M_n = \sum[(a_n + b_n) - m_n]$ it follows that $\sum M_n$ converges since the difference of convergent series is convergent. If, however, both $\sum a_n$ and $\sum b_n$ diverge, then so do $\sum(a_n + b_n)$ and $\sum M_n$, but in this case $\sum m_n = \sum[(a_n + b_n) - M_n]$ and nothing can be inferred since the difference of divergent series need not diverge.

Meaningful Multiple Vector Products

E 2439 [1973, 1058]. *Proposed by Edmund Umberger, Pennsylvania State University*

Let $\cdot$ and $\times$ denote the usual dot and cross product of vectors in 3-space, and let $*$ denote any of the following operations: the usual scalar multiplication of scalar and vector, multiplication of vector and scalar with the obvious interpretation, or ordinary multiplication of scalar and scalar. How many of the 3^n ways of inserting the symbols $\cdot$, $\times$, and $*$ between consecutive vectors of the string $\mathbf{v}_1\mathbf{v}_2\cdots\mathbf{v}_{n+1}$ will

result in meaningful expressions by suitable insertion of parentheses? For example, $\mathbf{v}_1 \times \mathbf{v}_2 * \mathbf{v}_3 \cdot \mathbf{v}_4$ can be made meaningful, whereas $\mathbf{v}_1 * \mathbf{v}_2 \cdot \mathbf{v}_3 \cdot \mathbf{v}_4$ cannot.

Solution by Kenneth Schilling, University of California, Davis. The number of ways that will result in meaningful expressions is

$$\sum_{k=0}^{n} \frac{n!}{[\frac{1}{2}k]!\,[\frac{1}{2}(k+1)]!\,(n-k)!}\,. \tag{1}$$

This follows from the fact that a given placement of the symbols can be made meaningful if and only if $s \leqq d \leqq s+1$ where s is the number of occurrences of $*$ and where d is the number of occurrences of $\cdot$.

Suppose first that $s \leqq d \leqq s+1$. The proof is by induction; the cases $n = 1,2$ are trivial. For larger n, if some two vectors are separated by a $\times$, then they can be combined to form a single vector, thus reducing the situation to the previous case. If there are no occurrences of $\times$, then somewhere there must be a $*$ adjacent to a $\cdot$. Then $(\mathbf{v}_1 \cdot \mathbf{v}_2) * \mathbf{v}_3$ or $\mathbf{v}_1 * (\mathbf{v}_2 \cdot \mathbf{v}_3)$ likewise form a single vector and the induction is complete.

Conversely, note that each time $*$ is performed, one scalar is lost, and each time $\cdot$ is performed, one scalar is gained. When all computations are done, there must remain exactly one or zero scalars. Since we begin with no scalars, we must have $s \leqq d \leqq s+1$.

The number of ways of having k occurrences of both $*$ and $\cdot$, and $(n-2k)$ occurrences of $\times$ is the multinomial coefficient

$$\frac{n!}{k!\,k!\,(n-2k)!}$$

and the number of ways of having k occurrences of $*$, $(k+1)$ occurrences of $\cdot$, and $(n-2k-1)$ occurrences of $\times$ is

$$\frac{n!}{k!\,(k+1)!\,(n-2k-1)!}$$

Summing these possibilities and manipulating the resulting sums yields (1).

Also solved by T. J. Grilliot, Myron Hlynka, Ralph Jones, Carolyn MacDonald, and J. G. Mauldon.

A Tournament Scheduling Problem

E 2441 [1973, 1058]. *Proposed by Cornelius Groenewoud, Snyder, N.Y., and F. K. Hwang, Bell Telephone Laboratories*

One has n locations and m teams of n players each. Every week, each team is to send one player to each location where a resolvable round-robin tournament is

conducted. Show that it is possible to construct an n-week schedule such that every player goes to every location exactly once, and each player plays against every other player on every other team exactly once, whenever n is a prime power which exceeds m.

Solution by O. P. Lossers, Technological University, Eindhoven, Netherlands. It is well known (H. J. Ryser, *Combinatorial Mathematics*, Carus Mathematical Monograph No. 14, p. 81) that for n a prime power, m mutually orthogonal Latin squares of order n exist whenever $n > m$. Let $A(1), \cdots, A(m)$ be m such mutually orthogonal Latin squares of order n. Write $A(p) = (a_{ij}(p))$ and let $a_{ij}(p) = k$ mean that player number k of the pth team is sent to location j in the ith week. Since each row and each column of $A(p)$ contains every element $1, 2, \cdots, n$ exactly once, it follows that each player of team p goes to every location exactly once (rows), and that in each week exactly one player of team p is sent to each location (columns). Since $A(p)$ and $A(q)$ are orthogonal for $1 \leqq p, q \leqq m$ and $p \neq q$, each player of the pth team plays against each player of the qth team exactly once.

Also solved by Peter de Buda, Michael Goldberg, R. A. Jacobson, Carolyn MacDonald, J. G. Mauldon, Robert Patenaude, J. H. Pitt, Ken Rebman, Paul Smith, G. W. Valk, M. R. Vitale, and the proposers.

A Sum Involving Roots of Unity

E 2442 [1973, 1058]. *Proposed by J. C. Hemperly, University of Maryland*

Let $\omega_1, \omega_2, \cdots, \omega_n$ denote the nth roots of unity. Evaluate

$$\Sigma \left| \omega_i - \omega_j \right|^{-2},$$

the sum being taken over all distinct i, j.

I. *Solution by E. H. Umberger, Pennsylvania State University.* The set $|\omega_i - \omega_j|$, $1 \leqq i < j \leqq n$, represents the lengths of all sides, diagonals, and diameters (if any) of the regular n-gon inscribed in the unit circle.

When $n = 2m + 1$, there are n sides and n diagonals of each distinct length; using the Law of Cosines and a half-angle formula, we have

$$\begin{aligned}\sum_{i<j} \left| \omega_i - \omega_j \right|^{-2} &= n \sum_{k=1}^{m} \left(2 - 2\cos\frac{2\pi k}{n}\right)^{-1} = \frac{n}{4} \sum_{k=1}^{m} \csc^2 \frac{\pi k}{n} \\ &= \frac{n}{4} \cdot \frac{n^2 - 1}{6} = \frac{n^3 - n}{24}.\end{aligned}$$

When $n = 2m$, there are in addition m diameters of length 2; hence

$$\sum_{i<j} \left| \omega_i - \omega_j \right|^{-2} = \frac{m}{4} + \frac{n}{4} \sum_{k=1}^{m-1} \csc^2 \frac{\pi k}{n} = \frac{n}{8} + \frac{n}{4} \cdot \frac{n^2 - 4}{6} = \frac{n^3 - n}{24}.$$

It follows that the required sum is

$$2\left(\frac{n^3-n}{24}\right)=\frac{n(n^2-1)}{12}.$$

The formulas for the sum of the finite trigonometric series are given in Bromwich, *Theory of Infinite Series* (2nd edition), p. 210, and (incompletely) in L. B. W. Jolley, *Summation of Series* (2nd edition), formulas 439, 440.

II. *Solution by M. R. Murty and V. K. Murty* (*undergraduates*), *Carleton University*. Let $\omega_n = 1$. Then

$$\sum_{i\neq j}\left|\omega_i-\omega_j\right|^{-2}=n\sum_{i=1}^{n-1}\left|1-\omega_i\right|^{-2}=-n\sum_{i=1}^{n-1}\omega_i(1-\omega_i)^{-2},$$

the first equality following from symmetry and the second from the fact that

$$\left|1-\omega_i\right|^{-2}+\omega_i(1-\omega_i)^{-2}=0$$

since $\left|\omega_i\right| = 1$. Now consider the following function of two variables:

$$F(x,y)=x^{n-1}+x^{n-2}y+\cdots+xy^{n-2}+y^{n-1}=\begin{cases}\dfrac{x^n-y^n}{x-y} & \text{if } x\neq y\\ nx^{n-1} & \text{if } x=y.\end{cases}$$

We see that the partial derivative F_1 is given by

$$F_1(x,y)=\begin{cases}\dfrac{nx^{n-1}}{x-y}-\dfrac{x^n-y^n}{(x-y)^2} & \text{if } x\neq y\\ \frac{1}{2}n(n-1)x^{n-2} & \text{if } x=y,\end{cases}$$

and so, if $G = F_1/F$, then

$$G(x,y)=\begin{cases}\dfrac{nx^{n-1}}{x^n-y^n}-\dfrac{1}{x-y} & \text{if } x^n\neq y^n\\ \dfrac{n-1}{2x} & \text{if } x=y\neq 0,\end{cases}$$

the function G being defined for all (x,y) except when $x=\omega_i y$ for some nth root $\omega_i\neq 1$ of 1. Now evaluating the partial derivative $G_2(1,1)$, we have

$$G_2(1,1)=\lim_{h\to 0}\frac{G(1,1+h)-G(1,1)}{h}=\frac{1-n^2}{12}.$$

But we have also that

$$F(x,y)=\prod_{i=1}^{n-1}(x-\omega_i y),$$

and so

$$F_1(x,y) = \sum_{i=1}^{n-1} \prod_{j\neq i} (x - \omega_j y)$$

and therefore

$$G(x,y) = \frac{F_1(x,y)}{F(x,y)} = \sum_{i=1}^{n-1} (x - \omega_i y)^{-1}$$

implying that $G_2(x,y) = \sum_{i=1}^{n-1} \omega_i (x - \omega_i y)^{-2}$. Therefore

$$\sum_{i\neq j} |\omega_i - \omega_j|^{-2} = -n \sum_{i=1}^{n-1} \omega_i (1-\omega_i)^{-2} = -nG_2(1,1) = \frac{n(n^2-1)}{12}.$$

Also solved by Bruce Berndt, Robert Breusch (New Zealand), Brother Alfred Brousseau, Leonard Carlitz, Chin-Hung Ching (Australia), Thomas Elsner, Clark Givens & Otto Ruehr, Michael Goldberg, M. G. Greening (Australia), Rev. William Habakkuk, A. C. Hindmarsh, M. S. Klamkin, M. J. Knight, Sinan Kunt, J. R. Kuttler, O. P. Lossers (Netherlands), Bernard Martin, L. E. Mattics, J. G. Mauldon, Roger Nelson, Keith Powls, R. V. Pulskamp, St. Olaf College Students, Michael Shimshoni (Israel), and the proposer. Partial solutions by Jonathan Jankus, L. A. Ringenberg, Michael Skalsky, and Allen Stenger.

ADVANCED PROBLEMS

All solutions of Advanced Problems should be sent to J. Barlaz, Rutgers — The State University, New Brunswick, N. J. 08903. Solutions of Advanced Problems in this issue should be typed (with double spacing) on separate, signed sheets and should be mailed before February 28, 1975.

An asterisk (*) means neither the proposer nor the editors supplied a solution.

5952* [1974, 175]. *Proposed by J. Gilles, University of Charleroi, Belgium*

Correction. The formulas to be proved were intended by the proposer to read as follows:

$$\sum_{n=1}^{\infty} \frac{1}{n^3 \sin n\pi\sqrt{2}} = -\frac{13\pi^3}{360\sqrt{2}}. \tag{3}$$

$$\Gamma\left(\frac{8}{9}\right) = \frac{9 - \sqrt{14} + \sqrt{75 - 32\sqrt{3}}}{33} \cdot \sqrt[4]{182}. \tag{4}$$

5994. *Proposed by J. J. Higgins, University of Missouri, Rolla*

Let F and G be right-continuous distribution functions corresponding to the Lebesgue-Stieltjes measures μ_F and μ_G respectively. If F and G have no common points of discontinuity, the well-known integration by parts formula gives

$$\int_{(a,b]} F(x)\mu_G(dx) = F(b)G(b) - F(a)G(a) - \int_{(a,b]} G(x)\mu_F(dx).$$

Give the corresponding integration by parts formula when F and G are allowed to have common points of discontinuity.

5995. *Proposed by L. A. Harris, University of Kentucky*

Let A be a bounded linear operator on normed linear space. Suppose A satisfies a quadratic equation with roots λ_1 and λ_2 and that A is not a scalar multiple of the identity operator. Show that λ_1 and λ_2 are eigenvalues of A, $\sigma(A) = \{\lambda_1, \lambda_2\}$, and

$$f(A) = \begin{cases} f(\lambda_1)I + f'(\lambda_1)(A - \lambda_1 I) \text{ for } \lambda_1 = \lambda_2, \\ \dfrac{1}{\lambda_1 - \lambda_2}\left[f(\lambda_1)(A - \lambda_2 I) - f(\lambda_2)(A - \lambda_1 I)\right] \quad \text{for} \quad \lambda_1 \neq \lambda_2, \end{cases}$$

for any function f which is analytic in a disk containing both λ_1 and λ_2. (For example, any 2×2 matrix satisfies its characteristic equation, which is quadratic.)

5996. *Proposed by Arthur Smith, Carleton University*

A well-known problem posed by Kuratowski concerns the maximum number of sets obtainable from a subset A of a topological space T by the operations of closure $(^-)$, taking interiors $(^0)$ and complementation. Suppose we do not allow complementation but do allow arbitrary unions.

Show that at most 13 sets can be constructed from A and give an example of a set A for which these 13 sets may be constructed, when T is the real line with the usual topology.

5997. *Proposed by M. S. Klamkin, Ford Motor Company*

Prove that

$$\left| x^{p-1}(x-1)^p(x-2)^p \cdots (x-n)^p \right| \leqq \Gamma(1+n)^p$$

where $0 \leqq x \leqq n$ and p, n are real and $\geqq 1$. (This inequality has been given by A. Ostrowski for integral p, n. See Mitrinović and Vasić, *Analytic Inequalities*, Springer-Verlag, 1970, p. 198.)

5998*. *Proposed by D. E. Daykin, Reading University, England*

Let n be a positive integer and let N be the set $\{1, 2, \cdots, n^2 + n + 1\}$. Suppose F is a family of distinct subsets of N such that

(i) each member of F contains more than n^2 integers of N, and (ii) each integer of N is in more than n^2 members of F. Prove that two members of F have union N.

5999*. *Proposed by R. D. McKelvey, University of Rochester*

Let μ be a finite measure on the σ-algebra of sets in R^n generated by the half spaces defined by hyperplanes through the origin. I.e., for any $\alpha \in R^n$, let $H_\alpha = \{x \in R^n \mid x \cdot \alpha > 0\}$ and let $H = \{H_\alpha \mid \alpha \in R^n\}$. Further, let B be the σ-algebra

generated by H, and $\mu\colon B \to R$ be a finite positive measure on B. Prove or disprove the conjecture: If $\mu(A) = \mu(-A)$ for all $A \in H$, then $\mu(A) = \mu(-A)$ for all $A \in B$. (Here, of course, $-A = \{x \in R^n \mid -x \in A\}$.)

SOLUTIONS OF ADVANCED PROBLEMS

Factoring Unimodular Matrices

5876 [1972, 913]. *Proposed by C. H. Kimberling, University of Evansville*

In the ring of 2×2 matrices over the reals, is every unimodular matrix a product of matrices of finite order? If so, generalize.

Solution by W. C. Waterhouse, Cornell University. All conjugates of the matrix $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ have order 4; let N be the subset of $\mathrm{SL}(2,\mathbf{R})$ consisting of products of them. Then N is a normal subgroup not consisting purely of scalar matrices, and therefore [E. Artin, *Geometric Algebra*, p. 165] it is all of $\mathrm{SL}(2, \mathbf{R})$.

The same proof is valid for $\mathrm{SL}(n,k)$ with any $n \geqq 2$ and any field k, except for $\mathrm{SL}(2,\mathbf{Z}/2\mathbf{Z})$ and $\mathrm{SL}(2, \mathbf{Z}/3\mathbf{Z})$ — and these, of course, are finite groups.

Note: It is also possible to give a brief computational proof of the fact that every unimodular matrix is the product of matrices of determinant -1 of order two, valid over any division ring k: The unimodular group is generated by the matrices $I + ae_{ij}$, $i \neq j$, where a runs through non-zero elements of k and e_{ij} denotes the usual matrix units (loc. cit., p. 151). Now it is easily verified that if $i \neq j$, $(I + e_{ij} - 2e_{jj})(I + (a+1)e_{ij} - 2e_{jj}) = I + ae_{ij}$ and $(I + be_{ij} - 2e_{jj})^2 = I$, for any b in k. Hence $I + ae_{ij} - 2e_{jj}$, $i \neq j$, is a set of matrices of order two such that every unimodular matrix is a product of elements of this set.

Also solved by D. Ž. Djoković, Gertrude Ehrlich, A. A. Jagers (Netherlands), Marijo LeVan, and J. B. Sunday.

Lengths in Quasi-Frobenius Rings

5891 [1973, 82]. *Proposed by V. Dlab and C. M. Ringel, Carleton University, Ottawa, Canada*

Prove or disprove that the left and right lengths of every factor ring R/I coincide, where R is a local quasi-Frobenius ring and I is an ideal of R. A local quasi-Frobenius ring is an Artinian local ring with a unique minimal left and a unique minimal right ideal (which necessarily coincide).

Solution by T. A. Hannula, University of Maine at Orono. Let K be a field with isomorphic subfield K' such that $[K\colon K'] = 2$. Let R be the ring with additive group $K \times K$ and multiplication given by $(k,m)(l,n) = (kl, kn + ml')$. Note that

R is the ring obtained from the $K-K$ bimodule K with scalar multiplication given by $k * m * l = kml'$, where l' is the image of l in K' under the isomorphism from K onto K'. R is an example of an exceptional ring given by Dlab and Ringel ([**2**], p. 139). Clearly, R has left dimension equal to 2 and right dimension equal to 3. We show that R is a homomorphic image of a quasi-Frobenius local ring yielding the necessary counterexample.

Note that R is a finite dimensional algebra over the field K. Thus the functor $\mathrm{Hom}_K(-,K)$ induces a Morita Duality between ${}_R\mathscr{F}$, the category of finitely generated left R-modules and $\mathscr{F}_R$, the category of finitely generated right R-modules. It now follows ([**1**], Th. 5.2, p. 56) that there exists an R–R bimodule U such that (1) $\mathrm{End}_R({}_RU) \cong R$ with the isomorphism induced by right multiplication and (2) U is a finitely generated injective left R-module. The ring $R \times U$ obtained from the R–R bimodule U by letting $(r,u)(s,v) = (rs, rv + us)$ is a left self-injective ring ([**4**], Th. 1.9(c), p. 132) since the right annihilator of U in R is 0 when $R \cong \mathrm{End}_R({}_RU)$. Since both R and U have left composition series, $R \times U$ has finite length. It follows that $R \times U$ is a left self-injective, left Artinian ring, hence is quasi-Frobenius ([**3**], Th. 18, pp. 11–12).

It is easy to check that $R \times U$ is a local ring with $\mathrm{rad}(R \times U) = \mathrm{rad}(R) \times U$. Moreover, $0 \times U$ is an ideal with $R \cong R \times U / 0 \times U$. Thus $R \times U$ is the required local quasi-Frobenius ring.

References

1. P. M. Cohn, Morita Equivalence and Duality, Queen Mary College Lecture Notes, 1966.
2. V. Dlab and C. M. Ringel, Balanced Rings, in Lectures on Rings and Modules (Tulane University Ring and Operator Year 1970–1971), pp. 72–143. Lecture Notes in Mathematics, v. 246 Springer, New York–Berlin–Heidelberg, 1972.
3. S. Eilenberg and T. Nakayama, On the dimension of modules and algebras, II (Frobenius Algebras and Quasi-Frobenius Rings), Nagoya Math. J. 9(1955), 1–16.
4. R. Fossum, P. Griffith and I. Reiten, Trivial Extensions of Abelian Categories and Applications to Rings: An Expository Account, in Ring Theory, Edited by Robert Gordon, Academic Press, N. Y., 1972.

T. A. Hannula submitted a second solution. Also solved by the proposers.

Decompositions of Abelian Groups

5924 [1973, 814]. *Proposed by Donald Girod, Canisius College, Buffalo, N.Y.*

A standard exercise in an introductory algebra course is to show that no group G can ever be the set theoretic union of two proper subgroups H_1, H_2. It is possible, however, for a group to be the union of some finite number (> 2) of proper subgroups. For example, $Z \oplus Z$ is the union of three proper subgroups. Characterize those abelian groups G having a finite set of proper subgroups $\{H_1, \cdots, H_n\}$ such that $G = H_1 \cup \cdots \cup H_n$.

Solution by Robert Gilmer, Florida State University. The following conditions on an abelian group G are equivalent:

(1) G is the set theoretic union of a finite family of proper subgroups.

(2) There exists a proper subgroup H of G such that G/H is finite and is not cyclic.

(3) There exists an integer $n > 1$ such that G/nG is not cyclic.

(4) There exists a prime integer p such that G/pG is not cyclic.

Proof. (1) $\rightarrow$ (2): Assume that $\{H_i\}_{i=1}^n$ is a finite family of proper subgroups of G such that $G = \bigcup_{i=1}^n H_i$, and G is not the union of a proper subfamily of $\{H_i\}_1^n$. By a result of D. S. Passman [*Infinite group rings*, Marcel Dekker, New York, 1971; Lemma 5.2, p. 17], each H_i has finite index in G, and hence $H = \bigcap_{i=1}^n H_i$ also has finite index in G. Since the group G/H is the union of its family $\{H_i/H\}_{i=1}^n$ of proper subgroups, G/H is not cyclic.

(2) $\rightarrow$ (3): If H is a proper subgroup of G such that G/H has order n and G/H is not cyclic, then nG is contained in G, and since G/H is a homomorphic image of G/nG, it follows that G/nG is not cyclic.

(3) $\rightarrow$ (4): Assume that G/nG is not cyclic, where $n > 1$, and let $n = p_1^{e_1}p_2^{e_2}\cdots p_k^{e_k}$ be the prime factorization of n. Since G/nG is isomorphic to the direct sum of the family $\{G/p_i^{e_i}G\}_{i=1}^k$ of primary groups, it follows that $G/p_i^{e_i}G$ is not cyclic for some i. This implies that G/p_iG is not cyclic, for if $G = p_iG + \langle g\rangle$, where $\langle g\rangle$ denotes the cyclic group generated by the element g, then it also follows that $G = p_i^rG + \langle g\rangle$ for each positive integer r.

(4) $\rightarrow$ (1): Assume that p is a prime integer such that G/pG is not cyclic. Thus, as a vector space over $Z/(p)$, G/pG has a basis $\mathscr{B}$ containing more than one element. Choose distinct elements v_1, v_2 of $\mathscr{B}$ and let H be the subgroup of G generated by $\mathscr{B} - \{v_1, v_2\}$. The factor group $G/(pG + H)$ is elementary abelian of order p^2, and hence is the union of its (finite) family of proper subgroups (there are $p + 1$ such subgroups); taking inverse images under the natural homomorphism of G onto $G/(pG + H)$ it follows that G is the union of $p + 1$ proper subgroups.

REMARKS. In view of (4), it is clear that G *cannot* be expressed as a finite union of proper subgroups if G is cyclic or if G is divisible. But there are other families of groups G that cannot be so expressed. For example, if D is a subring of the rationals containing Z, then the additive group of D cannot be expressed as a finite union of proper subgroups, and if $S = \{p_\alpha\}$ a set of distinct primes, than the direct product of the family $\{C(p_\alpha^{n_\alpha})\}$ of cyclic groups, where each n_α is positive, is an abelian group that cannot be expressed as a finite union of proper subgroups.

Also solved by David Fried, A. A. Jagers (Netherlands), R. C. Lyndon, and the proposer.

A Positive Definite Matrix

5925 [1973, 814]. *Proposed by A. G. O'Farrell, Brown University*

Show that the matrix (a_{ij}), where $a_{ij} = 1/(1 + |j - i|)$, is positive definite.

Solution by F. Gerrish, The Polytechnic, Kingston-upon-Thames, England. Suppose that (for $i,j = 1,\cdots,n$) each $b_{ij}(t)$ is Riemann integrable on some interval $I \subseteq \mathbb{R}$, and that the symmetric matrix $(b_{ij}(t))$ is positive definite for each $t \in I$. Then (a_{ij}) is positive definite, where

$$a_{ij} = \int_I b_{ij}(t)\,dt.$$

For when $x \neq 0$ we have $0 < \sum_{i,j} b_{ij}(t)x_i x_j$ (all $t \in I$), and so

$$0 < \int_I \Big(\sum_{i,j} b_{ij}(t)x_i x_j\Big)dt = \sum_{i,j} a_{ij}x_i x_j,$$

i.e., (a_{ij}) is positive definite.

The choice $I = (0,1)$ and $b_{ij}(t) = t^{|i-j|}$ solves the original problem because (for each $r = 2,\cdots,n$) the leading minor of order r in $(b_{ij}(t))$ is reducible by the successive column operations $c_j \to c_j - tc_{j-1}$ $(j = r,\ r-1,\cdots,2)$ to a triangular determinant with value $(1-t^2)^{r-1}$, which is positive when $t \in (0,1)$, and hence $(b_{ij}(t))$ is positive definite.

Also solved by Per Asinoff & Tom Høholdt (Denmark), P. R. Chernoff, L. E. Clarke (England), Jane Evans, E. W. Ewing, David Fried, Clark Givens, M. L. T. Hautus (Netherlands), Finbarr Holland (Eire), A. A. Jagers (Netherlands), C. R. Johnson, J. C. Kieffer, I. I. Kotlarski, F. W. Steutel (Netherlands), J. H. Wells, and the proposer.

Bound for a Convolution

5926 [1973, 814]. *Proposed by R. P. Boas, Northwestern University*

If f and g are nonnegative, bounded, and integrable over $(-\infty,\infty)$, does it follow that

$$\int_{-\infty}^{\infty} \sup_t\,[g(x-t)f(t)]dx \geqq \sup_x \int_{-\infty}^{\infty} g(x-t)f(t)dt?$$

Solution by P. R. Chernoff, University of California, Berkeley. Since $\sup_t[g(x-t)f(t)]$ need not be measurable, we shall prove instead the modified, and formally stronger, inequality

$$\int_{-\infty}^{\infty} \operatorname{ess\,sup}_t\,[g(x-t)f(t)]dx \geqq \sup_x \int_{-\infty}^{\infty} g(x-t)f(t)dt.$$

First note that $h(x) = \operatorname{ess\,sup}_t[g(x-t)f(t)]$ is a measurable function. Indeed, if D is a countable dense subset of the unit ball of $L^1(R)$, we have

$$h(x) = \sup_{\phi \in D} \int_{-\infty}^{\infty} g(x-t)f(t)\phi(t)dt;$$

thus h is the supremum of a countable family of measurable functions.

Now we can prove the inequality:

$$\sup_x \int_{-\infty}^{\infty} g(x-t)f(t)dt \leqq \|g\|_{L^1}\|f\|_{L^\infty} = \sup_{\phi \in D} \|g\|_{L^1}\int_{-\infty}^{\infty} f(t)\phi(t)dt$$

$$= \sup_{\phi \in D} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x-t)f(t)\phi(t)dtdx$$

$$\leqq \int_{-\infty}^{\infty} dx\left(\sup_{\phi \in D}\int_{-\infty}^{\infty} g(x-t)f(t)\phi(t)\right)dt$$

$$= \int_{-\infty}^{\infty} h(x)dx.$$

Also solved by P. O. Frederickson (Germany), David Fries, G. A. Heuer (Germany), Finbarr Holland (Eire), A. A. Jagers (Netherlands), O. P. Lossers (Netherlands), Roy Olson, Edward Rozema, and the proposer.

Note. The proposer has noted a point by Frederickson that the inequality is not best possible. In fact, according to A. Prékopa (Acta Sci. Math. (Szeged) 32 (1971) 301–316; generalized by L. Leindler, *Ibid.* 33 (1972) 217–223),

$$\int_{-\infty}^{\infty} \sup_x [g(x-t)f(t)]dt \geqq 2\left\{\int_{-\infty}^{\infty} f^2dx \int_{-\infty}^{\infty} g^2dx\right\}^{1/2} \geqq 2\int_{-\infty}^{\infty} f(t)g(x-t)dt.$$

The proof of Prékopa's inequality is complicated. It would be interesting to have a simple proof of the stronger version of the original inequality in which the right hand member is doubled.

REVIEWS

EDITED BY J. ARTHUR SEEBACH, JR. AND LYNN A. STEEN

with the assistance of the mathematics departments of St. Olaf and Carleton Colleges

COLLABORATING EDITOR FOR FILMS: SEYMOUR SCHUSTER, CARLETON COLLEGE

We invite readers to submit reviews of significant recent college-level mathematics books. We especially encourage reviews based on classroom use, or comparative reviews of several related books. Reviews should ordinarily not exceed two pages (per book) typed double spaced. Manuscripts of reviews as well as books submitted for review should be sent to: Book Review Editor, American Mathematical Monthly, St. Olaf College, Northfield, MN 55057.

Linear Optimization. By W. Allen Spivey, Robert M. Thrall. Holt, Rinehart and Winston, New York, 1970. xii + 530 pp. $15.75. (Telegraphic Review, June/July 1971.)

Having used *Linear Optimization* several times in an advanced undergraduate

course on linear programming for students in mathematics, operations research, and systems engineering, I have found this text to be an excellent one for classroom use. The fundamentals of linear programming are presented in a pedagogically sound manner that is well received by students. At the same time, there is no loss of rigor in that the basic theory of the simplex algorithm is fully developed with a particularly nice treatment of the degeneracy problem.

The first two chapters give an introduction to the geometry and the model building aspects of linear programming. A collection of modeling problems is given and reference is made to these throughout the rest of the text. Chapter three contains a constructive development of the simplex algorithm. This development differs from most text book presentations in that artificial variables are not used to obtain an initial basic feasible solution. The constructive approach given involves only the notion of a pivot which consists of elementary row operations. Given any linear programming problem, a systematic procedure is presented which shows how to pivot from one form of the problem to equivalent forms of the problem until a terminal form is obtained in a finite number of pivots. Even if certain variables are not constrained to be nonnegative (free variables), a systematic pivoting procedure is presented for handling such problems. Actually, this manner of handling free variables is one of the few places I disagree with the pedagogy of the authors. I usually demonstrate first how to replace n free variables with $n + 1$ nonnegative variables and then, if time allows, proceed with the pivoting procedure as an alternative method.

Chapters four, five, and eight complete the development of general theory including the revised simplex method, duality, postoptimality analysis, and introduction of the elegant Tucker schema. The remaining chapters of the text are concerned with more specialized topics including transportation and assignment problems, matrix games, and decomposition techniques for large structured linear programs. I have found it best for classroom use to use the material on the transportation and assignment problems in a slightly different manner. The text first develops algorithms for the assignment problem (Chapter 6) and then presents an algorithm for the capacitated transportation problem (Chapter 7). This latter algorithm is reasonably complicated and seems to be a stumbling block for most students. I usually combine the appropriate material in Chapters 6 and 7 to present first an algorithm for the uncapacitated transportation problem from which the algorithm for the assignment problem is easily obtained as a special case. Students are then in a better position to see the extension of the method to the capacitated case and this is usually given as an outside reading assignment.

There are over 200 exercises included in the text. Most of these exercises are straightforward but do illustrate points made in the text. I usually supplement these with a few exercises to challenge the better students. The text also contains appendices giving necessary prerequisite material including topics in linear algebra.

The authors state that their major purpose in writing the text was to combine a constructive development of the simplex algorithm with a presentation that is both mathematically sound and intended for a reader who is not primarily a mathematician. They have certainly achieved this goal and have written a very usable classroom text for students in varying disciplines.

J. G. ECKER, Rensselaer Polytechnic Institute

Introduction to Probability Theory. By Paul G. Hoel, Sidney C. Port, Charles J. Stone. Houghton Mifflin, Boston, Massachusetts, 1971. xi + 258 pp. $10.95. (Telegraphic Review, November 1971.)

Compared to other current texts this appears pleasantly small. When using this text in a three semester hour course, this instructor found himself adding some material from other texts but paying for it by having to skip material in this text. In outline this book contains a short course in discrete probability (chapters 1–4), analogous material for continuous distributions (chapters 5–7.4), the central limit theorem (section 7.5), and optional material on characteristic functions and Poisson processes (chapters 8–9). Upon closer inspection the following features stand out.

The idea of a random variable is introduced early. The definition is later modified to handle continuous and general random variables. Although no mention is made of measure theory on the real line, the requirement that the inverse image of the half infinite interval $(-\infty, x]$ be a member of the σ-field of events in Ω is completely rigorous and correct.

The central limit theorem is stated for sums of independent identically distributed random variables. The proof is not given until after characteristic functions have been introduced. Although the elementary proof for the binomial case is missing, there are graphs and numerical illustrations which make the theorem plausible. Further, this is one situation where the "elementary" proof is far less simple than the "complex" one.

Chapter four deserves special commendation. A glance at the heading promises only some properties of discrete expectation. However, after covering variance of a sum of random variables, Chebyshev's inequality (easily generalized proof without sums), and the weak law of large numbers, I had the pleasant realization that all the topics that make the last two lectures of a class harried had already been done. This I enjoyed.

At this point, one may feel that the sections on continuous distributions would be repetitious. They are not. The super theorem validating chapter four in the continuous case is on page 176 without a boldface number and without proof. The rest of chapters five and six is devoted to examples, brand-name densities, change of variable formulas, density of a sum formula, and conditional densities.

Introduction to Probability Theory is the first of a three volume series by the same authors. The second and third volumes cover statistics and stochastic processes. The first volume makes me eager to see the others.

RICHARD GISSELQUIST, Minneapolis, Minnesota

Elements of Set Theory, Second Edition. By Peter W. Zehna and Robert L. Johnson. Allyn and Bacon, Boston, Massachusetts, 1972. ix + 197 pp. $8.95. (Telegraphic Review, August-September 1972.)

The authors of this book express three purposes: to provide elements of set theory for upper division and graduate students planning to teach mathematics; to facilitate the transition from the computational methods of calculus to the abstract mathematics in higher level courses; to provide an alternative to calculus in the twelfth grade. We used this book in a sophomore course falling under the second classification. The intention of the course (and the book, I assume) is to study the fundamental concepts such as sets, relations, and functions that occur in all areas of mathematics and to investigate the general techniques used in proofs, such as mathematical induction, DeMorgan's laws, and the method of indirect proof.

The first goal is admirably met by *Elements of Set Theory*. Fundamental concepts are clearly explained with good examples and intriguing exercises. The definitions are carefully worded, and the notation is generally excellent, although it was difficult to remember which of f^* and f_* is the image function and which is the inverse image function.

The later chapters on infinite sets, cardinal numbers and ordering relations were well written but, unfortunately, beyond the level of many sophomore students. To be sure, these notions are difficult at first, but the authors could have provided more examples and made compromises in some of the theory. The students generally liked the book but complained that it was too brief with not enough examples, not enough exercises, and only one page explaining mathematical induction.

One criticism that was heard is probably leveled at all books of this type: "It didn't show how to do the proofs." Of course, no book can entirely satisfy all students, but some improvements could be made. The first chapter had a fine, concise presentation of propositional logic with truth tables and of predicate logic, defining the quantifiers and giving rules for introducing and dropping them. But when logical rules, that is, informal rules of logic, were applied later in proofs, attention was not drawn to the procedures being used. It was as if the proofs were disjoint from the preceding logic. The student studying abstract theoretical concepts for the first time needs as much structuring of methods and listing of proof "tricks" as possible. Techniques used in proofs need to be identified and labeled so that they are recognized the next time and can be applied by the student. What this book lacked was an overall strategy for presenting the proof-techniques in a coordinated manner so that the student would be prepared for abstract algebra and analysis.

KENNETH R. SLONNEGER, State University College, Fredonia, N. Y.

TELEGRAPHIC REVIEWS

Telegraphic reviews are designed to give prompt notice of new books with sufficient information to assist our readers in deciding whether to order an examination copy or to suggest library purchase. Possible uses are indicated as follows:

T = textbook P = professional reading
S = supplementary reading L = undergraduate library purchase
13 to 18 = freshman to second year graduate level usage
1 to 4 = appropriate time in semesters to cover text

Asterisks (*) or question marks (?) denote special positive or negative emphasis, respectively. Publishers are denoted by standard abbreviations; complete addresses may be found in *Books in Print*.

GENERAL, P*, L*, *The Influence of Computing on Mathematical Research and Education*. Ed: Joseph P. LaSalle. Proc. of Symp. in Appl. Math., V. 20. AMS, 1974, viii + 205 pp, $16.50. Six invited and fourteen contributed papers from an AMS-MAA symposium at Missoula, August, 1973. Altogether, an excellent, easily readable exposition of the present interface between computing and mathematics. LAS

GENERAL, T(13: 1), *An Introduction to Mathematics, Second Edition*. Bevan K. Youse. Allyn, 1974, xii + 337 pp, $10.95. For the "cultural" math course. Students and teachers alike will enjoy using this book. High school teachers will find some fascinating supplementary ideas. Pictures and biographical sketches of ten great mathematicians begin the chapters: The Nature of Mathematics, Numbers and Numerals, Checks and Pseudo-checks, The Expanding World of Numbers, Number Patterns, A Look at Topology, Probability and Statistics, Logic and the Axiomatic Approach, The Principles of Computer Programming, and Number Theory. Second edition contains some new topics and many new exercises. Chapter on computers is new. RBK

GENERAL, T(13-14: 2), *Mathematics for Managerial Decisions*. Robert L. Childress. P-H, 1974, xiv + 689 pp, $12.95. Matrix algebra, linear and integer programming, single and multivariable calculus, probability theory, each surrounded by business and economic illustrations. A readable text for a year's survey course. LAS

GENERAL, T*(14-15: 1), S, L, *An Introduction to Axiomatic Systems*. Burnett Meyer. Prindle, 1974, xiii + 190 pp, $10.50. For a course to bridge the gap between calculus and "abstract proof-oriented advanced courses." Axiom systems, the affine plane, informal logic, informal set theory, groups, rings, fields, the number system, topology. An interesting approach to a common problem. FLW

GENERAL, S?(13), *Thinking Metric*. Thomas F. Gilbert, Marilyn B. Gilbert. Wiley, 1973, xii + 142 pp, $2.95 (P). Self-instructional booklet put out in anticipation of favorable congressional action on converting to the metric system. Goes somewhat overboard in its attempt to justify the change and to make the metric system meaningful. One wonders, for example, whether the teaching of fractions and mixed numbers" presumably can be eliminated from the curriculum," and what one would do with the "well over a billion dollars a year" that "the time spent in teaching just those skills may [now] cost us." RSK

GENERAL, S(13-16), L, *Le Nombre Langage de la Science.* Tobias Dantzig. Blanchard, 1974, 248 pp, 28F (P). A translation of Dantzig's classic (Macmillan, N.Y., 1930) which was revised in this *Monthly* 38 (1931) 164-66. The translation occasionally loses a little. Example: In the original, "As I am writing these lines there rings in my ears the old refrain: Reading, 'riting and 'rithmetic/Taught to the tune of a hickory stick." Translation: "...of reading, writing and arithmetic, it is the last which humanity has had the most difficulty learning." Except for a few cases like the above, a good translation. PJM

GENERAL, S(16-18), P, *Überblicke Mathematik.* Detlef Laugwitz. Bibliographisches Inst. *Band 4: 1971,* 123 pp; *Band 5: 1972,* 172 pp; *Band 6: 1973,* 242 pp, (P). A series of expository and historical articles of general professional interest at a high level, e.g., an introduction to homology theory, an historical overview of topology. JAS

GENERAL, S*(17-18), L, *Meyers Handbuch über die Mathematik.* Ed: Herbert Meschkowski. Bibliographisches Inst, 1972, 1234 pp, 36DM. "Das Buch nicht mit 1 + 1 = 2, sondern mit den Elementen der Mathematik, mit der Einführung in die Sprache des Mathematikers." Es bietet ein Begriffswörterbuch der Mathematik und über tausend Seite mit einer "Leicht lesbare Darstellung der Grundlagen und der verschiedenen Arbeitsweisen der modernen Mathematik." Einschliesende sind Topologie, Wahrscheinlichkeitsrechnung, Informationstheorie, Theorie der transfiniten Mengen, usw. *An equally literate translation of this unique volume (at the same cost) would be most welcome.* JAS

BASIC, *Practical Applications in Mathematics.* Edwin I. Stein. Allyn, 1972, iv + 220 pp, $5.50 (P). Consists only of a large number of drill problems in arithmetic and practical applications or word problems. No examples are worked out and no answers are provided. RBK

BASIC, T(13: 1), *Business Mathematics for Colleges.* Andrew Vazsonyi, Richard Brunell. Irwin, 1974, ix + 404 pp, $8.50 (P). Very basic mathematics, starting with the four arithmetic operations on decimal numbers and moving up to interest and percentages. Semi-programmed.PJM

BASIC, T(13: 1), S, *Practical Algebra.* Peter H. Selby. Wiley, 1974, x + 326 pp, $4.95 (P). In programmed format, for self-study or classroom use. The content of a high school algebra course through quadratics. Frequent self-tests and worked examples. No trig function or logs. Especially appropriate for review work. TAV

BASIC, T(13: 1), *Modern Elementary Algebra for College Students, Second Edition.* Vivian Shaw Groza, Susanne M. Shelley. Rinehart Pr, 1974, viii + 519 pp, $9.95. Factoring, graphing, quadratic equations. Intuitive, informal explanations, many verbal problems. Summary after each chapter. Historical notes. LH

PRECALCULUS, T(13: 1, 2), *Fundamental Mathematics, Fourth Edition.* Thomas L. Wade, Howard E. Taylor. McGraw, 1974, xiii + 604 pp, $11.95. First half of text for those whose algebraic skills are insufficient for standard pre-calculus or finite math course. Second half includes functions, graphs, basic trigonometry, logarithms, interest, probability and statistics. Many drill exercises. LH

PRECALCULUS, T(13: 1), *College Trigonometry*. Mustafa A. Munem, William Tschirhart, James P. Yizze. Worth, 1974, ix + 386 pp, $8.95; *Study Guide to Accompany College Trigonometry*, 243 pp, $3.95 (P). A good, solid introduction to trigonometry. After the necessary definition of sets and functions, the authors define trigonometric functions in terms of angles, and then as "circular" functions defined on real numbers. Trigonometric identities and graphs of the trig and inverse trig functions follow. Finally, all of the machinery is applied: first to solving triangles, then to complex numbers and vectors in the plane. A slower paced version of the appropriate sections of the first and third authors' precalculus book. The study guide is semi-programmed (Q and A but no branching) and contains more exercises. If a course in trigonometry is given, this is a good book; but for a pre-calculus course, or a combined algebra-trig course, to use this book and another would be too expensive. PJM

PRECALCULUS, T(13: 1), *College Algebra*. John M. Peterson, Floyd E. Haupt. Prindle, 1974, ix + 346 pp, $11.50. Intuitive approach to concepts. Emphasis on mechanical skills with applications to business, economics, biological and social sciences. Functions, systems of equations, matrices, linear programming, complex numbers, sequences, series, probability. LH

PRECALCULUS, T(13: 1), *Modern Intermediate Algebra for College Students, Second Edition*. Vivian Shaw Groza, Susanne M. Shelley. Rinehart Pr, 1974, x + 533 pp, $9.95. Functions, graphs of quadratics, systems of equations, matrices, sequences and series. Detailed examples of each concept, chapter summaries, many exercises. LH

PRECALCULUS, T(13: 1, 2), *Algebra and Calculus for Business*. Thomas R. Dyckman, L. Joseph Thomas. P-H, 1974, xii + 450 pp, $10.95. Basic mechanics of differentiation and integration with business applications, aimed at the mathematically unsophisticated. Algebra review includes logarithms, simultaneous equations, inequalities and linear programming. LH

EDUCATION, S(13, 16), P, *Learning Mathematics through Activities: A Resource Book for Elementary Teachers*. S. Jeanne Kelley. Freel, 1973, vi + 121 pp, $3.50 (P). A collection of simple yet effective activities to make mathematical concepts interesting for primary school students. Must reading for new and experienced teachers; highly recommended for parents of young children. Most activities look like more fun than TV. TAV

EDUCATION, T(15-16: 1), *Multiple Methods of Teaching Mathematics in the Elementary School, Second Edition*. Charles H. D'Augustine. Har-Row, 1973, xv + 400 pp, $6.95 (P). Revision of earlier methods text. Updated bibliographies. Increased attention to applications. Behavioral objectives listed for each chapter. First three chapters designed for self study. PSJ

EDUCATION, P*, *Mathematics Library--Elementary and Junior High School*. Clarence Ethel Hardgrove, Herbert F. Miller. NCTM, 1973, v + 70 pp, $1.50 (P). Companion bibliography to *The High School Mathematics Library, Fifth Edition*, by William L. Schaaf (TR, December 1973). Revision of the authors' 1968 pamphlet. Annotated, with grade-placement suggestions. RSK

EDUCATION, P, *Rechnerkunde*. Ed: Günter Lobin. Hermann Schroedel, 1973, 218 pp, (P). A series of papers on various aspects of computer science education sponsored by the "Forschungs-und Entwicklungszentrum für objektivierte Lehr-und Lernverfahren" (FEoLL). This is one of the Paderborn conferences held in October 1972. The approach is very "scientific." JAS

EDUCATION, P, *Schulfernsehen im Unterricht*. Gerhard Tulodziecki. Hermann Schroedel, 1973, 121 pp, (P). The proceedings of the fifth Paderborn conference from October 1972. Part of it concerns a fifth school year television program "Einführung in die Mengenlehre." JAS

HISTORY, P, L**, *Leibniz in Paris 1672-1676: His Growth to Mathematical Maturity*. Joseph E. Hofmann. Cambridge U Pr, 1974, xi + 372 pp, $23.50. A thoroughly revised translation of the 1949 original *Die Entwicklungsgeschichte der Leibnizschen Mathematik*. A minutely detailed study of the meteoric ascendence of Leibniz from a "deplorable" breadth of knowledge to one of the greatest mathematicians of all time. "Seldom, if ever, has a decisive period in the history of mathematics been described with such intensity"--D.J. Struik, reviewing the original edition. LAS

HISTORY, P, *La Vita di Copernico di Bernardino Baldi*. Bronislaw Biliński. PWN, 1973, 109 pp, (P). A scholarly study of Baldi's "Life of Copernicus." This early (1588) biography suffers from acute manuscript difficulties, so the emphasis is on textual research based on Biliński's extensive study of the manuscripts. JAS

HISTORY, S(14-16), L*, *Le Programme D'Erlangen*. Félix Klein. Gauthier-Villars, 1974, xiv + 72 pp, 19F (P). A translation into French of Klein's 1872 address on the relationship of groups and geometry. PJM

HISTORY, P, L*, *The Development of Mathematics in China and Japan, Second Edition*. Yoshio Mikami. Chelsea, 1974, x + 389 pp, $9.50. Corrected reprint of 1913 edition, originally published as an appendix to Moritz Cantor's *Geschichte der mathematischen Wissenschaften*, together with an appendix on soroban calculation by Rikitaro Fujisawa. Like all Chelsea mathematics books, printed on durable, acid-free paper. LAS

HISTORY, S*, P, L**, *Niels Henrik Abel: Mathematician Extraordinary*. Oystein Ore. Chelsea, 1974, 277 pp, $8.50. Reprint of a modern classic, first published by U. Minnesota in 1957. A sympathetic biography, unencumbered by mathematical detail. LAS

FOUNDATIONS, T(13: 1), S, *Logic in Mathematics: An Elementary Approach*. Arthur E. Hallerberg. Hafner Pr, 1974, vi + 90 pp, $2.75 (P). A programmed text covering implications, *modus ponens* (by means of a diagram), connectives, truth tables, tautologies, proofs, and definitions. Very little on quantification. FLW

FOUNDATIONS, T(13: 1), S, *Mathematical Proof: An Elementary Approach*. Arthur E. Hallerberg. Hafner Pr, 1974, viii + 104 pp, $3.95 (P). Programmed text to follow the author's *Logic in Mathematics*. Proofs, definitions, formal proofs, and a little on quantifiers. Appendices on proof strategies, induction, and other topics. FLW

FOUNDATIONS, T(13: 1), S, *Introduction to Symbolic Logic.* Karl J. Smith. Brooks/Cole, 1974, viii + 119 pp, $3.95 (P). Connectives, truth tables, tautologies, quantifiers, proofs, and Boolean algebra. Not much consideration of the negation of quantified statements which many students seem to find difficult. FLW

FOUNDATIONS, P, *Quantoren-Modalitäten-Paradoxien.* Horst Wessel. VEB, 1972, 524 pp, M29,80. The stated purpose is to "popularize the aims, results and methods of modern logic." The result is an unusual collection of papers, all in German, some translated from Russian; highly technical papers on a wide variety of modern research topics are preceded by several articles of an historical nature. JAS

FOUNDATIONS, P, L, *Berkeley's Philosophy of Science.* Richard J. Brook. Martinus Nijhoff, 1973, 210 pp, $18. "We take seriously Berkeley's claim that he is separating the 'wheat' of Newtonian science from the 'chaff' of metaphysical additions to it. We claim, however,...that much of what Berkeley would eliminate is more essential than he believed to a proper understanding of the Newtonian edifice." Chapters on signification, vision, physics and mathematics, including a lengthy commentary on *The Analyst.* LAS

FOUNDATIONS, T(16-18: 1), S, P, L, *Lectures on Model Theory, Part I.* Richard Gostanian. Lect. Notes Ser., No. 39. Aarhus U, 1974, ii + 207 pp, $3.75 (P). A presentation of the main methods used in the model theory of first order languages. Presumes a good familiarity with set theory and some knowledge of modern algebra; proofs are quite detailed. LCL

FOUNDATIONS, S, P, L, *The Principles of Inductive Logic.* John Venn. Chelsea, 1973, xx + 604 pp, $15. Unaltered reprint of the second edition published in Cambridge in 1907. The philosophical foundations of mathematics continue to be confronted with fundamental problems regarding the external world of phenomena "pursuing its undeviating course" and the internal world of "the observing and thinking mind." Here is a beautifully reasoned analysis that is hard to put down.LCL

FOUNDATIONS, P, L, *Logic, Methodology and Philosophy of Science IV.* Ed: Patrick Suppes, *et al.* Stud. in Logic and Found. of Math., V. 74. North-Holland, 1973, x + 981 pp, $56. An impressive and diverse array of papers presented in 1971 in Bucharest, Romania, at the Fourth International Congress. Topics range from mathematical logic to economics, psychology and quantum mechanics; level of articles range as broadly, from elegant expositions for informed laymen to concise research reports for specialists. LAS

FOUNDATIONS, P, *Lecture Notes in Mathematics-337: Cambridge Summer School in Mathematical Logic.* Ed: A.R.D. Mathias, H. Rogers. Springer-Verlag, 1973, ix + 660 pp, $17.30 (P). Papers from an August, 1971 conference highlighted by an extensive exposition of intuitionism by D. van Dalen. LAS

COMBINATORICS, P, *Lecture Notes in Mathematics-382: Combinatorial Theory Seminar Eindhoven University of Technology.* Jacobus H. van Lint. Springer-Verlag, 1974, vii + 131 pp, $7.40 (P). Notes of a seminar which was based on Marshall Hall's *Combinatorial Theory.* Contains generalizations of theorems in Hall's book, references to recent results, and problems which arose during the seminar. A valuable book for researchers. SG

COMBINATORICS, P. *Map Color Theorem.* Gerhard Ringel. Grund. math. Wissenschaften, B. 209. Springer-Verlag, 1974, xii + 191 pp, $22.20. An account of the history and ultimate proof (in 1968 by the author and the late J.W.T. Youngs) of the map color theorem first stated (with incomplete proof) in 1890 by P.J. Heawood: The chromatic number of an orientable surface of genus $p \geq 1$ equals $\lfloor (7 + \sqrt{1 + 48p})/2 \rfloor$ (where $\lfloor x \rfloor$ denotes the largest integer not exceeding x). The proof employs new combinatorial analyses of current graphs. LAS

NUMBER THEORY, P. *Lecture Notes in Mathematics-334: The Metrical Theory of Jacobi-Perron Algorithm.* Fritz Schweiger. Springer-Verlag, 1973, 111 pp, $7.20 (P). Studies aspects of Jacobi algorithm related to measure theory, ergodic theory, dimension theory and diophantine approximation. Treats the algorithm as a model of an f-expansion. Complements the 1972 monograph *Lecture Notes-207* (TR, February 1972) by L. Bernstein on the algebraic and arithmetic aspects. DFA

NUMBER THEORY, P. *Aspects of the Greatest Integer Function, Part I: The greatest integer function in the domain of the rational numbers.* Folke Ryde. Almqvist & Wiksell, 1973, 109 pp, Sw.kr. 70. A report on the author's investigation of the greatest integer function, concentrating on Diophantine equations involving this function. SG

NUMBER THEORY, T(18), P*. *Uniform Distribution of Sequences.* L. Kuipers, H. Niederreiter. Wiley, 1974, xiv + 390 pp, $24.50. An excellent graduate-level text and reference. The book covers the classical theory of uniform distribution mod one (i.e., Weyl's paper), a quantitative approach to the classical theory (discrepancy), uniform distribution in compact spaces and topological groups, and sequences of integers and polynomials. Extensive bibliography; informative historical notes on each topic; many, many exercises. SG

ALGEBRA, S(18), P. *Lecture Notes in Mathematics-357: Primideale in Einhüllenden auflösbarer Lie-Algebren.* Walter Borho, Peter Gabriel, Rudolf Rentschler. Springer-Verlag, 1973, 182 pp, $7 (P). An informally written account, for specialists, of the theory of prime ideals in enveloping algebras of solvable Lie algebras. Deals largely with the work of Dixmier. JD-B

ALGEBRA, T(17-18), S, P, L. *Introduction to Quadratic Forms, Second Printing, Corrected.* O.T. O'Meara. Grund. math. Wissenschaften, B. 117. Springer-Verlag, 1971, x + 342 pp, $21.60. See review by G. Whaples in this *Monthly*, February 1965, pp. 211-212. LCL

ALGEBRA, S(18), P. *Homological Dimensions of Modules.* Barbara L. Osofsky. CBMS Reg. Conf. in Math., No. 12. AMS, 1973, viii + 89 pp, $4.40 (P). Theorems about homological dimension for non-Noetherian (large) rings and modules involve set theoretic considerations such as the continuum hypothesis. A number of such results are presented here together with the necessary set-theoretic and homological backgrounds. JAS

ALGEBRA, P. *Lecture Notes in Mathematics-356: Infinitesimally Central Extensions of Chevalley Groups.* W.L.J. van der Kallen. Springer-Verlag, 1973, vii + 147 pp, $6.20 (P). A study of the connection between infinitesimally central extensions of Chevalley groups and universal central extensions of their Lie algebras. JAS

ALGEBRA, T**(16-17: 2, 3), S*, P*, L. *Discrete Mathematics: Applied Algebra for Computer and Information Science.* Leonard S. Bobrow, Michael A. Arbib. Saunders, 1974, xiii + 719 pp, $16.50. A massive exposition of applied modern algebra designed to do for the budding computer scientist or systems engineer what texts like Courant and Hilbert did for the physicist and engineer several decades ago. It begins with ideas based solely on high school algebra, yet concludes at the fontier of algebraic automata theory; in between are the makings of many different courses ranging from Modern Algebra with Applications to Automata Theory. Some topics: graphs, automata, semigroups, computability, context-free languages, normal subgroups, universal algebras, Markov chains, rings, modules, polynomial rings, cyclic codes, category theory. LAS

ALGEBRA, S, P, L. *Problems in Group Theory.* John D. Dixon. Dover, 1973, xv + 176 pp, $3 (P). Unabridged republication of the 1967 edition (TR, January 1968). LCL

ALGEBRA, P. *Lecture Notes in Mathematics-319: Conference on Group Theory.* Ed: R.W. Gatterdam, K.W. Weston. Springer-Verlag, 1973, 188 pp, $6.70 (P). 22 papers from a June 1972 conference sponsored by U. Wisconsin, Parkside. LAS

ALGEBRA, P, L**. *Reviews on Finite Groups.* Ed: Daniel Gorenstein. AMS, 1974, xi + 706 pp, $50 (P). Reprints of 3052 reviews originally published in *Mathematical Reviews* from 1940 through 1970, organized and cross referenced by subject. LAS

ALGEBRA, P. *Lecture Notes in Mathematics-340: Groupes de Monodromie en Géométrie Algébrique.* P. Deligne, N. Katz. Springer-Verlag, 1973, x + 438 pp, $18 (P). Lectures ten thru twenty-two of the 1967-69 Séminaire de Géométrie Algébrique du Bois-Marie. Earlier lectures appear in *Lecture Notes No. 288.* JAS

ALGEBRA, P. *Number Theory, Algebraic Geometry and Commutative Algebra: In Honor of Yasuo Akizuki.* Ed: Y. Kusunoki, *et al.* Kinokuniya Books, 1973, 528 pp, $63.60. A collection of papers published in honor of Yasuo Akizuki on the occasion of his seventieth birthday. Includes a listing of the contents of the Journal of Mathematics of Kyoto University, Vol. 13, No. 1, another volume dedicated to Professor Akizuki. JAS

ALGEBRA, S(18), P*. *Introduction to Some Methods of Algebraic K-Theory.* Hyman Bass. CBMS Reg. Conf. in Math., No. 20. AMS, 1974, v + 68 pp, $4.40 (P). Definition, discussion, computations, and applications of K_1. A good presentation of the sort that might help keep mathematicians in better communication with one another. From a conference at Colorado State University in August, 1973. JAS

FINITE MATHEMATICS, T*(13-16: 1, 2), S, L*. *Applicable Finite Mathematics.* David S. Moore, James W. Yackel. HM, 1974, xi + 398 pp, $10.95. Superior. More serious level than usual for such books but without sacrificing simplicity, clarity. Computer projects, suggestions for further development at end of each chapter. Probability, Markov chains, linear programming, game theory, decision theory. LH

FINITE MATHEMATICS, T(13-14: 1). *Finite Mathematics.* Hugh G. Campbell, Robert E. Spencer. Macmillan, 1974, x + 326 pp, $9.95. Clear, well-motivated with applications every few pages to social, management, life sciences. Numerous examples and illustrations. List of new vocabulary at end of each chapter. Logic, sets, probability, matrices, linear programming, game theory. LH

FINITE MATHEMATICS, T(13-14: 1, 2). *Finite Mathematics and Calculus with Applications to Business.* Paul G. Hoel. Wiley, 1974, ix + 446 pp, $10.95. Functions, systems of linear equations, matrices, linear programming, probability, and a compact and straightforward introduction to calculus. FLW

FINITE MATHEMATICS, T(13-14: 1, 2). *Finite Mathematics.* David G. Crowdis, Susanne M. Shelley, Brandon W. Wheeler. Rinehart Pr, 1974, ix + 417 pp, $11.95. Logic (without quantifiers), sets (without order relations and equivalence classes), probability (without Bayes theorem and subjective probabilities), matrix algebra (with determinants in an appendix), linear programming, game theory, and BASIC programming.FLW

FINITE MATHEMATICS, T*(13: 2). *Mathematics: With Applications in the Management, Natural, and Social Sciences.* Margaret L. Lial, Charles D. Miller. Scott F, 1974, 564 pp, $10.95. In four essentially independent parts: I: Review and elementary functions; II: Matrix algebra, linear programming; III: Probability, decision theory, elementary statistics; IV: Elements of calculus. Unique feature is actual case studies from medicine and business that illustrate mathematical concepts. For the majority of students who take only one year of college mathematics, this provides a more realistic course than a detailed calculus course. TAV

CALCULUS, T(13: 2, 3). *Calculus and Analytic Geometry.* Philip S. Clarke, Jr. Heath, 1974, xii + 930 pp, $13.95. Very clear presentation. Thorough coverage of topics from review of inequalities and analytic geometry to vectors in n-space and multi-variable differentiation and integration. Some rigorous argument but avoids more difficult proofs. Wide range of applications in exercises and examples.LH

CALCULUS, T(13: 2, 3), L. *Mathematical Analysis, Business and Economic Applications, Second Edition.* Jean E. Draper, Jane S. Klingman. Har-Row, 1972, xi + 691 pp, $13.95. A calculus text that is well conceived and executed. The topics covered include the standard calculus topics, partial derivatives, differential equations, difference equations, matrix algebra, linear programming, game theory. Extensive and frequent sections on applications, mostly economic. The authors' claim that the material can be covered in a year is very doubtful. TAV

CALCULUS, T(13: 1). *A Short Calculus for Students of Business, the Social Sciences and Biology.* Watson Fulks. Boulder Book, 1974, xiii + 242 pp, $9.95. The author calls this a "what-it-means and how-to-do'it book." Topics: differentiation (emphasis on linear approximation), integration, functions of 2 variables through Lagrange multipliers. Applications are well chosen and the author's informal prose style is very attractive. TAV

CALCULUS, T(13: 2). *Engineering Mathematics.* A.C. Bajpai, L.R. Mustoe, D. Walker. Wiley, 1974, xiii + 793 pp, $22.50; $11.95 (P). Differentiation and integration, partial differentiation, ordinary differential

equations, some probability and statistics. Flow charting, numerical methods throughout. Problem oriented, with few proofs. From p. 495: "Definition. An *ordinary differential equation* is an equation which contains only total derivatives and no partial derivatives." DFA

CALCULUS, T(2), S, P, L. *Mathematical Methods for Economists*. Stephen Glaister. Lect. in Econ., No. 4. Gray-Mills, 1972, 207 pp, $14.50. A compact introductory survey of linear algebra and calculus, focusing on economic applications. Some problems, with answers to odd-numbered ones at the back. A good reference for college mathematics teachers in search of significant economic illustrations. LAS

REAL ANALYSIS, T(18: 1), P. *Lecture Notes in Mathematics-315: Integration Theory (with Special Attention to Vector Measures)*. Klaus Bichteler. Springer-Verlag, 1973, vi + 357 pp, $9.70 (P). Extends M.H. Stone's unified treatment of set functions and Radon measures. Definition of measurability includes those of Bourbaki and Caratheodory-Halmos. Concept of "upper gauge" leads to a Daniell integral even for measures and linear maps that do not have finite variation. Chapters: Riesz spaces and elementary integrals; Upper gauges and extension theory; Measurability; Elementary operations on measure spaces; Liftings and derivatives. Exercises for each section. Subject index, notation index. Bibliography. RSK

COMPLEX ANALYSIS, T*(16: 1), L. *Complex Variables*. George Polya, Gordon Latta. Wiley, 1974, xiv + 327 pp, $12.95. As befits a text by the first author the emphasis is on problems. The reader must be an active participant, for much of the material is developed in the exercises. A casual reader will find the textual material skimpy but enough is written for the real purpose of this text: learning by doing. TAV

COMPLEX ANALYSIS, S(17-18), P. *The General Theory of Dirichlet's Series*. Ed: J.G. Leathem, G.H. Hardy. Cambridge Tracts in Math. and Math. Physics, No. 18. Hafner, 1972, 78 pp, $5.95. The classical theory as originally published in 1915 by Cambridge University Press. LCL

COMPLEX ANALYSIS, T(16-17: 1, 2), L. *Introduction to Complex Variables*. E.A. Grove, G. Ladas. HM, 1974, x + 221 pp, $11.95. After discussing the algebra and topology of the complex numbers, the authors define analytic functions, talk about complex integration and the residue theorem, and give applications to such areas as 2-dimensional flows and Fourier transforms. Additional topics (e.g., conformal maps, doubly periodic functions) are relegated to appendices. Biographies of mathematicians are scattered throughout the book giving an historical flavor to the text which is all too rare. Exercises and index, but no bibliography. PJM

COMPLEX ANALYSIS, T(15-16: 1). *Complex Analysis with Applications*. Richard A. Silverman. P-H, 1974, x + 274 pp, $12.95. Very similar in content to many other texts for a first course, with some topics omitted to provide what the author calls "the irreducible minimum every scientist and engineer should know about the techniques of complex analysis." Extensive well chosen exercise sets and comments to provide insight close each chapter. Its very readable style should appeal to students. TAV

COMPLEX ANALYSIS, P. *Lecture Notes in Mathematics-348: Generalized Hypergeometric Functions with Applications in Statistics and Physical Sciences.* A.M. Mathai, R.K. Saxena. Springer-Verlag, 1973, vii + 314 pp, $10.70 (P). Integrals, finite and infinite series, logarithmic cases and computable representations of Meijer's G-functions. Applica tions to statistics, dual integral equations, communication theory, heat conduction. References to and exercises from 360 research papers DFA

COMPLEX ANALYSIS, P. *Lecture Notes in Mathematics-320, 349, 350: Modular Functions of One Variable.* Springer-Verlag, 1973. *V. I,* Ed: Willem Kuyk, 195 pp, $6.70 (P); *V. II,* Ed: W. Kuyk, P. Deligne, 598 pp, $14.70 (P); *V. III,* Ed: Willem Kuyk, J.-P. Serre, 350 pp, $10.70 (P). Proceedings of an international summer school at U. Antwerp, 1972. A fourth volume is also planned. LAS

COMPLEX ANALYSIS, T*(15: 1), L. *Complex Variables and Applications, Third Edition.* Ruel V. Churchill, James W. Brown, Roger F. Verhey. McGraw, 1974, x + 332 pp, $11.95. No major changes from the earlier edition, long the classic in the field. The revisions have been mainly for clarity and precision, with minor organizational changes. Includes new exercises and some new material on the argument principle and sequences. One of the best is now better. TAV

DIFFERENTIAL EQUATIONS, P. *Differentialgleichungen: Losungsmethoden und Losungen, Third Unaltered Edition.* E. Kamke. *Band I: Gewöhnliche Differentialgleichungen,* 1971, xxvi + 666 pp, $12.50; *Band II: Partielle Differentialgleichungen Erster Ordnung fur eine Gesuchte Funktion,* 1974, xii + 243 pp, $7.50. A reprint of the corrected (in 1948) 1944 third edition of Volume I, and of the unaltered 1944 third edition of Volume II. Includes a massive and encyclopaedic solutions manual for ordinary differential equations (1500 individually solved equations in Volume I). JAS

DIFFERENTIAL EQUATIONS, T(18: 1), P. *Introduction to Singular Perturbations.* Robert E. O'Malley, Jr. Appl. Math. and Mech., V. 14. Acad Pr, 1974, viii + 206 pp, $16.50. Techniques for obtaining asymptotic solutions to boundary value problems for linear and nonlinear ordinary differential equations. Includes problems arising in chemical reactor theory and in optimal control theory. For applied mathematicians, engineers, students. References to current literature throughout.DFA

DIFFERENTIAL EQUATIONS, T(17-18: 1, 2), S, P*. *Asymptotics and Special Functions.* F.W.J. Olver. Acad Pr, 1974, xvi + 572 pp, $39.50. An excellent and thorough introduction to both asymptotics and special functions, a natural and appealing combination. Prerequisites: ordinary differential equations, advanced calculus and complex variable theory. Oriented about the solution of difficult differential equations. Includes recent results on error analysis. The first half of the book, which could serve as a good text, is available separately under the title *Introduction to Asymptotics and Special Functions* for $10. RWN

NUMERICAL ANALYSIS, T(14-16: 1), L. *Theory and Applications of Numerical Analysis.* G.M. Phillips, P.J. Taylor. Acad Pr, 1973, x + 380 pp, £3.90 (P). A very useable introductory text. Topics include interpolation, approximation, numerical integration and numerical solution of nonlinear equations, linear systems, ordinary differential

equations and two point boundary value problems. The coverage varies: generally it's quite thorough, including the relevant theory; at times it's cursory. Good examples, exercises, and bibliography. RWN

NUMERICAL ANALYSIS, T*(16: 1, 2), P, L. *Numerical Methods.* Germund Dahlquist, Ake Björck. Transl: Ned Anderson. P-H, 1974, xviii + 573 pp, $15.95. Appears to be a very good text and reference. Presumes some differential equations and linear algebra but no functional analysis. Covers standard topics (function approximation, numerical linear algebra, nonlinear equations, interpolation, numerical integration and differential equations) plus chapters on accuracy considerations, Fourier methods, optimization and simulation. Bibliography and a guide to published algorithms. RWN

NUMERICAL ANALYSIS, T(15-17: 1, 2), L. *Introduction to Numerical Analysis, Second Edition.* F.B. Hildebrand. McGraw, 1974, xiii + 669 pp, $14.95. Same organization of topics as in the successful first edition (1956). Several new sections on recent developments, e.g., splines. The substantial selection of problems is augmented by 150 new problems. RWN

FUNCTIONAL ANALYSIS, P. *Lecture Notes in Mathematics-287: Hyperfunctions and Pseudo-Differential Equations.* Ed: Hikosaburo Komatsu. Springer-Verlag, 1973, vii + 529 pp, $13.40 (P). Papers from the conference at Katata, Japan in October 1971, together with three related papers from a conference at Kyoto University in September of that year. Nearly half the volume is a paper, "Micro-functions and Pseudo-differential Equations" by M. Sato, T. Kawai, and M. Kashiwara. JAS

FUNCTIONAL ANALYSIS, P, L. *Complete Normed Algebras.* F.F. Bonsall, J. Duncan. Ergebnisse der Math., B. 80. Springer-Verlag, 1973, x + 301 pp, $26.20. An account of the principal methods and results in the theory of Banach algebras, both commutative and noncommutative. Chapters: concepts and elementary results, commutativity, representation theory, minimal ideals, star algebras, cohomology, miscellany. Bibliography of 488 items. Index, symbol index. RBK

OPTIMIZATION, P. *Lecture Notes in Economics and Mathematical Systems-77: Problèmes de Minimax via l'Analyse Convexe et les Inégalités Variationnelles: Théorie et Algorithmes.* A. Auslender. Springer-Verlag, 1972, vii + 132 pp, $5.10 (P). For a general class of minimax problems, convex analysis is used to establish the existence theory and the variational inequalities are used to derive several numerical algorithms. RWN

ANALYSIS, T(17), P. *Differential Games.* Avner Friedman. CBMS Reg. Conf. in Math., No. 18. AMS, 1974, 66 pp, $4 (P). Develops the concepts of value, strategy and saddle points. The question of existence is treated as well as connections with the Hamilton-Jacobi equations. Bibliography. TAV

ANALYSIS, T*(16: 2), P*. *Absolute Analysis.* F. and R. Nevanlinna. Transl: Phillip Emig. Grund. math. Wissenschaften, B. 102. Springer-Verlag, 1973, 270 pp, $36.10. A compelling text. An attempt is made to present a systematic basis for a general, absolute coordinate and dimension free infinitesimal vector calculus. A joyous wedding of classical analysis and modern functional analysis. TAV

ANALYSIS, S(14-16), L*. *Les Séries Mathêmatiques.* Gaston Casanova. Pr U France, 1974, 128 pp, (P). Almost everything you want to know about series. Starting from the definition of limit of a sequence, the author discusses positive term series, real and complex series, and proceeds up to Fourier series and series of orthogonal functions. The last chapter on generalizations includes series with values in a topological group and some applications to physics. Short bibliography consisting entirely of French texts, no index or exercises. A good, handy reference book. The French is eminently readable. PJM

ALGEBRAIC GEOMETRY, P. *Lecture Notes in Mathematics-339: Toroidal Embeddings I.* G. Kempf, *et al.* Springer-Verlag, 1973, viii + 209 pp, $8.20 (P). A systematic presentation of results about local analytic coordinates of monomials, which turn out to yield embedded tori. After a general introduction, the major development of the theory is given in chapters on semi-stable reduction and applications by D. Mumford and a chapter on polyhedral subdivisions by F. Knudsen. JAS

GEOMETRY, S(16), P, L. *The Axioms of Projective Geometry.* A.N. Whitehead. Cambridge Tracts in Math. and Math. Physics, No. 4. Hafner, 1971, viii + 64 pp, $4. A delightful discussion of axiomatics and geometry precedes the development of a projective axiom system. Several of the axioms are presented after discussion of the properties requiring these assumptions. Not intended as a complete text on projective geometry, this tract requires some previous exposure to geometry. No index. Reprint of the 1906 original edition. JNC

GEOMETRY, T(13: 1). *Basic Concepts in Geometry: An Introduction to Proof.* Frank B. Allen, Betty Stine Guyer. Dickenson, 1973, xii + 253 pp, $9.95. Chapters on sets and logic, reasoning and proof together with extensive symbolism and flow-diagram formats are used to convey the reasoning involved in some of the elementary proofs of Euclidean geometry. Unfortunately, most of the proofs are left in outline form and the geometrical implications are lost in the symbolism. JNC

GEOMETRY, P. *Differential Geometry: In Honor of Kentaro Yano.* Ed: S. Kobayashi, M. Obata, T. Takahashi. Kinokuniya Books, 1972, viii + 541 pp, $46.80. A wide variety of papers presented in honor of Kentaro Yano on the occasion of his sixtieth birthday. JAS

TOPOLOGY, P. *Proceedings of the International Symposium on Topology and its Applications.* Duro R. Kurepa. Savez Drustava Mate., Beograd, 1973, 272 pp. These proceedings from August 1972 contain sixty-one papers and questions covering every conceivable area related to topology. JAS

TOPOLOGY, S(17-18), P. *Connected Orderable Spaces.* H. Kok. Math. Centre Tracts, No. 49. Math Centrum, 1973, 91 pp, Dfl. 10 (P). A space is (strictly) orderable if its topology contains (is equal to) the order topology given by some order relation on the underlying set. This research monograph is a study of topological properties related to the property of orderability and ends with a substantial list of (counter)examples. JAS

TOPOLOGY, T(17-18). *Mengentheoretische Topologie.* Boto von Querenburg. Springer-Verlag, 1973, viii + 195 pp, $6.10 (P). Sophisticated point set topology. A thorough text that starts with basics but moves

rapidly to cover a number of major results, e.g., the Stone-Weierstrass theorem, the Bing-Nagata-Smirnov metrization theory, Stone-Čech compactification and a number of special advanced topics. The text includes a number of not-too-difficult exercises, an adequate index, and a family tree of topology on the back cover. JAS

TOPOLOGY, P. *Theory of Sets and Topology: In Honour of Felix Hausdorff (1868-1942).* Ed: Günter Asser, *et al.* VEB, 1972, 525 pp. A large collection of papers on a wide variety of topics collected by the editors at the University of Greifswald where Hausdorff was from 1913 until 1921. JAS

PROBABILITY, P. *Lecture Notes in Mathematics-360: Laws of Large Numbers for Normed Linear Spaces and Certain Fréchet Spaces.* W.J. Padgett, R.L. Taylor. Springer-Verlag, 1973, vi + 111 pp, $6.20 (P). "The aims of these notes are to present a unified theory of weak and strong laws of large numbers for random elements [random variables taking their values in a function space] in abstract spaces, and give some applications..." Basically self-contained for the qualified reader. TAV

PROBABILITY, T(17: 1), P. *Random and Restricted Walks: Theory and Applications.* Michael N. Barber, B.W. Ninham. Gordin, 1970, xiii + 176 pp, $17.95. Principles of random walks are developed from generating functions, with emphasis on lattice walks and self-avoiding walks. A second section treats applications to polymer chemistry. Bibliography. TAV

PROBABILITY, S*(17), P, L. *Stochastic Models for Social Processes, Second Edition.* D.J. Bartholomew. Wiley, 1973, xi + 410 pp, $19.50. Valuable supplementary reading for a course in applied probability. Greatly expanded from the first edition (1967). An excellent example of the increasing use of sophisticated mathematical models by social scientists. Contains an extensive bibliography, no exercises. TAV

PROBABILITY, T?(18), P. *Random Fields and Interacting Particle Systems.* F. Spitzer. MAA, 1971, 126 pp, $2.35 (P). From lectures at the 1971 MAA Summer Seminar, Williams College. The author moves quickly from finite Markov chains to advanced models. No table of contents, or index. TAV

PROBABILITY, T(18: 2), P. *Independent and Stationary Sequences of Random Variables.* I.A. Ibragimov, Yu. V. Linnik. Wolters-Noordhoff, 1971, 443 pp, $38. A comprehensive analysis of the behavior of the distribution function $F_n(x)$ for a sum of n random variables from a stationary sequence. Four chapters on independent identically distributed R.V.'s, nine on the theory of large deviations, four on weakly stationary sequences and a chapter on unsolved problems. The English version reads smoothly and fluently. TAV

PROBABILITY, P. *Monotone Transformations and Limit Laws.* A.A. Balkema. Math. Centre Tracts, No. 45. Math Centrum, 1973, 170 pp, Dfl. 18 (P). Essentially the author's thesis, the monograph studies the question of convergence of distribution types. The basic question is under what conditions does a sequence of random variables converge to a r.v. with either a logarithmic, exponential or power distribution. A few sketchy applications are included. TAV

STATISTICS, T(17). *Digital Time Series Analysis.* Robert K. Otnes, Loren Enochson. Wiley, 1972, x + 467 pp, $17.75. Primarily a reference for users. Mathematical preliminaries, including Fourier transforms and an introduction to data acquisition are included. Computational methods for filtering, Fourier transforms, spectral densities, transfer and coherence functions, and probability density functions. RWN

STATISTICS, T(13-14: 1). *Statistics for Teachers.* Estelle S. Gellman. Har-Row, 1973, vii + 239 pp, $5.95 (P). Presupposes only high school algebra. Emphasizes examples of interest to classroom teachers. Very light on probability. FLW

STATISTICS, T(13-14: 1), S. *Statistics.* Donald J. Koosis. Wiley, 1972, x + 282 pp, $2.95 (P). A programmed workbook to be used alone or along with a standard text which presupposes only high school algebra. FLW

STATISTICS, T(15-17: 1, 2), S, L. *Probability and Statistical Inference.* Richard G. Krutchkoff. Gordon, 1970, xiv + 291 pp,$24.50. A compact and yet quite comprehensive text for a post-calculus course.FLW

STATISTICS, P. *Characterization Problems in Mathematical Statistics.* A.M. Kagan, Yu. V. Linnik, C. Radhakrishna Rao. Wiley, 1973, xii + 499 pp, $22.50. In the Wiley Series in Probability and Mathematical Statistics. Theoretical presentation of properties that characterize the principal distributions of mathematical statistics. Contains an extensive bibliography and a collection of approximately fifty unsolved problems related to the discussion. RSK

STATISTICS, T*(14-15: 2, 3). *Basic Statistics with Business Applications, Second Edition.* Richard C. Clelland, *et al.* Wiley, 1973, xii + 691 pp, $14.50. In the Wiley Series in Probability and Mathematical Statistics. A consolidation of the material in the 1966 *First Edition* (TR, January 1967), which was designed for sophomores at the Wharton School of Business having a background in calculus and the elements of set theory and matrix algebra. Contains more material on probability than most texts of this type, including a chapter on the moment generating function. Devotes half of section on inference to decision theory and Bayesian inference. Concludes with a section on regression, time series and econometrics. RSK

STATISTICS, P. *Distribution Theory for Tests Based on the Sample Distribution Function.* J. Durbin. CBMS Reg. Conf. in Appl. Math., No. 9. SIAM, 1973, vi + 64 pp, $4.80 (P). Based on ten lectures given at SUNY at Buffalo in August/September 1971. Primarily concerned with tests based on the two classes of statistics typified by the Kolmogorov-Smirnov statistic and the Cramér-von Mises statistic. Applications limited to goodness-of-fit tests. RSK

STATISTICS, S**(13), L*. *Flaws and Fallacies in Statistical Thinking.* Stephen K. Campbell. P-H, 1974, viii + 200 pp, $4.95 (P). Entertaining and informative presentation of the misuses of statistics, illustrated by hundreds of examples, most of which are real. Reminiscent of Darrell Huff's *How to Lie with Statistics*, but more comprehensive. Unfortunately marred by several sexist remarks and cartoons. RSK

STATISTICS, P, *Selected Tables in Mathematical Statistics, Volume I.* Coeditors: H.L. Harter, D.B. Owen. AMS, 1973, vii + 403 pp, $8.60. Second printing with minor revisions of book originally published by Markham Publishing Company in 1970 (TR, April 1971). RSK

STATISTICS, S*(15-17), P, L*, *Comparative Statistical Inference.* Vic Barnett. Wiley, 1973, xv + 287 pp, $16.50. In the Wiley Series in Probability and Mathematical Statistics. Well-documented comprehensive treatment surveying and contrasting the different approaches to statistical inference and decision making from a conceptual or philosophical viewpoint. Primarily concerned with the relationships and differences between classical inference, Bayesian inference and decision theory, it also mentions some alternative approaches, and discusses prerequisite concepts such as the nature of probability and utility. RSK

STATISTICS, S(16), P, L*, *Sequential Analysis.* Abraham Wald. Dover, 1973, xii + 212 pp, $3.50 (P). Reprint of Wald's 1947 classic. Presents the theory of the sequential probability ratio test, a procedure in which the number of observations is not predetermined but depends at each stage on the results of the previous observations. RSK

STATISTICS, P, *Selected Topics in Statistical Theory.* Geoffrey S. Watson. MAA, 1971, v + 167 pp, $2.35 (P). Notes produced at the 1971 Cooperative Summer Seminar at Williams College, covering one of the two series of lectures given. Contains no table of contents or index. Topics are outlined in the preface, and range from a short elementary chapter on sampling from a finite population to chapters on stationary processes and spectral estimation, and goodness-of-fit tests of the Cramér-von Mises type using Fourier series methods. RSK

STATISTICS, T(13: 1), *Educational Statistics: Use and Interpretation, Second Edition.* W. James Popham, Kenneth A. Sirotnik. Har-Row, 1973, xi + 413 pp, $11.95. Revision of Popham's 1967 text, designed to help educators become "professionally literate." Topics covered include analysis of covariance and factor analysis, but the approach is more common-sense than mathematical. Statistical techniques involving considerable computation are each dealt with in two chapters, one general, the other giving computational details. RSK

STATISTICS, T(13), S*, L, *Statistics by Example.* Ed: Frederick Mosteller, *et al.* A-W, 1973, $3 (P) each. *Exploring Data,* xvi + 125 pp; *Weighing Chances,* xiv + 145 pp; *Detecting Patterns,* x + 166 pp; *Finding Models,* xiv + 146 pp; *Teachers' Commentary and Solutions Manual,* Martha Zelinke, *et al,* $2 (P) each. Prepared by the Joint Committee on the Curriculum in Statistics and Probability of the ASA and the NCTM, which also prepared a companion collection of applications of statistics, *Statistics: A Guide to the Unknown,* Judith Tanur, *et al,* editors (ER, April 1974). Consists of sets of interesting examples based upon real problems, and includes exercises and projects related to these examples. The mathematical prerequisites range from arithmetic, rates and percentages for *Exploring Data* to elementary probability and intermediate algebra for *Finding Models.* A must for anyone planning to teach high school mathematics, at which level it is most appropriate. RSK

STATISTICS, T(16-17: 2), P*, L*. *The Design and Analysis of Experiments.* Oscar Kempthorne. Krieger, 1973, xix + 631 pp, $17.50. Reprint, with corrections, of the author's well-known and often referenced 1952 text. It remains the definitive work in many areas of design. RSK

COMPUTER SCIENCE, P. *Complexity of Computer Computations.* Ed: Raymond E. Miller, James W. Thatcher. Plenum Pr, 1972, x + 225 pp, $16.50. Proceedings of the 1972 IBM Research Symposium: 14 papers, panel discussion, bibliography, indices. LAS

COMPUTER SCIENCE, T(15-17: 1), L. *Computability Theory: An Introduction.* Neil D. Jones. Acad Pr, 1973, xiv + 154 pp, $9.50. Develops in a relatively elementary way the theory of computability for Turing machines using S-rudimentary formulas. This approach is syntatic rather than numeric. Also discusses related problems and other formulations of computability. RWN

COMPUTER SCIENCE, T*(15-17: 1), L. *Systems Programming.* John J. Donovan. McGraw, 1972, xviii + 488 pp, $13.95. One of the best texts on systems programming: assemblers, loaders, macros, compilers, and operating systems. Somewhat oriented toward the 360. Best used with a background in PL/I and assembly. Gets at the heart of the problem without being strapped by peculiar details. Exercises. RWN

COMPUTER SCIENCE, T(15). *Design Methods for Digital Systems.* J. Chinal. Transl: A. Preston, A. Summer. Springer-Verlag, 1973, xvii + 506 pp, $33.90. Translated from the French. An introduction to switching theory. Includes number systems, arithmetic, codes, hardware logic, boolean algebra, boolean functions, switching and sequential networks and their simplification and synthesis. RWN

COMPUTER SCIENCE, T(15-16: 1), L. *Translations of Computer Languages.* Frederick W. Weingarten. Holden-Day, 1973, xi + 180 pp, $10.95. A good introductory book. Concentrates on techniques for parsing: bottom-up, top-down and left-right. Includes preliminary material on formal grammars. Also considers restricted, bounded context, and precedence grammars. "Algorithmic rather than mathematical." RWN

COMPUTER SCIENCE, P. *ABC ALGOL: A Portable Language for Formula Manipulation Systems.* R.P. Van de Riet. Math Centrum, 1973. *Part 1: The Language*, Math. Centre Tracts, No. 46, iv + 173 pp, Dfl. 18 (P); *Part 2: The Compiler*, Math. Centre Tracts, No. 47, 116 pp, Dfl. 12 (P). A language which allows the declaration and use of complex data structures and operations on these structures for processing formulas (similar to Formula Algol and Formac). A well-documented listing of the pre-processor is provided. Applications include the manipulation of polynomials, determinants and differential equations. RWN

Reviewers Whose Initials Appear Above

David F. Appleyard, Carleton; Judith N. Cederberg, St. Olaf; John Dyer-Bennet, Carleton; Steven Galovich, Carleton; Loren Haskins, Carleton; Paul S. Jorgensen, Carleton; Roger B. Kirchner, Carleton; Richard S. Kleber, St. Olaf; Loren C. Larson, St. Olaf; Pierre J. Malraison, Carleton; R.W. Nau, Carleton; J. Arthur Seebach, Jr., St. Olaf; Lynn A. Steen, St. Olaf; T.A. Vessey, St. Olaf; Frank L. Wolf, Carleton.

NEWS AND NOTICES

EDITED BY RAOUL HAILPERN, SUNY at Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to Mathematical Association of America, 1225 Connecticut Avenue, NW, Washington, D. C. 20036. Items must be submitted at least two months before publication can take place.

PERSONAL ITEMS

Gonzaga University: Instructor Barbara Harras has been promoted to Assistant Professor; Assistant Professor K. E. Martin has been appointed Chairman of the Mathematics Department.

Mr. Jack Barone, Bernard Baruch College, CUNY, has been promoted from Lecturer to Assistant Professor.

Professor H. F. Bohnenblust, Caltech, has retired with the title of Professor Emeritus.

Assistant Professor D. E. Cameron, The University of Akron, has been promoted to Associate Professor.

Assistant Professor W. H. Ford, Clemson University, has been appointed Assistant Professor at the University of the Pacific.

Assistant Professor W. R. Klinger, Marion College, has been promoted to Associate Professor.

Assistant Professor Larry Krajewski, Viterbo College, has been promoted to Associate Professor and appointed Chairman of the Department of Mathematics.

Vice President for Academic Affairs D. W. Lick, Russell Sage College, has been appointed Dean of the School of Sciences and Professor of Mathematics at Old Dominion University.

Associate Professor R. G. McDermot, Duquesne University, has been appointed Chairman of the Mathematics Department.

Dr. Maria Steinberg, California State University, Northridge, has retired with the title of Professor Emeritus.

Associate Professor E. C. Young, Florida State University, has been promoted to Professor.

Mrs. Beatrice P. DeLany, Research Associate, Illinois Institute of Technology, died on October 14, 1973. She was a member of the Association for twenty-four years.

Mr. Eugene B. Duffy, Arlington, Virginia, died on May 18, 1974, at the age of 71. He was a member of the Association for twenty-two years.

Professor Abraham Robinson, Yale University, died on April 11, 1974, at the age of 55. He was a member of the Association for five years.

CANADIAN SOCIETY FOR HISTORY AND PHILOSOPHY OF MATHEMATICS
SOCIÉTÉ CANADIENNE D'HISTOIRE ET DE PHILOSOPHIE DES MATHÉMATIQUES

At a founding meeting on June 3, 1974, in Toronto, the C.S.H.P.M./S.C.H.P.M. adopted a constitution stating that "the aim of the Society is to promote throughout Canada discussion, research, teaching, and publication in the history and philosophy of mathematics." The first Executive Council was elected as follows: President: C. V. Jones (York University); Vice-President: T. W. Settle (Guelph University); Secretary-Treasurer: J. L. Berggren (Simon Fraser University); Council Members: W. S. H. Crawford (Mount Allison University), N. T. Gridgemen (National Research Council), and Fred Ustina (University of Alberta at Edmonton).

Membership is open "to any person with interest in the history and philosophy of mathematics." Annual dues of $4.00 were set for 1974 and 1975. *Historia Mathematica* was named the official journal of the Society, and members were offered the option of paying a single fee of $10.00 for membership and subscription to the journal, starting with 1974 and Volume I. Those interested in membership should write to the Secretary-Treasurer, Professor J. L. Berggren, Mathematics Department, Simon Fraser University, Burnaby, British Columbia.

The first annual meeting was divided between sessions with the Canadian Society for the History and Philosophy of Science in connection with the Learned Societies Conference in Toronto (3–4 June) and with the Canadian Mathematical Congress at Laval University (7 June). In Toronto there was a joint meeting with the C.S.H.P.S./S.C.H.P.S. on the role of mathematics in the history of physical science, at which the speakers were J. L. Berggren, Stillman Drake, and H. S. M. Coxeter. There was also a session with papers by Tyrone Lai, Stephen Regoczei, Byron Wall, V. Linis, G. H. Moore, and P. K. Schotch. At Laval University there was a session with papers by Wei-Ching Chang, K. O. May, and G. H. Moore. In addition, H. Zassenhaus and E. A. Barbeau gave invited addresses on historical topics.

MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

OCTOBER MEETING OF THE INDIANA SECTION

The fall meeting of the Indiana Section of the MAA was held at Ball State University, Muncie, on Saturday, October 20, 1973, with approximately 120 persons in attendance.

The morning program consisted of papers by Professor M. A. Nyman, Manchester College, on "Some Remarks on a Generalization of Haar Series," and by Professor F. W. Owens, Ball State University, "On Some Graph Theoretical Research Problems from the Monthly." There followed a panel discussion on mathematics as viewed by representatives from business and industry, with Professor M. D. Thompson, Indiana University, as chairman, and consisting of Kenneth Clark, Lincoln National Life Insurance Co., Fort Wayne; Donald Holland, Operations Research Group, Detriot Diesel, Allison Division of General Motors, Indianapolis; David Sears, Computer Science–Numerical Analysis Division, General Motors, Indianapolis; and Lealon Tomkinson, Statistics Group, Eli Lilly Company, Indianapolis.

A business meeting followed the luncheon. In addition to reports on current plans of the MAA, the MAA High School Contest in Indiana, and the Indiana School Mathematics Journal, the members approved unanimously the motion of Professor M. C. Gemignani, Indiana-Purdue at Indianapolis, that the Section adopt a registration fee of one dollar at each meeting, with the proviso that the Executive Committee be delegated authority to exempt certain classes of attendants, such as speakers and students, from payment of this fee. In the absence of any news on the bill on science and technology, there was no report made.

An invited address followed on "Mass Balance and Interdisciplinary Models," by Professor Robert Pingry, Purdue University.

R. T. Hood, *Secretary-Treasurer*

APRIL MEETING OF THE INDIANA SECTION

The spring meeting of the Indiana Section of the MAA was held at Rose-Hulman Institute of Technology, Terre Haute, on Saturday, April 27, 1974, with approximately forty-five members present. The morning program consisted of the following papers: "A Constructive Theory of Completeness," by Professor John Lennes, Valparaiso University; "Newton's Power Sum Formula and Some Related Multisections," by Professor Clark Kimberling, Evansville University; "Fixed Point Iteration Using Infinite Matrices," by Professor B. E. Rhoades, Indiana University; "When do the Periodic Elements of a Group Form a Subgroup?" by Professor Gary Sherman, Rose-Hulman Institute.

Following lunch an MAA film was shown of an address by Professor R. H. Bing: "Some Challenging Conjectures." During the business meeting, Professor L. J. Cote was chosen to be the Indiana co-chairman of the MAA High School Contest for 1975. The formation of a speakers' bureau (of members who would be willing to speak at sister institutions) was discussed. Professor Harold Hanes, Earlham College, as chairman of the Nominating Committee, presented the following slate of officers for 1974–75 (which was unanimously approved): Chairman, Professor M. D. Thompson, Indiana University; Vice-chairman, Professor R. T. Hood, Franklin College; Secretary-Treasurer, Professor David Wilson, Wabash College.

Following the business meeting, Mr. Dale Van Laningham, a Rose-Hulman student and member of Pi Mu Epsilon, spoke on "The Sinbad Steamship Company: A Program to Schedule Freighter Traffic Optimally." An invited address followed, by Professor Casper Goffman, Purdue University: "Everywhere Convergence of Fourier Series."

R. T. HOOD, *Secretary-Treasurer*

APRIL MEETING OF THE IOWA SECTION

The 61st regular meeting of the Iowa Section of the MAA was held at Upper Iowa University, Fayette, Iowa, on April 19, 1974. Chairman Donald Bailey presided. Total attendance was 33, including 32 members of the Section and one visitor from the Kansas Section.

Following the invited address, "Ramsey Theory and the Problem of Eccentric Hosts," by Seymour Schuster, Carleton College, and Governor Hogg's report, the business meeting was held. A progress report of the Visiting Lectures Program conducted by the Section was given by Chairman Bailey, and by general consent, it was agreed that we should continue the program. Attendance and ways to increase participation in the section affairs were discussed, but no definitive action was taken.

Lawrence Hart, Loras College, Dubuque, Iowa, was elected as Chairman-Elect.

The following contributed papers completed the program:

1. *Isoderivative curves*, by George Bridgman, Wartburg College.
2. *A mean value theorem for integrals*, by Donald Bailey, Cornell College.
3. *A simplified proof of Bezout's theorem*, by Arnold Adelberg, Grinnell College.
4. *A small college cooperative seminar*, by E. T. Hill, Cornell College.

B. E. GILLAM, *Secretary-Treasurer*

APRIL MEETING OF THE SEAWAY SECTION

The Spring Meeting of the Seaway Section of the MAA was held at Union College, Schenectady, N. Y. on April 27, 1974, with an attendance of 83 people, including 73 members of the Association. Professor W. C. Stone of Union College, President of the Section, presided.

At the morning session, Dr. H. C. Martin, President of Union College, welcomed the members of the Seaway Section on behalf of the host institution.

Professor Malcolm Pownall of Colgate University, Governor of the Section, gave an invited lecture, "Recent Trends in Undergraduate Mathematical Programs."

Professor Arthur Danese, State University College at Fredonia, presented the Harry S. Gehman Lecture, his topic being "The Mathematician as a Teacher."

At the business meeting the following officers were elected: Chairman, D. O. McKay, University of Western Ontario; First Vice-Chairman, Mabel D. Montgomery, State University College at Buffalo; Second Vice-Chairman, Peter Lindstrom, Genesee Community College.

It was announced that P. G. deBuda, student at the University of Toronto, had received the highest score in the Section in the 1973 William Lowell Putnam Mathematical Competition, and that he would receive a check for $10 from the Section.

The Annual Report on the MAA High School Mathematics Contest was presented as a written report from H. B. Foisy, State University College at Potsdam, read by J. F. Smith.

During the afternoon the following contributed papers were presented:

Fuzzy Sets Theory, by W. E. Hartnett, State University College at Plattsburgh.

Mathematical Models — or — Should Mathematics be just a Set of Classroom Exercises? by A. C. Green, State University College at Buffalo.

Report on the National Conference for Personalized Instruction in Higher Education, Washington, D. C., by W. H. Reynolds, State University College at Cortland.

Elementary Linear Algebra-Polynomial Evaluation of Functions Using Real Matrices, by Donald Fama, Auburn Community College.

Multi-structured College Courses in General Mathematics and Calculus, by Larry Copes, Syracuse University.

A Variable Instruction Program for Mathematics, by J. A. Voytuk, Rensselaer Polytechnic Institute.

Examples of Two-Dimensional Symmetry by the Graphic Artist, M. C. Escher, by S. C. Van Orden, Eisenhower College.

Fibonacci Bases of Egyptian Fractions, by John McKibben, Skidmore College.

Some Reasons to Doubt the Validity of Goldbach's Conjecture, by R. D. Larsson, Schenectady County Community College.

A Note on $\int_a^b x^k\,dx$, by P. A. Lindstrom, Genesee Community College.

Normality and Relative Density in Groups, by G. T. Frey, St. Bonaventure University.

Irreducibility of Polynomials over Finite Fields, by C. W. Kohls, Syracuse University.

EMMET STOPHER, *Secretary-Treasurer*

APRIL MEETING OF THE TEXAS SECTION

The annual meeting of the Texas Section of the MAA was held at the University of Texas at Austin on April 5 and 6, 1974. There were 220 registered persons attending, including 193 members of the Association and 26 students.

Professor D. E. Edmondson, University of Texas at Austin, Vice-Chairman of the Section, became Chairman. The following officers were elected: First Vice-Chairman, Professor J. E. Hodge, Angelo State University; Second Vice-Chairman, Professor G. R. Blakley,

Texas A & M University; Level I Director, Professor Amogene Devaney, Amarillo College; Director-at-Large, Professor R. G. Dean, Stephen F. Austin State University; Director of MAA High School Contest, Professor J. R. Boone, Texas A & M University.

Professor Dorothy Bernstein spoke on "How to Make and Break Codes"; Professor W. T. Reid spoke on "Anatomy of the Ordinary Differential Equation"; Professor Emeritus H. J. Ettlinger spoke on "The Cauchy Initial Value Problem"; Professor Amogene Devaney arranged a panel on Basic Mathematics; Professor J. L. Poirot arranged for a series of papers on Computer Science in Mathematics Departments.

The following contributed papers were presented:

1. *A Progress Report on Phase A of MAP*, by D. M. Kulvicki, University of Texas at Austin.

2. *The Volume of the Convex Hull of a Set of Vectors*, by A. R. Amir-Moez, R. E. Long, Texas Tech University.

3. *Finite Reductive Modular Lattices*, by D. E. Edmondson, University of Texas at Austin.

4. *A Class of Reflexive Lattices of Subspaces*, by Gretchen Mooningham, Texas Tech University.

5. *Projective Representations of Wreath Products*, by Bolling Farmer, University of Texas at Austin.

6. *Holomorphic Extension for Certain Non-CR-Submanifolds*, by J. W. Mooningham, Texas Tech University.

7. *On the Structure of Certain Linear Extensor Operators via Multitensor Transform Analysis*, by D. M. Kulvicki, University of Texas at Austin.

8. *Linear Best Approximation Mappings onto C_0*, by Russell Bilyeu, North Texas State University.

9. *Generalized Lipschitz Conditions*, by F. N. Huggins, University of Texas at Arlington.

10. *Irreducibly Confluent Mappings*, by D. R. Read, Lamar University.

11. *Continued Fractions and Totally Monotone Sequences*, by D. F. Dawson, North Texas State University.

12. *On Canonical Tangential Lie Group Resolutions and Associated Lie Algebra Resolutions*, by D. M. Kulvicki, University of Texas at Austin.

13. *Cubic Programming*, by S. W. McGuire, J. F. Harvill, Lamar University.

14. *A Nonlinear Least-Squares Model for Spectral Analysis*, by S. W. McGuire, B. D. Read, Lamar University.

15. *Computing and Elementary Linear Algebra*, by C. B. Murray, University of Houston.

16. *Academic and Financial Aspects of a Time-Sharing Network*, by D. A. Caughfield, Abilene Christian College.

17. *A Multi-Discipline Computer Science Course*, by Fred Wright, Tyler Junior College.

18. *Computer Science Service Courses for the Math Student*, by G. G. Early, J. L. Poirot; Southwest Texas State University.

19. *Computer Science Department Organization and Its Relationship to Other Disciplines*, by A. R. Goddard, East Texas State University.

20. *A Computer Science Degree within the Math Department*, by J. L. Poirot, G. G. Early, Southwest Texas State University.

21. *The Self-Paced Calculus Course at Rice University*, by M. L. Curtis, Rice University.

22. *Self-Paced Instruction and the Transfer Student*, by J. A. Nickel, University of Texas of the Permian Basin.

23. *A Liberal Arts Mathematics Course with Laboratory*, by B. J. Dulin, McMurry College.

24. *'Proof' that All Triangles are Isosceles*, by John Lamb, Jr., East Texas State University.

25. *Rational Triangles with a 60° Angle*, by George Berzsenyi, Lamar University.

26. *Eccentricity — As Appolonius Might Have Defined It*, by Margaret R. Hutchinson, University of St. Thomas.

27. *Cuisenaire Rods in College Algebra*, by J. A. Bell, Texas A & I University at Laredo.

28. *The Game of Tick-Tack-Toe*, by Jack Hardy, Texas A & I University.

29. *Some Sufficient Conditions for Prime Regular Rings to be Primitive*, by E. P. Armendariz, University of Texas at Austin.

30. *On Gaussian Integers All of whose Prime Factors are Large*, by D. G. Hazlewood, Southwest Texas State University.

31. *A Gap Theorem for Complex Taylor Series*, by J. M. Stark, Lamar University.

32. *The Radius of Convergence of a Generalized Power Series*, by Kenny Zuber and Bill Anderson, East Texas State University.

33. *Summation of Certain Infinite Series by Using the Gamma Functions*, by Russell Cowan, Lamar University.

34. *Experience with Self-Paced Computer Courses*, by K. W. Kennedy, Rice University.

35. *n-th Order Vector Differential Equations with Matrix Coefficients*, by A. D. Stewart, Prairie View A & M University.

36. *Separable Quotients of Banach Spaces*, by Elton Lacey, University of Texas at Austin.

J. C. BRADFORD, *Secretary-Treasurer*

MAY MEETING OF THE ALLEGHENY MOUNTAIN SECTION

The spring meeting of the Allegheny Mountain Section of the MAA was held at Allegheny College in Meadville, Pennsylvania, on May 3 and 4, 1974, with Dr. Charles Cable, Chairman of the Section, as host. Approximately 150 members registered at the meeting.

On Friday evening, a panel discussion on "The Problem of Attracting and Stimulating Genuine Student Interest in Mathematics" was moderated by Dr. Earle Myers of the University of Pittsburgh. Other participants included Father Socher of Canevin High School, "Computers in High School Classes," Dorothea Peeler of Allegheny County Community College, "Calculators in the Classroom," and Dr. Raymond Voltz of Grove City College, "A Student Centered Curriculum."

A special feature of the meeting was a program for students. A panel discussion on "Job Opportunities for Mathematics and Computer Science Majors" was moderated by Professor R. Lundgren of Allegheny College. Panel members included Dr. Robert Dannels, Westinghouse Electric Corp.; Ms. Kathleen Seech, Mellon Bank; and Dr. John Forman, IBM Corp. The panel discussion on "Graduate School Programs in the Allegheny Mountain Section" was chaired by Professor Richard McDermot of Allegheny College. Representatives from each college with a graduate program discussed their respective programs.

Fifteen-minute talks were presented as follows:

Faculty

On linear algebra and multilinear algebra, by J. F. Kim, West Virginia University.

Equivalence classes of polynomials over finite fields, by G. L. Mullen, Pennsylvania State University.

Flat Banach spaces, by Ronald Harrell, Allegheny College.

Multiplicative functions, by Sahib Singh, Clarion State College.

Computer Programming as a tool to understanding calculus, by Ronald Larson, Behrend Campus, Pennsylvania State University.

Experimentation in the mathematics requirement for general education, by John Broughton, Indiana University of Pennsylvania.

Recent research in the teaching of mathematical induction, by R. A. Ward, Bethany College.

Applications of $D = \sum_i r_i t$ *in college freshman mathematics*, by Shu-Shen Sah, Slippery Rock State College.

Students

Numerical solution of the plate bending problem, by Sharon Trimpey, University of Pittsburgh.

The modeling of an ecological system, by Bruce Gavett, Allegheny College.

The tower of Hanoi, Nim game, and the wonderful world of base two, by Bill Means, Edinboro State College.

A ring of number theoretic functions, by Kathleen Kreep, Indiana University of Pennsylvania.

Square trigonometry, by Sharon Dittmer, Slippery Rock State Collge.

Relaxing point-differentiability as a sufficient condition for point continuity, by Frank Ditraglia and Jeffrey Sheaffer, Bethany College.

Selected MAA films were also shown.

Invited lecturers for the meeting included Dr. J. C. Eaves of West Virginia University, "Selected Topics for the Undergraduate"; Dr. Frederick Steen, Allegheny College, "Cross Country Run;" and Dr. Alex Rosenberg, Cornell University, Editor of the MONTHLY, "Current Directions of Research in Algebra."

Professor Charles Cable, Chairman of the Section, presided at the Business Meeting. The Secretary's report included the list of Putnam Competitions winners, and new books for the MAA Library. The top five students in the Putnam Competition were C. B. Croke, J. P. Smith, S. J. Spector, R. C. Valentini, of Carnegie Mellon University, and Karen Weinstein of Grove City College. A one year subscription to the MATHEMATICS MAGAZINE will be awarded to these students. A report on the High School Mathematics Contest was given by Professors Frank Kocher of Penn State and I. D. Peters of West Virginia University for Western Pennsylvania and West Virginia respectively. A report on the Visiting Lecturers Program was given by Dr. Earle Myers. Reports were also given by Professor R. G. Ayoub, Governor of the Section, and by Dr. Alex Rosenberg, the guest speaker for the Association. Officers elected were Robert McDermot, Duquesne University—Second Vice Chairman, and M. R. Woodard, Indiana University of Pennsylvania — Secretary-Treasurer. Continuing as Chairman is Charles Cable of Allegheny College and Charles Hall of the University of Pittsburgh as First Vice Chairman.

M. R. WOODARD, *Secretary-Treasurer*

MAY MEETING OF THE ILLINOIS SECTION

The Fifty-third annual meeting of the Illinois Section of the MAA was held on the campus of Knox College, Galesburg, on Friday and Saturday, May 10–11, 1974, with approximately 100 members in attendance. The program featured Professor Ralph Boas of Northwestern University, President of the MAA, who spoke following the Friday banquet on "Anomalous Cancellation."

The following papers were presented:

Density questions in additive number theory, by M. B. Nathanson, S. I. U. — Carbondale.

Differential geometry and convexity, by Stephanie Alexander, University of Illinois.

Matrices of matrices, by Paul Halmos, Indiana University.

Curves which are rated X-Y, by William Andrews, Triton College.

Didactics of mathematics in teacher education, by Peter Braunfeld, University of Illinois.

Functional analysis and the calculus of finite differences, by A. J. Insel, Illinois State University.

At the annual business meeting with Professor Neal Foland, Chairman, presiding, committee reports were presented, the by-laws amended, and Professor Robert Bryan of Knox College elected Chairman for 1974–1975. Professor Jon Laible of Eastern Illinois University, was named Chairman-elect, and Professor Robert Johnson of Augustana College was elected First Vice-chairman.

H. C. SAAR, *Secretary-Treasurer*

CALENDAR OF FUTURE MEETINGS

Fifty-eighth Annual Meeting, Washington, D. C., January 25–27, 1975.

Fifty-fifth Summer Meeting, Western Michigan University, Kalamazoo, August 18–20, 1975.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Editorial Director.

ALLEGHENY MOUNTAIN, Duquesne University, Pittsburgh, Pennsylvania, April 25–26, 1975.
FLORIDA, Manatee Junior College, Bradenton, March 7–8, 1975.
ILLINOIS, Rockford College, Rockford, May 9–10, 1975.
INDIANA
IOWA, Iowa State University, Ames, April 18–19, 1975.
KANSAS
KENTUCKY
LOUISIANA-MISSISSIPPI, Centenary College, Shreveport, Louisiana, February 1975.
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA
METROPOLITAN NEW YORK
MICHIGAN
MISSOURI, Missouri Western College, St. Joseph, Spring 1975.
NEBRASKA, Nebraska Wesleyan University, Lincoln, April 18–19, 1975.
NEW JERSEY
NORTH CENTRAL, Hamline University, St. Paul, Minnesota, April 28, 1975.
NORTHEASTERN
NORTHERN CALIFORNIA, Menlo College, Menlo Park, February 8, 1975.
OHIO
OKLAHOMA-ARKANSAS, Central State University, Edmond, Oklahoma, April 4–5, 1975.
PACIFIC NORTHWEST
PHILADELPHIA
ROCKY MOUNTAIN, Mesa College, Grand Junction, Colorado, April 11–12, 1975.
SEAWAY
SOUTHEASTERN, University of South Alabama, Mobile, March 21–22, 1975.
SOUTHERN CALIFORNIA
SOUTHWESTERN
TEXAS, Angelo State University, San Angelo, April 1975.
WISCONSIN, Univeisity of Wisconsin-Superior, April or May 1975.

FUTURE MEETINGS OF OTHER ORGANIZATIONS

AMERICAN ASSOCIATION FOR THE ADVANCEMENT OF SCIENCE
AMERICAN MATHEMATICAL SOCIETY, Washington, D. C., January 23–26, 1975.
AMERICAN SOCIETY FOR ENGINEERING EDUCATION, Colorado State University, Fort Collins, June 16–19, 1975.
ASSOCIATION FOR COMPUTING MACHINERY, Radisson Hotel, Minneapolis, Minnesota, October 21–23, 1975.
ASSOCIATION FOR SYMBOLIC LOGIC, Shoreham Hotel, Washington, D. C., January 23–24, 1975.
ASSOCIATION FOR WOMEN IN MATHEMATICS, Washington, D.C., January 24, 1975.
FIBONACCI ASSOCIATION
INSTITUTE OF MATHEMATICAL STATISTICS
MU ALPHA THETA
NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS, Washington, D. C., January 25–26, 1975 (joint meeting with MAA).
OPERATIONS RESEARCH SOCIETY OF AMERICA, Chicago, April 30—May 2, 1975.
PI MU EPSILON, Western Michigan University, Kalamazoo, August 19–20, 1975.
SCHOOL SCIENCE AND MATHEMATICS ASSOCIATION
SOCIETY FOR INDUSTRIAL AND APPLIED MATHEMATICS

Page 1

U S POSTAL SERVICE

STATEMENT OF OWNERSHIP, MANAGEMENT AND CIRCULATION
(Act of August 12 1970 Section 3685 Title 39 United States Code)

SEE INSTRUCTIONS ON PAGE 2 (REVERSE)

1 TITLE OF PUBLICATION
AMERICAN MATHEMATICAL MONTHLY

2 DATE OF FILING
SEPT. 17, 1974

3 FREQUENCY OF ISSUE
10 issues per year

4 LOCATION OF KNOWN OFFICE OF PUBLICATION *(Street, city, county, state, ZIP code) (Not printers)*
1225 Connecticut Ave., N.W., Washington, DC 20036

5 LOCATION OF THE HEADQUARTERS OR GENERAL BUSINESS OFFICES OF THE PUBLISHERS *(Not printers)*
The Mathematical Association of America, 1225 Connecticut Ave.,NW, Washington, DC 20036

6 NAMES AND ADDRESSES OF PUBLISHER, EDITOR AND MANAGING EDITOR

PUBLISHER *(Name and address)*
The Mathematical Association of America, 1225 Connecticut Ave.,NW, Washington, DC

EDITOR *(Name and address)*
Prof. Alex Rosenberg, Cornell Univ., Dept. of Math., Ithaca, NY 14850

MANAGER EDITOR *(Name and address)*
Dr. Raoul Hailpern, Math. Assn of America, SUNY at Buffalo, Buffalo, NY 14214

7 OWNER *(If owned by a corporation its name and address must be stated and also immediately thereunder the names and addresses of stockholders owning or holding 1 percent or more of total amount of stock If not owned by a corporation the names and addresses of the individual owners must be given If owned by a partnership or other unincorporated firm its name and address as well as that of each individual must be given)*

NAME	ADDRESS
The Mathematical Association of America, Inc.	1225 Connecticut Ave.NW Washington, DC 20036

8 KNOWN BONDHOLDERS, MORTGAGEES AND OTHER SECURITY HOLDERS OWNING OR HOLDING 1 PERCENT OR MORE OF TOTAL AMOUNT OF BONDS MORTGAGES OR OTHER SECURITIES *(If there are none so state)*

NAME	ADDRESS
none	

9 FOR OPTIONAL COMPLETION BY PUBLISHERS MAILING AT THE REGULAR RATES (Section 132 121 Postal Service Manual)

39 U S C 3626 provides in pertinent part No person who would have been entitled to mail matter under former section 4359 of this title shall mail such matter at the rates provided under this subsection unless he files annually with the Postal Service a written request for permission to mail matter at such rates '

In accordance with the provisions of this statute I hereby request permission to mail the publication named in item 1 at the reduced postage rates presently authorized by 39 U S C 3626

(Signature and title of editor publisher business manager or owner)

10 FOR COMPLETION BY NONPROFIT ORGANIZATIONS AUTHORIZED TO MAIL AT SPECIAL RATES *(Section 132 122 Postal Manual)*
(Check one)

The purpose function and nonprofit status of this organization and the exempt status for Federal income tax purposes ☒ Have not changed during preceding 12 months ☐ Have changed during preceding 12 months *(If changed, publisher must submit explanation of change with this statement)*

11 EXTENT AND NATURE OF CIRCULATION	AVERAGE NO COPIES EACH ISSUE DURING PRECEDING 12 MONTHS	ACTUAL NUMBER OF COPIES OF SINGLE ISSUE PUBLISHED NEAREST TO FILING DATE
A TOTAL NO COPIES PRINTED *(Net Press Run)*	23,350	23,350
B PAID CIRCULATION 1 SALES THROUGH DEALERS AND CARRIERS STREET VENDORS AND COUNTER SALES	none	none
2 MAIL SUBSCRIPTIONS		
C TOTAL PAID CIRCULATION	20,741	20,290
D FREE DISTRIBUTION BY MAIL CARRIER OR OTHER MEANS 1 SAMPLES COMPLIMENTARY AND OTHER FREE COPIES	60	60
2 COPIES DISTRIBUTED TO NEWS AGENTS, BUT NOT SOLD		
E TOTAL DISTRIBUTION *(Sum of C and D)*	20,801	20,350
F OFFICE USE LEFT OVER UNACCOUNTED SPOILED AFTER PRINTING	2,549	3,000
G TOTAL *(Sum of E & F—should equal net press run shown in A)*	23,350	23,350

I certify that the statements made by me above are correct and complete ▶ *(Signature of editor publisher, business manager, or owner)* Alfred B. Willcox

PS Form 3526 July 1971

Most Math and Statistics texts are adequate... for ideal norms. Wiley introduces 4 texts...for people.

This fresh, student-oriented edition will make your job easier. . . .

1. COLLEGE ALGEBRA AND TRIGONOMETRY
Second Edition
By **Margaret F. Willerding**
Simplified and expanded, the second edition of COLLEGE ALGEBRA AND TRIGONOMETRY has over 3500 problems, and features more review material. Maintains the original aim of presenting an integrated approach to intermediate algebra and trigonometry. Stress on manipulative skills prepares students for advanced algebra or calculus. Instructor's manual also available.
1975

Containing over 3000 exercises, this is the most up-to-date intermediate text available. . . .

2. MODERN INTERMEDIATE ALGEBRA,
Second Edtion
By **Margaret F. Willerding**
This completely revised edition of MODERN INTERMEDIATE ALGEBRA combines an informal exposition style with in-depth explanations of all concepts and definitions. Like the popular first edition, it exposes students with mathematical deficiencies to a step-by-step approach to intermediate algebra. Every theoretical and computational aspect is explained by at least one example. An instructor's manual suggests methods of presentation, and includes the answers to all even-numbered exercises.
1975

A solid presentation of all the principles of introductory algebra. . . .

3. INTRODUCTORY ALGEBRA: A College Approach
Third Edition
By **Milton D. Eulenberg, Theodore S. Sunko,** and **Howard A. James**
This new edition of INTRODUCTORY ALGEBRA retains the basic features that made the previous editions so successful. Many field-tested exercise sets have been redesigned to make this an even more challenging introductory presentation. Emphasis on sound mathematical practice builds a firm foundation for further mathematical study. A corresponding instructor's manual contains a comprehensive testing program, and provides answers to all the exercises.
1975

4. STATISTICS MADE RELEVANT: A Casebook of Real-Life Examples
By **Paul Baum** and **Ernest M. Scheuer**
Your students will appreciate this one-of-a-kind statistics text. Using real newspaper articles, it shows how to apply statistical analysis to the interpretation of raw data. This common-sense approach enables students to relate statistical techniques to their own life experiences. Equally important is what you won't find in this text: no more synthetic problems, mind-boggling formulas, or extraneous theorems.
1975

For your complimentary examination copies, contact your Wiley representative or write Bob McConnin, Dept. 238, N.Y. office. Please include course title, enrollment, and present text.

JOHN WILEY AND SONS, INC.
605 Third Avenue, New York, N.Y. 10016
In Canada: 22 Worcester Road, Rexdale, Ontario

A4639RM

THE MATHEMATICAL ASSOCIATION OF AMERICA
1225 Connecticut Avenue, N.W.
Washington, DC 20036

INDEX TO VOLUME 81, 1974

THE AMERICAN MATHEMATICAL MONTHLY

AUTHOR INDEX

SUBJECT INDEX

This index uses the AMS (MOS) *Subject classification scheme* (1970), Math. Rev., 39 (1970) A1–A42, where a complete version can be found. Underlined entries indicate that the subject heading is secondary. In the (MOS) classification scheme the third level entries ending in 99 are "None of the above, but in this section." In this index the 99's carry the second level classification in which they lie.

QUERIES

Numbers in **boldface** type refer to pages of replies

PROBLEMS AND SOLUTIONS

PROBLEMS PROPOSED

PROBLEMS SOLVED

SOLUTIONS

Numbers in **boldface** type refer to problems, those in lightface, to pages

* Editor's Comment only

REVIEWS

Names of authors are in ordinary type, those of reviewers in capitals

FILMS

TELEGRAPHIC REVIEWS

NEWS AND NOTICES

PERSONAL ITEMS

GENERAL INFORMATION

NECROLOGY

REPORTS AND ANNOUNCEMENTS OF THE ASSOCIATION AND ITS SECTIONS

MEETINGS AND ANNOUNCEMENTS OF THE ASSOCIATION

MEETINGS OF ITS SECTIONS

THE AMERICAN MATHEMATICAL MONTHLY

(FOUNDED IN 1894 BY BENJAMIN F. FINKEL)

THE OFFICIAL JOURNAL OF

THE MATHEMATICAL ASSOCIATION OF AMERICA

VOLUME 81 NUMBER 10

CODEN: AMMYAE

CONTENTS

(*Continued on inside cover*)

DECEMBER 1974

NOTICE TO AUTHORS

Research papers per se are unsuitable; see statement of policy (Vol. 81, p. 1).

Please follow the format in current issues of the Monthly. Manuscripts must be legibly typewritten or reproduced from typewritten copy, double spaced with wide margins. Three copies should be submitted to the appropriate editor and one kept by the author as protection against loss. The author's full address *must* appear at the end of the manuscript.

Backlog: Main Articles 16 months, Math. Notes 18 months, Research Problems 10 months, Classroom Notes 21 months, Math. Education 20 months.

EDITORIAL CORRESPONDENCE AND MAIN ARTICLES: to Alex Rosenberg, Department of Mathematics, Cornell University, Ithaca, N.Y. 14853; NOTES, etc.: to the corresponding Associate Editor; REPRINT PERMISSION: to Leonard Gillman, Mathematical Association of America, University of Texas, Austin, Texas 78712 (see also the copyright notice below); ADVERTISING CORRESPONDENCE: to Raoul Hailpern, Mathematical Association of America, SUNY at Buffalo, Buffalo, N. Y. 14214; CHANGE OF ADDRESS and SUBSCRIPTIONS: to A. B. Willcox, Mathematical Association of America, 1225 Connecticut Ave., N.W., Washington, D.C. 20036.

Annual dues for members of the Association (including a subscription to the American Mathematical Monthly) are $14.00 for each of the first two years of membership and $18.00 thereafter. Student Membership is available with annual dues of $9.00. For nonmembers the subscription price is $18.00.

Published by the Association at Washington, D. C., and Menasha, Wisconsin, during the months of January, February, March, April, May, June–July, August–September, October, November, December.

Second-class postage paid at Washington, D. C., and additional mailing offices.

NOTICE TO OUR CONTRIBUTORS

We regret to announce that due to ever increasing production costs, it will no longer be possible, beginning with the January 1975 issue, to supply our contributors with 50 free reprints of their articles. All reprints of articles appearing in Volume 82 and future volumes will have to be purchased.

EULER AND THE ZETA FUNCTION

RAYMOND AYOUB

1. Introduction. Mathematics in general appeals to the intellect; great mathematics, however, has, in addition, a kind of perceptual quality which endows it with a beauty comparable to that of great art or music. In this category belongs much of the work of the great 18th century Swiss mathematician, Leonhard Euler (1707–1783).

One of the most enchanting episodes is his work on the zeta function, to which this article is devoted. In anticipation of the later notation of G.F.B. Riemann (1826–1866), let

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s},$$

where for the moment no specification will be made on s. Euler's work on $\zeta(s)$ began about 1730 with approximations to the value of $\zeta(2)$, continued with the evaluation of $\zeta(2n)$, where n is a natural number $\geqq 1$, and resulted about 1749 in the discovery of the functional equation *almost* 110 *years before Riemann.*

Before beginning the story, we shall give a brief sketch of Euler's life. Most of the facts are taken from a eulogy delivered by Nicholas Fuss (1755–1825), the husband of one of Euler's granddaughters.

The city of Basel in Switzerland was one of many free cities in Europe and by the 17th century had become an important center of trade and commerce. The University became a noted institution, largely through the fame of an extraordinary family — the Bernoullis. This family had come from Antwerp to Basel and the founder of the mathematical dynasty was Nicholas Bernoulli. He had 3 sons, two of whom, James (often referred to as Jacob) (1654–1705), and John (1667–1748), became noted mathematicians. Both were pupils of G. Leibniz (1646–1716) with whom John carried on an extensive correspondence and with whose work both James and John became familiar. James was a professor at Basel until his death in 1705. John, who had been a professor at Groningen, replaced him. To give an indication of the mathematical activity of this period, it is worthwhile pointing out that I. Newton (1642–1727) had started his work on the theory of fluxions about 1664, publishing his great work,

Principia Mathematica, in 1687. On the continent G. Leibniz began his studies on the calculus about 1672 and published much of his work in the journal *Acta Eruditorum*. This was a monthly periodical published in Leipzig and devoted to miscellaneous articles, books and book reviews.

Paul Euler (1670–1745) was a Lutheran Pastor who was mathematically talented and who had studied mathematics with James Bernoulli at the University of Basel. Into this intellectually rich and stimulating environment, Leonhard was born in 1707. He was a precocious child who received much encouragement from his father. He entered the University of Basel and displayed such remarkable talent for mathematics that John Bernoulli gave him special instruction on Saturdays. He graduated with a kind of master's degree in 1724 at the age of 17. He had enrolled in the Faculty of Theology and had written a thesis in Latin on a comparison between Newtonian and Cartesian philosophy. Although Paul expected his son to study theology, he did not discourage Leonhard's interest in mathematics. (Still, mathematics was fine as a hobby, but surely not as a profession!)

At this period there were 3 famous centers of learning, the academies at Berlin, Paris, and St. Petersburg, and it was frequently the case that a young scholar would journey to one of these.

John Bernoulli had 3 sons. Two of them, Nicholas II (1695–1726) and Daniel (1700–1782), were mathematicians who befriended Euler. They both went to St. Petersburg in 1725 and both had a high regard for their younger colleague. After some effort, Daniel wrote to Euler that he had secured for him a stipend in the Academy. The appointment was actually in the physiology section but Euler quickly drifted into the mathematics section. He then left Basel and arrived in St. Petersburg in 1727, remaining there until 1741.

The period had its troubles. Tsarina Catherine I was committed to carrying out the policy of her husband, Peter the Great, in establishing a strong Academy. Unfortunately she died the day Euler set foot in Russia. The throne passed to Peter I's grandson, Peter II, who was only 12 and Russia was ruled by despotic regents who declared that the Academy was very costly and was of little use to the state. Euler despaired of being able to pursue his interests and decided to join the navy. Admiral Sievers saw in him a valuable asset to the navy and offered him a position as lieutenant, with promises of rapid promotion. From the sources available to the author it is not clear to what extent, if any, Euler was active in naval affairs. The death of Peter II brought to an end the despotic regency and the Academy's condition improved, but the despotism had discouraged some foreign scholars, who returned to their homeland. An opportunity arose when Bullfingr left Russia and in 1731 Euler was appointed professor of natural sciences. Two years later, when Daniel decided to return to Basel in 1733, he recommended that Euler be appointed his successor as professor of mathematics. Euler remained in this position until 1741 when he was summoned by Frederick the Great of Prussia to the Berlin Academy. He was in Berlin until 1766. Catherine II, the Great, acceded to the throne of Russia in 1762 and in 1766 sum-

moned Euler back to the Academy in St. Petersburg where he remained until his death in 1783.

Euler did significant work in all areas of mathematics and his work in any one of these would have assured him a place in history. He was a prodigious writer whose collected works run currently to 70 quarto volumes with more to come. In editing Euler's works shortly after his death N. Fuss listed 756 articles distributed in time as follows: 1727–33 : 24; 1734–43 : 49; 1744–53 : 125; 1754–63 : 99; 1764–72 : 104; 1773–82 : 355. The most astonishing feature is the phenomenal number written in the last 10 years of his life, during which years he was blind. Since Fuss's editing activities, numerous additional manuscripts have been found and the total will run to almost 900. In addition to his articles he wrote several books, among the most noted and influential of which was his *Introductio in Analysin Infinitorum*. Some have criticized his writings as being repetitive but it is proper to ignore this kind of pedantry.

Euler's articles were mostly in Latin which is unfortunate in view of our present day ignorance of the classics. On the other hand, the Latin is comparatively simple and, with a rudimentary knowledge, together with a dictionary, the reader will be rewarded for his efforts. It is especially fortunate that the notation is familiar, and where the language is difficult, the mathematics comes to the rescue. It is customary to be surprised at how "modern" his notation is; the truth is that his influence was so profound that we still use much of the notation he helped to establish.

Reading his papers is an exhilarating experience; one is struck by the great imagination and originality. Sometimes a result familiar to the reader will take on an original and illuminating aspect, and one wishes that later writers had not tampered with it.

Euler's personal life, though relatively uneventful, was marred by several tragedies. Though apparently of a strong constitution, he developed a massive infection which resulted in the loss of one eye in 1735. The second eye developed a cataract about 1766 which rendered him blind. He could still distinguish lights and shadows and sometimes wrote mathematics in very large symbols on a blackboard. Despite this handicap, he continued unabated his mathematical activities with the help of young assistants. He once met with J. d'Alembert (1717–1783) who was utterly astonished at Euler's ability to carry out in his head the most complicated analytical calculations.

Euler married Catherine Gsell in 1733. She was the daughter of a well-known artist. She had 13 children 8 of whom tragically died in childhood. Catherine died in 1776, Euler then married her half sister.

His character was that of a kind and gentle man. He had a phenomenal memory, had studied the classics, and is said to have known the Aeneid by heart. Though the recipient of numerous honors during his lifetime, he retained his modesty and humility and it was said of him that he took as much pleasure in the discoveries of others as he did in his own.

He carried on an extensive correspondence with various mathematicians, especially Christian Goldbach (1690–1764). He also wrote a series of letters on various

subjects in natural philosophy addressed to a German princess. The quality of all his letters reflects his pleasant personality.

2. Early history of the function $\zeta(s)$. In elementary courses in calculus, one of the first examples of an infinite series is that given by $\zeta(s)$.

The student quickly learns, mainly via the integral test, that

$$\sum_{n=1}^{\infty} \frac{1}{n^s}$$

converges if $s > 1$ and diverges if $s \leqq 1$. Some enthusiastic teachers will point out that, in fact,

$$\zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}, \tag{1}$$

and perhaps remark that this relation is difficult to prove and that students who go on in mathematics will eventually learn at least one proof. More enthusiastic teachers will further point out that if k is an integer $k \geqq 1$, then

$$\zeta(2k) = \frac{(-1)^{k-1} B_{2k} (2\pi)^{2k}}{2(2k)!}, \tag{2}$$

where B_{2k} is a *rational* number, viz. a Bernoulli number, a fact first proved by Euler. The generating function for these numbers is given by

$$\frac{x}{e^x - 1} = 1 - \frac{x}{2} + \sum_{n=2}^{\infty} \frac{B_n}{n!} x^n.$$

However, it is easily seen that $B_{2m+1} = 0$ and that $B_2 = 1/6$, $B_4 = -(1/30)$, $B_6 = 1/42, \cdots$. They might further point out that if m is odd, $m = 2k + 1$ $(k \geqq 1)$, then no such formula is known for $\zeta(m)$, and despite considerable efforts over the years, the arithmetic nature of even $\zeta(3)$ remains an unsolved problem.

Before proceeding, it is interesting to note that Euler often worked with

$$\theta(s) = \sum_{n=0}^{\infty} \frac{1}{(2n+1)^s},$$

with

$$\phi(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^s},$$

and with

$$\psi(s) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^s}.$$

The first two are related to $\zeta(s)$ by

$$\zeta(s) = \theta(s) + \frac{1}{2^s} \zeta(s) ;$$

hence $\theta(s) = (1 - (1/2^s))\zeta(s)$, while

$$\phi(s) = \zeta(s) - \frac{2}{2^s}\zeta(s) = (1 - 2^{1-s})\zeta(s).$$

Thus $\phi(n)$ and $\theta(n)$ can be evaluated if $\zeta(n)$ can be. One important advantage of $\phi(s)$ over $\zeta(s)$ is that the series for $\phi(s)$ converges if $s > 0$, while that for $\zeta(s)$ only for $s > 1$.

By contrast $\psi(s)$ has a superficial resemblance to $\phi(s)$ but although $\psi(2n+1)$ has been evaluated, $\phi(2n+1)$ has not. In fact Euler proved that

$$\psi(2n+1) = (-1)^n \frac{E_{2n}}{2^{2n+2}(2n)!}\pi^{2n\ \ 1},$$

where

$$\sec x = \sum_{n=0}^{\infty} (-1)^n \frac{E_{2n}}{2n!} x^{2n}$$

and E_{2n} are called Euler numbers.

Let us begin the story and go back ... Infinite series have occurred sporadically in mathematics for centuries — in fact Archimedes (287–212 B.C.), when he derived his famous theorem on the quadrature of the parabola, proved in effect that the series

$$\sum_{n=1}^{\infty} 4^{-n}$$

converges. As far as the harmonic series is concerned, however (despite Plato's interest), the earliest recorded appearance appears to be in the works of Nicholas of Oresme (1323–1382) who proved that the series

$$\sum_{n=1}^{\infty} \frac{1}{n}$$

diverges.

The problem occurs again in 1650 in a book *Novae Quadraturae Arithmeticae* by a professor of mechanics in Bologna named Pietro Mengoli (1625–1686). He related the series to the logarithm and posed the problem of finding the sum of the series

$$\sum_{n=1}^{\infty} \frac{1}{n^2},$$

if it converges.

Whether through the book of Mengoli or (what seems likely) independently, the problem became known in France and England. In fact, the English mathematician John Wallis (1616–1703), professor at Oxford, commented on the problem in 1655 in his book *Arithmetica Infinitorum*. He had computed the value of $\zeta(2)$ to 3 decimal

places but it does not appear that he recognized this value, 1.645, as being about $\pi^2/6$.

In a letter to John Bernoulli in 1673, Leibniz wrote: "let

$$dy = \frac{1}{1} + \frac{x}{2} + \frac{x^2}{3} + \cdots$$

then $dy = (-[\log(1-x)]/x)\, dx$, thus

$$x + \frac{x^2}{2^2} + \frac{x^3}{3^3} + \cdots = -\int \frac{\log(1-x)}{x}\, dx.$$

As $\log(1-x)$ is infinite when $x = 1$, consider instead

$$dy = \frac{x}{1} - \frac{x^2}{2} + \frac{x^3}{3} - \cdots$$

and get $y = \int [(\log(1+x)]/x)dx$."

He now integrates by parts and deduces that the evaluation of the sum

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$$

reduces to the evaluation of integrals of the form $\int x^e(1+x)^n dx$. He continues: "If perhaps it were possible to consider all the cases in order, some light would be shed upon the problem."

In a letter to James Bernoulli in 1691, his brother John wrote, "I see now the route for finding the sum $\frac{1}{1} + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots$." No further work, however, was forthcoming from him until 1742 when he published a proof similar to that given by Euler in 1734.

In the St. Petersburg Academy, the members were drawn to the problem and took a great interest in the evaluation of $\zeta(2)$. That it is a tantalizing problem stems in part from the fact that the series has a superficial resemblance to the series

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)},$$

whose value is easily seen to be

$$\sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1}\right) = 1.$$

This fact was early recognized by the academicians. In 1728, Daniel Bernoulli wrote to Goldbach that he had a method for computing quickly an approximation to $\zeta(2)$ and gave as an approximate value 8/5. In reply Goldbach wrote that he could show that

$$1\tfrac{16}{25} = 1.64 < \zeta(2) < 1\tfrac{2}{3} = 1.66.$$

Neither gave indications of their computations. As noted above, Daniel Bernoulli and Euler both lived in St. Petersburg between 1727 and 1733 and it seems very probable that they discussed the problem together.

3. Euler's early contributions. There the problem lay. Euler's first contribution came in 1731 when he gave an original method for computing $\zeta(2)$. His method appeared in a paper *De summatione innumerabilium progressionum.* He deals with sums of the type

$$\sum_{k=1}^{\infty} \frac{x^k}{(ak+b)^m} .$$

In the special case of $\zeta(2)$ his argument is as follows: Since

$$\frac{\log(1-x)}{x} = -\sum_{n=1}^{\infty} \frac{x^{n-1}}{n}, \text{ it follows}$$

that $-\zeta(2) = \int_0^1 (\log(1-x)/x)dx$. Replacing $1-x$ by t and splitting the integral, it follows that

$$-\zeta(2) = \int_0^1 \frac{\log t}{1-t}\,dt = \int_0^x \frac{\log t}{1-t}\,dt + \int_x^1 \frac{\log t}{1-t}\,dt = I_1 + I_2.$$

In I_2, put $u = 1-t$, expand in a power series and integrate termwise; then if $y = 1-x$,

$$I_2 = \sum_{n=1}^{\infty} \frac{y^n}{n^2} .$$

On the other hand, in I_1, expand $(1-t)^{-1}$ in a series, and integrate by parts getting

$$I_1 = -\log x \log(1-x) - \sum_{n=1}^{\infty} \frac{x^n}{n^2} .$$

Hence $\zeta(2) = \log x \log(1-x) + \sum_{n=1}^{\infty} x^n/n^2 + \sum_{n=1}^{\infty} (1-x)^n/n^2$. Putting $x = \frac{1}{2}$, we conclude that

$$\zeta(2) = (\log 2)^2 + \sum_{n=1}^{\infty} \frac{1}{2^n n^2} .$$

What has been achieved by this next argument? The series $\sum_{n=1}^{\infty} 1/2^n n^2$ converges much more rapidly than does the series for $\zeta(2)$. Knowing that

$$\log 2 = -\log\left(1-\frac{1}{2}\right) = \sum_{n=1}^{\infty} \frac{1}{n2^n} \sim .480453,$$

and that

$$\sum_{n=1}^{\infty} \frac{1}{n^2 2^n} \sim 1.164482,$$

Euler concludes that $\zeta(2) \sim 1.644934$.

It should be remarked that in 1730 James Stirling (1692–1770) had computed $\zeta(2)$ to 9 decimal places, of which 8 were correct, but Euler was unaware of these calculations.

Euler's next contribution came in 1732/33 in a paper entitled *Methodus Generalis Summandi Progressiones.* In this he states the "Euler-McLaurin" formula (Colin McLaurin (1698–1746)). In a later paper *Inventio summae cuiusque seriei ex dato Termino generali,* published in 1736, he gives a proof. Although the paper was published in 1736, it is reasonable to assume that the work was done before 1734. We shall give Euler's argument which we modify slightly. Moreover, we shall ignore a few technicalities. Let

$$S(x) = \sum_{n \leqq x} f(n).$$

The object is to approximate $S(x)$ by an integral. We have

$$\text{(A)} \qquad f(x) = S(x) - S(x-1).$$

Using the Taylor (Brooke Taylor, 1685–1731) expansion, it follows that

$$\text{(B)} \qquad f(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} S^{(n)}(x)}{n!} \;;$$

(the difficulty, of course, is that in writing (A) we are assuming x to be an integer while in (B), we assume x to be any real number).

Assume now that this series can be inverted; that is, assume there exist constants $b_0, b_1, b_2, \cdots$ such that

$$\text{(C)} \qquad S^{(1)}(x) = \sum_{n=0}^{\infty} b_n f^{(n)}(x).$$

Differentiating (B),i nserting in (C), and equating coefficients, gives recurrence formulae for the b's, viz.,

$$b_0 = 1\,,\; b_1 = \frac{b_0}{2},\; b_2 = \frac{b_1}{2!} - \frac{b_0}{3!},\; b_3 = \frac{b_2}{2!} - \frac{b_1}{3!} + \frac{b_0}{4!}, \text{ etc.}$$

Hence $S(x) = b_0 \int f(x)dx + \sum_{n=1}^{\infty} b_n f^{(n-1)}(x)$. The b's turn out to be essentially the Bernoulli numbers. This fact can be intuitively gleaned from the following argument: let D denote the operator d/dx, then (B) can be written as

$$f(x) = \left(1 - \frac{D}{2!} + \frac{D^2}{3!} - \cdots\right) S^{(1)}(x) = \left(\frac{e^{-D} - 1}{D}\right) S^{(1)}(x),$$

or inverting,

$$S^{(1)}(x) = \left(\frac{D}{e^{-D} - 1}\right) f(x).$$

On the other hand, the generating function for the Bernoulli numbers as noted above, gives

$$\frac{x}{e^x - 1} = 1 - \frac{x}{2} + \frac{B_2 x^2}{2!} + \frac{B_3 x^3}{3!} + \cdots.$$

Hence, replacing x by $-D$ gives the desired result. Euler is evidently excited by this discovery (as which of us would not be!) and proceeds to apply it with great enthusiasm in the paper which appeared in 1736, *Inventio summae cuiusque seriei ex dato Termino generali*, referred to above.

He derives a formula for

$$\sum_{m=1}^{n} m^k \qquad (k \geqq 1)$$

and painstakingly computes the necessary constants $B_2, \cdots, B_{16}$ and writes out at length the results for $k = 1, \cdots, 16$. Then he applies it to the harmonic series, showing that

$$\sum_{n \leq x} \frac{1}{n} = \text{const} + \log x + \frac{1}{2x} - \frac{1}{12x^2} + \cdots,$$

and performs calculations for $x = 10^l$ for $l = 1, 2, 3, 4, 5, 6$. Finally among other things, he computes $\zeta(2)$ and $\zeta(3)$ with great accuracy. For $\zeta(2)$, he writes

$$\zeta(2) = \sum_{n=1}^{10} \frac{1}{n^2} + \sum_{n=11}^{\infty} \frac{1}{n^2}.$$

He computes the first term by hand and then estimates the remainder by the formula. His result is that approximately

$$\zeta(2) = 1.64493406684822643647.$$

Still the evaluation of $\zeta(2)$ in closed form eluded him. Needless to say, this method of approximation opened a whole new area of research.

4. First triumph. Euler's first triumph came in 1734. Having previously done work on the roots of polynomials, he conceived the idea of generalizing the factorization of polynomials to transcendental functions. Euler communicated his result to Daniel Bernoulli and, while unfortunately this letter has been lost, the reply does exist. Daniel says: "The theorem on the sum of the series

$$1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots = \frac{pp}{6} \text{ and } 1 + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \cdots = \frac{p^4}{90}$$

is very remarkable. You must no doubt have come upon it *a posteriori*. I should very much like to see your solution."

Here is a sketch of it as it appears in *De summis serierum reciprocarum*. Consider the expression $f(x) = 1 - (\sin x / \sin \alpha)$ with α fixed and α not a multiple of π. Leibniz

had derived the power series expansion for sin x, so write

$$f(x) = 1 - \frac{x}{\sin\alpha} + \frac{x^3}{3!\sin\alpha} - \cdots .$$

The right hand side is now viewed as a polynomial of infinite degree. If $a_1, a_2, \cdots, a_n, \cdots$ are the roots, then write

$$f(x) = \left(1 - \frac{x}{a_1}\right)\left(1 - \frac{x}{a_2}\right) \cdots \left(1 - \frac{x}{a_n}\right) \cdots = \prod_{k=1}^{\infty} \left(1 - \frac{x}{a_k}\right).$$

The roots of $f(x)$ however, are evident from the left hand side, viz.,

$$x = \begin{cases} 2n\pi + \alpha \\ 2n\pi + \pi - \alpha \end{cases} \qquad n = 0 \pm 1, \pm 2, \cdots;$$

thus

$$\text{(F)} \quad \begin{aligned} f(x) &= \prod_{n=-\infty}^{\infty} \left(1 - \frac{x}{2n\pi + \alpha}\right)\left(1 - \frac{x}{2n\pi + \pi - \alpha}\right) \\ &= \left(1 - \frac{x}{\alpha}\right)\prod_{n=1}^{\infty}\left(1 - \frac{x}{(2n-1)\pi - \alpha}\right)\left(1 + \frac{x}{(2n-1)\pi + \alpha}\right)\left(1 - \frac{x}{2n\pi + \alpha}\right)\left(1 + \frac{x}{2n\pi - \alpha}\right). \end{aligned}$$

We can now expand the right hand side in a power series and equate coefficients. The expansion on the right involves the "infinite" elementary symmetric functions and Euler now derived the infinite analogues of Newton's formulae, viz., if $a_1, \cdots, a_n, \cdots$ is a sequence and

$$\sigma_m = \sum_{i_1, \cdots, i_{...}} a_{i_1} \cdots a_{i_m}$$

while $S_m = \sum_{i=1}^{\infty} a_i^m$, then in particular,

$$S_1 = \sigma_1, \quad S_2 = \sigma_1^2 - 2\sigma_2, \quad S_3 = \sigma_1^3 - 3\sigma_1\sigma_2 + 3\sigma_3.$$

The other relations may be similarly derived.

Applying these facts to (F) we get (since $\sigma_2 = 0$),

$$\frac{1}{\alpha} + \sum_{n=1}^{\infty} \left(\frac{1}{(2n-1)\pi - \alpha} - \frac{1}{(2n-1)\pi + \alpha} + \frac{1}{2n\pi + \alpha} - \frac{1}{2n\pi - \alpha}\right) = \frac{1}{\sin\alpha},$$

$$\frac{1}{\alpha^2} + \sum_{n=1}^{\infty} \left(\frac{1}{((2n-1)\pi - \alpha)^2} + \frac{1}{((2n-1)\pi + \alpha)^2} + \frac{1}{(2n\pi + \alpha)^2} + \frac{1}{(2n\pi - \alpha)^2}\right) = \frac{1}{\sin^2\alpha},$$

$$\frac{1}{\alpha^3} + \sum_{n=1}^{\infty} \left(\frac{1}{((2n-1)\pi - \alpha)^3} - \frac{1}{((2n-1)\pi + \alpha)^3} + \frac{1}{(2n\pi + \alpha)^3} - \frac{1}{(2n\pi - \alpha)^3}\right) = \frac{1}{\sin^3\alpha} - \frac{1}{2\sin\alpha}.$$

Putting $\alpha = \pi/2$, the first gives $(4/\pi)(1 - \frac{1}{3} + \frac{1}{5} \cdots) = 1$ — a fact already known to James Gregory (1638–1675). The second, however, leads to the long sought after objective, for it gives

$$\frac{8}{\pi^2}\left(1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right) = 1.$$

However, as Euler remarks,

$$\zeta(2) = \left(1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right) + \frac{1}{4}\zeta(2)$$

and this, then, gives $\zeta(2) = \pi^2/6$. Similar arguments give

$$1 - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \cdots = \frac{\pi^3}{32},$$

$$1 + \frac{1}{3^4} + \frac{1}{5^4} + \cdots = \frac{\pi^4}{96},$$

and, hence, $\zeta(4) = \pi^4/90$.

Likewise Euler computes the corresponding series with exponents 5, 6, 7, and 8.

If $\alpha = \pi/4$, the first relation gives

$$\frac{\pi}{2\sqrt{2}} = 1 + \frac{1}{3} - \frac{1}{5} - \frac{1}{7} + \frac{1}{9} + \frac{1}{11} - \cdots$$

— a fact he attributes to Newton.

This elegant discovery gave him one of his earliest successes and established him as a mathematician of the first rank.

One is naturally tempted to ask why, if Euler intends to use infinite products, he does not simply use sin x itself? In fact he does; as a postscript to this paper, he notes that

$$\text{(G)} \qquad \frac{\sin x}{x} = \prod_{n=1}^{\infty}\left(1 - \frac{x^2}{(n\pi)^2}\right)$$

and deduces more directly, $\zeta(2n)$ for $n = 1, 2, 3, 4, 5, 6$. (G), however, does not give the flexibility of (F) and clearly has no hope of yielding anything about $\zeta(2n+1)$. One might surmise that he first proved (G) and then the more general result (F).

Two objections were raised to this proof by Daniel Bernoulli. In the first place, one can't compute with infinite series in the same way that one does with polynomials, and in the second place, it is not evident that all the roots of $\sin x = \sin \alpha$ are real. Euler recognizes the second objection as being valid and proceeds to prove that, in fact, all the roots are real. As to the first objection, he rightfully insisted in 1740 that the method is as well founded as any other method and, moreover, it is based upon a principle of which adequate use had not been made. Indeed, it opened up the theory

of infinite products and partial fraction decomposition of transcendental functions and its importance goes far beyond the immediate application.

5. Connections with arithmetic. Having achieved his objective of evaluating $\zeta(2)$, Euler now turned to the arithmetic properties of $\zeta(s)$. In 1737 he communicated a paper entitled *Variae Observationes circa series infinitas.*

Here for the first time he proved the famous Euler product decomposition in the form

$$\zeta(s) = \frac{2^s \cdot 3^s \cdot 5^s \cdot 7^s \cdot 11^s \cdots}{(2^s - 1)(3^s - 1)(5^s - 1)(7^s - 1)(11^s - 1)\cdots}.$$

One of his theorems is the statement that

$$\sum_p \frac{1}{p} \sim \log \sum_n \frac{1}{n},$$

where the left hand side is summed over all p. Nowadays we would insist on writing that as $x \to \infty$

$$\sum_{p \leqq x} \frac{1}{p} \sim \log \sum_{n \leqq x} \frac{1}{n}.$$

He also "proved" that if $n = p_1^{r_1} \cdots p_l^{r_l}$ and $\lambda(n) = (-1)^{r_1+r_2+\cdots+r_l}$, then

$$\sum_{n=1}^{\infty} \frac{\lambda(n)}{n} = 0$$

and the corresponding fact for $\mu(u)$ (what is now called the Möbius function) is stated in his "Introductio". Regretfully, we have put the word "proved" in quotation marks since the justification of this statement is as deep a result as the prime number theorem itself.

6. Return to $\zeta(s)$. He returned to $\zeta(s)$ in 1740 in a paper entitled *De Seribus Quibusdam Considerationes.* In this he developed the partial fraction decomposition of various functions. In particular, he proved that

$$\frac{\pi \cos[(b-a)/2n]\pi}{n \sin[(b+a)/2n]\pi - n \sin[(b-a)/2n]\pi} = \frac{1}{a} + \sum_{k=1}^{\infty} \frac{2b}{(2k-1)^2 n^2 - b^2} - \frac{2a}{(2kn)^2 - a^2}.$$

By specializing, once again he deduced the values of $\zeta(2), \zeta(4), \cdots$.

In the meantime what has happened to $\zeta(3)$? In this same paper he computed approximate values of $\zeta(2n+1)$ for $n = 1,2,3,4,5$ to which he added the known values of $\zeta(2n)$. He wrote these in the form

$$\zeta(n) = N\pi^n.$$

He says that if n is even, then N is rational, while if N is odd then he conjectures that N is a function of $\log 2$.

There is now a slight digression.

Apparently to respond to the earlier criticism concerning his first proof, Euler published a paper in an obscure journal, "Literary Journal of Germany, Switzerland and the North (The Hague)", entitled *Démonstration de la somme de la suite* $1 + \frac{1}{4} + \frac{1}{9} + \cdots$. Here he derived once again the formula for $\zeta(2)$.

Since this method is elementary, and is not generally known, and can be given in an elementary course, we present it in detail. We have

$$\tfrac{1}{2}(\arcsin x)^2 = \int_0^x \frac{\arcsin t}{\sqrt{1-t^2}}\,dt.$$

If we expand $(1-u^2)^{-\frac{1}{2}}$ by the binomial theorem and integrate termwise, we get

$$\arcsin t = \int_0^t \frac{du}{\sqrt{1-u^2}} = t + \sum_{n=1}^{\infty} \frac{1\cdot 3\cdots(2n-1)}{2\cdot 4\cdots 2n}\,\frac{t^{2n+1}}{2n+1}.$$

It follows that

$$\tfrac{1}{2}(\arcsin x)^2 = \int_0^x \frac{tdt}{\sqrt{1-t^2}} + \sum_{n=1}^{\infty} \frac{1\cdot 3\cdots(2n-1)}{2\cdot 4\cdots 2n}\,\frac{1}{2n+1}\int_0^x \frac{t^{2n+1}}{\sqrt{1-t^2}}\,dt.$$

Euler then writes, "Since the first term is integrable all the others will also be since the integration of each term reduces to that of the preceding. One can see this clearly if we reflect that in general

$$\int_0^x \frac{t^{n+2}}{\sqrt{1-t^2}}\,dt = \frac{n+1}{n+2}\int_0^x \frac{t^n}{\sqrt{1-t^2}}\,dt - \frac{x^{n+1}}{n+2}\sqrt{1-x^2}."$$

(Apparently the favorite phrases of mathematicians, "clearly etc.," are not of recent origin!) In fact, it takes a few steps to see this "clearly." Let

$$I_n(x) = \int_0^x \frac{t^{n+2}}{\sqrt{1-t^2}} = \int_0^x \frac{t^{n+1}tdt}{\sqrt{1-t^2}}.$$

Integration by parts gives $I_n(x) = -x^{n+1}\sqrt{1-x^2} + (n+1)\int_0^x t^n(\sqrt{1-t^2})dt$. Multiplying the integrand by $1 = \sqrt{1-t^2}/\sqrt{1-t^2}$, and splitting into 2 parts gives

$$I_n(x) = -x^{n+1}\sqrt{1-x^2} + (n+1)I_{n-2}(x) - (n+1)I_n(x),$$

and the result follows.

Thus

$$\text{(H)} \qquad \int_0^1 \frac{t^{2n+1}}{\sqrt{1-t^2}} = \frac{2n}{2n+1}\int_0^1 \frac{t^{2n-1}}{\sqrt{1-t^2}}$$

and as $\int_0^1 tdt/\sqrt{1-t^2} = 1$, we conclude that

$$\int_0^1 \frac{t^{2n+1}}{\sqrt{1-t^2}} = \frac{2n(2n-2)\cdots 2}{(2n+1)(2n-1)\cdots 3}.$$

Therefore $\pi^2/8 = \frac{1}{2}(\text{arc sin } 1)^2 = \sum_{n=0}^{\infty} 1/(2n+1)^2$, which as we know from above is equivalent to $\zeta(2) = \pi^2/6$. The same result may be obtained by first showing that $\frac{1}{2}(\text{arc sin } x)^2$ satisfies the differential equation

$$(1-x^2)y'' - xy' = 1,$$

then using undetermined coefficients to derive the series for $\frac{1}{2}(\text{arc sin } x)^2$, and finally integrating termwise to get $\frac{1}{6}(\text{arc sin } x)^3$, after using the above result (H). The reader will find it interesting to carry out these steps. The method gives $\zeta(2) = \pi^2/6$ directly. Euler concludes with the remark that despite repeated efforts, he was unable to use this technique to find $\zeta(2n)$ for $n \geqq 2$. The reader will note that we have glossed over the mild difficulties associated with the point $x = 1$.

Since the time of Euler, there have been many proofs giving the value of $\zeta(2n)$. The interested reader is urged to consult K. Knopp's book on "Infinite Series."

7. The functional equation and $\zeta(3)$. In the middle of the paper *De Seriebus*$\cdots$ referred to above, Euler began a highly interesting new development. There he states that

$$1 - 3 + 5 - 7 + \cdots = 0$$

$$1 - 3^3 + 5^3 - 7^3 + \cdots = 0,$$

etc., whereas,

$$1 - \tfrac{1}{2} + \tfrac{1}{3} - \tfrac{1}{4} + \cdots = \log 2,$$

$$1 - 2 + 3 - 4 + \cdots = \tfrac{1}{4},$$

$$1 - 2^3 + 3^3 - 4^3 + \cdots = -\tfrac{1}{8},$$

$$1 - 2^5 + 3^5 - 4^5 + \cdots = \tfrac{1}{4},$$

$$1 - 2^7 + 3^7 - 4^7 + \cdots = -17/16$$

On the other hand,

$$1 - 2^2 + 3^2 - 4^2 + \cdots = 0,$$

$$1 - 2^4 + 3^4 - 4^4 + \cdots = 0,$$

$$1 - 2^6 + 3^6 - 4^6 + \cdots = 0.$$

Where do these come from? They are derived as follows. Let

$$f(x) = 1 + x + x^2 + \cdots + x^n + \cdots = 1/(1-x) \text{ if } |x| < 1.$$

Euler has no reluctance to put $x = -1$; then $1 - 1 + 1 - 1 + \cdots = \frac{1}{2}$.

To $f(x)$ apply the operator $x(d/dx)$. Then

$$x \frac{d}{dx} f(x) = x + 2x^2 + 3x^3 + \cdots = \frac{x}{(1-x)^2} ;$$

putting $x = -1$, gives $1 - 2 + 3 - 4 + \cdots = \frac{1}{4}$.

Apply the operator again:

$$x + 2^2x^2 + 3^2x^3 + \cdots = \frac{x(1+x)}{(1-x)^2} .$$

Putting $x = -1$, gives $1 - 2^2 + 3^2 - \cdots = 0$.

As the series converges at each stage of this process for $|x| < 1$, we have Euler anticipating "Abel summability" by some 75 years. Then Euler notes that

$$1 - 2 + 3 - 4 + \cdots = \frac{1}{4} = \frac{2 \cdot 1}{\pi^2}\left(1 + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right) ,$$

$$1 - 2^3 + 3^3 - 4^3 + \cdots = -\frac{1}{8} = \frac{-2 \cdot 3!}{\pi^4}\left(1 + \frac{1}{3^4} + \frac{1}{5^4} + \cdots\right) ,$$

$$1 - 2^5 + 3^5 - 4^5 + \cdots = -\frac{1}{4} = \frac{2 \cdot 5!}{\pi^6}\left(1 + \frac{1}{3^6} + \frac{1}{5^6} + \cdots\right) ,$$

$$1 - 2^7 + 3^7 - 4^7 + \cdots = -\frac{17}{16} = \frac{-2 \cdot 7!}{\pi^8}\left(1 + \frac{1}{3^8} + \frac{1}{5^8} + \cdots\right) ,$$

as can be verified by an easy computation using the values of $\theta(2n)$:

As in Section 1, let $\theta(s) = \sum_{n=0}^{\infty} 1/(2n+1)^s$ and $\phi(s) = \sum_{n=1}^{\infty}(-1)^{n-1}/n^s$. These relations can be rephrased as

$$\theta(1-2n) = \frac{(-1)^{n-1} 2 \cdot (2n-1)!}{\pi^{2n}} \phi(2n) \quad (n = 1,2,3,4),$$

where, of course, $\theta(m)$, $(m = 0, \pm 1, \pm 2, \cdots)$ is to be understood as

$$\lim_{x \to 1^-} \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n^m} .$$

Although he does not explicitly say so, one gets the impression that Euler is trying energetically to develop a technique for evaluating $\zeta(3)$, and this impression is partially confirmed later, as we shall see.

In 1749 he gave a paper to the Berlin Academy entitled *Remarques sur un beau rapport entre les séries des puissances tant directes que réciproques.*

This time he considers

$$\phi(s) = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^s}$$

alone and notes the following relations:

$$\frac{1-2+3-4+5-6+\cdots}{1-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+\frac{1}{5^2}-\frac{1}{6^2}+\cdots} = +\frac{1\cdot(2^2-1)}{(2-1)\pi^2},$$

$$\frac{1^2-2^2+3^2-4^2+5^2-6^2+\cdots}{1-\frac{1}{2^3}+\frac{1}{3^3}-\frac{1}{4^3}+\frac{1}{5^3}-\frac{1}{6^3}+\cdots} = 0,$$

$$\frac{1^3-2^3+3^3-4^3+5^3-6^3+\cdots}{1-\frac{1}{2^4}+\frac{1}{3^4}-\frac{1}{4^4}+\frac{1}{5^4}-\frac{1}{6^4}+\cdots} = -\frac{1\cdot 2\cdot 3(2^4-1)}{(2^3-1)\pi^4},$$

$$\frac{1^4-2^4+3^4-4^4+5^4-6^4+\cdots}{1-\frac{1}{2^5}+\frac{1}{3^5}-\frac{1}{4^5}+\frac{1}{5^5}-\frac{1}{6^5}+\cdots} = 0,$$

or if $n \geqq 2$,

$$\text{(J)} \qquad \frac{\phi(1-n)}{\phi(n)} = \begin{cases} \dfrac{(-1)^{(n/2)+1}(2^n-1)(n-1)!}{(2^{n-1}-1)\pi^n} & \text{if } n \text{ is even,} \\ 0 \text{ if } n \text{ is odd.} \end{cases}$$

These relations are listed for $n = 2, 3, \cdots 10$. On the other hand, if $n = 1$, we see that

$$\frac{1-1+1-1+\cdots}{1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots} = \frac{1}{2\ln 2},$$

"whose connection with the others is entirely hidden." (J) is now rewritten in the form

$$\frac{\phi(1-n)}{\phi(n)} = \frac{-(n-1)!(2^n-1)}{(2^{n-1}-1)\pi^n}\cos\frac{\pi n}{2},$$

and Euler says "I shall hazard the following conjecture:

$$\text{(K)} \qquad \frac{\phi(1-s)}{\phi(s)} = \frac{-\Gamma(s)(2^s-1)\cos \pi s/2}{(2^{s-1}-1)\pi^s}$$

is true for all s". Isn't this derivation beautiful!?

Now taking the limit of the right hand side as $s \to 1$ gives exactly $1/2\ln 2$! Euler continues: "The validity of our conjecture for $s = 1$ (which case first appeared to deviate from the others) is already a strong justification[1] of the truth of our conjec-

[1] Although Euler uses the word "preuve," the original meaning in English (and presumably also in French) conveys the idea of testing an assumption or statement rather than proving in our sense. Compare, for example, the expression "the exception that 'proves' the rule."

ture since it appears unlikely that a false assumption could have upheld the truth of this case. We can therefore regard our conjecture as being solidly based but I shall give other justifications which are equally convincing."

He then checks the formula for $s = \frac{1}{2}, \frac{3}{2}$, and in general $s = (2k + 1)/2$.

We have seen in Chapter 1, that

$$\phi(s) = (1 - 2^{1-s})\zeta(s),$$

which leads at once from (K) to

$$\zeta(1 - s) = \pi^{-s}2^{1-s}\Gamma(s) \cos \frac{\pi s}{2} \zeta(s),$$

and this is the famous functional equation[2]. It was proved by Riemann in 1859.

It should be noted that Euler could not have used $\zeta(s)$ itself since

$$\lim_{x \to 1^-} \sum_{n=1}^{\infty} n^k x^n$$

does not exist for $k = 0, 1, 2, \cdots$ and therefore he could not have attached a meaning to

$$\sum_{n=1}^{\infty} n^{-(1-s)}$$

for $s = 2, 3, \cdots$.

On the other hand, it can be shown that the series

$$\phi(s) = \sum_{n=1}^{\infty} (-1)^{n+1} n^{-s} = (1 - 2^{1-s})\zeta(s)$$

converges for $s > 0$ (in fact if $s = \sigma + it$, for $\sigma > 0$), but as the pole of $\zeta(s)$ at $s = 1$ has been removed by the factor $(1 - 2^{1-s})$, there remains nothing in the nature of $\phi(s)$ to account for this limitation, and it turns out that

$$\sum_{n=1}^{\infty} (-1)^{n+1} n^{-s}$$

is Abel summable for every value of s.

One is naturally tempted to ask whether Riemann could have seen Euler's work. There is no evidence that he had[3].

Euler continues:

"As far as the sum of the reciprocals of powers (i.e., $\sum_{n=1}^{\infty} (-1)^{n+1}/n^k$) is concerned, I have already observed that their sum can be assigned a value only when k

---

[2] Since completing this article the author has found that E. Landau has given a rigorous proof of the functional equation in the form (K). See *Bibliotheca Mathematica*, vol. 7 (1906–1907) pp. 69–79.

[3] *Added in proof.* A. Weil remarks that the external evidence supports strongly the view that Riemann was very familiar with Euler's contributions.

is even and that when k is odd, all my efforts have been useless up to now."

Euler now observes as follows: If $s = 2\lambda + 1$, then

$$\phi(2\lambda + 1) = -\frac{(2^{2\lambda} - 1)\pi^{2\lambda+1}}{\Gamma(2\lambda + 1)(2^{2\lambda+1} - 1)} \frac{\phi(-2\lambda)}{\cos((2\lambda + 1)\pi/2)},$$

and $\phi(-2\lambda)$ as well as $\cos((2\lambda + 1)\pi/2)$ vanish if λ is an integer. Taking the limit as $\lambda \to m$ a positive integer with the help of l'Hospital's rule, we get

$$\text{(L)} \qquad \phi(2m + 1) = +\frac{2(2^{2m} - 1)\pi^{2m}}{(2m)!(2^{2m+1} - 1)} \frac{\sum_{n=1}^{\infty}(-1)^{n+1} n^{2m} \log n}{\cos \pi m}.$$

"It is necessary therefore to find the value of these sums

$$\sum_{n=1}^{\infty} (-1)^{n+1} n^{2m} \log n.$$

But this research is probably more difficult than the one we have in mind (meaning $\phi(2m + 1)$) and I perceive no method whatsoever which could lead us to the proposed objective."

He returned to the question for what appears to be the last time in 1772 in a paper entitled *Exercitationes Analyticae*. Through a striking and elaborate scheme, he proved that

$$1 + \frac{1}{3^3} + \frac{1}{5^3} + \cdots = \frac{\pi^2}{4} \log 2 + 2\int_0^{\pi/2} x \log \sin x dx.$$

Here is a sketch of the proof which invokes the extreme virtuosity of a master:

We know from (L) that

$$1 + \frac{1}{3^3} + \frac{1}{5^3} + \cdots = \frac{\pi^2}{2} Z,$$

where

$$Z = \sum_{n=2}^{\infty} (-1)^n n^2 \log n.$$

This follows from (L) as well as the relations cited in Section 1. Of course we continue to understand that if $\sum_{n=1}^{\infty} a_n$ does not converge but $\sum_{n=1}^{\infty} a_n x^n$ converges for $|x| < 1$, then $\sum_{n=1}^{\infty} a_n$ is defined by

$$\lim_{x \to 1^-} \sum_{n=1}^{\infty} a_n x^n, \text{ if this limit exists.}$$

Euler then shows that

$$Z = \sum_{n=1}^{\infty} n^2 \log \frac{(2n)^2}{(2n - 1)(2n + 1)} - \sum_{n=1}^{\infty} n(n + 1) \log \frac{(2n + 1)^2}{(2n)(2n + 2)}.$$

The expansion of the logarithm is carried out and the series rearranged. Letting

$\lambda(s) = \sum_{n=1}^{\infty} 1/(n(n+1))^s$, then

$$Z = \frac{1}{2 \cdot 2^2} + \sum_{n=2}^{\infty} \frac{1}{n2^{2n}} (\zeta(2n-2) + (-1)^n \lambda(n-1)).$$

$\lambda(n)$ is then expressed in terms of $\zeta(2k)\,(k = 1, 2, \cdots, n)$, and if

$$S(n) = \frac{1}{n2^{2n}} + \sum_{k=1}^{\infty} \frac{(n+k-1)(n+k)\cdots(n+2k-2)}{k!(n+k)2^{2n+2k}},$$

then

$$Z = -\frac{1}{8} + S(1) + 2\sum_{n=1}^{\infty} \zeta(2n)\left(\frac{1}{(2n+2)2^{n+2}} - S(2n+1)\right).$$

He now finds the sum $S(n)$ by showing that

$$S_x(n) = \frac{x^n}{n} + \sum_{k=1}^{\infty} \frac{(n+k-1)(n+k)\cdots(n+2k-2)x^{n+k}}{k!(n+k)}$$

satisfies a difference differential equation and that

$$S_x(1) = \frac{1 + 2x - \sqrt{1-4x}}{4}.$$

This is to be evaluated when $x = \frac{1}{4}$. The result of these intricate details is that

$$S(2n+1) - \frac{1}{(2n+2)2^{2n+2}} = \frac{1}{(2n+1)(2n+2)2^{2n+1}}.$$

$$Z = \frac{1}{2^2} - \sum_{n=1}^{\infty} \frac{\zeta(2n)}{(2n+1)(2n+2)2^{2n}}.$$

We know that $\zeta(2n) = \alpha_{2n}\pi^{2n}$, where α_{2n} is explicitly determined in terms of the Bernoulli numbers.

If then

$$f(x) = x^2 \sum_{n=1}^{\infty} \frac{\alpha_{2n}x^{2n}}{(2n+1)(2n+2)},$$

then by twice differentiating $f(x)$, we see that it satisfies a differential equation which can be solved in view of the fact that we can evaluate the generating function

$$\sum_{n=1}^{\infty} \alpha_{2n}x^{2n}.$$

Is not this derivation breathtaking, especially in the light of the fact that Euler was now blind and these calculations were performed mentally!

8. Conclusion. So end the main contributions of Euler to the zeta function. He

did, however, write a brief paper on the function $\sum_{n=1}^{\infty} x^n/n^2$ toward the end of his life (1779), which was published posthumously. We have given only the highlights of his work on $\zeta(s)$. Scattered throughout his papers on analysis and in his correspondence with Goldbach and the Bernoulli's are many results which are related to the problem.

While he did not succeed in every objective he set himself, his triumphs stand like a grand fresco — a monument to his extraordinary imagination and sense of beauty and harmony.

Acknowledgements

In addition to the original papers themselves, the author has found the following sources especially helpful:

1. The preface of Vol. 16 of series 1 of Euler's Collected Works is an article entitled "Übersicht über die Bände 14, 5.16, 16," by Georg Faber. Faber gives a summary of the contents, classified by topics.
2. The article by Paul Stäckel, "Eine vergessene Abhandlung Eulers." This first appeared in the now defunct journal Bibliotheca Mathematica, 83 (1907–1908) 37–54. In this, Stäckel discusses the article "Démonstration de la somme..." and gives numerous interesting historical facts. It is reprinted in Euler's Collected Works, Vol. 14, pp. 156–176.
3. Correspondence between Euler and Goldbach published by Deutsche Akademie der Wissenschaften and edited by A. Juskevic and E. Winter. The editors' comments on the letters were very helpful.
4. The paper of Landau referred to in the footnote.
5. The referee kindly suggested stylistic changes and pointed out some errors.

Department of Mathematics, Pennsylvania State University, University Park, PA 16802.

THE WILLIAM LOWELL PUTNAM MATHEMATICAL COMPETITION

A. P. HILLMAN

The following results of the thirty-fourth William Lowell Putnam Mathematical Competition, held on December 1, 1973, have been determined in accordance with the regulations governing the Competition. This competition is supported by the William Lowell Putnam Prize Fund for the Promotion of Scholarship left by Mrs. Putnam in memory of her husband and is held under the auspices of the Mathematical Association of America.

The first prize, five hundred dollars, is awarded to the Department of Mathematics of the **California Institute of Technology,** Pasadena, California. The members of the team were Arthur L. Rubin, James B. Shearer, and Michael F. Yoder; to each of these a prize of one hundred dollars is awarded.

The second prize, four hundred dollars, is awarded to the Department of Mathematics of the **University of British Columbia,** Vancouver, British Columbia. The

members of the team were Mark Latham, J. Bruce Neilson, and John L. Spouge; to each of these a prize of seventy-five dollars is awarded.

The third prize, three hundred dollars, is awarded to the Department of Mathematics of the **University of Chicago**, Chicago, Illinois. The members of the team were Franklin T. Adams, James E. McClure, and David A. Vogan; to each of these a prize of fifty dollars is awarded.

The fourth prize, two hundred dollars, is awarded to the Department of Mathematics of **Harvard University**, Cambridge, Massachusetts. The members of the team were Seth I. Breidbart, David C. Garlock, and Alan S. Grenadir; to each of these a prize of fifty dollars is awarded.

The fifth prize, one hundred dollars, is awarded to the Department of Mathematics of **Princeton University**, Princeton, New Jersey. The members of the team were Angelos J. Tsirimokos, Loring W. Tu, and Joseph L. Tupper III; to each of these a prize of fifty dollars is awarded.

The five persons ranking highest in the examination, named in alphabetical order, are **David J. Anick**, Massachusetts Institute of Technology; **Peter G. de Buda,** University of Toronto; **Matthew L. Ginsberg**, Wesleyan University; **Arthur L. Rubin,** California Institute of Technology; and **Angelos J. Tsirimokos,** Princeton University. Each of these has been designated as a Putnam Fellow by the Mathematical Association of America and is awarded a prize of two hundred and fifty dollars.

The next five highest ranking individuals, named in alphabetical order, are *Franklin T. Adams*, University of Chicago; *Seth I. Breidbart*, Harvard University; *Christopher L. Henley*, California Institute of Technology; *Gary A. Miller*, University of Chicago; and *John L. Spouge*, University of British Columbia. To each of these a prize of one hundred dollars is awarded.

The following teams, named in alphabetical order, won honorable mention: *Massachusetts Institute of Technology*, the members of the team were David J. Anick, Glenn A. Iba, and Frank E. Morgan; *Oberlin College*, the members of the team were Peter F. Garst, James A. Paget, and Craig D. Seeley; *Purdue University*, the members of the team were Neal E. Brand, Leslie P. Chew, and Paul M. Farmwald; *Swarthmore College*, the members of the team were Leslie Hogben, David S. Shucker, and Kin O. Tam; and *University of Waterloo*, the members of the team were Stephen C. Locke, John M. MacDonald, and Jan F. Verster.

Honorable mention is given to the following thirty-one individuals, named in alphabetical order: *Mark D. Anderson*, Lehigh University; *Anders E. Carlsson*, Harvard University; *Julius P. Collins*, Polytechnic Institute of New York; *Frederic G. Commoner*, Harvard University; *Mark S. Fischler*, Massachusetts Institute of Technology; *Joe Y. Halpern*, University of Toronto; *Steven H. Herman*, Rensselaer Polytechnic Institute; *Jesse O. Hobbs*, Michigan State University; *David S. Jerison*, Harvard University; *Kevin J. Karplus*, Michigan State University; *Sheldon H. Katz*, Massachusetts Institute of Technology; *Albert S. Kyle*, Davidson College; *Mark Latham*, University of British Columbia; *Paul S. Lemke*, Rensselaer Polytechnic

Institute; *Frank M. Liang*, California Institute of Technology; *James M. Lyon*, Princeton University; *Armando Manduca*, University of Connecticut; *Ross E. Millikan*, University of California-Berkeley; *Ian L. Morrison*, University of Toronto; *James A. Paget*, Oberlin College; *Robert S. Rumely*, Grinnell College; *James B. Saxe*, Union College; *Edward A. Severn*, University of Waterloo; *James B. Shearer*, California Institute of Technology; *Stephen C. Spriggs*, University of Maryland; *Jacob A. Sturm*, Columbia University; *Stephen T. Tappel*, California Institute of Technology; *Jeffrey D. Tiedeman*, Reed College; *David A. Vogan*, University of Chicago; *Edward L. Wimmers*, Massachusetts Institute of Technology; *Michael F. Yoder*, California Institute of Technology.

The other individuals who were ranked in the top one hundred, listed in alphabetical order of their schools are: Acadia University, *Michael A. Stockdale*; Amherst College, *Mark I. Heiligman*; Bishop's University, *Neil J. Redding*; University of British Columbia, *J. Bruce Neilson*; Brown University, *Russell B. Campbell*, *William R. Monach*, *Hisao Nakanishi*; California Institute of Technology, *David S. Dummit*, *Douglas B. Tyler*; California Polytechnic State University, *Martin C. Weiss*; University of Chicago, *Thomas P. Branson*, *James E. McClure*, *Andrew M. McLennan*; Clarkson College of Technology, *Amos R. Newcombe*; Columbia University, *Meir Shinnar*; Dartmouth College, *David S. Pearson*, *Daniel J. Velleman*; Denison Univresity, *Clifford T. Thomas*; Drexel University, *Thomas J. Leidigh*; Grinnell College, *James P. Fernow*; Harvard University, *Jeffrey M. Dielle*, *David C. Garlock*, *Alan S. Grenadir*, *David Harbater*, *George P. Mair*, *Philip E. Moore*, *Lloyd N. Trefethen*; Indiana University, *John H. Boyd III*; University of Manitoba, *Terry H. Andres*, *Thomas G. Kucera*; Massachusetts Institute of Technology, *Frank E. Morgan*; McMaster University, *Donald T. Kersey*, *John R. Lawson*; Michigan State University, *David M. Bowen*, *Mark Hersey*, *Mark P. Merriman*; University of Michigan, *Gene D. Cooperman*, *Glenn Herteg*; University of Minnesota-Minneapolis, *Lowell D. Palecek*; City College of New York, *David A. Spear*; Northwestern University, *Gary M. Lieberman*; University of Pennsylvania, *James B. Orlin*; Pomona College, *Charles M. Grinstead*; Princeton University, *Karl C. Rubin*, *Roger S. Schlafly*; Purdue University-W. Lafayette, *Leslie P. Chew*; Queen's University, *Harold S. Wilson*; Reed College, *Daniel W. Bump*; Rice University, *John W. Myre*, *John F. Rudin*; Rose-Hulman Institute of Technology, *Leo H. Ringwald*; Stanford University, *Bruce A. Fast*, *John T. Robinson*; Swarthmore College, *Leslie Hogben*, *David S. Schucker*, *David H. Vanderbilt*; University of Waterloo, *Stephen C. Locke*, *Peter F. Schneider*; Wesleyan University, *Scott E. Brodie*; University of Wisconsin-Milwaukee, *Robert H. Marheine*; Yale University, *Robert C. Weissler*; Yeshiva College, *Joel Gross*.

Two thousand and fifty-three students from three hundred and sixty-two colleges and universities in the United States and Canada participated in the examination on December 1, 1973.

The Questions Committee, consisting of Nathan S. Mendelsohn (chairman),

Donald J. Newman, and J. Ian Richards prepared the problems (listed below) for the competition.

PROBLEMS, PART A

A–1. (a) Let ABC be any triangle. Let X, Y, Z be points on the sides BC, CA, AB respectively. Suppose the distances $\overline{BX} \leqq \overline{XC}$, $\overline{CY} \leqq \overline{YA}$, $\overline{AZ} \leqq \overline{ZB}$ (see Figure 1). Show that the area of the triangle XYZ is $\geqq (1/4)$ (area of triangle ABC).

(b) Let ABC be any triangle, and let X, Y, Z be points on the sides BC, CA, AB respectively (but without any assumption about the ratios of the distances $\overline{BX}/\overline{XC}$, etc.; see Figures 1 and 2). Using (a) or by any other method, show: One of the three corner triangles AZY, BXZ, CYX has an area $\leqq$ area of triangle XYZ.

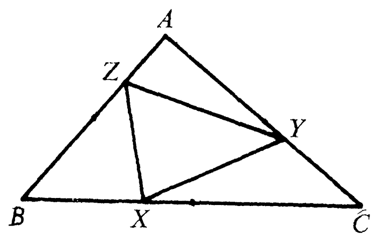


FIG. 1

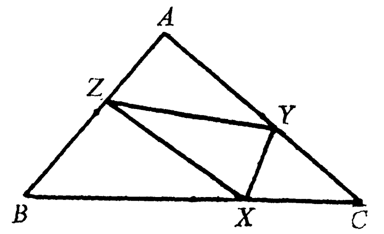


FIG. 2

A–2. Consider an infinite series whose nth term is $\pm (1/n)$, the $\pm$ signs being determined according to a pattern that repeats periodically in blocks of eight. [There are 2^8 possible patterns of which two examples are:

$$+ + - - - - + +,$$

$$+ - - - + - - - .$$

The first example would generate the series

$$1 + (1/2) - (1/3) - (1/4) - (1/5) - (1/6) + (1/7) + (1/8)$$

$$+ (1/9) + (1/10) - (1/11) - (1/12) - \cdots .]$$

(a) Show that a sufficient condition for the series to be conditionally convergent is that there be four "+" signs and four "−" signs in the block of eight.

(b) Is this sufficient condition also necessary?
[Here "convergent" means "convergent to a finite limit."]

A–3. Let n be a fixed positive integer and let $b(n)$ be the minimum value of

$$k + \frac{n}{k}$$

as k is allowed to range through all positive integers. Prove that $b(n)$ and $\sqrt{4n+1}$ have the same integer part. [The "integer part" of a real number is the greatest integer which does not exceed it, e.g. for π it is 3, for $\sqrt{21}$ it is 4, for 5 it is 5, etc.]

A-4. How many zeros does the function $f(x) = 2^x - 1 - x^2$ have on the real line? [By a "zero" of a function f, we mean a value x_0 in the domain of f (here the set of all real numbers) such that $f(x_0) = 0$.]

A-5. A particle moves in 3-space according to the equations:

$$\frac{dx}{dt} = yz, \quad \frac{dy}{dt} = zx, \quad \frac{dz}{dt} = xy.$$

[Here $x(t)$, $y(t)$, $z(t)$ are real-valued functions of the real variable t.] Show that:

(a) If two of $x(0)$, $y(0)$, $z(0)$ equal zero, then the particle never moves.

(b) If $x(0) = y(0) = 1$, $z(0) = 0$, then the solution is:

$$x = \sec t, \ y = \sec t, \ z = \tan t;$$

whereas if $x(0) = y(0) = 1$, $z(0) = -1$, then

$$x = 1/(t+1), y = 1/(t+1), z = -1/(t+1).$$

(c) If at least two of the values $x(0)$, $y(0)$, $z(0)$ are different from zero, then either the particle moves to infinity at some finite time in the future, or it came from infinity at some finite time in the past. [A point (x, y, z) in 3-space "moves to infinity" if its distance from the origin approaches infinity.]

A-6. Prove that it is impossible for seven distinct straight lines to be situated in the euclidean plane so as to have at least six points where exactly three of these lines intersect and at least four points where exactly two of these lines intersect.

PROBLEMS, PART B

B-1. Let $a_1, a_2, \ldots, a_{2n+1}$ be a set of integers such that, if any one of them is removed, the remaining ones can be divided into two sets of n integers with equal sums. Prove $a_1 = a_2 = \cdots = a_{2n+1}$.

B-2. Let $z = x + iy$ be a complex number with x and y rational and with $|z| = 1$. Show that the number $|z^{2n} - 1|$ is rational for every integer n.

B-3. Consider an integer $p > 1$ with the property that the polynomial $x^2 - x + p$ takes prime values for all integers x in the range $0 \leqq x < p$. (Examples: $p = 5$ and $p = 41$ have this property.) Show that there is exactly one triple of integers a, b, c satisfying the conditions:

$$b^2 - 4ac = 1 - 4p,$$

$$0 < a \leqq c,$$

$$-a \leqq b < a.$$

B–4. (a) On $[0, 1]$, let f have a continuous derivative satisfying $0 < f'(x) \leqq 1$. Also suppose that $f(0) = 0$. Prove that

$$\left[\int_0^1 f(x)dx\right]^2 \geqq \int_0^1 [f(x)]^3 dx.$$

[Hint: Replace the inequality by one involving the inverse function to f.]

(b) Show an example in which equality occurs.

B–5. (a) Let z be a solution of the quadratic equation

$$az^2 + bz + c = 0$$

and let n be a positive integer. Show that z can be expressed as a rational function of z^n, a, b, c.

(b) Using (a) or by any other means, express x as a rational function of x^3 and $x + (1/x)$. (Display your answer explicitly in a clearly visible form.)

[By a rational function of several variables, we mean a quotient of polynomials in those variables, the polynomials having rational numbers as coefficients, and the denominator being not identically zero. Thus to obtain x as a rational function of $u = x^2$ and $v = x + (1/x)$, we could write $x = (u + 1)/v$.]

B–6. On the domain $0 \leqq \theta \leqq 2\pi$:

(a) Prove that $\sin^2\theta \cdot \sin(2\theta)$ takes its maximum at $\pi/3$ and $4\pi/3$ (and hence its minimum at $2\pi/3$ and $5\pi/3$).

(b) Show that

$$|\sin^2\theta\{\sin^3(2\theta) \cdot \sin^3(4\theta) \cdots \sin^3(2^{n-1}\theta)\} \sin(2^n\theta)|$$

takes its maximum at $\theta = \pi/3$. (The maximum may also be attained at other points.)

(c) Derive the inequality:

$$\sin^2\theta \cdot \sin^2(2\theta) \cdot \sin^2(4\theta) \cdots \sin^2(2^n\theta) \leqq (3/4)^n.$$

SOLUTIONS, PART A

A–1. (a) If X, Y, Z are at the midpoints of the sides, the area of $\triangle XYZ$ is one fourth of the area of $\triangle ABC$. Also, as long as $\overline{BX} \leqq \overline{XC}$, $\overline{CY} \leqq \overline{YA}$ and $\overline{AZ} \leqq \overline{ZB}$, moving one of X, Y, Z to the midpoint of its side, while leaving the other two fixed, does not increase the area of $\triangle XYZ$ since the altitude to the fixed base of $\triangle XYZ$ decreases or remains constant.

(b) Under the hypothesis of (a) the three corner triangles have no more than three fourths of the total area and so one of them must have smaller area than $\triangle XYZ$. All other cases are similar to the one in which $\overline{XC} < \overline{BX}$ and $\overline{CY} < \overline{YA}$. Then consideration of the altitudes to base XY shows that $\triangle CYX$ has smaller area than $\triangle XYZ$.

A–2. The ideas in both parts are similar and the answer in (b) is "Yes." Let u_n be the nth term $\pm 1/n$ and $S_n = u_1 + \cdots + u_n$. Since $u_n \to 0$ as $n \to \infty$, $\{S_n\}$ will converge if and only if $\{S_{8m}\}$ does. Using the facts that

$$\frac{1}{n} - \frac{1}{n+k} = \frac{k}{n(n+k)},$$

that $\Sigma(1/n^2)$ converges, and that $\Sigma(1/n)$ diverges, one shows that with four "+" signs and four "−" signs in each block, $\{S_{8m}\}$ converges as the term-by-term sum of four convergent sequences while an imbalance of signs makes $\{S_{8m}\}$ divergent as the sum of a convergent and a divergent sequence.

A–3. Let $c(n) = \sqrt{4n+1}$ and let $[x]$ denote the greatest integer in x; then we wish to show that $[b(n)] = [c(n)]$. Let $k(n)$ be a value of k that minimizes $k + (n/k)$. Then

$$b(n-1) \leqq k(n) + \{(n-1)/k(n)\} < k(n) + \{n/k(n)\} = b(n),$$

i.e., $b(n-1) < b(n)$. Let m be a positive integer. Then

$$b(m^2) = 2m, b(m^2+m) = 2m+1. \tag{I}$$

It follows from formulas (I) and the strictly increasing nature of $b(n)$ that

$$[b(n)] = 2m \text{ for } m^2 \leqq n < m^2+m,\ [b(n)] = 2m+1 \text{ for } m^2+m \leqq n < (m+1)^2. \tag{II}$$

On the other hand, $c(n)$ is also an increasing function and

$$c(m^2-1) = \sqrt{4m^2-3} < 2m,\ c(m^2) = \sqrt{4m^2+1} > 2m,\ c(m^2+m) = \sqrt{4m^2+4m+1} = 2m+1.$$

These facts show that (II) remains true when $[c(n)]$ is substituted for $[b(n)]$.

A–4. Three; at 0, 1, and some $x > 1$. The first two are clear and the other follows from $f(4) < 0$ and $f(5) > 0$ or from $f'(1) < 0$ while $f(x) \to +\infty$ as $x \to +\infty$. There are no more zeros since four zeros of f would imply a zero of f''' using an extension of Rolle's Theorem; but $f'''(x) = (\log 2)^3 2^x \neq 0$ for all x.

A–5. (a) If two of $x(0)$, $y(0)$, $z(0)$ vanish, then $x'(0) = y'(0) = z'(0) = 0$, and the uniqueness theorem applies (the equations are clearly "Lipschitz").

(b) Clear (this was intended as a hint for part (c)).

(c) Now write the equations in the symmetric form:

$$xx' = yy' = zz' = xyz.$$

Thus $x^2 - c_1 = y^2 - c_2 = z^2 - c_3$ with constant c_i. Say without loss of generality that $c_1 \geqq c_2 \geqq c_3$, and then set $c_3 = 0$. Thus $z^2 \leqq y^2 \leqq x^2$, and:

$$z^2 = x^2 - c_1 = y^2 - c_2, \; c_i \geqq 0;$$

$$\frac{dz}{dt} = \pm \sqrt{(z^2 + c_1)(z^2 + c_2)}.$$

Now let time t move in the direction which makes $|z|$ increase (this depends on the sign of z and on the $\pm$ sign in the square root).

For simplicity assume that $z(0) \geqq 0$, and that the sign on the square root is $+$; then let time move positively. Since

$$t = \int dz/\sqrt{(z^2 + c_1)(z^2 + c_2)}$$

and the z-integral converges, a finite amount of time suffices to push z out to infinity.

A–6. Any two distinct lines in the plane meet in at most one point. There are altogether $\binom{7}{2} = 21$ pairs of lines. A triple intersection accounts for 3 of these pairs of lines, and a simple intersection accounts for 1.

Finally, $6 \cdot 3 + 4 \cdot 1 = 22 > 21$.

SOLUTIONS, PART B

B–1. Since the sum of the $2n$ integers remaining is always even, no matter which of the a_i is taken away, all of the a_i must have the same parity. Now a similar argument shows that they are all congruent (mod 4); for the property held by the a_i is shared by the integers $a_i/2$ or $(a_i - 1)/2$ (depending on whether the a_i are all even or all odd). Continuing in this manner, all of the a_i are congruent (mod 2^k) for every k. This is possible for integers only if they are equal.

B–2. Let $z = e^{\theta i}$ and $z^n = w = u + iv$ (with u and v real). Then $|z^{2n} - 1| = |w^2 - 1| = [(u^2 - v^2 - 1)^2 + (2uv)^2]^{1/2} = 2|v|$, using $u^2 + v^2 = 1$. [$|z^{2n} - 1| = 2|\sin n\theta|$ is also easily shown geometrically using an isosceles triangle.] Hence it suffices to show that $v = \sin n\theta$ is rational when $x = \cos\theta$ and $y = \sin\theta$ are rational. For $n \geqq 0$, this follows from $(x + iy)^n = u + iv$ or by mathematical induction using the addition formulas for the sine and cosine. Then the case $n < 0$ follows using $\sin(-\alpha) = \sin\alpha$.

B–3. One triple (a, b, c) satisfying the conditions is $(1, -1, p)$; it remains to show that this is the only solution. Clearly b must be odd since $b^2 \equiv 1 \pmod 4$. Also $b^2 = (-b)^2$, so write $|b| = 2x - 1$. Then $b^2 - 4ac = 1 - 4p$ gives

$$x^2 - x + p = ac.$$

If $0 \leqq x < p$, the hypothesis tells us that ac is prime; then $0 < a \leqq c$ implies that $a = 1$, it follows from $-a \leqq b < a$ and the oddness of b that $b = -1$, and

$1-4p = b^2 - 4ac = 1 - 4c$ gives us $c = p$. Since $x = (|b| + 1)/2 \geqq 0$, it suffices to show that $x < p$. Since $|b| \leqq a \leqq c$, $b^2 - 4ac = 1 - 4p$, and $p \geqq 2$, one sees that $x < p$ using

$$3a^2 = 4a^2 - a^2 \leqq 4ac - b^2 = 4p - 1,$$
$$|b| \leqq a \leqq \sqrt{(4p-1)/3},$$
$$x = (|b| + 1)/2 < \sqrt{p/3} + (1/2) < p.$$

B–4. We give two solutions; the first does not use the hint and the second does. The following theorem and proof submitted by Professor J. G. Mauldon of Amherst College provides the first solution. (This is similar to the solutions provided by all but one solver of the problem.)

THEOREM. *If f is continuous on $[0,1]$, $f(0) = 0$, and $0 \leqq f'(x) \leqq 1$ on $(0,1)$, then*

$$\left[\int_0^1 f(x)dx\right]^2 > \int_0^1 [f(x)]^3 dx$$

unless, identically on $[0,1]$, either $f(x) = x$ or $f(x) = 0$.

Proof. Define $G(t) = 2\int_0^t f(x)dx - [f(t)]^2$ for $t \in [0,1]$. Then $G(0) = 0$ and $G'(t) = 2f(t)[1 - f'(t)] \geqq 0$, so that $G(t) \geqq 0$ and consequently $f(t)G(t) \geqq 0$.

Now define $F(t) = [\int_0^t f(x)dx]^2 - \int_0^t [f(x)]^3 dx$ for $t \in [0,1]$. Then $F(0) = 0$ and $F'(t) = f(t)G(t) \geqq 0$, so that $F(t) \geqq 0$ and in particular $F(1) \geqq 0$.

Equality is possible only if $f(t)G(t) = F'(t) = 0$ for all t, which implies that, for some K, $f = 0$ on $[0,K]$ and $G' = 0$, with $f > 0$, on $(K,1)$. We then have $f' = 1$ on $(K,1)$, which is admissible only if $K = 0$ or $K = 1$, since otherwise $f'(K)$ is simultaneously defined and undefined.

The unique answer to (b) is $f(x) = x$. The following is an outline of a proof of (a) using the hint. Let $f(1) = c$. The hypothesis implies that f has an inverse g with $g'(y) \geqq 1$ on $0 \leqq y \leqq c$. Let

$$A = \left[\int_0^1 f(x)dx\right]^2 \text{ and } B = \int_0^1 [f(x)]^3 dx.$$

Then

$$A = \left[\int_0^c yg'(y)dy\right]^2 = \int_0^c \int_0^c yg'(y)zg'(z)dzdy = 2\int_0^c \int_0^z yg'(y)zg'(z)dydz$$

using the symmetry of the integrand about the line $y = z$. Now $g'(y) \geqq 1$ implies

$$A \geqq \int_0^c zg'(z)\left[\int_0^z 2ydy\right]dz = \int_0^c z^3 g'(z)dz = B.$$

B–5. (a) Let $r = -b/a$ and $s = -c/a$. Let polynomials p_n and q_n in r and s be defined by the initial conditions $p_0 = 0$, $p_1 = 1$, $q_0 = 1$, and $q_1 = 0$ and the

recursion formulas $p_n = rp_{n-1} + sp_{n-2}$ and $q_n = rq_{n-1} + sq_{n-2}$ for $n > 1$. Using $z^n = rz^{n-1} + sz^{n-2}$ and mathematical induction, one proves that $z^n = p_n z + q_n$ and that all the coefficients in $p_n(r,s)$ are positive. Then multiplying numerator and denominator of the right hand side of $z = [z^n - q_n(-b/a, -c/a)]/p_n(-b/a, -c/a)$ by the proper power of a leads to $z = F(z^n, a, b, c)/G(a, b, c)$, where F and G are polynomials with integer coefficients. Since all the coefficients in $p_n(r,s)$ are positive, the same is true of $G(a,b,c)$. Therefore $G(a,b,c)$ is not identically zero and F/G is the desired rational function.

(b) Let $v = x + (1/x)$. Then $x^2 - vx + 1 = 0$. Using (a) with z replaced by x, one finds that $x^3 = p_3 x + q_3$ with $p_3 = v^2 - 1$ and $q_3 = -v$. Then

$$x = (x^3 - q_3)/p_3 = (x^3 + v)/(v^2 - 1).$$

B–6. (a) Simple calculus.

(b) By induction: The case $n = 1$ is just (a).

Now the ratio of the expression for $n+1$ to the expression for n is equal to:

$$\left|\sin^2 2^n\theta \cdot \sin 2^{n+1}\theta\right|.$$

Since $\theta = \pi/3$ gives $2^n\theta \equiv 2\pi/3$ or $4\pi/3 \pmod{2\pi}$, this ratio is maximized at $\theta = \pi/3$, and by induction, then, the whole expression is maximized.

(c) Set $\theta = \pi/3$, and observe that the expression in part (b) is then exactly equal to $(3/4)^{3n/2}$; its 2/3 power is thus equal to $(3/4)^n$. That is the maximum; in general the 2/3 power of the expression in (b) is $\leqq (3/4)^n$. To get from that to the expression in (c), we would increase the powers of the end factors $\sin\theta$ and $\sin 2^n\theta$; this can only decrease the product, since $|\sin\theta| \leqq 1$.

Department of Mathematics, University of New Mexico, Albuquerque, NM 87131.

THE DEFINITE INTEGRAL SYMBOL

MARTIN G. BEUMER

The title of this note is borrowed from Franklin M. Turrell who wrote about this subject in 1960 [**1**]. I should like to present here an anecdote on the same subject; to the best of my knowledge this anecdote was never published.

In the beginning of the 20th century there were frequent and fruitful contacts between the leading chemists in Western Europe (including the Scandinavian countries and, of course, Great Britain); I consider Read's description [**2**] of these contacts to be the best on record.

At that time mathematics (esp. calculus) began to penetrate the circles of the adepts of physical chemistry. One of the leading personalities in that field was Jacobus

Henricus van't Hoff (1852–1911), the Nobel prize winner for chemistry in 1901. Since 1889 van't Hoff was a member of the *Deutsche chemische Gesellschaft* (German Chemical Association). At an informal meetıng with fellow members he gave the following explanation concerning the origin of the integral-symbol:

The word is derived from *Integer* (whole, entire) and *Aal* (German word for: eel).

This anecdote was related to me some 25 years ago by Professor Ernst Cohen, one of van 't Hoff's pupils and collaborators; in the excellent biography of van 't Hoff, written by Cohen [3] it is not mentioned. Shortly after relating this anecdote to me Cohen was killed. He was one of many thousands of Jewish compatriots who suffered the same fate in Germany just before the end of world war II. Since then I have not been able to trace this anecdote from any other source.

References

1. Franklin M. Turrell, The definite integral symbol, this MONTHLY, 67 (1960) 656.
2. John Read, Humour and humanism in chemistry, London (G. Bell and Sons), 1947.
3. Ernst Cohen, J. H. van 't Hoff, Leipzig (Akad. Verlagsgesellschaft), 1912.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF TECHNOLOGY, JULIANALAAN 132, DELFT, THE NETHERLANDS.

QUERIES

EDITED BY A. C. ZITRONENBAUM

This Department welcomes queries from readers about mathematics at the collegiate level, such as sources for exposition of a particular topic from a special point of view, references to vageluy remembered articles, descriptions of special kinds of courses or teaching methods, and methods for constructing illustrative examples for exercises of particular kinds (questions on research topics should, in general, be addressed to the "Queries Department" of the Notices of the American Mathematical Society). Replies will be forwarded to the questioner and may also be edited into a composite answer for publication in this Department. Consequently all items submitted for consideration for possible publication should include the name and complete mailing address of the person who is to receive the reply. Queries and answers should be sent to A. C. Zitronenbaum, Department of Mathematics, Cornell University, Ithaca, NY 14853.

Reply to Query 7. Part (i) asks for a discussion of the lens formulae of Newton and Gauss. A relevant reference is: A. A. Blank, *A Note on Gaussian Optics*, Proceedings of the UICSM Conference "The Role of Applications in a Secondary School Mathematics Curriculum," 1963.

Part (iii) asks for discussion of the Buckingham pi theorem of dimensional analysis. Three references were supplied: G. Birkhoff, *Hydrodynamics*, Dover, 1950, Ch. 3. P. W. Bridgman, *Dimensional Analysis*, Yale, 1963, Ch. 4. J. C. Oxtoby, *American Physics Teacher*, 2 (1934), 85–90. The last is particularly suitable.

16. Arthur Marshall. Yaglom and Yaglom, *Challenging Mathematical Problems with Elementary Solutions*, Vol. II, Holden-Day, 1967; Problem 171 (page 40) gives as "Mertens' First Theorem":

$$\frac{\ln 2}{2} + \frac{\ln 3}{3} + \frac{\ln 5}{5} + \frac{\ln 7}{7} + \frac{\ln 11}{11} + \cdots + \frac{\ln p}{p} \sim \ln N,$$

where p is the greatest prime $\leqq N$. Can the exact Mertens reference be supplied?

MATHEMATICAL NOTES

EDITED BY DAVID ROSELLE

Material for this Department should be sent to David Roselle, Department of Mathematics, Virginia Polytechnic Institute, Blacksburg, VA 24061.

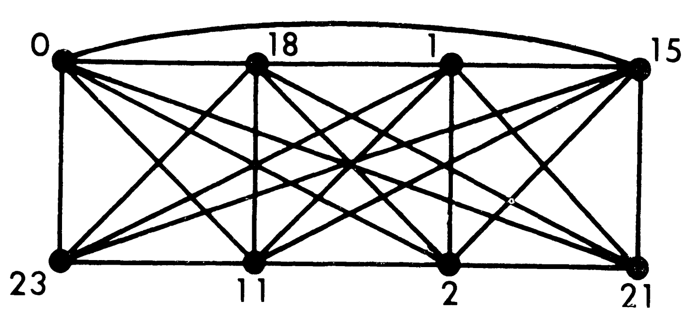
COMMENTS AND COMPLEMENTS

DAVID ROSELLE

This annual article is designed to provide our readers with an outlet for remarks on papers that have appeared in the Mathematical and Classroom Notes sections of the MONTHLY. Because of the nature of the material we seek to publish, it is inherent that there will be some duplication of results already in the literature; such duplication may not be undesirable depending upon accessibility, presentation, etc. In any case, we are happy to have readers' comments on articles we have published.

During the past year, we have received the following information.

Research problems. It is unusual to address this section in this annual article, but G. J. Simmons has asked that there be published a counter example to the conjecture given by Solomon W. Golomb in *The largest graceful subgraph of the complete graph* (this MONTHLY, 81 (1974), 499–501). The counterexample shows that $E(8) \geqq 23$ and Simmons has results which show $E(n) > [n^2/4] + n - 2$ for all $n \geqq 8$. The example is shown in the following graph:



Interested readers are referred to Simmons' article *Synch-sets; A variant of difference sets* (to appear in *Proceedings of the Fifth Southeastern Conference on Combinatorics, Graph Theory and Computing*, Boca Raton, Florida, Feb. 1974).

Calculus. Richard Johnsonbaugh has pointed out that the proof given by A. G. Fadell, *A proof of the chain rule for derivatives in n-space* (this MONTHLY, 80 (1973), 1134–5) also appears in Lang's *A Second Course in Calculus* (second edition), pages 527–8.

Analysis. The complete solution of the problem posed by D. P. Stanford in *Functions satisfying a mean value property at their zeros* (this MONTHLY, 80 (1973), 665–7) is given by Lawrence Zalcman in *Analyticity and the Pompeiu problem* (Arch. Rat. Mech. Anal. 47 (1972), 237–54). This reference was communicated by Professor Zalcman.

D. E. Sanderson has observed a more general version of the theorem given by Professors Wilansky and Heinen in their article *A theorem on set inclusion in metric spaces* (this MONTHLY, 80 (1973), 46–8):

THEOREM. *If $\partial A \subset B$ and B' is connected, then $A \subset B$ or $B' \subset A$.*

Proof. Since $\partial A \subset B$ and $C(A) = A \cup \partial A$, then $A \cap B' = A^\circ \cap B'$ and $A' \cap B' = C(A)' \cap B'$ and both are open in B'. Since B' is connected, either $A \cap B'$ or $A' \cap B'$ must be empty, i.e., either $A \subset B$ or $B' \subset A$.

Topology. Robert Herrmann points out that the theorem proved by R. J. St. Andre in *A classification of ultrafilters* (this MONTHLY, 78 (1971), 1126–7) also appears in P. M. Cohn's *Universal Algebra* as problem 16 on p. 199.

H. Gonshor has indicated a simpler proof of the theorem on p. 788 of G. P. Barker's article *Topological properties of the row echelon form* (this MONTHLY, 80 (1973), 787–9). From the fact stated on p. 787 that $\mu(A-B) < \varepsilon$ implies that A and B are pattern equivalent, it is immediate that $P(A)$ is a closed open set. Hence $C(A) \subset A(A) \subset P(A)$.

Professors Eric K. van Douwen, James T. Smith, and Elliott S. Wolk have independently pointed out that the content of C. Metelli's and L. Salce's note *A note on the well ordering of cardinals* (this MONTHLY, 81 (1974), 501–2) was given previously by C. S. Honig (*Proof of the well ordering of cardinal numbers*, Proc. Amer. Math. Soc., 5 (1954), 312).

Algebra. J. L. Brenner has commented again on the articles by Bartlow (79 (1972) 776–9) and Parkinson (80 (1973), 190–2) and indicates that it is possible to show that a necessary and sufficient condition that a permutation in A_n be expressible as a product of two permutations of period 2 is that it can be conjugate to its own inverse. Brenner also calls readers attention to treatments of determinants which avoid the introduction of permutations. Lang's text *Algebra* (p. 330) gives the sort of treatment to which Brenner refers.

General. Bruce Berndt has sent a very short proof of Lemma 2 of Tom M. Apostol's article *Another elementary proof of Euler's formula for* $\zeta(2n)$ (this MONTHLY, 80 (1973), 425–31). We recall first that

$$\cot y = \sum_{n=0}^{\infty} (-1)^n 2^{2n} B_{2n} \frac{y^{2n-1}}{(2n)!} \qquad (0 < y < \pi),$$

where B_n is the nth Bernoulli number. Thus, since $\cos y = \sin y \cot y$, we obtain

$$\frac{(-1)^r}{(2r)!} = \sum_{n=0}^{r} (-1)^r \frac{2^{2n} B_{2n}}{(2(r-n)+1)!\,(2n)!}$$

which is the same as the identity in question.

Albert A. Mullin writes to this author that the results in Hemminger's article (this MONTHLY, 73 (1966), 1001–2) can be generalized. The generalization involves replacing Hemminger's condition (2) by the weaker condition that the elements of the sequence be generalized relatively prime (for definition see A. A. Mullin, *Problems on the density of arithmetic sequences*, this MONTHLY, 79 (1972), 1118–9). It can then be shown that any generalized acceptable sequence can be used to establish the infinitude of primes.

Professor A. Zirahzadeh calls attention to the relation between his article (Mathematics Student, 30 (1962)) and the article *Generalized Pythagorean Theorem* (March, 1974, 262–4) by Beyer and Conant. The earlier paper deals with ordinary Euclidean volume while the latter deals with m-dimensional Lebesgue measure.

Leon Gerber has sent an annotated list of references related to the Beyer-Conant article with references as early as Tinseau, *Solution de quelques problèmes relatifs à la théorie des surfaces courbes et des courbes à double courbure*, Acad. Sci. Paris, Mem. Math. Phys., 9 (1780), 593–624. Henry S. Tropp refers readers to *The Pythagorean Proposition* by Elisha Scott Robinson (National Council of Teachers of Mathematics, 1940, republished in 1968) where a survey of the elementary aspects of this theorem appears. Finally, I. J. Good refers readers to elementary problem 1923 (this MONTHLY, 73 (1966), 891) both for the solution and the references given.

MATHEMATICS DEPARTMENT, VIRGINIA POLYTECHNIC INSTITUTE AND STATE UNIVERSITY, BLACKSBURG, VA 24061.

RESEARCH PROBLEMS

EDITED BY RICHARD GUY

In this Department the Monthly presents easily stated research problems dealing with notions ordinarily encountered in undergraduate mathematics. Each problem should be accompanied by relevant references (if any are known to the author) and by a brief description of known partial results. Manuscripts should be sent to Richard Guy, Department of Mathematics, Statistics, and Computing Science, The University of Calgary, Calgary, Alberta, Canada, T2N 1N4.

ON THE MAXIMUM OF THE SUM OF SQUARED DISTANCES UNDER A DIAMETER CONSTRAINT

H. S. WITSENHAUSEN

Let $x_1, \cdots, x_n$ be points in E^d chosen under the constraint $\| x_i - x_j \| \leqq 1$. We seek the maximum $M(d, n)$ of $\Sigma_{i,j} \| x_i - x_j \|^2$.

CONJECTURE: *The maximum is attained when the points are distributed as evenly as possible among the $d + 1$ vertices of a regular simplex of edge-length* 1.

The following theorem gives an upper bound which shows that the conjecture is correct at least when n is a multiple of $d + 1$. Otherwise a small interval of uncertainty is left between the conjectured value and the upper bound.

THEOREM. $M(d, n) \leqq n^2 d/(d + 1)$.

Proof. By translation one may assume $\Sigma_i x_i = 0$, i.e., the origin is the centroid of the n points. Then

$$S \equiv \sum_{i,j} \| x_i - x_j \|^2 = \sum_{i,j} \langle x_i - x_j, x_i - x_j \rangle = 2n \sum_i \| x_i \|^2.$$

The centroid is well known to have the property of minimizing $\Sigma_i \| x_i - y \|^2$ over y in E^d. Let c be the center of the unique sphere of minimum radius R, containing the n points. Then

$$S \leqq 2n \sum_i \| x_i - c \|^2 \leqq 2n \sum_i R^2 = 2n^2 R^2.$$

In 1901, Jung [2] established the following inequality between R and the diameter D, for sets in E^d

$$R \leqq \sqrt{\frac{d}{2(d + 1)}}\, D.$$

Since $D \leqq 1$ in the present case, the theorem is established.

Note that the theorem and conjecture carry over to $\Sigma_{i,j} \| x_i - x_j \|^p$ for $p \geqq 2$ because $\| x_i - x_j \| \leqq 1$ implies

$$\| x_i - x_j \|^p \leqq \| x_i - x_j \|^2$$

with equality when $\| x_i - x_j \| = 0$ or 1, as is the case for all distances in the arrangements considered above.

J. Seidel has pointed out that the diameter graph of the conjectured configuration is a Turán graph, it has the largest possible number of edges for maximum clique size $d + 1$. However, this has not led to a proof of the conjecture.

A completely analogous problem, with similar results but different methods, has been considered by J. B. Kruskal and the author [3], following a conjecture of W. Kruskal. The problem is to choose n unit vectors u_i in E^d subject to $\langle u_i, u_j \rangle \geqq 0$ so as to minimize $\| \Sigma u_i \|$. The conjectured optimal configuration is to distribute the vectors as evenly as possible among the d members of an orthonormal set.

The problem of maximizing $\Sigma_{i,j} \| x_i - x_j \|$ for $\| x_i - x_j \| \leqq 1$ appears much more difficult, even in the plane. Regular polygons are only optimal for $n = 3$ or 5. Vertices of the equilateral triangle repeated k times ($n = 3k$) are not optimal for large k. Even the limit for $n \to \infty$ is not known. We conjecture that it is a distribution on the boundary of the Reuleaux triangle.

For the maximum product of the distances, in the plane, Danzer and Pommerenke [1] have shown that regular polygons are not optimal for n odd.

References

1. L. Danzer and Ch. Pommerenke, Über die Diskriminante von Mengen gegebenen Durchmessers, Monatsh. Math., 71 (1967) 100–113.

2. H. W. E. Jung, Über die kleinste Kugel, die eine räumliche Figur einschliesst, J. Reine Angew. Math., 123 (1901) 241–257.

3. J. B. Kruskal and H. S. Witsenhausen, An inequality for positively correlated variables, J. Amer. Statist. Assoc., 69 (1974) 540–542.

4. P. Erdös, A. Meir, V. T. Sós, and P. Turán, On some applications of graph theory, I. Discrete Math., 2 (1972) 207–228.

BELL LABORATORIES, MURRAY HILL, NEW JERSEY, 07974.

SPHERICAL SETS WITHOUT ORTHOGONAL POINT PAIRS

H. S. WITSENHAUSEN

Let S_n denote the boundary of a sphere in E^{n+1} with unit n-dimensional content. Two points of S_n are called orthogonal if they subtend a right angle at the center of the sphere.

PROBLEM: *Find a_n, the supremum of the content of the measurable sets in S_n which contain no pair of orthogonal points.*

Observe that if a set A satisfies this condition then so does the union of A with its antipodal set. Thus it suffices to consider sets symmetric with respect to the center of the sphere. This implies that: (i) If the condition is strengthened to exclude antipodal pairs, the supremum becomes $a_n/2$. (ii) The problem could be stated in the

elliptic space $\mathscr{E}^n$ of unit content, with a_n the supremum of all measurable sets no two points of which have the diameter of the space as distance.

One has the bounds

$$\int_0^{\pi/4} \sin^{n-1} x \cdot dx \Big/ \int_0^{\pi/2} \sin^{n-1} x \cdot dx \leqq a_n \leqq (n+1)^{-1}. \tag{1}$$

The left side of (1) corresponds to an open spherical cap of radius $\pi/4$ and its antipode, and is of order $O(n^{-1/2}2^{-n/2})$. The right side comes from a probabilistic argument. Let $P_0, P_1, \cdots, P_n$ be pairwise orthogonal points in S_n. Then for τ selected at random in the real orthogonal group on E^{n+1}, under normalized Haar measure, the events $\tau P_t \in A$ have probability equal to the content of A and, if A satisfies the condition of the problem, these $n+1$ events are mutually exclusive.

From (1) follows $a_1 = 1/2$ and $1 - 2^{-1/2} \leqq a_2 \leqq 3^{-1}$, but even the value of a_2 does not appear to be known.

BELL LABORATORIES, MURRAY HILL, NEW JERSEY, 07974.

CLASSROOM NOTES

EDITED BY DAVID ROSELLE

Material for this Department should be sent to David Roselle, Department of Mathematics, Virginia Polytechnic Institute, Blacksburg, VA 24061.

THE GEOMETRY OF LIAPUNOV FUNCTIONS

ALFRED INSELBERG AND GIORA DULA

A real-valued function V defined on a region Ω containing the origin θ of R^n is called **positive definite** if

(i) It is continuous and has continuous first partial derivatives on Ω.

(ii) $V(X) \geqq 0 \qquad \forall X \in \Omega$.

(iii) $V(X) = 0 \iff X = \theta$.

For an autonomous n-dimensional system

$$\dot{X} = F(X) \text{ with } F(\theta) = \theta, \tag{1}$$

and having Ω as its region of existence of unique solutions with prescribed initial values, V is a **Liapunov function** if in addition

$$\dot{V}(X) = F(X) \cdot \operatorname{grad} V(X) \leqq 0 \qquad \forall X \in \Omega \tag{2}$$

along the paths of (1).

The requirement on the first partials of V, in the definition of positive definiteness, guarantees the existence of the gradient of V which is needed in the second definition.

The importance of Liapunov functions in stability theory is well appreciated and does not require any elaboration.

There is a widespread, and not entirely harmless, misconception concerning the geometry of the surface $z = V(X)$. The fallacy seems to have originated from a remark made in several texts (even in some good ones [1]), that positive definite functions have an isolated relative minimum at the origin. This observation is then interpreted geometrically to depict the surface $z = V(X)$, in a sufficiently small neighborhood of the origin, as a smooth *cup*. Such a "picture" is often used to provide an intuitive background for some of the theory.

The purpose of this note is to point out that positive definite as well as Liapunov functions need not have an isolated relative minimum at the origin. Further, that the cup-like form of V is not essential and that V can be bounded above and below with continuous functions which are monotone increasing in the norm $\| X \|$. That is, V can always be "squeezed" between two smoothly behaving monotone functions. We have then a model which in many ways is more useful than the previous incorrect one in visualizing the behavior of Liapunov functions. Furthermore, this model can easily be generalized to apply to non-autonomous systems.

Proceeding now with an example let

$$f(x) = \begin{cases} x^{2N}\left(\sin\dfrac{1}{x} + 2\right) & x \neq 0 \\ 0 & x = 0, \end{cases}$$

where N is a positive integer. This function has continuous derivatives of order up to (and including) $N-1$ at $x = 0$.

Notice that

$$0 \leqq x^{2N} \leqq f(x) \leqq 3x^{2N} \qquad \forall x \in R,$$

hence f is positive definite for $N > 1$. However, f does not have an isolated relative minimum at the origin.

In R^n let

$$F(X) = \sum_{i=1}^{n} f(x_i),$$

where $x_1, \cdots, x_n$ are the components of the vector X. By construction, F has continuous partial derivatives of order $N-1$ at θ. It is also positive definite on

$$\Omega = \{X : |x_i| < 1, \quad i = 1, \cdots, n\},$$

where $0 \leqq F(X) \leqq 3 \| X \|^2$, $\forall X \in \Omega$. Clearly, F does not have an isolated minimum at the origin.

An autonomous system for which F serves as a Liapunov function can be easily constructed. In particular, for

$$\dot{X} = -\nabla F(X) = -\operatorname{grad} F(X) \tag{3}$$

we have along the paths of (3)

$$\begin{aligned}\dot{F}(X) &= -(\nabla F(X)) \cdot (\nabla F(X)) \\ &= -(\nabla F(X))^2 \leqq 0.\end{aligned}$$

Hence by Liapunov's First Stability Theorem the origin is a stable critical point of (3). For $N > 2$, the system (3) has unique solutions in Ω for prescribed initial values.

Evidently, not only positive definite but even Liapunov functions need not have an isolated relative minimum at the origin.

The pitfall of the cup-like representation of the surface $z = V(X)$ is altogether avoided in [2]. The development there, however, lacks an intuitively helpful geometrical model.

Hahn, [3], points out that there exist continuous functions $\psi_i(\| X \|)$, $i = 1,2$ such that

$$\psi_1(\| X \|) \leqq V(X) \leqq \psi_2(\| X \|), \tag{4}$$

where the ψ_i are monotone increasing in the norm of $\| X \|$.

In our example for f

$$\psi_1(x) = x^2,\ \psi_2(x) = 3x^2 \text{ and for } F$$

$$\psi_1(\| X \|) \equiv 0,\ \psi_2(\| X \|) = 3\| X \|^2$$

in Ω.

Appending Hahn's observation to the development in [2] provides a didactically very useful device, since geometrically the ψ_i can be easily visualized. Also the generalization [4] of Liapunov functions to non-autonomous systems can be made directly from (4).

References

1. J. La Salle and S. Lefschetz, Stability by Liapunov's Direct Method, Academic Press, New York, 1961, p. 33.

2. I. G. Malkin, Theory of Stability of Motion, Translation U.S.A.E.C. tr — 3352, (1967) 23–36.

3. W. Hahn, Stability of Motion, Springer-Verlag, New York, 1967, p. 98.

4. H. Hochstadt, Differential Equations, Holt, Rinehart and Winston, New York, 1965, p. 235.

Applied Mathematics Department, Technion — Israel Institute of Technology, Haifa, Israel.

Department of Mathematics, Tel Aviv University, Ramat Aviv, Israel.

CORRECTION TO "THE BOUNDED CONVERGENCE THEOREM"

W. R. WADE

The last inequality displayed on page 388 (this MONTHLY, 81 (1974)) should read

$$\int_E |f_n(x)-f(x)|^p dx \leqq \left\{\int_E |f_n(x)-f(x)|^r dx\right\}^{p/r} \cdot \left\{\int_E |1|^q dx\right\}^{1/q} < (2M')^p . \{m(E)\}^{1/q}$$

where

$$M' = \begin{cases} M, & \text{if } 1 < r < \infty \\ 2^{(1-r)/r} M, & \text{if } 0 < r \leqq 1. \end{cases}$$

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF TENNESSEE, KNOXVILLE, TN 37916.

MATHEMATICAL EDUCATION

EDITED BY SHIRLEY HILL AND PAUL T. MIELKE

Material for this Department should be sent to Shirley Hill, Department of Mathematics, University of Missouri, Kansas City, MO 64110, or to Paul T. Mielke, Department of Mathematics, Wabash College, Crawfordsville, IN 47933.

A COLLEGE TEACHING COURSE FOR FUTURE PH.D.'S IN MATHEMATICS

RICHARD BILLSTEIN

Most of the Ph.D. graduates in mathematics acquire jobs in the academic world where teaching becomes their main function. The nation's two- and four-year colleges, as well as the universities, need Ph.D.'s who are not only well trained in mathematics but also effective teachers in the classroom. Possibly the largest market at the present time for mathematics graduate students is at the opposite extreme of their training, i.e., the small two- and-four-year colleges. The majority of the students that Ph.D. graduates will be working with lack mathematical training. The students in the small colleges for the most part are not mathematics majors; they are physicists, chemists, economists, biologists, engineers, teachers, accountants, or anybody else who feels that he either needs to know, or would like to know, more mathematics. Perhaps the most difficult job for the majority of mathematics Ph.D. graduates will be teaching undergraduates who plan to use some mathematics in their disciplines but do not plan to become mathematics majors. Thus, it is the responsibility of the Ph.D.-granting institutions to help future Ph.D.'s teach as effectively as possible. In the words of the Commission on Undergraduate

Education in the Biological Sciences, "The university is the only place where future teachers in universities and in colleges of all types can learn to teach undergraduates. If the job is not done by the universities, it is not done."

Since there is little or no attention paid to teaching in the graduate school, one may ask where a college professor acquires skill as a teacher. The training of faculty members usually does not contain any special course or seminar in any aspect of college teaching. Ann Heiss is very critical of the way graduate education is managed: ". . .the emphasis in most Ph.D. programs has been heavily weighted in favor of preparing students to discover knowledge, and only incidentally, if at all, on how to impart to others the nature and value of that knowledge. As a result, the American college teacher is the only high level professional person who enters his career with no practice and no experience in using the tools of his profession." [1]

Now that the demand for researchers is slackening, an opportunity exists to attend to this problem. In the words of Herstein, "Indeed, if for no other reason than that most Ph.D.'s eventually earn their living by teaching, those seeking the Ph.D. should have some 'training' in their future profession."[2]

The Department of Mathematics of the University of Montana offers two options in its Ph.D. program:

1. The traditional Ph.D. program aimed at preparing research specialists in theoretical mathematics, and

2. The *Mathematical Sciences Ph.D. Option* intended to train college mathematics teachers.

Besides the subject matter curriculum in the *Mathematical Sciences Ph.D. Option*, several courses and seminars are offered to meet the special needs of the program. Among the courses and seminars being offered are: a course sequence on mathematical modeling in the sciences; a course in the history of contemporary mathematics; a seminar on college teaching; a current topics seminar, and a teaching internship.

Since the majority of the graduate students at the University of Montana will eventually be college teachers, a seminar in College Teaching seemed appropriate and, in fact, necessary. It is with this thought in mind that the Mathematics Department set up two sections of the seminar. Section One considered topics that students should be aware of before starting on their careers as teachers while Section Two considered actual classroom techniques and procedures in teaching a mathematics class.

Section One took the form of a formal class meeting the first two weeks with lectures and materials presented by the writer and another staff member. The remainder of the term was filled with informal lectures by each of the participants in the class over areas which the class and instructors agreed on as important in the life of a college teacher. Discussions and questions followed each presentation. It was hoped that each participant would gain from the research and individual experiences of the other members of the class. A formal written presentation by each member of the class was required with copies to be given to each member of the class. An impor-

tant feature of this formal write-up was an annotated bibliography concerning the topic being discussed. The purpose of the annotated bibliography was to give the members of the class a permanent record of sources of additional information which would be helpful to the students when out in the field teaching.

Topics which were selected as appropriate for discussion included the following: the role of mathematics in the physical, biological, and social sciences; curriculum planning in various size colleges; particular course planning; measurement and evaluation of student achievement and instructor effectiveness; teaching as a profession; job situations in mathematics inside and outside of teaching; different organizations for mathematics teachers; recommendations for the training of college mathematics teachers; trends in mathematics; new ideas in the teaching of mathematics; maintaining mathematical momentum while teaching; the role of research *vs.* teaching; computers in mathematics education; alternatives to the Ph.D.; obligations of the college professor outside the classroom; attitudes of students and teacher in the mathematics classroom; a basic mathematics library; a look at how and where to publish; promotion, tenure and administrative policies of various colleges; educational and psychological theories involving mathematics; and other topics of interest to the participants.

Several films such as "Challenge in the Classroom—The Methods of R. L. Moore," "Let's Teach Guessing," "John von Neumann," and "Göttingen and New York — Reflections on a Life in Mathematics," were shown to generate new ideas and discussions. Guest lecturers in certain special areas were welcomed in the course. Other students and members of the department were encouraged to attend the class meetings and contribute to the discussion if they wished. This served the purpose of getting everyone in the department interested in topics involved in teaching.

At the beginning of the quarter many handouts were distributed to the students. The class made extensive use of the CUPM booklets such as "Suggestions on the Teaching of College Mathematics," "Qualifications for a College Faculty in Mathematics," "A Beginning Graduate Program in Mathematics for Prospective Teachers of Undergraduates," and CUPM newsletters such as "The Beginning Teacher of College Mathematics," "New Methods for Teaching Elementary Courses and for the Orientation of Teaching Assistants," and "Maintaining Mathematical Momentum."

Copies of many articles of interest which appeared in journals such as *The American Mathematical Monthly*, *The Two-Year College Mathematics Journal*, or *Educational Studies of Mathematics* were distributed to the participants. Several books on the topics mentioned above were purchased by the department and kept in the periodical room for reference.

Topics which seemed to generate the most interest during the quarter included various grading techniques, setting up a mathematics curriculum in various size colleges, the uses of the computer in teaching mathematics, the life of the mathematics teacher both inside and outside of the classroom, maintaining mathematical momentum in a small college, alternatives to the traditional Ph.D., and job situations for mathematicians inside and outside of teaching.

The class found that each member had his own technique and philosophy of grading. Techniques in grading varied from computing straight percentages and using natural breaks for assigning grades to scaling the grades using a slide rule. Out of curiosity, the grading patterns of the faculty were compared with the grading patterns of the teaching assistants who taught the same undergraduate courses the previous quarter. A discussion of Pass/Non-Pass grading *vs.* the traditional letter grades also generated interest. The discussion during the week centered around the hypothesis: "Students suffer because of faculty ineptness in evaluating student performances."

The discussion of computers in mathematics education was centered in two general areas: (1) the use of computers as a computational device and a means of assisting instruction in the present courses and (2) individualization. Individualization was further broken down into the classifications of computer-managed instruction and computer-assisted instruction. Several of the more interesting programs that depend on the computer as a computational device were examined.

In the week devoted to the topic of the undergraduate mathematics curriculum, the class members put themselves in the position of being teachers in a small four-year college and were faced with the problem of designing an optimal mathematics program to be taught by a small staff of three-to-five faculty members. The question of what collection of mathematics courses should be considered as a prerequisite for a successful career and/or further study undertaken after completion of the undergraduate program was a point of major discussion. The class relied heavily on the Committee on the Undergraduate Program in Mathematics (CUPM) literature for guidance. Not only the number of courses but specific course content, order of offering, and course prerequisites were considered. One conclusion reached by the majority of the participants was that the breadth of a mathematics teacher in these schools is at least as important as his specialty.

"Why should graduate programs in mathematics consider an alternative to the Ph.D. degree?" was a topic which students considered as essential to the discussions. Two proposals were examined: (1) change the present Ph.D. program and (2) institute a degree, comparable to the Ph.D. intended for those whose primary function will be teaching rather than research. The pros and cons of each proposal were examined. The Doctor of Arts Program (D.A.) was examined in detail and compared and contrasted with the Montana program.

The following statement from a CUPM report generated another week of discussion:

> ..his (the instructor's) attitudes toward his students and his subject set the tone for his total teaching activity, and are often as influential on the subsequent mathematical development of his students as anything he does. [3]

The function of the teacher beyond that of simply providing information concerning mathematics was the major topic during one week of class. It was pointed out

that students in some classes use their teachers as models of what they might become. Thus the teacher's actions and responsibilities are of extreme importance. The teacher's personality can affect the atmosphere for learning. The teacher who fails to create a situation in which the student is motivated to learn fails in carrying out his role as a teacher. The class agreed that the great teacher was one who gives frequent consideration to the ways in which he is accomplishing his *total* task.

Another week was spent talking about the college teacher's duties and responsibilities outside of the classroom. During the week we considered some of the specific tasks a college teacher might be expected to perform, the special skills and knowledge needed to perform them, and sources of information that might be helpful in gaining this knowledge.

The discussion consisted of departmental-related duties such as textbook selection and committee work, university-related duties, community-related activities, student-related activities and finally, the duty of a mathematician in a small college to himself, i.e., staying mathematically alive.

The job situation in mathematics, salary schedules, and the best ways for college graduates in mathematics to find jobs was another topic of discussion. Many statistics and conjectures pertaining to the job situation inside and outside of teaching were presented. We found that jobs do exist in areas such as operations research or systems analysis but the training that most graduates receive in mathematics does not qualify them for the positions.

This has been a quick overview of several of the topics included in Section One of the College Teaching Course.

Section Two of the College Teaching Seminar was designed to help students improve their own teaching skills in the classroom. The class met formally the first two weeks to discuss problems which teachers confront in their own teaching assignments as teaching assistants. Each participant was videotaped several times teaching his own class. The graduate assistant whose presentation was videotaped viewed the tape later by himself and picked a 10-to-20 minute portion for showing to the other participants in the seminar. The presentation and classroom activities were then constructively criticized by the participants. There was strong emphasis on heuristics, plausible reasoning and intuition, and on trying to show why in posing a theorem or solving a problem a certain direction was taken and not another.

Spanier reports in this MONTHLY that CUPM interviewed several students to determine mathematical attitudes and backgrounds [**4**]. It was found that they exhibited "narrowness of training, lack of mathematical taste, lack of intellectual curiosity, and no sense of responsibility for their own education. Spanier felt that these faults are caused primarily by the graduate schools, both through their requirements for entering students and through the training that they give to future college teachers. It is hoped that seminars such as the ones described above can help prepare graduate mathematics students to become better teachers and help future students in mathematics to avoid some of the difficulties and attitudes of the students interviewed by CUPM.

References

1. Ann Heiss, Challenge to Graduate Schools, Jossey-Bass, San Francisco, 1970.
2. I. N. Herstein, On the Ph.D. in mathematics, this MONTHLY, 76 (August-September, 1969) 818.
3. CUPM, Suggestions on the teaching of college mathematics, A report of the Panel on College Teacher Preparation, CUPM, Berkeley, California, 1973.
4. E. Spanier, The undergraduate program in mathematics, this MONTHLY, 77 (1970) 752–755.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MONTANA, MISSOULA, MT 59801.

PROBLEMS AND SOLUTIONS

EDITED BY EMORY P. STARKE

All problems (both elementary and advanced) proposed for inclusion in this Department should be sent to E. P. Starke, 1000 Kensington Ave., Plainfield, NJ 07060. Proposers of problems are urged to enclose any solutions or information that will assist the editors. Ordinarily, problems in well-known textbooks and results in generally accessible sources are not appropriate for this Department. No solutions (except those accompanying proposals) should be sent to Professor Starke.

ELEMENTARY PROBLEMS

Solutions of Elementary Problems should be sent to Problems Group, Mathematics Department, University of Maine, Orono, ME 04473. To facilitate their consideration, solutions of Elementary Problems in this issue should be typed (with double spacing) and should be mailed before March 31 1975.

An asterisk (*) means neither the proposer nor the editors supplied a solution.

E 1030* [1952, 465]. *Proposed by Charles Salkind*

Consider the polygon formed by the internal trisectors of the angles of a given n-gon, intersecting in neighboring pairs.

(1) Prove that a necessary and sufficient condition that the trisector polygon be regular is that the parent n-gon be regular when $n \geqq 4$.

(2) Prove that the area ratio between the parent and trisector polygons is always irrational.

E 2509. *Proposed by C. W. Dodge, University of Maine at Orono*

The Texas Instruments SR-10 calculator has no memory, but it does have a squaring key $[x^2]$. To calculate n^2 for a given n, one must either enter n, $[\times]$, n, $[=]$ (which requires entering the value of n twice) or enter n, $[x^2]$ (which is more efficient since the value of n is entered only once); it is not possible to calculate n^2 by entering n, $[\times]$, $[=]$ as it is on some calculators.

To find n^3 efficiently, one can either enter n, $[x^2]$, $[\times]$, n, $[=]$ or enter n, $[x^2]$, $[x^2]$, $[\div]$, n, $[=]$. In either case, the value of n must be entered twice. It is not possible to calculate n^3 without entering n twice.

Find the smallest power of n that cannot be computed without entering the value of n at least k times in the keyboard.

SOLUTIONS OF ELEMENTARY PROBLEMS

A Problem in Zero-One Matrices

E 2429 [1973, 808]. *Proposed by E. T. Wang, University of British Columbia*

Let k_n denote the least integer such that every $n \times n$ matrix of zeros and ones with exactly k_n ones in each row and in each column contains a 2×2 submatrix without zero. Obtain a lower estimate for k_n and discuss the case of equality.

Solution by R. K. Guy, University of Calgary. If there are k ones in each column, there are $\binom{k}{2}$ pairs of ones, and $n\binom{k}{2}$ pairs in all n columns. By the pigeon-hole principle, two pairs will coincide, and form a 2×2 matrix of ones, if

$$n\binom{k}{2} > \binom{n}{2}. \tag{1}$$

This gives an *upper* bound for k_n,

$$k_n \leqq \left[\tfrac{1}{2}(3 + \sqrt{4n-3})\right], \tag{2}$$

where brackets denote "greatest integer not greater than."

To show that this upper bound is sometimes attained, we must construct an $n \times n$ matrix with $k-1$ ones in each row and column, containing no 2×2 submatrix of ones. When $n = q^2 + q + 1$ and q is a prime power, it is known that there is a projective plane of order q, with $q + 1$ points on each line, whose incidence matrix contains no such submatrix, since a pair of points determine a unique line. Moreover, since every pair of points determine a line, we have equality in place of inequality in (1) and the situation is optimal; $k-1 = q + 1$ and

$$k_n \geqq q + 2 = \tfrac{1}{2}(3 + \sqrt{4n-3}) \geqq k_n;$$

we have equality and the upper bound (2) is attained. The accompanying illustration has $q = 4$.

11111	0000	0000	0000	0000
10000	1111	0000	0000	0000
10000	0000	1111	0000	0000
10000	0000	0000	1111	0000
10000	0000	0000	0000	1111
01000	1000	1000	1000	1000
01000	0100	0100	0100	0100
01000	0010	0010	0010	0010
01000	0001	0001	0001	0001
00100	1000	0100	0001	0010
00100	0100	1000	0010	0001
00100	0010	0001	0100	1000
00100	0001	0010	1000	0100
00010	1000	0010	0100	0001
00010	0100	0001	1000	0010
00010	0010	1000	0001	0100
00010	0001	0100	0010	1000
00001	1000	0001	0010	0100
00001	0100	0010	0001	1000
00001	0010	0100	1000	0001
00001	0001	1000	0100	0010

Incidence matrix for the projective plane of order 4.

Unfortunately, n is rarely of the form $q^2 + q + 1$, and even when it is, q is not often a prime power. If q is not a prime power, many people suspect that projective planes do not exist, but the only case which is settled is $q = 6$ (Tarry, 1900), Euler's famous "36 officers" problem. Here we know that it is not possible to construct an analog of the above illustration, so $k_{43} = 7$ and the upper bound (2) is not attained.

To find a lower bound, let q be the largest prime power with $q^2 + q + 1 \leqq n$. Then we can construct an $n \times n$ matrix with $q + 1$ ones in each row and column and no 2×2 submatrix of ones: if $n > q^2 + q + 1$ proceed inductively from $n-1$ to n; replace q ones, selected from q different rows $i_1, \cdots, i_q$ and q different columns $j_1, \cdots, j_q$, by zeros and insert $2q + 1$ ones in positions $(i_1, n), \cdots, (i_q, n), (n, j_1), \cdots, (n, j_q)$, (n, n). Then $k_n \geqq q + 2$, and Rosser and Schoenfeld (Illinois J. Math., 6 (1962), 64–94) tell us that there is a prime, and *a fortiori* a prime power, between x and $x + x^{5/8}$, so that

$$k_n > n^{1/2} - n^{5/16}. \tag{3}$$

The correct answer is presumably much nearer to (2) than to (3): $k_2 = 2$, $k_3 = k_4 = k_5 = k_6 = 3$, $k_7 = k_8 = \cdots = k_{12} = 4$, $k_{13} = k_{14} = \cdots = k_{20} = 5$, $k_{21} = k_{22} = \cdots = k_{30} = 6$, $k_{31} = k_{32} = \cdots = k_{43} = 7$.

A partial solution was submitted by the proposer.

A Continuous Moment Problem

E 2443 [1973, 1058]. *Proposed by T. M. Apostol, California Institute of Technology*

Let f_1 and f_2 be two linearly independent functions which are continuous on the bounded interval $[a, b]$. Show that for every pair of constants c_1 and c_2 there exists a continuous function h on $[a, b]$ such that

$$\int_a^b h(x)f_1(x)\,dx = c_1 \text{ and } \int_a^b h(x)f_2(x)\,dx = c_2.$$

I. *Solution by D. J. Eustice, Ohio State University.* With the inner product

$$\langle f, g\rangle = \int_a^b f(x)g(x)\,dx$$

on the space of continuous functions on $[a, b]$, Problem E 2443 is a special case of the following: *If $\{v_1, v_2, \cdots, v_n\}$ is linearly independent in a (real or complex) inner product space V and if $c_1, c_2, \cdots, c_n$ are constants, then there exists $v \in V$ such that $\langle v, v_i\rangle = c_i$, for $i = 1, 2, \cdots, n$.* To prove this, note that if we write $v = \sum_{i=1}^n a_i v_i$, then there is a unique solution $a_1, a_2, \cdots, a_n$ to the system of linear equations $\langle v, v_i\rangle = c_i$ $(i = 1, 2, \cdots, n)$, since the matrix of coefficients is the Gramian $G = (g_{ij})$ where $g_{ij} = \langle v_i, v_j\rangle$, which is known to be nonsingular because $\{v_1, v_2, \cdots, v_n\}$ is linearly independent.

This result appears as problem 28, p. 193 in J. T. Moore, *Elements of Linear Algebra and Matrix Theory*, McGraw Hill, 1968.

II. *Comment by J. B. Conway, Indiana University.* Necessary and sufficient conditions can be given that an infinite set of such equations can be solved. This is the classical moments problem. See S. Banach, *Théorie des Opérations Linéaires*, Chelsea, 1955, Chapter IV, §7.

Also solved by K. F. Andersen, B. C. Anderson, Barefoot Carolina Hillbilly, Jon Barkhurst, H. J. Bremer & Richard Kerns (Germany), Robert Breusch (New Zealand), Fr. A. Brousseau, J. L. Brown, Jr., J. E. Chance, Bill Chewning, Chin-hung Ching (Australia), J. B. Conway, S. H. Cox, Jr. (Puerto Rico), S. C. Currier, Jr., P. deBuda, A. R. DiDonato, C. R. Diminnie, Robert Ellis, David Foley, Mary Ellen Foley, R. M. Gasper, R. Gentry, Michael Goldberg, Richard Gosselin, Robert Griswold, Richard Groeneveld, S. Gudder, W. Habakkuk, Ellen Hertz, A. C. Hindmarsh, Joe Howard & Dennis Bertholf, Richard Johnsonbaugh, M. S. Klamkin, Robert Kopp, D. Laugwitz (Germany), K. Levasseur, Peter Lindstrom, O. P. Lossers (Netherlands), Carolyn MacDonald, S. P. Marin, J. G. Mauldon, P. D. McCray, R. K. Meany, Glen Meyers, L. F. Meyers, Michael Moore, D. B. Price, Richard Pulskamp, Otto Ruehr, St. Olaf College Students, William Sánchez, Harvey Schmidt, Jr., SCSC Problem Solving Group, Michael Skalsky, Allen Stenger, Dorothea Stillinger, Jacob Sturm, F. E. Tidmore, Stephen Tillman, R. J. Wagner, E. C. Waymire, Gordon Williams, Dale Worley, J. K. Yates, Ken Yocom, and the proposer.

Editor's comment: The Gramian is always positive semidefinite and is positive definite (thus

nonsingular) if and only if $\{v_1,\cdots, v_n\}$ is linearly independent. In the case $n = 2$, this fact reduces to the classical inequality of Schwarz.

Several queries were received as to why your editors printed such an easy problem. We were aware of the generalization from two functions to n functions, but did not print the more general problem because we hoped that our solvers would discover it for themselves, and perhaps generalize the result in a way that we had not foreseen. Unfortunately, not too many solvers did generalize.

Note that in applying the problem to n complex-valued functions $f_1,\cdots, f_n$ and n complex scalars $c_1\cdots, c_n$ we must construct $h = a_1\bar{f}_1 + \cdots + a_n\bar{f}_n$ and then choose a_i as in Solution I in order that

$$\int_a^b h(x)f_i(x)dx = \langle h, \bar{f}_i\rangle = c_i.$$

A Consequence of Gronwall's Inequality

E 2444 [1973, 1138]. *Proposed by Ray Redheffer, University of California, Los Angeles*

Let S be an open connected subset of real Euclidean space R^n and suppose that $f: S \to R^m$ is differentiable. Let the Jacobian matrix $Df(x)$ at $x \in S$ satisfy

$$\| Df(x) \| \leqq \sigma \| f(x) \|$$

for some constant σ and all $x \in S$, where the norm of a matrix is the sum of the absolute values of its entries. Show that, if $x_1, x_2 \in S$ can be connected by a path of length d lying wholly within S, then

$$\| f(x_1) \| \leqq \| f(x_2) \| e^{\sigma d}.$$

(Note that a consequence of this is that f cannot vanish anywhere on S unless it vanishes everywhere on S.)

I. *Solution by I. N. Katz, Washington University.* If A is a matrix, then the norm defined in the statement of the problem is only one of many that can be defined. Another useful norm is the *operator norm* defined by

$$\| A \|_0 = \sup\{ \| A\mathbf{x} \| / \| \mathbf{x} \| : \| \mathbf{x} \| \neq 0\}.$$

It is known that $\| A \|_0 \leqq \| A \|$. We shall prove the conclusion of the problem under the weaker assumption

$$\| D\mathbf{f}(\mathbf{x}) \|_0 \leqq \sigma \| \mathbf{f}(\mathbf{x}) \|. \tag{1}$$

Let $\boldsymbol{\phi}(t)$ be a parameterization of a path of length d from $\mathbf{x}_2$ to $\mathbf{x}_1$ lying wholly within S. Assume that $\mathbf{x}_2 = \boldsymbol{\phi}(0)$, $\mathbf{x}_1 = \boldsymbol{\phi}(1)$ and let $\mathbf{F}(t) = \mathbf{f}(\boldsymbol{\phi}(t))$. Then for every $t \in [0, 1]$,

$$\mathbf{F}(t) = \mathbf{F}(0) + \int_0^t \mathbf{F}'(s)ds$$

so that

$$(2)\qquad \|\mathbf{F}(t)\| \leqq \|\mathbf{F}(0)\| + \int_0^t \|D\mathbf{f}(\boldsymbol{\phi}(s))\boldsymbol{\phi}'(s)\|\,ds.$$

Since $\mathbf{x} = \boldsymbol{\phi}(s) \in S$, by the definition of operator norm and (1) we have that

$$\begin{aligned}\|D\mathbf{f}(\boldsymbol{\phi}(s))\boldsymbol{\phi}'(s)\| &\leqq \|D\mathbf{f}(\boldsymbol{\phi}(s))\|_0\,\|\boldsymbol{\phi}'(s)\| \\ &\leqq \sigma\|\mathbf{f}(\boldsymbol{\phi}(s))\|\,\|\boldsymbol{\phi}'(s)\| = \sigma\|\mathbf{F}(s)\|\,\|\boldsymbol{\phi}'(s)\|\end{aligned}$$

so that (2) becomes

$$\|\mathbf{F}(t)\| \leqq \|\mathbf{F}(0)\| + \sigma\int_0^t \|\mathbf{F}(s)\|\,\|\boldsymbol{\phi}'(s)\|\,ds.$$

Gronwall's inequality [Philip Hartman, *Ordinary Differential Equations*, Wiley, New York, 1964, p. 24] now gives

$$(3)\qquad \|\mathbf{F}(t)\| \leqq \|\mathbf{F}(0)\|\exp\left(\sigma\int_0^t \|\boldsymbol{\phi}'(s)\|\,ds\right).$$

Setting $t = 1$ in (3) and noting that $\mathbf{F}(0) = \mathbf{f}(\boldsymbol{\phi}(0)) = \mathbf{f}(\mathbf{x}_2)$ and $\mathbf{F}(1) = \mathbf{f}(\boldsymbol{\phi}(1)) = \mathbf{f}(\mathbf{x}_1)$ we have

$$\|\mathbf{f}(\mathbf{x}_1)\| \leqq \|\mathbf{f}(\mathbf{x}_2)\|\exp\left(\sigma\int_0^1 \|\boldsymbol{\phi}'(s)\|ds\right) = \|\mathbf{f}(\mathbf{x}_2)\|\,e^{\sigma d}.$$

II. *Solution by G. D. Chakerian, University of California, Davis.* Suppose first that $\|\mathbf{f}(\mathbf{x})\| \neq 0$ for $\mathbf{x} \in S$. Define $g\colon S \to R$ by

$$g(\mathbf{x}) = \log\|\mathbf{f}(\mathbf{x})\| = \tfrac{1}{2}\log(\mathbf{f}(\mathbf{x})\cdot\mathbf{f}(\mathbf{x})).$$

Taking partial derivatives with respect to $\mathbf{x}_i$, we have

$$D_i g(\mathbf{x}) = \frac{\mathbf{f}(\mathbf{x})\cdot D_i\mathbf{f}(\mathbf{x})}{\mathbf{f}(\mathbf{x})\cdot\mathbf{f}(\mathbf{x})}$$

and therefore

$$\|D_i g(\mathbf{x})\| \leqq \frac{\|\mathbf{f}(\mathbf{x})\|\,\|D_i\mathbf{f}(\mathbf{x})\|}{\|\mathbf{f}(\mathbf{x})\|^2} = \frac{\|D_i\mathbf{f}(\mathbf{x})\|}{\|\mathbf{f}(\mathbf{x})\|}.$$

It follows that

$$\begin{aligned}\|\nabla g(\mathbf{x})\| &\leqq \sum_{i=1}^n |D_i g(\mathbf{x})| \leqq \|\mathbf{f}(\mathbf{x})\|^{-1}\sum_{i=1}^n \|D_i\mathbf{f}(\mathbf{x})\| \\ &\leqq \|\mathbf{f}(\mathbf{x})\|^{-1}\sum_{i=1}^n\sum_{j=1}^m |D_i\mathbf{f}(\mathbf{x})\cdot\mathbf{e}_j|,\end{aligned}$$

where $\mathbf{e}_j$ is the jth coordinate vector in R^m, so that the quantity $D_i\mathbf{f}(\mathbf{x}) \cdot \mathbf{e}_j$ is simply the jth coordinate of $D_i\mathbf{f}(\mathbf{x})$. That is,

$$\|\nabla g(\mathbf{x})\| \leqq \|\mathbf{f}(\mathbf{x})\|^{-1}\|D\mathbf{f}(\mathbf{x})\| \leqq \|\mathbf{f}(\mathbf{x})\|^{-1}\sigma\|\mathbf{f}(\mathbf{x})\| = \sigma.$$

Now if $\mathbf{x}_1, \mathbf{x}_2 \in S$ and if C is a rectifiable path of length d joining $\mathbf{x}_2$ to $\mathbf{x}_1$ lying wholly within S, we have

$$g(\mathbf{x}_1) - g(\mathbf{x}_2) = \int_C \nabla g(\mathbf{x}) \cdot d\mathbf{x} \leqq \int_C \|\nabla g(\mathbf{x})\| \, ds \leqq \sigma d$$

from which follows

(*) $$\|\mathbf{f}(\mathbf{x}_1)\| \leqq \|\mathbf{f}(\mathbf{x}_2)\| e^{\sigma d}.$$

This gives the inequality in the case that $N = \{\mathbf{x} \in S : \|\mathbf{f}(\mathbf{x})\| = 0\} = \varnothing$. Suppose that $N \neq \varnothing$; if $\mathbf{x}_1 \in S \setminus N$, then by (*), $\mathbf{x}_1$ cannot be joined to points $\mathbf{x}_2 \in S\setminus N$ with $\|\mathbf{f}(\mathbf{x}_2)\|$ arbitrarily small by arcs lying in $S\setminus N$. It follows that $N = S$ in this case, and the inequality holds trivially.

Also solved by T. S. Bolis, Bill Chewning, W. J. Gorman III, Peter Klein (Sweden), and Robert Kopp.

A Triangle Inequality

E 2445 [1973, 1138]. *Proposed by F. Leuenberger, Feldmeilen, Switzerland*

Let P be a point in the interior of a triangle ABC. Let R_1, R_2, R_3 denote the distances from P to the vertices of ABC and let r_1, r_2, r_3 denote the perpendicular distances from P to the sides of ABC. Show that

$$\Sigma \frac{r_2 + r_3}{r_2 + 2R_1 + r_3} \leqq 1 \leqq \frac{1}{3} \Sigma \frac{R_1}{r_2 + r_3}$$

with equality if and only if the triangle is equilateral and P is its center.

I. *Solution by the proposer.* By considering the area of quadrilateral $ABPC$, we obtain

(1) $$aR_1 \geqq br_2 + cr_3$$

with equality if and only if AP is perpendicular to BC. Reflecting P in the bisector of angle A, we get

(2) $$aR_1 \geqq br_3 + cr_2,$$

and by adding (1) and (2), get

(3) $$R_1 \geqq \frac{b+c}{2a}(r_2 + r_3)$$

with equality if and only if AP is perpendicular to BC and bisects angle A. Now letting s denote the semiperimeter of triangle ABC, we have

$$R_1 + \tfrac{1}{2}(r_2 + r_3) \geqq \frac{s}{a}(r_2 + r_3).$$

If h_1 is the altitude to side a, and r the inradius, then from this follows

$$\frac{R_1}{r_2 + r_3} + \frac{1}{2} \geqq \frac{h_1}{2r},$$

whence

$$\frac{r_2 + 2R_1 + r_3}{r_2 + r_3} \geqq \frac{h_1}{r},$$

and

$$\frac{r_2 + r_3}{r_2 + 2R_1 + r_3} \leqq \frac{r}{h_1}. \tag{4}$$

By adding (4) and the two corresponding inequalities and using $r\Sigma\, h_i^{-1} = 1$, we obtain the left-hand inequality, with the desired conditions for equality.

If we divide (3) by $r_2 + r_3$, we get

$$\frac{R_1}{r_2 + r_3} \geqq \frac{b}{2a} + \frac{c}{2a}.$$

Adding to it the corresponding inequalities for R_2 and R_3 and using $b/2a + a/2b \geqq 1$ three times, we get

$$\Sigma\ \frac{R_1}{r_2 + r_3} \geqq \Sigma\ \frac{b + c}{2a} \geqq 3,$$

which is equivalent to the right-hand inequality. Equality occurs when ABC is equilateral and P is its center.

II. *Solution to right-hand inequality by Leonard Goldstone, Watervliet, N.Y.* We prove the stronger inequality

$$3 \leqq \Sigma\ \frac{r_2 + r_3}{R_1} \tag{1}$$

with equality if and only if ABC is equilateral and P is its center. From $3/2 \geqq \Sigma \sin(A/2)$ [Bottema *et al.*, *Geometric Inequalities*, Item 2.9], we get

$$3 \geqq 2\ \Sigma \sin\frac{A}{2} \geqq \frac{r_2 + r_3}{R_1}$$

with the desired equality conditions. Now, by the arithmetic-geometric-harmonic mean inequalities,

$$\frac{1}{3}\sum\frac{R_1}{r_2+r_3} \geqq \left[\prod\frac{R_1}{r_2+r_3}\right]^{1/3} \geqq \frac{3}{\sum\dfrac{r_2+r_3}{R_1}} \geqq 1$$

from which both (1) and the desired inequality follow. The fact that

$$\left[\prod\frac{R_1}{r_2+r_3}\right]^{1/3} \geqq 1$$

provides a new proof of E 1433 [1961, 380], which states that $\prod R_1 \geqq \prod (r_2+r_3)$.

Also solved by T. S. Bolis, A. G. Ferrer (Mexico), M. G. Greening (Australia), Leonard Goldstone, J. N. Lillington (England), O. P. Lossers (Netherlands), and Carolyn MacDonald. Most solvers used trigonometry to some extent and some used the calculus with or without Lagrange multipliers to obtain extrema.

Point-Inverses with Compact Boundaries

E 2449 [1973, 1139]. *Proposed by Frank Siwiec, John Jay College*

Let f be a continuous mapping of $\mathbb{R}$ onto $\mathbb{R}$ with the property that for every $y \in \mathbb{R}$, the boundary of the set $f^{-1}(y) = \{x \in \mathbb{R}: f(x) = y\}$ is compact. Show that f is a closed mapping.

I. *Solution by Robert Smith, Swarthmore College.* We first show that the given hypotheses imply that $f^{-1}(y)$ is compact for every $y \in \mathbb{R}$. Since the boundary of $f^{-1}(y)$ is nonempty (f is onto and the reals are connected) and compact (by assumption) it has a smallest element p and a largest element q. By application of the Intermediate Value Theorem we see that either (i) $f(x) > y$ for all $x > q$, (ii) $f(x) = y$ for all $x > q$, or (iii) $f(x) < y$ for all $x > q$, since otherwise there would be a boundary point of $f^{-1}(y)$ in (q, ∞). A similar statement can be made about the behavior of f on $(-\infty, p)$. It is clear that case (ii) cannot hold, for suppose it did. Since f is continuous, it is bounded on the compact set $[p, q]$ and since f is onto, it would have to take on values both greater than y and less than y on $(-\infty, p)$, a contradiction. Similarly, the case that $f(x) = y$ for all $x < p$ is impossible. Thus $f(x) = y$ holds for *no* $x > q$ and *no* $x < p$ and so $f^{-1}(y) \subseteq [p, q]$; i.e., $f^{-1}(y)$ is bounded. Since $\{y\}$ is a closed set and f is continuous, $f^{-1}(y)$ is also closed and so is compact.

Let A be a closed subset of $\mathbb{R}$ and let $\{y_n\}$ be a sequence in $f(A)$ which converges to y. We must show $y \in f(A)$. We know that for each n, $f^{-1}(y_n) \cap A$ is nonempty and compact and so has a greatest element a_n; observe that $a_n \in A$ and $f(a_n) = y_n$. If we can show that $\{a_n\}$ has a subsequence which converges to some a, then $a \in A$ since A is closed. The continuity of f will then imply that $f(a) = y$, so that $y \in f(A)$ and $f(A)$ will be shown to be closed.

Suppose to the contrary that $\{a_n\}$ has no convergent subsequence. Then $\{a_n\}$ is unbounded and we can choose an unbounded monotonic subsequence; there is no loss of generality in assuming that $a_1 < a_2 < \cdots$. Let $f^{-1}(y)$ be bounded below

by p and above by q, and assume, again without loss of generality, that $f(x) < y$ for all $x > q$. Fix some $\varepsilon > 0$; then there is an N such that if $n > N$, then $y - \varepsilon < f(a_n) < y$. With each a_n we can associate a point $x_n > a_n$ such that $f(x_n) < y - \varepsilon$, for if this were not so, we would have $y - \varepsilon \leqq f(x) < y$ for all $x > a_n$ and this is impossible for the same reason that f cannot be constant on an unbounded interval. Between a_n and x_n there is a boundary point of $f^{-1}(y-\varepsilon)$, implying that the boundary of $f^{-1}(y-\varepsilon)$ is unbounded, hence not compact. This contradiction implies the existence of a convergent subsequence of $\{a_n\}$ and the proof is complete.

II. *Comment by B. R. Wenner, University of Missouri, Kansas City.* This problem can be viewed as a partial converse to the result of K. Morita and S. Hanai which asserts that if f is a closed continuous mapping of metric spaces, then the boundary of every point-inverse $f^{-1}(y)$ is compact. [*Closed mappings and metric spaces*, Proc. Japan Acad. 32 (1956), 10–14.] It is not possible to strengthen the statement of E 2449 to make a complete converse: for example, the identity mapping from the reals with the discrete topology to the reals with the usual topology is continuous, and the boundary of every point-inverse is void, hence compact, but the mapping is not closed.

Also solved by Kenneth Abernathy & Lee Hagglund, C. L. Belna, A. J. Berner, S. C. Currier, Jr., G. A. Heuer (Germany), Paul Ilacqua, T. C. Lominac, P. D. McCray, M. R. Modak (India), Roy Olson, Kenneth Schilling, Arthur Solomon, B. R. Wenner, J. K. Yates, and the proposer.

Editor's comment. Berner and the proposer note that f is actually a perfect mapping: the preimage of every compact set is compact.

ADVANCED PROBLEMS

All solutions of Advanced Problems should be sent to J. Barlaz, Rutgers — The State University, New Brunswick, N.J. 08903. Solutions of Advanced Problems in this issue should be typed (with double spacing) on separate, signed sheets and should be mailed before March 31, 1975.

An asterisk (*) means neither the proposer nor the editors supplied a solution.

6000. *Proposed by Siemion Fajtlowicz and Jan Mycielski, University of Colorado*

Let F be the space of all complex complex valued functions f: $[0,1] \to \{z: |z| \leqq 1\}$ such that $f(0) \neq 0$, $|f(t_2) - f(t_1)| \leqq t_2 - t_1$ for all $0 \leqq t_1 < t_2 \leqq 1$, with the distance $\operatorname{dist}(f_1, f_2) = \max\{|f_1(t) - f_2(t)|: 0 \leqq t \leqq 1\}$. We define a function $\phi: F \to \{z: |z| = 1\}$ putting $f^*(t) = f(t)/|f(t)|$ and $\phi(f) = f^*(\min\{t: |f(t)| = t\})$. Prove that ϕ is continuous.

Is the result true when the space of complex numbers is replaced by a Banach space so that f maps $[0,1]$ into a ball?

6001*. *Proposed by J. A. Eidswick, University of Nebraska-Lincoln*

When Taylor's theorem is applied to a power series f and a point x in its interval of convergence, a sequence $\{t_n\}$ is obtained such that

$$f(x) = \sum_{k=0}^{n-1} \frac{f^{(k)}(0)}{k!}x^k + \frac{f^{(n)}(t_n)}{n!}x^n$$

for every $n = 1, 2, \cdots$. Can $\{t_n\}$ always be chosen so that $\liminf |t_n| = 0$?

6002*. *Proposed by Bertram Ross, University of New Haven*

Examine for convergence, and if convergent evaluate

$$I = \int_0^x (x-t)^{p-1} \ln\Gamma(t)dt,$$

where $\Gamma(t)$ is the gamma function and $0 < p < 1$.

6003. *Proposed by Emeric Deutsch, Polytechnic Institute of Brooklyn*

Let A be a complex $n \times n$ matrix. Show that

$$r(e^A e^{A^*}) \leqq r(e^{A+A^*}),$$

where r denotes spectral radius.

6004. *Proposed by Oto Strauch, Bratislava, Czechoslovakia*

Find a concrete example of an injective function $f: R \times R \to R$ such that for every two real numbers x, y it will be true that

$$x < y \Rightarrow x < f(x, y) < y.$$

6005.* *Proposed by C. W. Anderson, University of California at Berkeley*

The probability that n, $m \in N$ are coprime is $6/\pi^2$. What is the probability that the product of two deficient numbers is deficient?

SOLUTIONS OF ADVANCED PROBLEMS

Boundary Approaches of Maps in a Euclidean Space

5905 [1973, 325]. *Proposed by J. G. Wendel, University of Michigan*

Let x be a mapping of $(0,1)$ into Euclidean space R^d and let $\{u_n\}$ be a countable dense set of vectors on the unit sphere of R^d. Suppose that for each n, $\limsup_{t\to 0} u_n \cdot x(t) = 1$. (a) Prove that $\limsup_{t\to 0} \|x(t)\| = 1$. (b) Can the denseness of the set $\{u_n\}$ be dispensed with? (c) Is the result true in Hilbert space?

Solution by A. A. Jagers, Technische Hogschool Twente, Enschede, Netherlands. We prove (a) and answer (b) and (c) in the negative.

(a) Denote $\limsup_{t\to 0} \|x(t)\|$ by α. Then $\alpha \geqq 1$, since $\|x(t)\| \geqq \|u_n\| \cdot \|x(t)\| \geqq u_n \cdot x(t)$ and $\limsup_{t\to 0} u_n \cdot x(t) = 1$. Now suppose $\alpha \geqq 1 + \varepsilon$ with $\varepsilon > 0$ and,

for technical reasons, $\varepsilon < 1$. Then there exists a sequence $(x(t_i))_{i=1}^{\infty}$ with $t_i \downarrow 0$ and $\| x(t_i) \| \geqq 1 + \varepsilon/2$. The unit sphere being compact in finite dimensional space, the normalized sequence given by $y_i = x(t_i)/\| x(t_i) \|$ has a convergent subsequence with limit y, say. By change of notation let $(y_i)_{i=1}^{\infty}$ be this subsequence. We then have

$$u_n \cdot x(t_i) \geqq \| x(t_i) \| (y \cdot y_i - \| u_n - y \|),$$

where, because of the denseness of $\{u_n\}$ in the unit sphere, $\| u_n - y \|$ can be made less than $\varepsilon/4$. It follows that for some n, $\limsup_{t \to 0} u_n \cdot x(t) \geqq (1 + \varepsilon/2)(1 - \varepsilon/4) > 1 + \varepsilon/8$. This contradiction shows that $\alpha = 1$.

(b) The above proof shows that it is not necessary to suppose that $\{u_n\}$ is dense in the whole unit sphere. It is sufficient to suppose that its boundary, i.e., the set of all x with $\| x \| = 1$, is contained in the closure of $\{u_n\}$. The latter condition is also necessary. In fact, suppose there exists $y \in R^d$ with $\| y \| = 1$ such that, for some $\delta > 0$, $\| y - u_n \|^2 \geqq 2\delta$ for all n. To avoid trivialities assume $\sup \| u_n \| = 1$. For $k = 1, 2, \cdots$, put $x(2^{-k}) = (1 + \delta)y$ and $x(3^{-k}) = u_{f(k)}$ where $f \colon \mathbb{N} \to \mathbb{N} \times \mathbb{N}$ is a bijection and $u_{(n,m)}$ is set equal to u_m for all $n \in \mathbb{N}$. For other values of t, put $x(t) = 0$. Then $\limsup_{t \to 0} \| x(t) \| = 1 + \delta > 1$ whereas $\limsup_{t \to 0} u_n \cdot x(t) = 1$.

(c) We suppose Hilbert space to be *ipso facto* separable. Note however, that the result is trivially true in nonseparable complete inner product spaces in the absence of a set $\{u_n\}$ with the prescribed properties.

Let $\{e_k\}$ be an orthonormal set in Hilbert space. For t not equal to a negative integer power of 2 define $x(t)$ by a similar construction as under (b). Otherwise, set $x(2^{-k}) = 2e_k$. Then $\limsup_{t \to 0} \| x(t) \| = 2$, whereas $\limsup_{t \to 0} u_n \cdot x(t) = 1$.

Also solved by E. A. Herman, Paul Milnes, Nicholas Passell, Phil Tracy, A. C. Yorke, and the proposer.

Extension of the Parallelogram Law

5919 [1973, 697]. *Proposed by L. A. Harris, University of Kentucky*

Show that if x and y are two vectors in a complex Hilbert space and if n is a positive integer, then

$$\sum_{k=1}^{2n} \| x + \lambda^k y \|^{2n} \leqq n \binom{2n}{n} (\| x \|^{2n} + \| y \|^{2n}),$$

where $\lambda = \exp(i\pi/n)$. Note that this generalizes the parallelogram law.

Solution by Finban Holland, University College, Cork, Ireland. We prove: *If x, y are vectors in a complex Hilbert space and m, n are integers, $0 \leqq m < 2n$ then*

$$\sum_{k=1}^{2n} \| x + \lambda^k y \|^{2m} = n \binom{2m}{m} 2^{1-m} (\| x \|^2 + \| y \|^2)^m,$$

where $\lambda = \exp(i\pi/n)$. *Equality holds if and only if* $x = zy$, *where* z *is a complex number of unit modulus.*

Proof. The inequality is trivial if $x = y = 0$. In what follows we suppose that $\|x\|^2 + \|y\|^2 > 0$ and set

$$a = 2|\langle x,y\rangle|/(\|x\|^2 + \|y\|^2),$$

so that $0 \leqq a \leqq 1$. Note first that

$$\|x + e^{i\theta}y\|^{2m} = \sum_{s=-m}^{m} c(s)e^{is\theta}, \qquad (c(s) = \overline{c(-s)}).$$

For any integer s

$$(1/2n)\sum_{k=1}^{2n} \lambda^{sk} = \begin{cases} 1, & \text{if } s \equiv 0 \pmod{2n} \\ 0, & \text{otherwise.} \end{cases}$$

It can now be verified that

$$(1/2n)\sum_{k=1}^{2n} \|x + \lambda^k y\|^{2m} = (1/2\pi)\int_{-\pi}^{\pi} \|x + e^{i\theta}y\|^{2m} d\theta$$

$$= (\|x\|^2 + \|y\|^2)^m (1/2\pi)\int_{-\pi}^{\pi} (1 + a\cos\theta)^m \, d\theta.$$

Now

$$(1/2\pi)\int_{-\pi}^{\pi} (1 + a\cos\theta)^m d\theta = \sum_{k=0}^{[m/2]} \binom{m}{2k} a^{2k} (1/2\pi)\int_{-\pi}^{\pi} \cos^{2k}\theta d\theta$$

$$\leqq (1/2\pi)\int_{-\pi}^{\pi} (1 + \cos\theta)^m d\theta$$

$$= 2^{-m}\binom{2m}{m},$$

and the inequality is strict unless $a = 1$. The stated result follows.

Also solved by L. E. Clarke (England), L. E. Mattics, and the proposer.

Distribution of Squares in $Z/(n)Z$

5928 [1973, 949]. *Proposed by Carl Pomerance, University of Georgia*

If n is an integer greater than 1, let $t(n)$ equal the number of squares in the ring $\mathbf{Z}/(n)\mathbf{Z}$. Find a formula for $f(n) = \lim_{k\to\infty} t(n^k)n^{-k}$.

Solution by L. E. Clarke, University of East Anglia, England. $t(n)$ is the number of those residues c in a complete set of residues $(\bmod n)$ for which the congruence

(1) $$x^2 \equiv c \pmod n$$

is solvable. Suppose that $n = p^r$, where p is an odd prime and r is a positive integer. Then if $p \nmid c$ the congruence

(2) $$x^2 \equiv c \pmod{p^r}$$

is solvable if and only if c is a quadratic residue of p [Landau, *Vorlesungen*, p. 45]. There are $\frac{1}{2}p^{r-1}(p-1)$ such residues in a complete set of residues $(\operatorname{mod} p^r)$. If $p \mid c$ then (2) is solvable if and only if either $r = 1$ or $r \geqq 2$, $p^2 \mid c$, and $y^2 \equiv cp^{-2}$ $(\operatorname{mod} p^{r-2})$ is solvable. Thus $t(p) = \frac{1}{2}(p-1) + 1$, and

$$t(p^r) = \tfrac{1}{2}p^{r-1}(p-1) + t(p^{r-2}) \qquad (r \geqq 2,\ t(1) = 1).$$

Induction yields

$$t(p^{2r-1}) = \tfrac{1}{2}(p-1)(p^{2r-2} + p^{2r-4} + \cdots + p^2 + 1) + 1$$

and

$$t(p^{2r}) = \tfrac{1}{2}(p-1)(p^{2r-1} + p^{2r-3} + \cdots + p^3 + p) + 1,$$

and so

$$p^{-k}t(p^k) \to \frac{p}{2(p+1)} \quad \text{as } k \to \infty.$$

Now suppose that $n = 2^r$. Clearly $t(2) = t(4) = 2$. Suppose next that $r \geqq 3$. If c is odd then

(3) $$x^2 \equiv c \pmod{2^r}$$

is solvable if and only if $c \equiv 1 \pmod 8$, and there are 2^{r-3} such c in a complete set of residues $(\operatorname{mod} 2^r)$. If c is even then (3) is solvable if and only if $4 \mid c$ and $y^2 \equiv \frac{1}{4}c \pmod{2^{r-2}}$ is solvable. Thus

$$t(2^r) = 2^{r-3} + t(2^{r-2}).$$

Again induction yields

$$t(2^{2r-1}) = \tfrac{1}{3}(2^{2r-2} + 5) \quad \text{and} \quad t(2^{2r}) = \tfrac{1}{3}(2^{2r-1} + 4),$$

and so $2^{-k}t(2^k) \to \frac{1}{6}$ as $k \to \infty$.

Finally, suppose that $n = p_1^{a_1} \cdots p_m^{a_m} = \prod p^a$, say, where $p_1, \cdots, p_m$ are distinct primes. Then (1) is solvable if and only if each of the m congruences

$$x^2 \equiv c \pmod{p_i^{a_i}} \qquad (i = 1, \cdots, m)$$

is solvable, whence (by the Chinese Remainder Theorem) $t(n) = \prod t(p^a)$. Therefore as $k \to \infty$,

$$n^{-k}t(n^k) = \prod p^{-ak}t(p^{ak}) \to \begin{cases} \prod \dfrac{p}{2(p+1)}, & n \text{ is odd} \\ \frac{1}{2}\prod \dfrac{p}{2(p+1)}, & n \text{ is even.} \end{cases}$$

Also solved by D. M. Bloom, Eric Grosswald, Joel Spencer, and the proposer.

A Partition Identity

5929 [1973, 949]. *Proposed by Frederick Stern, California State University at San Jose*

Let $0 \leqq x < 1$. Show that

$$\left(\prod_{k=1}^{\infty}(1-x^k)\right)^{-1} = 1 + \sum_{m=1}^{\infty} \sum_{S} \frac{x^{i_1+i_2+\cdots+i_m}}{(1-x^{i_1})(1-x^{i_2})\cdots(1-x^{i_m})},$$

where $S = (i_1, i_2, \cdots, i_m)$ is any set of positive integers such that $i_1 < i_2 < \cdots < i_m$.

Solution by R. C. Read, University of Waterloo. The left hand side can be rewritten as

$$\prod_{k=1}^{\infty}\left(1 + \frac{x^k}{1-x^k}\right).$$

In forming the terms in the expansion of this product let $i_1, i_2, \cdots, i_m$ be, in order, the values of k for which the term $x^k/(1-x^k)$ is chosen from the kth factor. The corresponding term is then

$$\frac{x^{i_1+i_2+\cdots+i_m}}{(1-x^{i_1})(1-x^{i_2})\cdots(1-x^{i_m})}.$$

The right hand side results on collecting the terms for each value of m, and noting that the term "1" corresponds to $S = \varnothing$. Convergence questions present no difficulty for $0 \leqq x < 1$.

L. E. Clark (England) gives three distinct solutions.

Also solved by M. T. Bird, L. Carlitz, M. G. Greening (Australia), Emil Grosswald, A. A. Jagers (Netherlands), O. P. Lossers (Netherlands), Albert Nijenhuis, Joel Spencer, Allen Stenger, Gordon Williams, and the proposer.

Graphs of Measurable Functions

5930 [1973, 949]. *Proposed by J. J. Buckley, University of South Carolina*

Let $\mathscr{A}$ be the Lebesgue subsets of R and let $\mathscr{B}$ be the Borel subsets of R. A problem in Halmos, *Measure Theory* ([**1**], p. 143) implies that if $f: R \to R$ is Lebesgue measurable, then the graph of f belongs to the product σ-algebra $\mathscr{A} \times \mathscr{B}$. Is the converse true?

Solution by the proposer. Yes; if the graph of $f: R \to R$ belongs to the product σ-algebra, then f is measurable. Let $\mathscr{N}$ be all sets of Lebesgue measure zero.

1. If $E \in \mathscr{A} \times \mathscr{B}$, then there are sets $E_1 \in \mathscr{B} \times \mathscr{B}$ and E_2 in the product σ-ring $\mathscr{N} \times \mathscr{B}$ such that $E = E_1 \cup E_2$ because if $\mathscr{S}$ is the set of all such unions, then $\mathscr{S}$ contains all the measurable rectangles in $\mathscr{A} \times \mathscr{B}$ and $\mathscr{S}$ is a monotone class.

2. Let P be the projection of R^2 into R. If $E \in \mathscr{A} \times \mathscr{B}$, then $P(E) \in \mathscr{A}$ because

(i) $P(E) = P(E_1) \cup P(E_2)$; (ii) $P(E_1) \in \mathscr{A}$ since $P(E_1)$ is an analytic set ([2], p. 482), and (iii) $P(E_2) \in \mathscr{A}$ because E_2 is contained in a measurable rectangle ([1], p. 141) and so $P(E_2)$ has Lebesgue measure zero.

3. Construct a sequence of simple functions S_n in the usual way so that $S_n \to f$ ([3], p. 237). Each S_n is Lebesgue measurable because

$$\left\{x \,\middle|\, \frac{i-1}{2^n} \leqq f(x) < \frac{i}{2^n}\right\} = P\left[\left(Rx\left[\frac{i-1}{2^n}, \frac{i}{2^n}\right)\right) \cap \{(x,y) \,|\, x \in R, y = f(x)\}\right]$$

is in $\mathscr{A}$.

The result generalizes the fact that $f: R \to R$ is Borel measurable if and only if the graph of f is a Borel subset of the plane ([2], p. 365, Proposition 4, and p. 398, Proposition 2).

References

1. P. R. Halmos, Measure Theory, D. Van Nostrand, New York, 1950.
2. C. Kuratowski, Topology, Vol. I, Academic Press, New York, 1966.
3. A. E. Taylor, General Theory of Functions and Integration, Blaisdell, Massachusetts, 1965.

Also solved by Josef Dravecký, (Czechoslovakia), and by A. G. P. M. Nijet & D. A. Overdijk (Netherlands).

REVIEWS

EDITED BY J. ARTHUR SEEBACH, JR. AND LYNN A. STEEN
with the assistance of the mathematics departments of St. Olaf and Carleton Colleges

COLLABORATING EDITOR FOR FILMS: SEYMOUR SCHUSTER, CARLETON COLLEGE

We invite readers to submit reviews of significant recent college-level mathematics books. We especially encourage reviews based on classroom use, or comparative reviews of several related books. Reviews should ordinarily not exceed two pages (per book) typed double spaced. Manuscripts of reviews as well as books submitted for review should be sent to: Book Review Editor, American Mathematical Monthly, St. Olaf College, Northfield, MN 55057.

Calćulus. By Leonard Gillman and Robert H. McDowell. W.W. Norton, New York, 1973. 674 pp. $12.95. (Telegraphic Review, March 1974.)

Here's news — a careful, mathematically impeccable *Calculus* with these two happy properties (believed up to now by this reviewer to be incompatible): It is so well-written that even the innocent freshman can read and understand it on his own; it is so full of fresh ideas and new approaches that even the experienced professor will

find to his surprise that there is much to calculus he never knew, and there are ways of introducing topics he never considered.

One reads with the suspicion and scorn born of years of frustration with pedestrian, repetitive texts the authors' statement on the book-jacket that they "have taken a fresh look at every concept and proof, as well as at layout and design." But as the months wear on, these brave words hold up.

A major departure from tradition is the authors' definition — essentially as suggested by Howard Levi — of the integral of a continuous function as that function (shown unique) which satisfies two special properties, (A) and (B). Are these undistinguished labels chosen by the authors in a lapse into unimaginative notation? No. By good luck or good management, (A) denotes Additivity and (B) denotes Betweenness. The Additivity referred to here is not that $\int(f+g) = \int f + \int g$, but rather that $\int_a^b f = \int_a^c f + \int_c^b f$ whenever $a \leqq c \leqq b$; and Betweenness is that $\int_a^b f$ lies between $(b-a)\cdot\min f$ and $(b-a)\cdot\max f$. That's about all there is to the integral — no upper sums, no choosing ξ_k in $[x_{k-1}, x_k]$, no passing to the limit as the norm of the partition goes to zero.

Another innovation is the Fundamental Lemma on Closed Intervals, due to L. R. Ford: If $\mathscr{I}$ is a family of closed subintervals of $[a,b]$ whose interiors cover $[a, b]$, and if $I_0 \cup I_1 \in \mathscr{I}$ whenever $I_0, I_1 \in \mathscr{I}$ and $I_0 \cap I_1 \neq \varnothing$, then $[a,b] \in \mathscr{I}$. From this result, whose proof is admittedly difficult for students at this stage, follow all the "theorems of analysis" concerning continuous functions on closed intervals essential for elementary calculus: the theorems on intermediate value, boundedness, maximum value and uniform continuity. The ease and speed with which these important results unfold is best appreciated by the professor who for years has struggled at length and in vain to explain their meanings and proofs to unseeing students.

There are other novelties, both mathematical and pedagogical, put forth tastefully and exploited systematically, too numerous to describe in this short review.

Is there nothing to criticise here? (1) It is no surprise that the heights of expository excellence achieved frequently in this text are separated by plateaus of routine competence. Expressed less positively: the writing is uneven. Specifically, this teacher and his class found the sections on maxima and minima over non-closed intervals, on length of arc, on fluid force and on work to be relatively uninspired. (2) The authors' Axiom of Completeness, heralded in bold-face type and set in italics for emphasis, asserts that every subset of $\mathscr{R}$ with an upper bound has a least upper bound. This statement, read aloud in class by a confused student, strikes the professor as careless and false; the bell rings amidst general confusion and a loss of faith in the Greater Mathematical Establishment. Only after class is it noted that some paragraphs earlier, in another subsection, the authors remark (in parentheses) that their subsets of $\mathscr{R}$ are taken to be non-empty. The authors' integrity is preserved, but it is a near thing. (3) It is perhaps more a personal failing than a failure of the text, but here is the simple truth: More than once in the past year I lost confidence in my ability and the authors' to put their new ideas across to the class effectively, and I reverted to

time-worn, standard methods of approach. (The authors make it easy to base a traditional course on their text. The method of approximating sums, for example, though logically inessential to their development of the integral, is presented in detail.)

By way of preparing these remarks I asked my freshman class to comment on the text (the question was worth 8 points of 100 on the final examination). The reader of this Monthly, if he is accustomed to activities of this sort, will not be surprised to learn that in the individualized judgments of members of the class of 1977 at Wesleyan University, the book is both "a gem" and "a dud"; the authors are at once "masterful expositors" and "creatures without sympathy"; the problems are "nicely varied and well-chosen—very helpful" and "excessively numerous in their number"; and in sum the book is "a real pleasure to read, especially after Th*m*s" and "too obscure—the Prof had to explain the jokes to us." (This last refers to the authors' statements that an inessential theorem concerning greatest lower bounds is included "for the sake of completeness" and that "a little reflection shows that the graph of f^{-1} is the mirror image ... of the graph of f.") When appropriate discount is made for the ridicule and distaste normally visited by liberal arts students upon their calculus texts, these comments add up to a nearly unmitigated hymn of praise.

I hold with the first judgment: it's a gem. I enjoyed it and so did my students. I predict the same for you and yours.

W. WISTAR COMFORT, Wesleyan University

Systems of Ordinary Differential Equations: An Introduction. By Jack L. Goldberg and Arthur J. Schwartz. Harper-Row, New York, 1972. xiii + 315 pp. $9.95. (Telegraphic Review, January 1973.)

I used this book as a text for a mixed class, mostly junior and senior mathematics or physics majors, who had studied matrix-linear algebra. I felt that it worked exceptionally well and I certainly intend to continue using it. Without this prerequisite it would be a rush to get to some of the more interesting material. There is a review of basic linear algebra in Chapter 1 and eigenvalues and eigenvectors are covered with enough care in context in the later chapters that they need not have been covered in prior work, if Chapter 1 is to be omitted.

The comments by the students during the course and in their evaluations near the end can be briefly summarized as stating that the book gives a readable, quite satisfactory, well-written presentation of the material, but some sections are a bit difficult. The numerous misprints are but a minor annoyance.

The basic material of linear systems and nth order equations in the early chapters (2 through the middle of 6) could easily be covered in a one-quarter course, assuming a year of calculus and the matrix-linear algebra as prerequisites. In a semester, quite a selection from the remaining topics on series, non-linear equations, the Euler

method, orbits, stability, and perturbations can be included. My students were sent off to work on individual projects, chosen from these last topics, according to their interests, and they seemed to need but little help in reading the material.

The models are well-chosen to illustrate how systems of differential equations may occur, although some students would like to have seen an example from economics and an additional example from biology, or one from the social sciences. I felt a need to supplement the problem sets associated with the models in order to obtain an adequate number of exercises.

I strongly recommend the use of this book for any class with the appropriate prerequisites.

R. G. BUSCHMAN, University of Wyoming

The Nature of Modern Mathematics. By Karl J. Smith. Brooks/Cole, Monterey, Calif., 1973. xiv + 466 pp.

This is another book intended for a terminal (one or two semester) course for liberal arts students: while it could possibly be used with prospective elementary teachers, it is most appropriate for students who will not have to "use" mathematics.

The first two chapters and chapters four through nine follow an outline which is by now fairly common: (1) Induction and Set Theory, (2) Number Systems (base n), (4) Algebraic Laws (groups, modular arithmetic), (5) Real Numbers, (6) Number Theory, (7) Logic, (8) Permutations and Combinations, and (9) Probability. Clearly, the individual chapters are tightly packed: for instance, in 49 pages Chapter 5 moves from the natural numbers through the integers (via the number line), the rationals (ordered pairs, but "equivalence relation" is not defined, and the question whether operations are well-defined is not raised), and the irrationals (infinite decimals, with some appeal to geometry also) to get the real numbers. The author suggests that a one-semester course can cover five chapters, but four might be more realistic for most students.

The most attractive features of the book are Chapter 3, on computers, and the exercises. Chapter 3 begins with a brief history of calculating devices, and then talks about modern computer programming on three levels: flow charting, assembly language/machine language programming (for some strange reason, decimal notation is used in the latter), and BASIC programming. The material on BASIC is in fact adequate to enable students to write simple programs: the only additions necessary are telling the students how to cope with the local monitor system and whether local keyboard conventions are the same. If you have available a small computer designed for or usually used for BASIC, even that may not be necessary. The specific machine the author apparently has most in mind is the PDP–8, but others are as appropriate. In each chapter after Chapter 3, several of the optional exercises employ

a computer either in the form "to solve this, run the following BASIC program..." or in the form "write a program to solve this." These problems appear to be well designed and helpful both as programming practice and for learning the mathematics at issue.

The computer exercises are not the only interesting ones. About a third of the material in the book is problems: a generous supply of elementary-to-intermediate exercises (with a limited number of answers provided in the back), and two other categories of problems called "Mind Bogglers" and "Problems for Individual Study." (In addition, each Chapter has review questions and an outline.) These two other categories introduce a great deal of mathematics beyond what is in the body of the text, usually motivating it (sometimes loosely) as an offshoot of something in the previous section. E.g., the Königsberg Bridges problem appears in the chapter on number systems; the four-color problem in the section on the number line; Fermat's Last Theorem in the section on the reals; and the St. Petersburg Paradox in the section on mathematical expectation (probability). These problems provide ample material for brighter classes or for allowing students to go off on individual projects. They can thus be of great assistance in dealing with one of the major problems involved in teaching a course of this type: the wide variety of talent and background within a single class.

All in all, Karl Smith's book seems well worth examining for a course of this sort.

EDWARD T. ORDMAN, University of Kentucky

Basic Statistics for Business and Economics. By George W. Summers and William S. Peters. Wadsworth Publishing Company, Belmont, California, 1973. 445 pp. $11.95.

I have taught precalculus service courses in statistics several times and this text, despite many flaws, is the most teachable one I have used. Obviously a great deal of time and effort went into this material. The result is a very flexible package consisting of no less than five parts: (1) the main text, a hardbound book of the traditional sort, (2) a paperbound book, *Self-Correcting Exercises for Basic Statistics in Business and Economics*, (3) a paperbound *Instructor's Manual*, (4) a paperbound *Self-Instruction Supplement for Basic Statistics in Business and Economics*, written by F. F. Elzey and C. P. Armstrong, and (5) a card file of multiple-choice test items.

I did not examine the card file. Solutions to problems in the text are given in the instructor's manual but not in the text. However, the 184 page book of "self-correcting exercises" has solutions to many problems which are virtually the same as those in the text.

The self-instruction supplement by Elzey and Armstrong is programmed and was very useful. It emphasizes basic material from the first fourteen out of eighteen

chapters in the text. Several students expressed thanks that it was available. An instructor who wanted to develop his own lecture material without being tied too closely to a text might use this supplement, along with a book of tables, in place of any other text.

The main text has many good features. It covers many topics in a short space; it has a pleasant uncluttered appearance, and considerable thought seems to have gone into selecting and arranging topics, examples, and problems. It also has several bad features.

First, the quality of the exposition is uneven. Chapter 1 is especially confusing. For example, in explaining (pp. 4–6) the differences among nominal, ordinal, interval, and ratio scales the authors appeal to "the order principle", "the distance principle", and "the origin principle" without saying what they take these principles to be. The concept of a random sample is used without explanation in the first exercise set.

Throughout the text important theorems are called "statements," and proofs for all but the last of the statements in Chapter 1 are given. But after that proofs are almost never again mentioned. It would be better to leave out proofs entirely.

Notation is poorly used. For example, the meanings of ambiguous symbols, such as "X", are changed without adequate warning. Explicit indication of the limits of summation in expressions using sigma notation is dropped so soon that students are confused, for example, about how to calculate means of distributions.

In their discussion of correlation the authors are not very careful about the distinction between mathematical and causal relations. They also make occasional wild claims such as the "rule" (p. 60) that in cross-classification tables, taking percentages within classes of the independent variable, will expose "any" relationship between the independent and the dependent variable. They also miss at this point a perfect opportunity to discuss the Humean problem of induction in the context of using past correlations as a basis for predictions about future correlations.

Their discussion of probability glosses over important distinctions among observed relative frequencies, *a priori* relative frequencies, limits of relative frequencies, and their relations to random selection procedures and probability assignments. As a result, the student, quite literally, does not know what to expect.

In summary, this text has many infuriating blunders, confusions, and infelicities. But students are able to learn from it and most of its problems could be easily corrected. So while I would not choose to use it again as it stands, I might choose to use an improved second edition.

R. E. Lover, University of North Carolina at Charlotte

TELEGRAPHIC REVIEWS

Telegraphic reviews are designed to give prompt notice of new books with sufficient information to assist our readers in deciding whether to order an examination copy or to suggest library purchase. Possible uses are indicated as follows:

T = textbook P = professional reading
S = supplementary reading L = undergraduate library purchase
13 to 18 = freshman to second year graduate level usage
1 to 4 = appropriate time in semesters to cover text

Asterisks (*) or question marks (?) denote special positive or negative emphasis, respectively. Publishers are denoted by standard abbreviations; complete addresses may be found in *Books in Print*.

GENERAL, L. *Handbook of Mathematical Calculations for Science Students and Researchers*. Karen Assaf, Sáid A. Assaf. Iowa St U Pr, 1974, xv + 309 pp, $10.50. A collection of mathematical facts, formulae, tables, and applications. An incredible potpourri ranging from elementary algebra tricks through calculus tables with the Russian alphabet somewhere in between. Something for everybody. TAV

GENERAL, S*(13). *The Stanford Mathematics Problem Book with Hints and Solutions*. George Polya, Jeremy Kilpatrick. Teachers College Pr, 1974, 68 pp, $3.50 (P). All 20 sets of problems from the Stanford Mathematics Exams (1946-1965), followed by a section of hints, then complete solutions. Many problems are intriguing, and their solutions instructive. TAV

GENERAL, S(13). *Metric Math: The Modernized Metric System (SI)*. James R. Smart. Brooks/Cole, 1974, x + 85 pp, $3.50 (P). Designed to help people "understand the metric system" and "to use it with confidence." A short historical introduction is followed by chapters on units of length, area and volume, and mass (weight). A final chapter introduces additional units of a more scientific nature. May be used as a text for a short course or unit on the metric system. RSK

GENERAL, S?(13). *The Penguin Book of Tables*. Penguin Education, 1974, 39 pp, $.95 (P). Brief precalculus tables of minimal accuracy (usually 4 or 5 places): squares and square roots, log, log trig, trig, exponential, etc. Second exponential table has algebraically incorrect headings. RSK

GENERAL, P. *Transactions of the Nineteenth Conference of Army Mathematicians*. US Army, 1973. *Part 1:* xxi + 586 pp; *Part 2:* xi + 605 pp. Papers from a May, 1973 conference in Orlando, Florida. LAS

GENERAL, P. *Transactions of the Moscow Mathematical Society for the Year 1972, V. 26*. AMS, 1974, iii + 239 pp, $35.50.

GENERAL, S. *Mathematical Games and Puzzles*. Trevor Rice. St. Martin's Pr, 1973, 95 pp, $5.95. A collection of about 40 old chestnuts (e.g., tangrams, nim, hex) suitable for construction and use in school mathematics clubs. No mention of the underlying mathematical principles. LAS

BASIC, T(13: 1, 2). *Introduction to Technical Mathematics*. Gilbert M. Peter, Daniel R. Peterson. Scott F, 1974, 416 pp, $8.95. Arithmetic review, the slide rule, algebra, informal geometry, trigonometry, vectors. FLW

BASIC, T(13: 1). *Developing Skills in Algebra: A Lecture Worktext, 2 volumes.* J. Louis Nanny, John L. Cable. Allyn, 1974. *V. I:* vii + 239 pp, $4.50 (P); *V. II:* v + 346 pp, $6.25 (P). Designed for students proficient in arithmetic, but who lack manipulative skills in algebra. May be used with or without a teacher. Examples are well chosen, and rules of procedure are clearly stated. Answers to all exercises appear in the back. Volume I: fundamental concepts, first degree equations--one unknown, products and factoring, algebraic fractions. Volume II: exponents and radicals, quadratic equations, functions and systems of equations, logarithms. RBK

BASIC, T(13). *Plane Trigonometry with Tables, Fourth Edition.* E. Richard Heineman. McGraw, 1974, xvi + 325 pp, $10.50. A standard treatment of trigonometry, starting with the definitions in terms of angles. Not much emphasis on the notion of a function. Includes chapters on logarithms and complex numbers. Ample exercises, index, no bibliography. PJM

BASIC, S(13). *Basic Mathematical Skills.* Thomas G. Smithsi. P-H, 1974, x + 332 pp, $6.95 (P). A "semi-programmed" text to teach basic arithmetic skills. Informal style should appeal to students and the 4200 exercises should teach them to add, subtract, multiply and divide. TAV

BASIC, T(13: 1), S. *Arithmetic.* Sandra Preis, George Cocks. P-H, 1974, xiii + 444 pp, $9.50 (P). Programmed text containing twenty-two chapters, with ten diagnostic pretests to determine what material to study. First twenty chapters cover the four fundamental operations on integers, fractions and decimals. Last two cover percent, ratio and proportion. RSK

BASIC, T(13: 1, 2). *Algebra Text.* Robert H. Alwin, Robert D. Hackworth, Joseph W. Howland. P-H, 1974. *Elementary,* xi + 700 pp, $8.95 (P); *Intermediate,* xi + 654 pp, $8.95 (P). Instructional texts containing objectives, summaries, progress tests, self-tests, exercises, and marginal notes. LLK

BASIC, T(13). *Intermediate Algebra, Third Edition.* Roy Dubisch, Vernon E. Howes. Wiley, 1974, xi + 384 pp, $9.95. A well written book, mostly on polynomials, with chapters on sequences, series, complex numbers and matrices. Those last two chapters seem out of place, but otherwise a good book. PJM

PRECALCULUS, T(13: 1). *College Algebra.* Mustafa A. Munem, William Tschirhart, James P. Yizze. Worth, 1974, ix + 518 pp, $9.95; *Study Guide,* 346 pp, $3.95 (P). Many combinations of pre-calculus topics are being published by combinations of these authors in a successful format of text and study guide. This one is for students with one year of high school geometry and one year of high school algebra.LLK

PRECALCULUS, T(13: 1, 2). *College Algebra and Trigonometry, Second Edition.* Steven J. Bryant, *et al.* Goodyear, 1974, 474 pp, $11.95. Revision of First Edition (TR, June 1970) includes miscellaneous problems at the end of each chapter, expanded trigonometry chapter, and a section on linear programming. New two-color format with wide margins, sometimes used for graphs. LLK

PRECALCULUS, T(13: 1). *College Algebra with Trigonometry.* Raymond A. Barnett. McGraw, 1974, xvi + 557 pp, $11.95. "Another College Algebra", justified by the author with ordered exercises, Level A, B, and C. LLK

PRECALCULUS, T(13: 1). *Algebra and the Elementary Functions with Included Instructor's Guide: Special Edition.* William L. Hart. Goodyear, 1974, xv + 478 pp, $13.95. Insertion of grey sections with comments for the teacher, suggested tests, and answers to even problems sets this apart. There are plenty of exercises and examples. LLK

EDUCATION, T(13: 1). *Mathematics: An Integrated View.* Roland F. Smith. Merrill, 1974, ix + 374 pp, $11.95. Sequential treatment of sets, logic and number systems. Seems appropriate for elementary education majors. LLK

EDUCATION, T(15-16), S, P, L. *How Children Learn Mathematics, Second Edition: Teaching Implications of Piaget's Research.* Richard W. Copeland. Macmillan, 1974, ix + 374 pp, $9.50. An apparently unquestioning account of Piaget's research and its implications for teaching elementary school mathematics. Intended for both pre- and in-service methods courses. Provides suggestions for implementing findings in the classroom. Awkward sentence structure sometimes impedes reading. New chapters focus on a child's conception of time and proportion, and on the relation between mathematics and genetic epistemology. No bibliography. PSJ

EDUCATION, T(13: 1, 2). *Essentials of Elementary School Mathematics.* Max D. Larsen, James L. Fejfar. Acad Pr, 1974, xx + 410 pp, $10.95. Truth tables, sets, relations, types of numbers (defined in terms of equivalence classes) and operations on them, informal motion geometry, coordinate geometry. FLW

EDUCATION, T(13-14: 1), S. *Arithematics: A Text for Elementary School Teachers.* Robert L. Johnson, Charles R. McNerney. Macmillan, 1974, xiv + 476 pp, $9.95. Interesting presentation of topics such as logic, number theory, etc. with historical inserts. Good resource book for elementary teachers. LLK

HISTORY, P, L*. *Galileo Galilei: Two New Sciences.* Transl: Stillman Drake. U of Wisc Pr, 1974, xxxix + 323 pp, $4.50 (P); $12.50. An entirely new translation (the first into English since the standard one of 1914) of the first (1638) edition together with additional notes dictated by Galileo. Includes a very useful introduction with appropriate biographical and editorial details, and a glossary of Galileo's technical vocabulary. LAS

HISTORY, L. *Oeuvres de Paul Lévy, V. I.* Ed: Daniel Dugué. Gauthier-Villars, 1973, ix + 497 pp, 140F. First of five projected volumes, this one contains papers on functional analysis, potential theory and symbolic calculus. LAS

HISTORY, P. *Mathematische Abhandlungen.* O. Bolza, *et al.* Chelsea, 1974, viii + 451 pp, $17.50. Unaltered reprint of a 1914 collection of papers in honor of H.A. Schwarz known informally as "Festschrift Schwarz." Contributors include Carathéodory, Féjèr, Hilbert, Hurwitz, Landau, Schur, Toeplitz. LAS

HISTORY, P. *Ancient Logic and Its Modern Interpretations.* Ed: John Corcoran. Reidel, 1974, x + 211 pp, $29. Ten papers (including three by Corcoran) plus a panel discussion from a symposium held at SUNY at Buffalo, April 1972. "Surely the most complete discussion of the current state of historical knowledge in the area of ancient logic." LAS

FOUNDATIONS, P. *Combinatorial Functors.* J.N. Crossley, Anil Nerode. Ergebnisse der Math., B. 81. Springer-Verlag, 1974, viii + 146 pp, $13.90. Combinatorial functors are defined in sufficient generality for a great variety of present and hoped-for future applications in the first short chapter. The rest of the book consists of applications to the classification of models. JAS

FOUNDATIONS, P. *Lecture Notes in Mathematics-369: Victoria Symposium on Nonstandard Analysis.* Ed: Albert Hurd, Peter Loeb. Springer-Verlag, 1974, xviii + 339 pp, $10.10 (P). Over two dozen papers from a May, 1974 symposium at U. Victoria. A sequel to previous symposia at Oberwolfach, 1970 and Cal Tech, 1968. A useful touch: the table of contents includes abstracts of the articles. LAS

FOUNDATIONS, T(16-18), P, L. *From Mathematics to Philosophy.* Hao Wang. Humanities Pr, 1974, xiv + 428 pp, $21. An extensive, personal philosophical analysis (called "substantial factualism") of contemporary mathematics (including two chapters on computers and machine theory) motivated by a concern that philosophy (and philosophers) take more seriously substantial parts of human knowledge, e.g. mathematics. Includes some direct and indirect (via prepublication critique) contributions by Kurt Gödel. LAS

ALGEBRA, P. *On the Theory and Applications of Differential Torsion Products.* V.K.A.M. Gugenheim, J. Peter May. Memoirs No. 142. AMS, 1974, ix + 93 pp, $3.20 (P). This memoir contains a new definition of the torsion product $\mathrm{Tor}^U(M,N)$, for U a differential algebra, and M and N differential U-modules. The new definition is computationally more convenient than the classical one of Eilenberg and Moore, and application is made to computation of homology groups of some topological spaces. PJM

ALGEBRA, P. *Lecture Notes in Mathematics-370: Universal Extensions and One Dimensional Crystalline Cohomology.* Barry Mazur, William Messing. Springer-Verlag, 1974, vii + 134 pp, $6.60 (P). A universal extension is an algebraic group associated to an Abelian variety. A crystal is a "nice" sheaf. In these notes, the Lie algebra of a universal extension is shown to be isomorphic to the one dimensional crystalline cohomology of an appropriate module. PJM

ALGEBRA, P. *La Série Génératrice Exponentielle Dans les Problèmes D'Enumération.* Dominique Foata. Pr U Montreal, 1974, 186 pp, $5 (P). Some recent results in applications of formal power series to enumeration problems, e.g., how many alternating permutations are there on a finite set? The answer is given for all sets by the coefficients of a formal power series. PJM

ALGEBRA, P. *Fully Ordered Groups.* A.I. Kokorin, V.M. Kopytov. Transl: D. Louvish. Halsted Pr, 1974, x + 147 pp, $23. Systematic study of the subject, suitable for an advanced graduate seminar (especially since open problems are mentioned throughout). Includes some material on order and topology, and on ordered division rings and modules. 250 references. DFA

ALGEBRA, P. *The Separable Galois Theory of Commutative Rings.* Andy R. Magid. Pure and Appl. Math., V. 27. Dekker, 1974, xvii + 134 pp, $11.75. An exposition of recent developments in separable algebras. Background chapters on profinite topological spaces and the Boolean spectrum of a commutative ring lead to the Galois theory of algebras with only trivial idempotents and to the study of the fundamental groupoid. The classification theorem is established and yields the Galois correspondence theorem for certain algebras over a commutative ring. A very readable reference. SG

ALGEBRA, T(16-17: 1, 2), P, L. *Modules: A Primer of Structure Theorems.* Tom Head. Brooks/Cole, 1974, x + 150 pp, $10.95. Designed for a second course in algebra at the senior level. Projective and injective modules, countably generated modules: all are here, with very little homological algebra and only one passing reference to category theory. In fact, no exact sequences. A good book, but not very modern in presentation. PJM

ALGEBRA, P. *Lecture Notes in Mathematics-366: Conjugacy Classes in Algebraic Groups.* Robert Steinberg. Springer-Verlag, 1974, vi + 159 pp, $7.40 (P). Lectures delivered at the Tata Institute. Begins with the basics of affine groups: representation theorems, Jordan decomposition, Kolchin's theorem, rigidity theorem, solvable groups, etc.; second half covers representation theory; classification of semisimple, unipotent, regular, and subregular elements. SG

ALGEBRA, P. *Radical and Semisimple Classes of Rings.* Richard Weigandt. Pure and Appl. Math., No. 37. Queen's U, 1974, iv + 248 pp, $6 (P). An introduction to the radical theory of associative rings. Emphasis on results obtained in last 10 years. Some topics: general theory of radical and semisimple classes, radical constructions, Wedderburn-Artin-Noether structure theorem, various radicals, homomorphically closed semisimple classes. SG

COMPLEX ANALYSIS, P. *n-Dimensional Quasiconformal (QCf) Mappings.* Petru Caraman. Editura Academiei Rep. Soc. Romania, 1974, 551 pp, $14.75. A study of a generalization of conformal maps. Several equivalent fairly technical definitions are given, none suitable for reproduction here. This is a slightly revised translation of an earlier Romanian edition of the book. Index and bibliography are excellent, and provide in particular references to less advanced introductions to the subject. PJM

COMPLEX ANALYSIS, P. *Lectures on Complex Analytic Varieties: Finite Analytic Mappings.* R.C. Gunning. Princeton U Pr, 1974, ii + 163 pp, $4 (P). A sequel to *Lectures on the Local Parametrization Theorem* (TR, January 1971). An analytic mapping is a mapping between germs of analytic varieties. It is finite if the inverse image of the basepoint is the basepoint. Prerequisites for this book are sheaf theory and functions of several complex variables. PJM

COMPLEX ANALYSIS, P. *Proceedings of the Symposium on Complex Analysis, Canterbury, 1973.* Ed: J. Clunie, W.K. Hayman. London Math. Soc. Lect. Notes, No. 12. Cambridge U Pr, 1974, 180 pp, $9.50 (P). 31 research papers followed by a survey of old and new research problems in function theory by W.K. Hayman. LAS

DIFFERENTIAL EQUATIONS, P. *Lecture Notes in Mathematics-362: Proceedings of the Conference on the Numerical Solution of Ordinary Differential Equations.* Ed: Dale G. Bettis. Springer-Verlag, 1974, viii + 490 pp, $14 (P). Papers from a 1972 conference in Austin, Texas with a special focus on the n-body problem. LAS

DIFFERENTIAL EQUATIONS, P. *Nonlinear Wave Motion.* Ed: Alan C. Newell. Lect. in Appl. Math., V. 15. AMS, 1974, viii + 229 pp, $19.10. Proceedings of the 1972 AMS summer seminar from Potsdam, N.Y. Includes major lectures by Benjamin, Benney, Kruskal, Lax and Whitham surveying significant advances of the last decade. LAS

DIFFERENTIAL EQUATIONS, P. *Spectral Theory and Asymptotics of Differential Equations.* E.M. DeJager. Math. Stud., V. 13. North-Holland, 1974, 208 pp, $10.80 (P). Invited addresses from a conference held in September, 1973 at Scheveningen, the Netherlands. LAS

DIFFERENTIAL EQUATIONS, T(15: 2). *An Introduction to Applied Mathematics, Second Edition.* J.C. Jaeger, A.M. Starfield. Oxford U Pr, 1974, xii + 504 pp, $27.25. Updates 1951 edition by adding material on difference equations, matrices, and numerical solution of ordinary and partial differential equations. Still provides a combination of the first course in differential equations and its "applied advanced calculus" successor. DFA

DIFFERENTIAL EQUATIONS, P. *Direct and Inverse Imbedding Theorems. Applications to the Solution of Elliptic Equations by Variational Methods.* L.D. Kudrjavcev. Transl. Math. Mono., V. 42. AMS, 1974, iv + 206 pp, $22.40. Weighted function extensions from the boundary onto the whole domain, weighted imbedding theorems, general variational principles concerning the first boundary problem for self-adjoint elliptic equations of second order, existence and uniqueness theorems for boundary problems of elliptic differential equations degenerate on the boundary of the domain. DFA

DIFFERENTIAL EQUATIONS, T(14: 1). *Elementary Differential Equations, Fifth Edition.* Earl D. Rainville, Phillip E. Bedient. Macmillan, 1974, xiii + 511 pp, $10.95. A new revision by Bedient (4th Edition, TR, June 1969) of the late Rainville's book. Revision is mainly the expansion of the chapter on systems of linear equations to use matrix methods. LLK

NUMERICAL ANALYSIS, P. *Numerische Methoden bei Differentialgleichungen und mit funktionalanalytischen Hilfsmitteln.* J. Albrecht, L. Collatz. Birkhauser, 1974, 231 pp, Sfr. 59. Lectures from two conferences with a common purpose, namely, forwarding contact between theoretical and applied mathematics. The conferences whose names generated the title took place at Oberwolfach and the Technical University of Clausthal-Zellerfeld in June 1972. JAS

NUMERICAL ANALYSIS, T(16: 1), S, L. *Discrete Numerical Methods in Physics and Engineering.* Donald Greenspan. Math. in Sci. and Eng., V. 107. Acad Pr, 1974, xi + 312 pp, $12.50. A useful, basic set of notes on the numerical solution of ordinary and partial differential equations, especially those arising in fluid dynamics. A brief, preliminary chapter on solving linear systems of algebraic equations. One appendix is an essay on the nature of mathematics; the other is a program for solving the Navier-Stokes equation. Illustrative exercises. RWN

NUMERICAL ANALYSIS, P. *Lecture Notes in Mathematics-363: Conference on the Numerical Solution of Differential Equations.* Ed: G.A. Watson. Springer-Verlag, 1974, ix + 221 pp, $7.70 (P). 20 invited papers from the fifth biennial conference at Dundee, Scotland, LAS

FUNCTIONAL ANALYSIS, P. *The Structure of Factors.* Salvatore Anastasio, Paul Mitchell Willig. Algorithmics Pr, 1974, iii + 116 pp, $15. A sequel to Schwartz' *W-* Algebras.* Constructs continua of mutually non-isomorphic factors of type II_1, of type II_∞, of hyperfinite type III, and of non-hyperfinite type III. Describes Araki-Woods results on infinite tensor product factors. Historical, bibliographical remarks. DFA

FUNCTIONAL ANALYSIS, P. *Product Formulas, Nonlinear Semigroups, and Addition of Unbounded Operators.* Paul R. Chernoff. Memoirs No. 140. AMS, 1974, v + 121 pp, $3.30 (P). Theory of linear and nonlinear operator semigroup formulas of the type $\lim_{n\to\infty} F(t/n)^n = G(t)$. Defines (using the general Trotter-Lie product formula) and discusses generalized addition of semigroup generators, and self-adjoint operators in particular. DFA

FUNCTIONAL ANALYSIS, S(16-17), P. *Distributionen und ihre Anwendung in der Physik.* F. Constantinescu. B.G. Teubner, 1974, 144 pp, DM 16,80 (P). An introduction, intended for mathematicians and physicists, to the theory of distributions and their applications in physics. Very tersely written. Assumes a knowledge of the elements of linear algebra, general topology and functional analysis. No exercises. JD-B

FUNCTIONAL ANALYSIS, P. *Physical Aspects of Lie Group Theory.* Robert Hermann. Pr U Montreal, 1974, 271 pp, $9.50. A sequel to the author's 1969 *Fourier Analysis on Groups and Partial Wave Analysis* dealing with the interconnection between operator theory and the Fourier analysis of functions on groups. LAS

FUNCTIONAL ANALYSIS, P. *Lecture Notes in Mathematics-384: Functional Analysis and Applications.* Ed: Leopoldo Nachbin. Springer-Verlag, 1974, 270 pp, $9.10 (P). Invited lectures from a July, 1972 symposium in Recife, Brasil, except for a series of lectures by F. Trèves which was published separately. LAS

OPTIMIZATION, T*(14: 1), L. *Sets, Matrices, and Linear Programming.* Robert L. Childress. P-H, 1974, xii + 356 pp, $10.95. The heart of this book is linear programming; the sections on sets (through $\cup, \cap$) and matrices (through inverses, transpose) are included as a background for the rest. The treatment of linear programming is lucid and complete with numerous examples, and sections on duality and sensitivity and analysis. Concludes with integer programming including the branch and bound algorithm. TAV

OPTIMIZATION, P, L. *Lecture Notes in Economics and Mathematicsl Systems-95: Linear Multiobjective Programming.* M. Zeleny. Springer-Verlag, 1974, x + 220 pp, $7.70 (P). The first complete extension of the simplex algorithm to non-comparable multi-valued objectives; the search for one optimal solution is replaced by the specification and description of a set of non-dominated solutions. Unfortunately, this set is often unmanageably large, so methods must yet be developed to find a small "representative" subset. LAS

ANALYSIS, P. *Lecture Notes in Mathematics-358: Tensor Products of Principal Series Representations.* Floyd L. Williams. Springer-Verlag, 1973, vi + 132 pp, $6.20 (P). Lecture notes on complex semisimple Lie groups from the viewpoint of induced representations. LAS

ANALYSIS, T*(17: 2), P. *Advanced Mathematical Analysis.* Richard Beals. Grad. Texts in Math., V. 12. Springer-Verlag, 1973, x + 230 pp, $9.50 (P). This attempt to present a unified treatment of advanced analysis for both mathematics and engineering students is eminently successful. The unifying feature is an early and complete treatment of distributions, e.g., L^2 theory of Fourier series is established with no measure theory by showing $L^2(0, 2\pi)$ is complete when viewed as a subspace of the space of periodic distributions, a Hilbert space. TAV

ANALYSIS, S(15-17), P, L. *The Fractional Calculus: Theory and Applications of Differentiation and Integration to Arbitrary Order.* Keith B. Oldham, Jerome Spanier. Math. in Sci. and Eng., V. 111. Acad Pr, 1974, xiii + 234 pp, $19.50. A thorough introductory monograph on the properties and applications of fractional derivatives (e.g., $d^{1/2}x/dx^{1/2}$) which may be defined in terms of the gamma function. Not only is the theory an intriguing and exotic variant of classical analysis (e.g., virtually all named transcendental functions are simple fractional derivatives of just three common functions), but it also has extensive applications to a variety of diffusion problems--discussed in the last chapter of this book. LAS

ANALYSIS, P. *Contributions to Analysis.* Ed: Lars V. Ahlfors, *et al.* Acad Pr, 1974, xvii + 441 pp, $36.50. 32 papers--some expository, some not--by leading analysts in honor of Lipman Bers' sixtieth birthday. Topics range broadly, including Teichmüler spaces, theta functions, quasiconformal maps and differential topology. LAS

ANALYSIS, T(18: 1, 2), S, P, L. *General Theory of Banach Algebras.* Charles E. Rickart. Krieger, 1974, xi + 394 pp, $15. Corrected reprint of well-known 1960 original edition, published at that time by Van Nostrand. LAS

ANALYSIS, P. *Lie Groups, Lie Algebras, and Their Representations.* V. S. Varadarajan. P-H, 1974, xiii + 430 pp, $19.95. General theory of Lie groups and their Lie algebras, structure theory of Lie algebras and also, specifically, of complex semisimple ones, plus theory of finite-dimensional representations of complex semisimple Lie groups. Comprehensive. Many carefully written exercises extend the text. DFA

GEOMETRY, T(13: 1). *A First Course in Geometry.* Edward T. Walsh. Rinehart Pr, 1974, xiii + 370 pp, $10.95. Attractive and well-written, this presentation of elementary Euclidean metric geometry is saved from being just another geometry text by the author's knack for explaining the algebra of logic and for elucidating the logic involved in the geometric proofs. Annotated bibliography. JNC

GEOMETRY, T(14: 2), L. *Elementary Geometry fron an Advanced Standpoint, Second Edition.* Edwin E. Moise. A-W, 1974, xv + 425 pp, $11.95. A rigorous presentation of Euclidean geometry including the necessary fragments of algebra and number theory. This corrected edition contains more problems and adds a short discussion of isometries and dilatations. JNC

GEOMETRY, T(18), P*, L*, *Noneuclidean Tesselations and Their Groups.* Wilhelm Magnus. Pure and Appl. Math., No. 61. Acad Pr, 1974, xiv + 207 pp, $17.50. An extremely well-written introduction to the group theory, geometry, and function theory associated with tesselations of noneuclidean planes. Concrete approach with a minimum of prerequisites. Based on the tesselations found in Klein and Fricke. Topics: various noneuclidean geometries; triangle tesselations in planar euclidean, hyperbolic, and elliptic goemetry; various discontinuous groups. Accessible to advanced undergraduates and beginning graduates. A beautiful blending of several branches of mathematics. SG

GEOMETRY, T*(14: 1), S, L*, *Projective Geometry, Second Edition.* H.S.M. Coxeter. U of Toronto Pr, 1974, xii + 162 pp, $7.50. Easily readable and spiced with historical anecdotes, this classic text emphasizes the synthetic approach to projective geometry. The second edition corrects errors in the first, contains a slight notational change and a few minor textual additions. JNC

GEOMETRY, T**(16-18), S**, L**, *Calculus of Several Variables and Differentiable Manifolds.* Carl B. Allendoerfer. Macmillan, 1974, xi + 227 pp, $12.95. At last a treasury of folklore for the beginning (or forgetful) differential geometer. A careful treatment of advanced calculus with the choice of topics and approach guided by geometric aims, namely a study of manifolds in Euclidean spaces. Lots of exercises, not enough pictures, and a reasonable index. As a course by itself without an enthusiastic instructor it might be a bit dry and formal but with lots of pictures and enthusiasm from an instructor it should give students some exciting insights into what is really going on in modern differential geometry. JAS

GEOMETRY, P, *Lecture Notes in Mathematics-392: Géométrie Différentielle.* Ed: Enrique Vidal. Springer-Verlag, 1974, vi + 225 pp, $8.20 (P). Proceedings of an October, 1972 conference at the University of Santiago de Compostela, Spain. LAS

GEOMETRY, S, L*, *Curves and Their Properties.* Robert C. Yates. NCTM, 1974, xiv + 245 pp, $6.40. A museum of mathematical morphology: astroids, caustics, glissettes, kieroids, nephroids, strophoids, etc. A photo-offset reprint--fourth in a series of such "Classics in Mathematics Education"--of a work first published in 1952. LAS

TOPOLOGY, P, *Lecture Notes in Mathematics-368: Unstable Homotopy from the Stable Point of View.* R. James Milgram. Springer-Verlag, 1974, 109 pp, $6.20 (P). By generalizing the EHP spectral sequence, the author obtains results about unstable homotopy of spaces by looking at the stable groups for related spaces. Example: The "related spaces" for spheres are truncated projective planes. For other spaces they are more complicated. PJM

TOPOLOGY, P, *Metrizability in Generalized Ordered Spaces.* M.J. Faber. Math. Centre Tracts, No. 53. Math Centrum, 1974, 120 pp, Dfl. 13 (P). A generalized ordered (GO-) space is a subspace of a linearly ordered topological space. This monograph translates traditional topological concepts into order-theoretic terms (e.g., jumps, gaps, convexity) and then characterizes metrizability for GO-spaces. LAS

TOPOLOGY, P. *Stable Homotopy and Generalized Homology*. J.F. Adams. U of Chicago Pr, 1974, x + 373 pp, $11.50 (P). Three separate lectures, the last of which deals with the subject of the title. The first two deal with work of Novikov and Quillen on complex cobordism. A great book for graduate students, if read backwards. PJM

TOPOLOGY, P. *Lecture Notes in Mathematics-378: TOPO 72--General Topology and its Applications*. Ed: Richard A. Alò, Robert W. Heath, Jun-iti Nagata. Springer-Verlag, 1974, xiv + 651 pp, $20.50 (P). 65 research notes from the Second Pittsburgh International Conference in December, 1972, including a memorial to J.H. deGroot by R.H. McDowell. LAS

TOPOLOGY, S(18), P. *Normal Topological Spaces*. Richard A. Alò, Harvey L. Shapiro. Tracts in Math., No. 65. Cambridge U Pr, 1974, xi + 306 pp, $18.50. Designed for a second course in general topology, this monograph uses normality as a vehicle for illustrating current research tools of set-theoretical topology. Focuses on normal covers, pseudometrics and uniformities. LAS

PROBABILITY, T(18: 2). *Stochastic Processes*. John Lamperti. Lect. Notes Ser., No. 38. Aarhus U, 1974, iii + 221 pp, $4.25 (P). The first half deals with stationary processes, the second with Markov processes. The treatment concentrates on the mathematical aspects, not the applications of the theory. Presumes familiarity with measure and Hilbert space theory. TAV

PROBABILITY, P. *Advances in Probability and Related Topics, V. 3*. Ed: Peter Ney, Sidney Port. Dekker, 1974, viii + 410 pp, $25.75. 4 survey papers: M.A. Pinsky on multiplicative operator functionals, L.C. Young on stochastic and Stieltjes integrals (part II), P. Jagers on random measures and B. Fristedt on sample functions of stochastic processes. LAS

PROBABILITY, T(17: 1, 2). *A Course in Probability Theory, Second Edition*. Kai Lai Chung. Prob. and Math. Stat., No. 21. Acad Pr, 1974, xii + 365 pp, $15. Revision of the 1968 *First Edition* (TR, August 1968). Mathematical treatment presuming elementary real variables and some measure theory. RSK

PROBABILITY, T, S*(15-17), P**, L**. *Theory of Probability: A Critical Introductory Treatment, Volume 1*. Bruno DeFinetti. Transl: Antonio Machí, Adrian Smith. Wiley, 1974, xix + 300 pp, $22.50. A personal, hortative treatise in a vivid idiosyncratic style setting forth the subjectivist position that "probability does not exist." Addressed to all who use probability, not necessarily to those (e.g., college sophomores) learning it for the first time. Translated from the 1970 original *Teoria Delle Probabilità*. According to D.V. Lindley, writing in the foreward, this book is "destined ultimately to be recognized as one of the great books of the world." LAS

PROBABILITY, T*(18: 2), P**. *Fundamentals of Queueing Theory*. Donald Gross, Carl M. Harris. Wiley, 1974, xvi + 556 pp, $22.50. A comprehensive modern treatment. Style and level will appeal to both mathematicians and non-mathematicians. Contains a thorough treatment of (M/M/1) queue and extensions to more involved settings. Useful appendices and huge bibliography make this one worth the price. TAV

PROBABILITY, P. *Lecture Notes in Mathematics-381: Séminaire de Probabilités VIII, Université de Strasbourg.* C. Dellacherie, P.A. Meyer, M. Weil. Springer-Verlag, 1974, 354 pp, $13.10 (P). 20 seminar lectures from 1972-73 focusing on various stochastic processes and on the "problem of Skorokhod." LAS

STATISTICS, S(13-16), L. *Experimental Design and its Statistical Basis.* D.J. Finney. U of Chicago Pr, 1974, xi + 169 pp, $2.95 (P). An informal treatment with many biological examples. FLW

STATISTICS, T(15-17: 1, 2), S. *Mathematical Methods of Statistical Quality Control.* K. Sarkadi, I. Vincze. Prob. and Math. Stat., No. 18. Acad Pr, 1974, 415 pp, $22. More than half devoted to a survey of standard post-calculus probability and statistics. The remainder considers control of production processes, acceptance sampling, and reliability theory. No exercises. Few proofs. FLW

STATISTICS, T(16-17: 1), S, L. *Multivariate Statistical Methods for Business and Economics.* Ben W. Bolch, Cliff J. Huang. P-H, 1974, xiv + 329 pp, $12.95. Presupposes calculus-based statistics (but reviews parts of this) and linear algebra. Includes analysis of covariance, nonlinear regression, discrimination, principal components, canonical correlations, spectral analysis, and computer programs in FORTRAN. Few proofs. FLW

STATISTICS, T(13-14: 1). *Business Statistics, A First Course.* John E. Freund, Benjamin M. Perles. P-H, 1974, viii + 359 pp, $10.95. Presupposes only high school algebra. A well written introduction to basic concepts and techniques, including time series. Helpful bibliography. FLW

STATISTICS, T*(15: 2). *Introduction to Probability Theory and Statistical Inference, Second Edition.* Harold J. Larson. Wiley, 1974, xi + 430 pp, $12.95. In the Wiley Series in Probability and Mathematical Statistics. Modest revision of the author's successful 1969 text (ER, November 1972). Additions include a discussion of sufficient statistics, and over 50% more problems. RSK

STATISTICS, T*(13-14: 1). *Elements of Probability and Statistics, Second Edition.* Frank L. Wolf. McGraw, 1974, xiv + 450 pp, $13.50. In the McGraw-Hill Series in Probability and Statistics. Revision of the author's 1962 text, providing a mathematical approach at an elementary level. Unstarred sections are suitable for a solid precalculus course, while starred sections contain optional material. Double-starred sections add mathematical topics which bring the book in line with CUPM's recommendations for a course following a term of calculus (Course 2P in *A General Curriculum in Mathematics for Colleges).* RSK

STATISTICS, T(18), P. *Nonlinear Parameter Estimation.* Yonathan Bard. Acad Pr, 1974, x + 341 pp, $22. Non-theoretical but technical presentation of the various aspects of the problem of estimating parameters when the equations of the models are not linear in the unknown parameters. Covers formulation of the problem, computational methods, interpretation, and design considerations, together with some special topics. Describes several computational algorithms. Matrix algebra is widely used throughout. Good set of references. RSK

STATISTICS, T(13-14: 1, 2). *Introductory Statistics.* Peter W. Zehna. Prindle, 1974, x + 470 pp, $12.50. Presupposes only high school algebra. The usual topics plus some non-parametric tests, good tables, and useful summaries of standard tests and confidence intervals. FLW

STATISTICS, T(13-14: 1). *Introduction to Statistics, Second Edition.* Ronald W. Walpole. Macmillan, 1974, xiii + 340 pp, $9.95. Presupposes only high school algebra. Probability and standard statistical topics plus a little on decision theory and Bayesian methods. FLW

STATISTICS, T*(14-15: 1, 2), L. *Applied Linear Statistical Models: Regression, Analysis of Variance, and Experimental Designs.* John Neter, William Wasserman. Irwin, 1974, xvii + 842 pp, $15.95. Presents a unified approach to the three topics, with approximately half the text devoted to regression, a third to analysis of variance and a sixth to experimental designs. Assumes a solid background in elementary statistical inference, but no calculus. Necessary matrix algebra for multiple regression is introduced in the text. Applications are mainly to business and economics. Contains many special topics not usually covered, such as a full chapter on indicator variables. May be useful as a reference, particularly in the area of regression. RSK

STATISTICS, T*(13: 1). *Introductory Statistical Analysis.* T.W. Anderson, Stanley L. Sclove. HM, 1974, xv + 499 pp, $11.95. Well-written, attractively printed elementary text. Contains three main sections, on descriptive statistics (longest), probability (shortest), and statistical inference. Mathematical level is low, with proofs relegated to appendices and separate sets of mathematical exercises, but the conceptual level of what is presented is high. Concludes with tests of differences between two populations. RSK

STATISTICS, T(14: 1). *Applied Statistical Methods.* Irving W. Burr. Acad Pr, 1974, xix + 479 pp, $15.95. Nontheoretical presentation of primarily standard first course topics. Some calculus is assumed, but is mainly required for a chapter on continuous probability distributions. Contains many realistic examples and problems, mostly industrial in nature. RSK

STATISTICS, T(1), P. *Design of Experiments: A Realistic Approach.* Virgil L. Anderson, Robert A. McLean. Statistics, V. 5. Dekker, 1974, xvii + 418 pp, $19.75. Introductory text assuming a statistical background through regression and one-way analysis of variance. Unique feature is its emphasis on "restriction error", the effect that restrictions on randomization have on an experiment, and the unrealistic conclusions that can be reached by not being aware of it. Concentrates on completely randomized, randomized complete block, nested, and split plot designs, and omits all designs involving concomitant variables. Contains many realistic examples from a variety of fields. RSK

Reviewers Whose Initials Appear Above

David F. Appleyard, Carleton; Judith N. Cederberg, St. Olaf; John Dyer-Bennet, Carleton; Steven Galovich, Carleton; Paul S. Jorgensen, Carleton; Lorraine L. Keller, St. Olaf; Roger B. Kirchner, Carleton; Richard S. Kleber, St. Olaf; Pierre J. Malraison, Carleton; R.W. Nau, Carleton; J. Arthur Seebach, Jr., St. Olaf; Lynn A. Steen, St. Olaf; T.A. Vessey, St. Olaf; Frank L. Wolf, Carleton.

NEWS AND NOTICES

EDITED BY RAOUL HAILPERN, SUNY at Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to Mathematical Association of America, 1225 Connecticut Avenue, NW, Washington, D. C. 20036. Items must be submitted at least two months before publication can take place.

PERSONAL ITEMS

University of Missouri — Rolla: Associate Professor L. J. Grimm has been promoted to Professor; Mr. George Luffel has been promoted from Instructor to Assistant Professor.

Dr. Jack Alanen, University of Nairobi, has been appointed Associate Professor in the Department of Computer Science, State University of New York at Buffalo.

Professor John Dyer-Bennet, Carleton College, will be on sabbatical leave in Zurich, Switzerland, during the 1974–75 academic year.

Professor M. S. Klamkin, Ford Scientific Laboratory, has been appointed Visiting Professor at the University of Waterloo for the academic year 1974–1975.

Brother Brendan Kneale, Chairman of the Department of Mathematics, Saint Mary's College of California, has been appointed Dean of Studies.

Dr. Ellen Torrance, Sterling College, has been appointed Visiting Assistant Professor at Kansas State University.

Professor A. W. Tucker, Princeton University, retired on July 1, 1974, with the title "Albert Baldwin Dod Professor of Mathematics, Emeritus."

Professor Emeritus Bancroft Huntington Brown, Dartmouth College, died on May 7, 1974, at the age of 79. He was a member of the Association for fifty-five years.

Dr. Paul E. Guenther, Case Western Reserve University, died on April 28, 1974, at the age of 58. He was a member of the Association for twenty-five years.

Professor Paul M. Hummel, University of Alabama, died on May 18, 1974, at the age of 66. He was a member of the Association for thirty-seven years.

Professor Emeritus S. Roscoe Smith, University of Wyoming, died on May 22, 1974. He was a member of the Association for thirty-seven years.

PROGRAM TO TRAIN MATHEMATICAL SCHOLARS TO SERVE NEEDS OF PRECOLLEGE SCHOOLS

The National Science Foundation has funded a program at Washington State University designed to acquaint mathematical scholars with most of the ingredients of precollege mathematics education. The scholars will engage in two summer sessions at Washington State University analysing pertinent education theory and an academic year attached to a major school system. Included among the staff are Calvin Long, Eldon Egbers, Jack Robertson, Duane De Temple and James Jordan (director). Anyone interested in being a participant should contact the director at Mathematics Department, Washington State University, Pullman, Washington 99163.

MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

APRIL MEETING OF THE METROPOLITAN NEW YORK SECTION

The thirty-third annual meeting of the Metropolitan New York Section of the MAA was held at the College of Mount St. Vincent, Riverdale, New York, on April 28, 1974. One hundred and forty-one persons registered.

Professor Israel Rose of Lehman College, CUNY, Chairman of the Section, presided over the morning session which began with the business meeting. After reports and discussion on matters of interest to the MAA in general, and the Metropolitan New York Section in particular, it was announced that (1) the recipient of the Section Award to that student in a Metropolitan New York Section School who scores the highest in the Putnam Competition was Julius Collins of the Polytechnic Institute of New York, and (2) Professor Howard Kleiman of Queensborough Community College, CUNY, and Professor William Zlot of New York University have received a grant from the MAA towards the development of the newly formed Section committee, the Mathematics Services Committee for Two-Year and Four-Year Colleges.

The morning session continued with the following lectures:

1. *Teaching mathematics in a preceptorial manner*, by Louis Auslander, Graduate Center, CUNY.
2. *Maximum principles in differential equations*, by David Ellis, Lehman College, CUNY.

Professor Gerald Freilich of Queens College, CUNY, Vice-Chairman for Four-Year Colleges, presided over the afternoon session which began with a panel discussion on *The New Math : Pro and Con*, moderated by Professor Freilich. The panelists were Professor Howard Fehr, Teachers College, Columbia University (Keynote Address); Ms. Cecile Cohen, John F. Kennedy H. S.; Professor Julius Hlavaty, Iona College; Professor Peter Lax, Courant Institute, NYU; and Professor Edwin Moise, Queens College, CUNY.

The following student and faculty papers were then given in parallel sessions:

1. *Summing series of the form* $\sum_{k=1}^{\infty} f(k)/n^k$, by Henry Ricardo, Manhattan College.
2. *An algorithm for extracting n-th roots*, by George Sheehan, student, Fordham University.
3. *Using the classical means to extract square roots*, by Joseph Ercolano, Baruch College, CUNY.
4. *Some aspects of infinite series*, by Arthur Schlissel, John Jay College, CUNY.
5. $1^k + 2^k + \ldots + n^k$ *and Bernoulli's Numbers*, by David Groisser, Math. Fair Winner, Midwood H. S., Brooklyn.
6. *A Hamiltonian simplex to solve the transportation algorithm*, by Turken Kumbaraci, Teka Medical Profiles, N.Y.C.
7. *Predicting future performance of stocks by computerized analysis of statistical factors*, by William Korn, student, Bronx H.S. of Science.
8. *On gravitational collapse in Schwartzchild geometry*, by Gregory Swain, Math. Fair Winner, Deer Park H.S.
9. *A Markov model for the "Dynamite Charge" to juries*, by Jeffrey Silver, Bronx Community College, CUNY, and Robert Silver, Harvard University.
10. *Geodesic domes*, by Robert Klitzman, Math. Fair Winner, Half Hallow Hills H.S., L.I.,
11. *Constructibility of angles whose degree measure is rational*, by Jacob Brandler and Ira Wolf, Brooklyn College, CUNY.

12. *On surface area*, by Leopoldo Toralballa, Bronx Community College, CUNY.

13. *Upper and lower bounds for the number of polysquares of order n in R^2 and generalizations in R^p*, by Joshua Massey, Math. Fair Winner, John Dewey H.S.

14. *Advanced differential equations by elementary geometry*, by H. Guggenheimer, Polytechnic Institute of N.Y.

15. *Mechanical proof that altitudes are associated*, by Leon Gerber, St. John's University, Jamaica, N.Y.

16. *Even numbers expressible as the sum of certain types of prime numbers*, by David Zwillinger, student, Bronx H.S. of Science.

17. *A "formula" for the nth prime*, by Robert Bumcrot, Hofstra University.

18. *Latin k-squares and k-cubes of order* 10, by Joseph Arkin, Spring Valley, N.Y.

19. *Quadratic partitions of selected large numbers*, by Daniel Finkel, Brooklyn, N.Y.

RORA IACOBACCI, *Secretary*

APRIL MEETING OF THE OKLAHOMA-ARKANSAS SECTION

The thirty-sixth annual meeting of the Oklahoma-Arkansas Section of the MAA was held at the University of Arkansas at Little Rock, Little Rock. Arkansas, on April 5–6, 1974. There were 91 members and 56 nonmembers attending for a total of 147. Professor Bill Spicer, of Seminole Junior College, presided over the meeting.

A dinner was held on Friday evening in the Student Union. Mr. Ernest Brickell of Oklahoma State University received an award for scoring the highest on the William Lowell Putnam Competition among the participants from Oklahoma and Arkansas. The invited address entitled "How to Make and Break Codes" was given of Friday afternoon by Professor Dorothy Bernstein of Goucher College. Friday evening, Professor L. W. Johnson, Professor Emeritus of Oklahoma State University, gave the second N. A. Court Lecture. The title was "A Geometric Derivation of the Basic Theorem for a Rigid Motion."

The following officers were elected: Chairman, Professor J. R. Hodges, University of Arkansas at Little Rock; Past-Chairman, Professor Bill Spicer, Seminole Junior College; First Vice-Chairman, Professor J. H. Yates, Central Oklahoma State University; Second Vice-Chairman, Gus Pekara, South Oklahoma City Community College; and Secretary-Treasurer E. K. McLachlan, Oklahoma State University. The following were elected as co-chairmen of the High School Mathematics Contest: For Arkansas, Professor E. E. McGehee, Jr., of the State College of Arkansas, and for Oklahoma, Professor Thomas Cairns of the University of Tulsa.

In Oklahoma 53 schools involving 1697 students and in Arkansas 41 schools participated in the high school contests.

The ,following papers were presented:

1. *A Non-Algebraic Approach to the Theory of Topological Linear Spaces*, by D. C. Kay, University of Oklahoma.

2. *Faces, Edges, Vertices on Some Polyhedra*, by C. H. Harbison. University of Arkansas at Little Rock.

3. *A Decomposition Theorem for m-Convex Sets*, by Marilyn Breen, University of Oklahoma.

4. *Game Theory: Two Person Zero-Sum Games*, by Susan Howard, State College of Arkansas.

5. *A Probability Model of a Conveyor Belt*, by Patricia L. Guinn, Hendrix College.

6. *The Arithmetic Theory of Continued Fractions*, by Cynthia Jackson, Langston University.

7. *On Using Filters to Generate Ideals in a Complete Direct Product of Fields*, by Kathy Belie, Cameron College.

8. *On When a Group Ring is an Integral Domain*, by David Goggans, Hendrix College.

9. *Control of Highway Vehicles for Minimum Fuel Consumption*, by A. B. Schwarzkopf, University of Oklahoma.

10. *Some Existence Theorems for n-th Order Boundary Value Problems*, by W. G. Kelley, University of Oklahoma.

11. *The Second Variation Functional and Conjugate Points*, by W. F. Denny, University of Oklahoma.

12. *Numerical Transform Methods for Solving the System of Partial Differential Equations*, by P. C. Lai, Tennessee State University.

13. *A Direction for Studying Oscillation Theory of Ordinary Differential Equations of Degree Greater than Two*, by M. S. Keener, Oklahoma State University.

14. *The Category of Sets is Epi-Coreflective in the Category of Relations*, by John Tiller, Hendrix College.

15. *Properties of Group Elements of Order Two*, by Beverly Burton, Langston University.

16. *On the Bernstein Theorem*, by Marilyn Martin, Hendrix College.

17. *The Use of the Fourier Transform in the Resolution of X-ray Spectra from Crystals*, by David Crowe, Oral Roberts University.

18. *A Derivation of Ext* (A, B), by F. F. Simpson, Jr., Hendrix College.

19. *An Introducation to Residually Finite Groups*, by Ernest Brickell, Oklahoma State University.

20. *The Adjunction of Roots to Groups*, by T. K. Teague, Hendrix College.

21. *What to Do with a 3-Manifold*, by Benny Evans, Oklahoma State University.

22. *Some Properties of the Symmetric Difference*, by Carolyn Bean, Hendrix College.

23. *Dilatation of Semigroups*, by Naoki Kimura, University of Arkansas.

24. *On Moments and the Characteristic Function in Hilbert Space*, by R. M. Norton, Oklahoma State University.

25. *On Error Reduction in Gaussian Elimination*, by L. W. Leavell and T. W. Cairns, University of Tulsa.

26. *A Probability Model to Determine Risk on an Instrument Landing System Approach*, by Donald Pate, Oklahoma Christian College.

27. *The Starlike Radius for Classes of Regular Bounded Functions*, by R. W. Sanders, Oral Roberts University.

28. *On Pairwise Compact and Pairwise Paracompact Bitopological Spaces*, by J. A. Wiley, University of Arkansas.

29. *A Note on When a Regular Category is Exact*, by T. H. Fay, Hendrix College.

30. *On the Termination of the Kurosh Lower Radical Construction*, by R. F. Rossa, Arkansas State University.

31. *Chains of Prime Ideals in a Commutative Noetherian Ring*, by E. G. Houston, Jr., University of Oklahoma.

32. *Fermat's Inequality in Polynomial Rings*, by John Woods, Oklahoma Baptist University.

33. *A Decomposition Theorem for a Class of Near-Rings*, by E. F. Ratliff, Jr., University of Oklahoma.

34. *Some Uses of Prime Ideals in the Characterizations of Commutative Rings*, by Charles McCamant, Oklahoma State University.

35. *A Comparative Study of an Advance Organizer in Mathematics to Determine its Effectiveness on Knowledge Acquisition and Retention*, by W. P. Caponecchi, Oscar Rose Junior College.

36. *Tic-Tac-Toe in Polar Coordinates*, by J. B. Browne, Oklahoma State University.

37. *The Effects on Achievement, Retention, and Attitude of an Individualized Instructional Program*

in Mathematics for Prospective Elementary Teachers, by J. T. Kontogianes, Oklahoma City Southwestern College.

38. *Properties of C-Continuous Functions*, by P. E. Long and M. D. Hendrix, University of Arkansas.

39. *Completeness in Topological Spaces*, by T. M. Phillips, University of Oklahoma.

40. *Limit Sets and Their Applications*, by T. R. Hamlett, University of Arkansas.

41. *Group Actions on Manifolds*, by J. W. Maxwell, Oklahoma State University.

42. *Singly Generated Uniform Convergence Spaces*, by A. C. Cochran, University of Arkansas, and R. B. Trail, Oklahoma City University.

43. *Path-Joined Spaces*, by L. L. Herrington, University of Arkansas.

E. K. McLachlan, *Secretary-Treasurer*

RETIRED MATHEMATICIANS

The List of Retired Mathematicians Available for Employment will be published and distributed to subscribers as part of the December 1974 issue of Employment Information for Mathematicians. It is available to non-subscribers who request it from the Mathematical Sciences Employment Register, P. O. Box 6248, Providence, R. I. 02940. Copies will also be available at the annual meeting in Washington, D.C., January 21–27, 1975.

Retired mathematicians interested in being included in future lists should submit the following information: name, date of birth, highest degree earned and where obtained, most recent employment, present address, date available, references, preference for academic or industrial employment, and geographic location preferred. Preprinted forms available from the Providence office may be used for this purpose, although they are not required.

MATHEMATICAL SCIENCES EMPLOYMENT REGISTER

The Empire Room of the Shoreham Hotel in Washington, D.C., will be the location of the Mathematical Sciences Employment Register Open Register during the annual meeting. In order to make it possible for applicants and employers to participate in as many interviews as possible, interviews are scheduled by computer on the basis of preference. Employers are encouraged to send more than one interviewer, making it possible to increase the number of interviews that may be scheduled.

The Open Register will operate for *four* days, Friday through Monday, January 24–27, 1975. Hours of operation will be from 9:00 A.M. to 4:00 P.M. on Friday for registration with interviews scheduled from 9:00 A.M. to 5:40 P.M. on Saturday, Sunday and Monday. The addition of a third day of interviews continues a previously adopted procedure recommended by the Joint Committee on Employment Opportunities in an attempt to expand the interview schedule and to eliminate the necessity for evening interviews.

The system of operation introduced in January 1971 will again be in effect in Washington, D.C. Applicants and employers must be registered for the general Mathematics Meetings before registering for the Open Register. There is no register fee for applicants participating in interview schedules; employers pay a $10 fee. Location of the general meeting registration area, fees, and hours of operation for the registration of participants for the Mathematics Meetings are listed in the program of the annual meeting.

Applicants and employers should plan to attend a short meeting called by the committee for Thursday, January 23, at 4: 30 P.M. in The Forum of the Shoreham Hotel. The purpose of this meeting is to explain Register procedures and the various printed forms which are used. It is hoped that both applicants and employers will attend the session in order to become familiar with the details in advance.

Instruction sheets will be available in the Empire Room registration area at 9: 00 A.M. on Friday; applicants and employers should secure this information as soon as possible. There will be no interviews scheduled for the first day. Appointments will be scheduled only for those people who have actually signed in at the Register and obtained a code number. Requests for appointments can be submitted on *Friday, Saturday, and Sunday only, and these interviews will be scheduled on Saturday, Sunday, and Monday, respectively.*

The Mathematical Sciences Employment Register publishes a bi-monthly bulletin, Employment Information for Mathematicians. Information on this publication and other Register services can be obtained by writing the MSER, P.O. Box 6248, Providence, Rhode Island 02940.

The Mathematical Sciences Employment Register is sponsored by the American Mathematical Society, the Mathematical Association of America, and the Society for Industrial and Applied Mathematics.

ACKNOWLEDGEMENT

The editors wish to thank the following mathematicians who have refereed manuscripts for Volume 81: L. V. Ahlfors, Henry Alder, C. B. Allendoerfer, F. Almgren, Nadhla Al-Salam, Waleed Al-Salam, M. Altman, R. D. Anderson, G. E. Andrews, Tom Apostol, Michael Arbib, Michael Artin, R. B. Ash, R. A. Askey, K. Atkinson, R. G. Ayoub, D. Babbitt, Reinhold Baer, Johnie Baker, T. F. Banchoff, P. T. Bateman, R. G. Bartle, Heinz Bauer, Gilbert Baumslag, G. E. Baxter, R. W. Beals, Ross Beaumont, Edwin Beckenbach, S. Bedrosian, Richard Bellman, John Benedetto, Russell Benson, Sterling Berberian, L. Berkovitz, E. R. Berlekamp, Gerald Berman, Bruce Berndt, A. Bernhart, Lipman Bers, R. H. Bing, Garrett Birkhoff, Frank Birtell, E. Bishop, R. L. Bishop, R. Bixby, J. R. Blum, R. Blumenthal, J. Boardman, R. P. Boas, E. D. Bolker, Carlos Borges, W. Bossert, W. E. Boyce, D. W. Boyd, C. Boyer, Fred Brauer, H. Bray, R. K. Brayton, J. W. Brewer, J. Brezin, John Brillhart, Felix Browder, William Browder, Dennison Brown, Tom Brown, R. C. Buck, J. D. Buckholtz, Peter Bullen, Martha Bunch, H. S. Butts, James Byrnes, P. J. Cahen, Hugh Campbell, Leonard Carlitz, J. B. Carrell, A. Cavaretta, Gulbank Chakerian, E. W. Cheney, Barton Childress, Kai Lai Chung, Dennis Clayton, Earl Coddington, W. J. Cody, Eckford Cohen, Haskell Cohen, Harvey Cohn, R. J. Collens, H. S. Collins, Peter Colwell, W. W. Comfort, Pierre Conner, William Connet, R. L. Cooley, Craig Cordes, H. S. M. Coxeter, Michael Crandall, Douglas Curtis, William Cutler, Jerome Dancis, N. G. De Bruijn, Karel DeLeeuw, Ronald Devore, J. F. Dillon, R. P. Dilworth, Richard Di Prima, Seymour Ditor, John Dixon, J. L. Doob, James Dorroh, Jim Douglas, R. G. Douglas, David Darsin, Dan Drew, Lester Dubins, R. Duffin, James Dugundji, R. A. Duke, John

Durbin, P. L. Duren, Meyer Dwass, P. Dwinger, V. H. Dyson, Michael Edelstein, C. H. Edwards, R. B. Eggleton, P. Eklof, Dennis Estes, Trevor Evans, Ky Fan, R. Farrell, S. Feferman, I. K. Feinstein, N. Findler, Nathan Fine, Daniel Finkbeiner, Harley Flanders, W. H. Fleming, Robert Fraser, Robert Fray, David Freedman, P. J. Freyd, Laszlo Fuchs, D. Fulkerson, Janos Galambos, P. R. Garabedian, Martin Gardner, F. Gehring, S. Gelbart, Leonard Gillman, Robert Gilmer, A. M. Gleason, Casper Goffman, M. Goldberg, Seymour Goldberg, L. J. Goldstein, I. J. Good, Basil Gordon, W. B. Gordon, H. W. Gould, J. Grabiner, R. Graff, Ronald Graham, W. F. Grams, F. K. Greenleaf, David Griffeath, E. Griffin, H. C. Griffith, L. Gross, Emil Grosswald, Branko Grünbaum, H. Guggenheimer, William Gustafson, W. H. Gustafson, Richard Guy, G. Habetler, Peter Hagis, Jr., Jack Hale, Marshall Hall, T. G. Hallam, P. R. Halmos, Larry Harper, T. Hawkins, David Hayes, Emilie Haynesworth, A. Heins, William Heinzer, G. A. Helzer, Melvin Henriksen, H. H. Herda, Israel Herstein, Edwin Hewitt, Graham Higman, John Hildebrant, Paul Hill, L. A. Hinricks, J. G. Hocking, R. E. Hodel, B. Hoffman, R. V. Hogg, A. S. B. Holland, John Horvath, B. Hubbard, Daniel Hughes, J. Humphreys, C. Hunter, H. G. Jacob, Nathan Jacobson, R. C. James, W. E. Jenner, Meyer Jerison, David Johnson, Ellis Johnson, F. B. Jones, Mark Kac, R. Kadison, Irving Kaplansky, Nicholas Kazarinoff, A. D. Keedwell, J. E. Keisler, John Kelingos, J. L. Kelley, J. B. Keller, Paul Kelly, W. E. Kirwan, Joseph Kitchen, M. S. Klamkin, V. L. Klee, A. Knapp, Martin Kneser, D. Knuth, Robert Koch, R. R. Korfage, H. F. Kreimer, E. O. Kreyszig, Chris Lacher, R. Lakein, Eric Langford, J. LaSalle, Jim Lawson, Peter Lax, David Lay, Solomon Leader, Derrick Lehmer, Joseph Lehner, T. Lehrer, Walter Leighton, Kenneth Leland, William LeVeque, Mark Levins, Norman Levinson, Daniel Lewis, S. Lichtenbaum, Tai Feng Lin, D. A. Lind, W. Lindstrom, C. Liu, Edgar Lopez-Escobar, David Lovelady, Morton Lowengrub, W. F. Lucas, R. Luce, Wilhelmus Luxemburg, R. Lyndon, Bernie Madison, Marvin Marcus, A. D. Mathias, J. G. Mauldon, Kenneth May, Charles McArthur, Paul McArthur, R. H. McDowell, O. C. McGehee, J. McGregor, David McGuinness, E. H. McShane, Ralph McWilliams, A. Meir, Gary Meisters, Z. A. Melzak, Nathan Mendelsohn, D. Mermin, Robert Meyers, E. A. Michael, E. P. Miles, R. Millman, E. C. Milner, Edwin Moise, C. Moler, H. Montgomery, J. W. Moon, M. H. Moore, R. T. Moore, C. Morawetz, Y. Moschovakis, W. O. Moser, G. D. Mostow, Francis Murray, Mark Nadel, C. St. J. A. Nash-Williams, Zeev Nehari, John Neuberger, C. Neugebauer, Morris Newman, Helen K. Nickerson, H. Niederreiter, A. Nijenhuis, Ivan Niven, R. Norman, M. Nosal, Phil Novinger, R. J. Nunke, T. H. O'Bierne, Stanley Ogilvy, Jack Ohm, C. D. Olds, J. M. H. Olmsted, B. O'Neill, J. E. Osborn, T. Ostrom, J. C. Owings, Gordon Pall, Thomas Parker, Emanuel Parzen, L. Payne, Daniel Pedoe, Anthony Peressini, B. J. Pettis, R. E. Phillips, G. Piranian, Robert Plemmons, M. D. Plummer, Harry Pollard, George Polya, M. W. Pownall, Bruce Prekowitz, J. J. Price, L. E. Pursell, Ronald Pyke, Frank Quigley, L. Redei, Ray Redheffer, Kenneth Reid, Irving Reiner, James

Retherford, W. Rheinboldt, J. Rhodes, Leonard Richardson, John Riordan, Fred Roberts, A. Robinson, R. M. Robinson, R. T. Rockafeller, E. R. Rodemich, C. A. Rogers, D. P. Roselle, Alex Rosenberg, Moshe Rosenfeld, Kenneth Ross, Gian-Carlo Rota, Klaus Roth, O. Rothaus, Bruce Rothschild, Joseph Rotman, H. L. Royden, Jean E. Rubin, W. H. Ruckle, Mary Ellen Rudin, Walter Rudin, D. C. Russell, H. J. Ryser, T. L. Saaty, G. E. Sacks, P. Sally, Hans Samelson, D. E. Sarason, D. W. Sasser, Richard Savage, Murray Schacher, Eugene Schenkman, A. M. Schinzel, Irwin Schochetman, I. J. Schoenberg, L. Schoenfeld, Dan Scholz, Steve Schonefeld, N. J. Schoonmaker, A. Schwartz, G. P. Scott, W. R. Scott, R. A. Scoville, C. L. Seebeck, R. T. Seeley, Sanford Segal, John Selfridge, Robert Serfling, H. N. Shapiro, J. H. Shapiro, Isador Sheffer, G. C. Shephard, A. L. Shields, J. R. Shoenfield, A. Leigh Silver, R. D. Sinkhorn, F. G. Slaughter, J. A. Sloane, Tom Slobko, Oberta Slotterbeck, Malcolm Smiley, D. A. Smith, William Smith, Ernst Snapper, L. J. Snell, J. Snover, Milton Sobel, Lee Sonneborn, E. H. Spanier, J. Spencer, F. L. Spitzer, B. Spivak, Michael Spivak, Olaf Stackelberg, I. Stakgold, R. P. Stanley, R. G. Stanton, Harold Stark, Emory Starke, Chris Steele, R. Steinberg, W. B. Stenberg, Rothwell Stephens, B. M. Stewart, Wilbur Stiles, Neal Stoltzfus, A. Stone, D. Stone, M. G. Stone, E. G. Straus, K. R. Stromberg, J. D. Swift, W. C. Swift, R. Switzer, Olga Taussky, H. Taylor, Wolfgang Thron, W. R. Tobler, John Todd, R. Troyer, David Trutt, W. Tutte, John Tyler, W. Utz, F. A. Valentine, H. R. van der Vaart, R. Varga, J. E. Vaughan, R. L. Vaught, Al Vekovius, J. W. Vick, Tom Wade, Robert Walker, D. W. Walkup, Robert Warner, Seth Warner, W. Waterhouse, D. Waterman, G. Weiss, J. H. Wells, J. G. Wendel, B. R. Wenner, O. Wesler, James West, W. C. Whitten, D. V. Widder, Albert Wilansky, H. Wilf, A. B. Willcox, Hugh Williams, Leland Williams, Lynn Williams, Stephen Williams, J. H. Williamson, T. J. Willmore, J. A. Wolf, J. Wood, Gordon Woodward, E. M. Wright, Perrin Wright, P. Yale, N. D. Ylvisaker, J. A. Yorke, E. C. Young, Gail Young, Lawrence Zalcman, Hans Zassenhaus, David Zeitlin, Daniel Zelinsky, Janice Zemanek, Anton Zettle, A. D. Ziebur, Antoni Zygmund.

CALENDAR OF FUTURE MEETINGS

Fifty-eighth Annual Meeting, Washington, D. C., January 25–27, 1975.

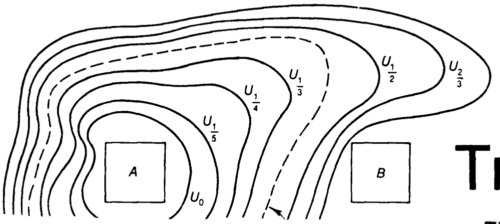
Fifty-fifth Summer Meeting, Western Michigan University, Kalamazoo, August 18–20, 1975.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Editorial Director.

ALLEGHENY MOUNTAIN, Duquesne University, Pittsburgh, Pennsylvania, April 25–26, 1975.
FLORIDA, Manatee Junior College, Bradenton, March 7–8, 1975.
ILLINOIS, Rockford College, Rockford, May 9–10, 1975.
INDIANA
IOWA, Iowa State University, Ames, April 18–19, 1975.
KANSAS
KENTUCKY
LOUISIANA-MISSISSIPPI, Centenary College, Shreveport, Louisiana, February 1975.
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA
METROPOLITAN NEW YORK
MICHIGAN
MISSOURI, Missouri Western State College, St. Joseph, April 18–19, 1975.
NEBRASKA, Nebraska Wesleyan University, Lincoln, April 18–19, 1975.
NEW JERSEY
NORTH CENTRAL, Hamline University, St. Paul, Minnesota, April 28, 1975.
NORTHEASTERN
NORTHERN CALIFORNIA, Menlo College, Menlo Park, February 8, 1975.
OHIO
OKLAHOMA-ARKANSAS, Central State University, Edmond, Oklahoma, April 4–5, 1975.
PACIFIC NORTHWEST
PHILADELPHIA
ROCKY MOUNTAIN, Mesa College, Grand Junction, Colorado, April 11–12, 1975.
SEAWAY
SOUTHEASTERN, University of South Alabama, Mobile, March 21–22, 1975.
SOUTHERN CALIFORNIA
SOUTHWESTERN
TEXAS, Angelo State University, San Angelo, April 1975.
WISCONSIN, University of Wisconsin-Superior, April or May 1975.

FUTURE MEETINGS OF OTHER ORGANIZATIONS

AMERICAN ASSOCIATION FOR THE ADVANCEMENT OF SCIENCE, New York City, January 26–31, 1975.
AMERICAN MATHEMATICAL SOCIETY, Washington, D.C., January 23–26, 1975.
AMERICAN SOCIETY FOR ENGINEERING EDUCATION, Colorado State University, Fort Collins, June 16–19, 1975.
ASSOCIATION FOR COMPUTING MACHINERY, Radisson Hotel, Minneapolis, Minnesota, October 21–23, 1975.
ASSOCIATION FOR SYMBOLIC LOGIC, Shoreham Hotel, Washington, D.C., January 23–24, 1975.
ASSOCIATION FOR WOMEN IN MATHEMATICS, Washington, D.C., January 24, 1975.
FIBONACCI ASSOCIATION
INSTITUTE OF MATHEMATICAL STATISTICS
MU ALPHA THETA
NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS, Washington, D.C., January 25–26, 1975 (joint meeting with MAA).
OPERATIONS RESEARCH SOCIETY OF AMERICA, Chicago, April 30–May 2, 1975.
PI MU EPSILON, Western Michigan University, Kalamazoo, August 19–20, 1975.
SCHOOL SCIENCE AND MATHEMATICS ASSOCIATION
SOCIETY FOR INDUSTRIAL AND APPLIED MATHEMATICS



New——for Topology and Trigonometry

TOPOLOGY: A FIRST COURSE
James R. Munkres, Massachusetts Institute of Technology

A one or two semester introduction for senior or graduate level courses in Topology. Requires only a strong background in Calculus, because an extensive introductory chapter on set theory and logic presents specific background material. Organization is flexible to permit adaptation to instructors' individual needs. Class tested, this new book contains exercises after each section and more than 130 figures. 1975, 432 pp., $14.95

TRIGONOMETRY: A UNITIZED APPROACH
Reuben W. Farley, David A. Schedler, and James A. Wood,
Virginia Commonwealth University

A combination text and programmed text that is proficiency-oriented and self-paced. Suitable for self-study or as a text for terminal or pre-calculus courses. Evaluative, diagnostic exercises and proficiency tests reinforce concepts and permit self-testing. Scripts for audio tapes and a useful **Instructor's Manual** are also available.
1974, 320 pp., $9.25

For further information please contact: Robert Jordan, Dept. J-982, College Division, Prentice-Hall, Englewood Cliffs, New Jersey 07632

APPLICATIONS OF UNDERGRADUATE MATHEMATICS IN ENGINEERING

written and edited by Ben Noble
Mathematics Research Center, U. S. Army, University of Wisconsin

Based on 45 contributions submitted by engineers in universities and industries to the Committee on Engineering Education and the Panel on Physical Sciences and Engineering of CUPM. About 400 pages.

Each member of the Association may purchase one copy of this book for $4.50. Orders with remittance should be addressed to:

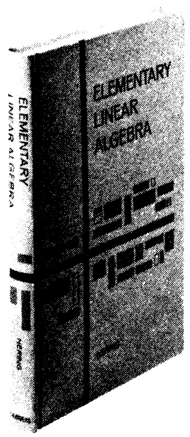
MATHEMATICAL ASSOCIATION OF AMERICA
1225 Connecticut Avenue, NW
Washington, D.C. 20036

Additional copies and copies for nonmembers may be purchased at $9.00 per volume from:

Macmillan Publishing Co., Inc.
Faculty Service Desk
100A Brown Street
Riverside, New Jersey 08075

]compute-ability[

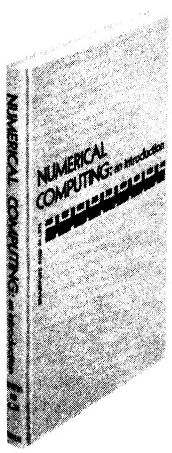
Ensure it with these Saunders texts:



NERING:
Elementary Linear Algebra

A careful integration of conceptual material with concrete computational methods, ideal for sophomores without a background in calculus. Dr. Nering uses the pivot operation to provide a foundation for development of both the numerical methods and conceptual aspects of linear algebra. Students can easily keep up with the text's gradual increase in sophistication through studying the examples and computational procedures which illustrate every abstraction. Significant topics covered include Gaussian elimination, special resolution, upper triangular form, and normal operators.

By **Evar D. Nering,** Arizona State Univ. 375 pp. $11.00. January 1974.



SHAMPINE & ALLEN:
Numerical Computing An Introduction

By examining only a single, effective algorithm for each basic problem, the authors develop fundamental theory understandable to students with only a background in calculus and a modest acquaintance with FORTRAN programming. The authors provide codes implementing the various algorithms presented, along with a number of realistic exercises and solutions that help link theory to practice. Coverage includes floating point arithmetic, numerical analysis, and numerical linear algebra. Department orders of ten or more copies qualify for a free FORTRAN card deck of the codes given in the text.

By **L. F. Shampine,** Sandia Laboratories; and **R. C. Allen, Jr.,** Univ. of New Mexico. 258 pp $11.95. April 1973.

For information, write
Textbook Marketing Div.,

W. B. Saunders Company
West Washington Square
Philadelphia, Pa. 19105